ERRATUM/ADDENDUM TO: FINITUDE GÉOMÉTRIQUE EN GÉOMÉTRIE DE HILBERT

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ABSTRACT. We amend Theorems 1.3 and 1.11 of [CM14a]: Finitude géométrique en géométrie de Hilbert. We seize the opportunity to show that in round Hilbert geometry, geometrical finiteness (gf) is equivalent to cusp-uniform action and to fill some small gaps that appear in two other proofs of [CM14a].

1. THE PUBLISHED STATEMENT

Let Ω be an open subset of $\mathbb{RP}^d$ which is properly convex, i.e. contained in an affine chart of $\mathbb{RP}^d$ where it is bounded. Suppose further Ω is *round*, in the sense that it has $\mathscr{C}^1$-boundary and is strictly convex (any segment contained in the boundary must be reduced to a point). Finally, let Γ be a discrete subgroup of $\mathrm{PGL}_{d+1}(\mathbb{R})$ that preserves Ω. Recall that, using a famous Γ-invariant metric on Ω, denoted d_Ω and called the *Hilbert metric*, one can check that Γ acts properly discontinuously on Ω, and Ω/Γ is called a convex projective orbifold. For more detailed reminders on convex projective geometry and the Hilbert metric, see [CM14a, §2].

The main goal of [CM14a] was to introduce a notion of geometrical finiteness for the action of Γ on Ω and for the underlying orbifold Ω/Γ, and then to study this notion, in particular by giving various characterisations of it, in the spirit of [Bow93, Bow95]. Before we describe these characterisations, let us recall some notations from [CM14a].

The *limit set* of Γ is $\Lambda_\Gamma = \overline{\Gamma x} \smallsetminus \Gamma x \subset \partial\Omega$, which is independent of the choice of an $x \in \Omega$. The *convex core* $\mathscr{C}(\Lambda_\Gamma)$ is the convex hull in Ω of the limit set. More generally, we will use the notation $\mathscr{C}(\cdot)$ to denote convex hulls.

Note that for any $p \in \mathbb{P}(\mathbb{R}^{d+1})$, the projective space $\mathbb{P}(\mathbb{R}^{d+1}/p)$ identifies with the space of lines of $\mathbb{P}(\mathbb{R}^{d+1})$ containing p. If $p \in \partial\Omega$ then we denote by $\mathscr{D}_p(\Omega)$ the space of lines containing p and intersecting Ω. Since $\partial\Omega$ is $\mathscr{C}^1$ at p, the space of lines $\mathscr{D}_p(\Omega)$ identifies with the affine space $\mathbb{A}_p = \mathbb{P}(\mathbb{R}^{d+1}/p) \smallsetminus \mathbb{P}(T_p\partial\Omega/p)$, where $T_p\partial\Omega$ is the tangent space to $\partial\Omega$ at p. The map $s_p : \overline{\Omega} \smallsetminus \{p\} \to \mathbb{A}_p$ given (through the former identification) by $q \mapsto (pq)$, will be called the *stereographic projection from p*.

Recall that $p \in \partial\Omega$ is a *parabolic point* if its stabilizer Γ_p is infinite and parabolic, which is equivalent to saying that $\gamma_n x \to p$ for any injective sequence $(\gamma_n)_n \subset \Gamma_p$ and any $x \in \Omega$ (see [CM14a, §3.5] for more characterisations of parabolicity); this implies $p \in \Lambda_\Gamma$. A parabolic point is *bounded* if the action of Γ_p on $\Lambda_\Gamma \smallsetminus \{p\}$ is cocompact. A parabolic point is *uniformly bounded* if the action of Γ_p on $s_p(\mathscr{C}(\Lambda_\Gamma))$ is cocompact. Note that $s_p(\mathscr{C}(\Lambda_\Gamma))$ is the convex hull in $\mathbb{A}_p$ of $s_p(\Lambda_\Gamma \smallsetminus \{p\})$, so uniformly bounded implies bounded.

2010 *Mathematics Subject Classification.* 20F55, 20F65, 20H10, 22E40, 51F15, 53C50, 57M50, 57S30.

Finally, a point $p \in \partial\Omega$ is called *conical* if there are $x \in \Omega$ and $(\gamma_n)_n \subset \Gamma$ such that $\gamma_n x \to p$ and $\gamma_n x$ stays at bounded Hilbert distance from the ray $[x,p)$; this implies $p \in \Lambda_\Gamma$.

We defines the following properties for the action of Γ on Ω:

(gf) : Every point of Λ_Γ is either conical or bounded parabolic.

(GF) : Every point of Λ_Γ is either conical or uniformly bounded parabolic.

(HC) : (gf) holds and for each parabolic point p, the group Γ_p is conjugate into $\mathrm{O}_{d,1}(\mathbb{R})$.

(TF) : There exists a Γ-invariant family $\mathscr{P}$ of points $p \in \Lambda_\Gamma$, a family of standard[1] regions R_p centered at p, such that $(R_p)_p$ is (Γ,Γ_p)-precisely equivariant (see Definition 6.1), the action of Γ on $\overline{\Omega} \smallsetminus (\Lambda_\Gamma \cup \bigcup_p R_p)$ is cocompact.

(PEC) : The thick part[2] of the convex core is compact.

(PNC) : The non-cuspidal part[3] of the convex core is compact.

(CU) : The action $\Gamma \curvearrowright \Omega$ is *cusp-uniform* i.e. there exists a Γ-invariant family $\mathscr{P}$ of points $p \in \Lambda_\Gamma$, a family of horoballs H_p center at p, such that $(H_p)_p$ is (Γ,Γ_p)-precisely equivariant, and the action of Γ on $\mathscr{C}(\Lambda_\Gamma) \smallsetminus \bigcup_p H_p$ is cocompact.[4]

$(\mathrm{VF})_R$: Γ is finitely generated and the uniform R-neighborhood (for the Hilbert metric) of the convex core $\mathscr{C}(\Lambda_\Gamma)/\Gamma$ is of finite volume.[5]

$(\mathrm{VF})_0$: Γ is finitely generated and the convex core $\mathscr{C}(\Lambda_\Gamma)/\Gamma$ is of finite volume for the Hilbert volume form from $\Omega \cap \mathrm{Span}(\mathscr{C}(\Lambda_\Gamma))$.

(Hyp) : The convex core $\mathscr{C}(\Lambda_\Gamma)$ is Gromov-hyperbolic[6] for the Hilbert metric of Ω.

(Gen) : The limit set Λ_Γ spans $\mathbb{RP}^d$, or its dual spans the dual projective space.[7]

Remark 1.1. Note that for all $R, R' > 0$, it is easy to check that $(\mathrm{VF})_R$ and $(\mathrm{VF})_{R'}$ are equivalent and imply $(\mathrm{VF})_0$. However, $(\mathrm{VF})_0$ does not implies $(\mathrm{VF})_1$, for example suppose we have two discrete subgroups $G,H < \mathrm{Isom}(\mathbb{H}^d)$ such that G stabilizes a proper subspace $V < \mathbb{H}^d$ and H is generated by a loxodromic element whose axis A is disjoint from V. If $G' < G$ and $H' < H$ are sufficiently small finite-index subgroups then one can play ping-pong (see e.g. [Mas88]) to prove that the group Γ generated by G' and H' is discrete, is isomorphic to the free product $G * H$, the convex cores $C_1 \subset V/G$ and $C_2 \subset A/H$ of G and H embed (disjointly) in the convex core $C \subset \mathbb{H}^d/\Gamma$ of Γ, and, fixing a point $x_0 \in C$, there is a constant $K > 0$ such that any point $x \in C$ at distance r from x_0 is at distance at most Ke^{-r} from $C_1 \sqcup C_2$. In particular, if V has codimension 1 and V/G is an abelian cover of a closed hyperbolic manifold, then one can check that C has finite volume, as was mentioned to us by D. Cooper.

[1] We do not recall the technical definition of standard parabolic regions since it will not be used here, see [CM14a, §7.3].

[2] The thick part consists of the projections of $x \in \Omega$ such that $\{\gamma \in \Gamma \mid d_\Omega(x,\gamma x) < \varepsilon\}$ generates a finite group, given a sufficiently small ε, see Section 8.1.

[3] For us the non-cuspidal part is the union of the thick part with the components of the thin parts that consists of tubular neighborhoods of short geodesics, see [CM14a, §6.2].

[4] See Section 7.1 for reminders on horoballs.

[5] Here our volume form is the Hausdorff measure of the Hilbert metric, see [CM14a, §2.1].

[6] Recall that a geodesic metric space is Gromov-hyperbolic if for some δ all geodesic triangles are δ-thin: any side is in the δ-neighborhood of the union of the two other sides, see e.g. [BH99, §III.H.1].

[7] Recall that Γ also preserves a properly convex open set Ω^* in the projective space of linear forms on $\mathbb{R}^{d+1}$, and hence has a limit set there too, see [CM14a, §2.3].

Note also, that the assumption Γ finitely generated cannot be removed from (VF)$_R$ has shown by [Ham98]. One can see (Gen) as a genericity assumption. It holds when Γ is irreducible, and a fortiori when it is Zariski-dense.

Those properties were linked by the following theorem, *which is wrong*.

Theorem 1.2. [CM14a, Thm. 1.3 & 1.11, Prop. 1.4] *For any* Γ *acting on a round convex* Ω*, the assertions* (GF)*,* (TF)*,* (HC)*,* (PEC)*,* (PNC) *,* (VF)$_1$ *and* ((gf)&(Hyp)) *are equivalent.*
Morever they all imply (gf) *but are not equivalent to it.*

The former proof used the following pattern, which is also recapitulated in Figure 1. We indicated in brackets where to find the proof in the original paper.

- (GF) ⇒ (TF) [Prop. 7.21 & 7.23], whose proof is correct.
- (TF) ⇒ (PEC) ⇒ (PNC)[8] [§8.2], whose proofs are corrects.
- (PNC) ⇒ (GF) [§8.2], this proof is wrong, and in fact the statement is wrong. However, the implication (PNC) ⇒ (gf) is true. The proof of the implication (PNC) ⇒ (gf) appears as a step in (PNC) ⇒ (GF). The proof of this implication is incomplete but can be corrected using the same strategy as the original paper. We will correct it with a slightly different strategy.
- (GF) ⇒ (VF)$_1$ [§8.3], whose proof is incomplete as only (VF)$_0$ is proved; we will fix this.
- (VF)$_1$ ⇒ (PEC) [§8.3], whose proof is correct (as stated the Lemma 8.5 used in the proof is incorrect but the proof is easy to fix, see Remark 1.5).
- (GF) ⇒ (HC) [Cor. 7.18], which is wrong in full generality but true if we ask (Gen); we will fix this.
- (HC) ⇒ (GF) [Prop. 7.21[9]], whose proof is correct.
- (GF) ⇒ ((gf)&(Hyp)) [Th. 9.1] is true but the proof has a small gap; we will fix this.
- ((gf)&(Hyp)) ⇒ (GF) [Th. 9.1] is false (Lemma 9.2 being wrong).
- (gf) $\not\Rightarrow$ (GF) [Prop. 10.6] whose proof by counter-example is correct; in fact this counter example satisfies (Hyp) and (VF)$_1$ as we explain in Section 13 and in [BM].

Remark 1.3. (The error in ((gf)&(Hyp)) ⇒ (GF)). The error is hidden in the sentence: “On peut identifier l’espace des droites $\mathscr{D}_p(\mathscr{C}(\Lambda_\Gamma))$ à sa trace sur l’horosphère $\mathscr{H}$.” meaning that we can identified $\mathscr{D}_p(\mathscr{C}(\Lambda_\Gamma))$ and $\mathscr{H}\cap\mathscr{C}(\Lambda_\Gamma)$, which is wrong. In fact, if $\mathscr{H}_t$ is family of horosphere such that the corresponding family of horoball decreases, then stereographic projection on $\mathbb{A}_p$ of $\mathscr{H}_t\cap\mathscr{C}(\Lambda_\Gamma)$ is a closed subset F_t of $\mathbb{A}_p$. For each t, the group Γ_p acts cocompactly on F_t but this family of closed subset is increasing and Γ_p may not act cocompactly on their union.

Remark 1.4 (The error in (PNC) ⇒ (GF)). The error is hidden in the sentence “nécessairement uniformément borné puisque la partie non cuspidale du cœur convexe est compacte.”, implicitly the authors had in mind that the intersection $\partial\Omega_\varepsilon(\Gamma_p)\cap\mathscr{C}(\Lambda_\Gamma)$ can be identified with $\mathscr{D}_p(\mathscr{C}(\Lambda_\Gamma))$. Similarly to the above mistake, the stereographic projection on $\mathbb{A}_p$ of $\partial\Omega_\varepsilon(\Gamma_p)\cap\mathscr{C}(\Lambda_\Gamma)$ is a closed subset E_ε of $\mathbb{A}_p$. For each ε, the group Γ_p acts cocompactly on

[8]Typo in the sentence “Preuve de (TF) ⇒ (PNC) ⇒ (PEC) ”, first proof of section 8.2. It should have been written: ”Preuve de (TF) ⇒ (PEC) ⇒ (PNC) ”. Note that the implication (PNC) ⇒ (PEC) is trivial since the thick part is a closed subset of the non-cuspidal part.

[9]Typo in the hypotheses: one needs to assume p is *bounded* parabolic.

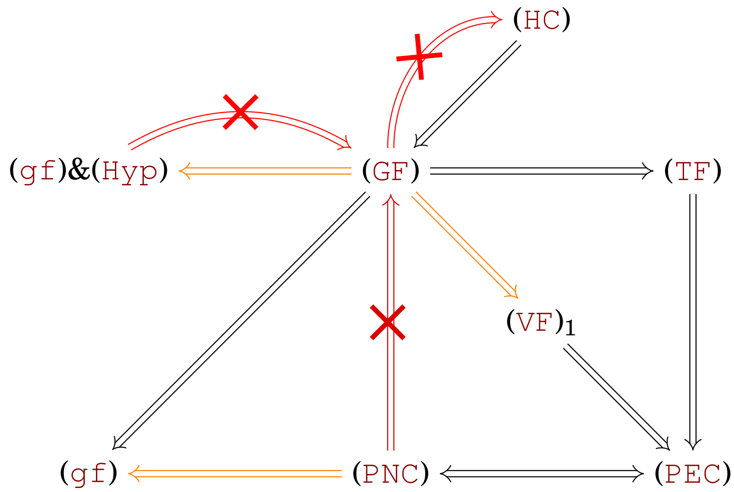

FIGURE 1. Old pattern of implications: black arrows were correctly proved in the former paper, the red arrows are mistakes of the former paper, the orange arrows need to be fixed (they are badly written, are incomplete, or have mistakes in the former paper).

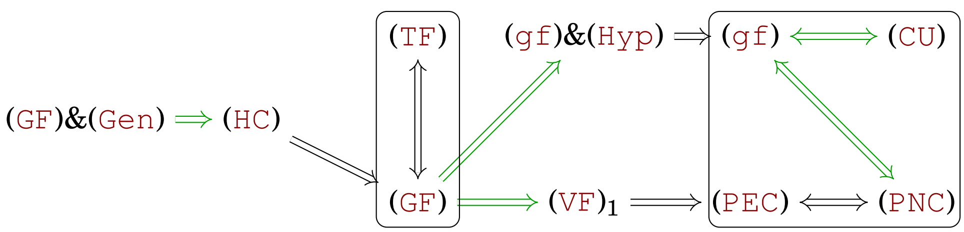

FIGURE 2. New pattern of implications: black arrows where correctly proved in the former paper, the green ones repair the mistakes of the former paper.

F_ε but this family of closed subset is increasing (as $\varepsilon \to 0$) and Γ_p may not act cocompactly on their union.

Remark 1.5. ((VF)$_1$ $\Rightarrow$ (PEC)) First, for any $R > 0$ there is a constant $c_R > 0$ independent of the convex Ω and of the point $x \in \Omega$ such that $\mathrm{Vol}_\Omega(B(x,R)) \geqslant c_R$ ([CV06, Thm.12] or see e.g. [CM14a, Lem.8.4]). Second, if x is in the ε-thick part then the ball $B(x,\varepsilon)$ of Ω embeds in Ω/Γ. Hence, if x is in the convex core then $B(x,\varepsilon)$ embeds in the 1-neighborhood of the convex core $\mathscr{C}(\Lambda_\Gamma)/\Gamma$ (assuming that $\varepsilon < 1$). Crampon and the second author conclude erroneously that such a ball embeds in $\mathscr{C}(\Lambda_\Gamma)/\Gamma$.

Leading to the erroneous conclusion that if the action of Γ satisfies (VF)$_0$ then the ε-thick part of the convex core can contain only finitely many disjoint balls of radius ε, hence is compact. When, in fact, one needs to assume (VF)$_1$ to conclude that the ε-thick part of the convex core can contain in its 1-neighborhood only finitely many disjoint balls of radius ε, and hence must be compact.

2. A CORRECT STATEMENT

In the present paper we prove the following result, which corrects Theorem 1.2. Figure 2 shows the new pattern of the proof.

Theorem 2.1. *Let Ω be a round convex of $\mathbb{RP}^d$ and $\Gamma \leqslant Aut(\Omega)$. Then:*

1. *((GF)&(Gen))* $\Longrightarrow$ (HC) $\Longrightarrow$ (GF) $\Longleftrightarrow$ (TF).
2. (PEC) $\Longleftrightarrow$ (PNC) $\Longleftrightarrow$ (gf) $\Longleftrightarrow$ (CU).
3. (GF) $\Longrightarrow$ $(\texttt{VF})_1$ $\Longrightarrow$ (gf).
4. (GF) $\Longrightarrow$ (gf)&(Hyp) $\Longrightarrow$ (gf).

A counter-example to the reciprocal of the implication (HC) $\Longrightarrow$ (GF) is given in Section 9. It is trivial to find an example satisfying (HC) but not (Gen). We will provide in a separate article counter-examples to all implications which are not equivalence in Theorem 2.1.(3-4), see Section 13 for an overview.

Theorem 2.2 ([BM]). *Let Ω be a round convex set of $\mathbb{RP}^d$ and $\Gamma \leqslant Aut(\Omega)$. Then:*

1. *The condition* $(\texttt{VF})_1$ *does not imply the condition* (GF).
2. *The condition* (gf)&(Hyp) *does not imply the condition* (GF).
3. *The condition* (gf) *does not imply the condition* $(\texttt{VF})_1$.
4. *The condition* (gf) *does not imply the condition* (gf)&(Hyp).

Indeed, for any non-uniform lattice Γ of $\mathrm{SL}_2(\mathbb{R})$, if $\rho : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}_5(\mathbb{R})$ is the 5-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{R})$ then, there exists $\rho(\Gamma)$-invariant round convex domains Ω_0, Ω_1 of $\mathbb{RP}^4$ such that :

- ***a**. $\rho(\Gamma) \curvearrowright \Omega_0, \Omega_1$ are (gf), but not (GF).*
- ***b**. The convex core of $\Omega_0/\rho(\Gamma)$ is of finite (nonzero) volume and*
- ***c**. $\mathscr{C}(\Lambda_\Gamma)$ is Gromov-hyperbolic for the Hilbert metric of Ω_0.*
- ***d**. While the convex core of $\Omega_1/\rho(\Gamma)$ is of infinite volume and*
- ***e**. $\mathscr{C}(\Lambda_\Gamma)$ is not Gromov-hyperbolic for the Hilbert metric of Ω_1.*

As we mentioned, in [CM14a, Prop. 1.4], the authors exhibit examples of pairs (Ω, Γ) which satisfies (gf) but not (GF). We use those examples to show the existence of Ω_0 in Theorem 2.2. The construction of Ω_1 is more involved.

Remark 2.3. Fix a discrete subgroup $\Gamma \leqslant \mathrm{Aut}(\Omega)$ preserving at least one round convex set of the projective space, such that Γ is non-elementary (not virtually nilpotent). As one can see from Theorem 2.2, the properties $(\texttt{VF})_0$, $(\texttt{VF})_1$ and (Hyp) depend on the choice of the Γ-invariant round convex set Ω: they might hold for one domain but not for another.

However, all the other properties studied in this paper are independent of the choice of Ω. This comes from the classical fact that the limit set Λ_Γ is independent of Ω. Indeed recall that an element $g \in \mathrm{SL}_{d+1}(\mathbb{R})$ is *proximal* if it has an attracting fixed point in $\mathbb{RP}^d$. Then one can check that the limit set Λ_Γ is the closure in $\mathbb{RP}^d$ of the set of attracting fixed points of proximal elements of Γ, also called proximal limit set [10]. From the definitions, one

[10] An element γ is proximal if and only if it is a hyperbolic automorphism of Ω in the sense of the classification theorem [CM14a, Th. 3.3]. This theorem also easily implies that any attracting fixed point of a proximal element is in the limit set, so the proximal limit set is contained in the limit set. To prove the other inclusion, consider p in the limit set and p_0 any point in the proximal limit set. Then $\gamma_n x \to p \in \partial\Omega$ for some sequence γ_n.

immediately sees that the properties (gf), (GF), (HC) and (Gen) are independent of Ω, and Theorem 2.1 implies the properties (TF), (PEC), (PNC) and (CU) are independent of Ω too.

The second author warmly thanks the first author for pointing out to him the mistake in the former paper and his help to find and write the proper statement. The second author also thanks A. Zimmer for pointing out to him the second point of Theorem 2.2 using the same Ω_0 that we will use. The authors thank B. Fléchelles and D. Cooper for interesting discussions and useful comments.

3. Plan of proof

The main results are (gf) $\iff$ (CU) and (gf) $\iff$ (PNC), whose proofs are extremely similar, so our goal is to prove them both at the same time, by proving a more general result.

In Section 4 we establish a short independent lemma, useful in the proofs of (gf) $\iff$ (CU) and (gf) $\iff$ (PNC).

In Section 5, we prove that (gf) is equivalent to a whole family of properties. More precisely, we show that, given any precisely equivariant family of star domains that satisfy a certain convexity condition, asking Γ to act geometrically finitely on $\partial\Omega$ (i.e. asking (gf)) is equivalent to asking that Γ acts cocompactly on the complement in the convex core of the family of star domains.

In Section 7, we check that horoballs satisfy the above convexity condition (because horoballs are convex), and obtain the equivalence (gf) $\iff$ (CU) as a consequence. The condition (CU) is not present in the original paper [CM14a] but it should have been, so we seize the opportunity to give a proof. A proof of the implication (gf) $\implies$ (CU) is also given in [BT, Prop. 3.3]

In Section 8, we check that the star domains obtained in the thick-thin decomposition of Ω also satisfy the above-mentioned convexity condition, and obtain the equivalence (gf) $\iff$ (PNC) as a consequence.

In Section 9 we give a counterexample to (GF) $\Rightarrow$ (HC), and then in Section 10 we prove that (GF) $\Rightarrow$ (HC) holds under the additional genericity assumption (Gen).

In Sections 11 and 12 we fill in the gaps in the proofs of respectively (GF) $\Rightarrow$ $(\mathrm{VF})_1$ and (GF) $\Rightarrow$ (Hyp).

The only missing implication of Theorem 2.1 is the implication (TF) $\Rightarrow$ (GF). This implication is not present in the original paper [CM14a]. Because it was done through the erroneous implication (PNC) $\Rightarrow$ (GF). Nevertheless, it is easy to check that (TF) $\Rightarrow$ (gf) (for instance using Proposition 6.3 below) and the proof of [CM14a, Prop. 7.21] shows that if (TF) holds then all bounded parabolic points are in fact uniformly bounded.

4. Dirichlet domain and conical limit points

This section only contains a short independent lemma saying that Dirichlet domains do not accumulate on conical limit point. The argument is standard, see for instance [Rob03,

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Up to extracting $\gamma_n y \to p$ for any $y \in \partial\Omega \smallsetminus \{q\}$, for some q (see[CM14a, Prop. 4.8]). Since Γ is non-elementary there is $\alpha \in \Gamma$ such that $\alpha p_0 \neq q$, so $\gamma_n \alpha p_0 \to p$, so p is in the proximal limit set.

Prop.1.10]. This lemma, as well as Dirichlet domains and the ideas in [Rob03, Prop.1.10], will be used to prove that (gf) implies cocompactness properties (see Proposition 6.2).

Let Ω be a round convex subset of $\mathbb{RP}^d$, $\Gamma \subset \mathrm{Aut}(\Omega)$ discrete. If o is a point of Ω, the *Dirichlet domain based at* o is

$$\mathscr{D} = \{x \in \Omega \,|\, \forall \gamma \in \Gamma, d_\Omega(x,o) \leqslant d_\Omega(x,\gamma o)\}$$

Note that $\mathscr{D}$ is a closed subset of Ω, and that the translates of $\mathscr{D}$ by Γ cover Ω.

Using the fact that Ω is strictly convex, one can check that if Γ is torsion-free then the translates of $\mathscr{D}$ by Γ have disjoint interiors and that those interiors have the form $\{x \in \Omega \,|\, d_\Omega(x,o) < d_\Omega(x,\gamma o), \forall \gamma \in \Gamma \smallsetminus \{1\}\}$, but we will not need this fact. Note that the translates of $\mathscr{D}$ by Γ may intersect on their interiors if Ω is not strictly convex: this happens for instance in the case of a $\mathbb{Z}$-action on a triangle generated by a diagonal 3×3 matrix with diagonal entries $2, 2, 1/4$.

In the following lemma we check that Dirichlet domains cannot contain conical limit points at their boundary at infinity. Compare with [CM14a, Lem. 8.2], which has a similar result for a different kind of fundamental domains.

Lemma 4.1. *Let Ω be a round convex subset of $\mathbb{RP}^d$, $\Gamma \subset Aut(\Omega)$ discrete. If $p \in \partial\mathscr{D} \cap \partial\Omega$ then p is not a conical limit point.*

Proof. There exists $(x_n)_n \in \mathscr{D}^{\mathbb{N}}$ such that $x_n \to p$. Assume p is a conical limit point. Then, there exists also $(\gamma_m)_m \in \Gamma^{\mathbb{N}}$ such that $\gamma_m(o)$ converges conically to p, i.e. there exists $(y_m)_m \subset [o,p)$ tending to p such that $(d_\Omega(\gamma_m o, y_m))_m$ is bounded. Since for any m we have

$$b_p(o,y_m) = \lim_{y\in[y_m,p)\to p} d_\Omega(o,y) - d_\Omega(y_m,y) = d_\Omega(o,y_m),$$

thus

$$b_p(o,\gamma_m(o)) = b_p(o,y_m) + b_p(y_m,\gamma_m(o)) \geqslant d_\Omega(o,y_m) - d_\Omega(y_m,\gamma_m(o)) \underset{m\to\infty}{\longrightarrow} +\infty$$

However, for any m, since $x_n \to p$, we also have:

$$b_p(o,\gamma_m(o)) = \lim_{n\to+\infty} d_\Omega(o,x_n) - d_\Omega(\gamma_m(o),x_n).$$

Since $x_n \in \mathscr{D}$ one has:

$$d_\Omega(o,x_n) - d_\Omega(\gamma_m(o),x_n) \leqslant 0.$$

So $b_p(o,\gamma_m(o)) \leqslant 0$ for any m, absurd. □

5. Cocompactness at parabolic points

5.1. **Strongly star-shaped domains.** We will need a class of well-behaved domains of Ω centered at parabolic points that encompasses both horoballs (see Section 7) and components of the thin part (see Section 8). Since the components of the thin part are not necessarily convex, we will use the larger class of star domains. Unfortunately star-shapedness alone will be too weak for our purposes: we will need an important extra convexity assumption which will be stated directly inside Lemma 5.3. Roughly, a star domain B satisfies this condition if it contains the convex hull of a smaller star domain.

Definition 5.1. Let Ω be a round convex subset of $\mathbb{RP}^d$ and $p \in \partial\Omega$. An open subset $B \subset \Omega$ is called *strongly star-shaped* at p if for every $x \in \partial\Omega \smallsetminus \{p\}$, the interval (x,p) intersects ∂B at exactly one point $y \in \Omega$, the interval (y,p) is contained in B, and the interval (x,y) is outside of $\overline{B}$.

Observe that this implies that

- **(i)** B is star-shaped at p,
- **(ii)** $\partial B \smallsetminus \{p\} \subset \Omega$,
- **(iii)** $\partial\Omega \smallsetminus \{p\}$ maps homeomorphically onto ∂B via the stereographic projection (in a G-equivariant way if B is invariant under some $G \subset \mathrm{Stab}(p) \subset \mathrm{Aut}(\Omega)$), and
- **(iv)** the stereographic projection from $\overline{\Omega} \smallsetminus (B \cup \{p\})$ to ∂B is surjective, continuous, G-equivariant and proper.

In other words, a strongly star-shaped open subset of Ω at p is the "interior" of a hypersurface of Ω that maps homeomorphically onto $\partial\Omega \smallsetminus \{p\}$ via the stereographic projection.

5.2. **A key cocompactness lemma about parabolic subgroups.** In this section we prove a cocompactness result for the parabolic subgroups, inspired by the argument in [BZ21, Prop. 8.12].

First we recall the following more classical properness result about parabolic subgroup. Note that in the reference we are using there is a typo: they define $\mathcal{O}_\Gamma := \Omega \smallsetminus \Lambda_\Gamma$ whereas it should be $\mathcal{O}_\Gamma := \overline{\Omega} \smallsetminus \Lambda_\Gamma$.

Fact 5.2 ([CM14a, Lem.4.5])**.** *Let Ω be a round convex subset of $\mathbb{RP}^d$ and $\Gamma \subset Aut(\Omega)$ discrete. Then Γ acts properly discontinuously on $\overline{\Omega} \smallsetminus \Lambda_\Gamma$.*

In particular, applying this to a parabolic subgroup Γ_p fixing $p \in \Lambda_\Gamma$ we get that Γ_p acts properly discontinuously on $\overline{\Omega} \smallsetminus \{p\}$.

Now comes the key cocompactness lemma. The formulation involving finite-index subgroups of the stabiliser of the parabolic point is an unfortunate necessary technicality. It will be used in Section 8.

Lemma 5.3. *Let Ω be a round convex subset of $\mathbb{RP}^d$, $\Gamma \subset Aut(\Omega)$ discrete non-elementary, and $p \in \Lambda_\Gamma$ be a bounded parabolic fixed point with stabilizer Γ_p. Consider $G \subset \Gamma_p$ a finite index subgroup and $B^- \subset B^+ \subset \Omega$ two G-invariant strongly star-shaped open subsets at p such that the convex hull of B^- is contained in B^+.*

Then the action of G on $\partial B^+ \cap \mathscr{C}(\Lambda_\Gamma)$ is cocompact.

Proof. Since p is bounded parabolic and G has finite index in Γ_p, there exists $K \subset \Lambda_\Gamma \smallsetminus \{p\}$ compact such that $G \cdot K = \Lambda_\Gamma \smallsetminus \{p\}$. Let L be the set of points $x \in \overline{\Omega}$ such that $[x,q]$ does not intersect B^- for some $q \in K$. To finish this proof, it suffices to check that $L' = L \cap \partial B^+ \cap \mathscr{C}(\Lambda_\Gamma)$ is compact and that $G \cdot L' = \partial B^+ \cap \mathscr{C}(\Lambda_\Gamma)$.

First we check L' is compact. It is clear that L is closed in $\overline{\Omega}$, and hence compact: Let $(x_n)_n \in L^{\mathbb{N}}$ such that $x_n \to x$, hence $[x_n,q_n] \cap B^- = \emptyset$ for some $q_n \in K$. Up to extracting, we can assume $q_n \to q \in K$, and to conclude that $x \in L$, we note that $[x,q]$ cannot intersect the open set B^-, otherwise the segments $[x_n,q_n]$ would also intersect it for n large enough.

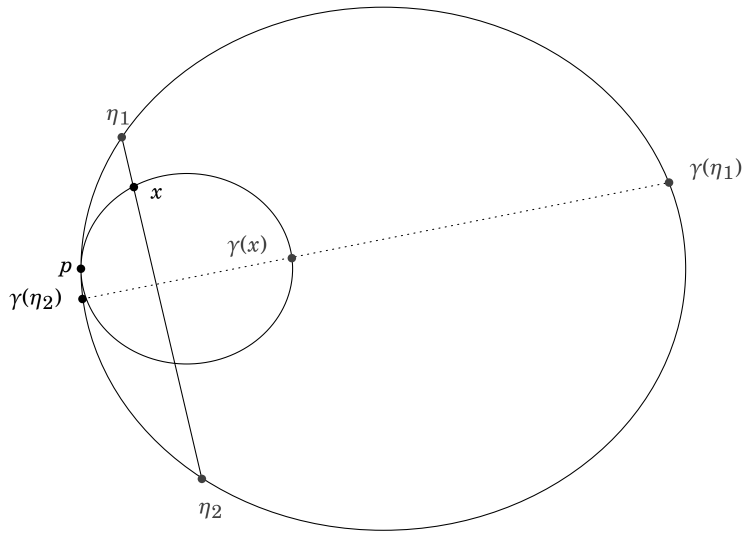

FIGURE 3. Illustration of the proof of Lemma 5.3
(For simplicity, x is here in the convex hull of only two points of the limit set)

Note that $\partial B^+ \cap \overline{\mathscr{C}(\Lambda_\Gamma)} = \partial B^+ \cap \mathscr{C}(\Lambda_\Gamma) \cup \{p\}$ since B^+ is strongly star-shaped (see **(ii)** in Definition 5.1), and p is not in L since B^- is strongly star-shaped. Thus

$$L' = L \cap \partial B^+ \cap \mathscr{C}(\Lambda_\Gamma) = L \cap \partial B^+ \cap \overline{\mathscr{C}(\Lambda_\Gamma)}$$

is compact.

It remains to check that for any $x \in \partial B^+ \cap \mathscr{C}(\Lambda_\Gamma)$ there exists $g \in G$ such that $gx \in L'$. Since $x \in \mathscr{C}(\Lambda_\Gamma)$, there exists $(\eta_i)_{i=1,\dots,d+1}$ in $\Lambda_\Gamma \smallsetminus \{p\}$ such that x is in the convex hull of p and the $(\eta_i)_{i=1,\dots,d+1}$. We claim that there exists i such that $[x,\eta_i] \cap B^- = \varnothing$. By contradiction, if this is not the case, then for each i, there exists $x_i \in [x,\eta_i] \cap B^-$. Since B^- is strongly star-shaped at p, there also exists $y \in [x,p] \cap B^-$. One can then check that x is in the convex hull of $\{x_i\}_i \cup \{y\}$, and hence that x lies in B^+ by our assumption that the convex hull of B^- is contained in B^+. This contradicts $x \in \partial B^+$.

Since $G \cdot K = \Lambda_\Gamma \smallsetminus \{p\}$, there is $g \in G$ such that $g\eta_i \in K$. Then $[gx, g\eta_i]$ does not intersect B^-, so $gx \in L'$, which concludes the proof. □

Lemma 5.4. *Let Ω be a round convex subset of $\mathbb{RP}^d$, $\Gamma \subset Aut(\Omega)$ discrete non-elementary, and $p \in \Lambda_\Gamma$ be a bounded parabolic fixed point with stabilizer Γ_p. Consider $G \subset \Gamma_p$ a finite index subgroup and $B \subset \Omega$ a G-invariant strongly star-shaped open subset at p such that the action of G on $\partial B \cap \mathscr{C}(\Lambda_\Gamma)$ is cocompact.*

Then the action of G on $\overline{\mathscr{C}(\Lambda_\Gamma)} \smallsetminus (B \cup \{p\})$ is cocompact.

Proof. This is an immediate consequence of **(iv)** and the fact that the image of $\overline{\mathscr{C}(\Lambda_\Gamma)} \smallsetminus (B \cup \{p\})$ under the stereographic map from $\overline{\Omega} \smallsetminus (B \cup \{p\})$ to ∂B is exactly $\partial B \cap \mathscr{C}(\Lambda_\Gamma)$. □

6. THE GENERAL RESULT

In this section we prove a general result that (gf) is equivalent to a whole family of properties which encompasses (CU) and (PNC). As a consequence, the equivalences (gf)⇔(CU) and (gf)⇔(PNC) will be particular cases of the results of this section.

Let us recall the definition of $(\Gamma,(\Gamma_p)_p)$-precisely equivariant.

Definition 6.1. Let $\Omega \subset \mathbb{RP}^d$ be a round convex subset, $\Gamma \subset \mathrm{Aut}(\Omega)$ a discrete subgroup and $\mathscr{P} \subset \partial\Omega$ a Γ-invariant subset. A $(\Gamma,(\Gamma_p)_p)$-*equivariant* family $(B_p)_{p\in\mathscr{P}}$ of domains of Ω is a family of domains such that $\gamma B_p = B_{\gamma p}$ for all $p \in \mathscr{P}$ and $\gamma \in \Gamma$.

It is called $(\Gamma,(\Gamma_p)_p)$-*precisely equivariant* if moreover $\overline{B}_p \cap \overline{B}_q = \varnothing$ for all distinct $p \neq q \in \mathscr{P}$.

6.1. Cocompactness consequences of (gf).

We can now state and prove one of the two main results of this section. The proof is standard, see for instance [Rob03, Prop. 1.10].

Proposition 6.2. *Let Ω be a round convex subset of $\mathbb{RP}^d$ and $\Gamma \subset Aut(\Omega)$ discrete, non-elementary, and geometrically finite on $\partial\Omega$ (Assumption (gf)). Let $\mathscr{P} \subset \Lambda_\Gamma$ be the set of parabolic points, and denote by $\Gamma_p \subset \Gamma$ the stabilizer of each $p \in \mathscr{P}$. Consider a $(\Gamma,(\Gamma_p)_p)$-equivariant family $(B_p)_{p\in\mathscr{P}}$ of domains. Suppose that Γ_p acts cocompactly on $\overline{\mathscr{C}(\Lambda_\Gamma)} \smallsetminus (B_p \cup \{p\})$ for every $p \in \mathscr{P}$. Then the action of Γ on $\mathscr{C}(\Lambda_\Gamma) \smallsetminus \bigcup_p B_p$ is cocompact.*

Proof. We fix a point $o \in \Omega$ and consider the (Dirichlet) domain:

$$\mathscr{D} = \{x \in \Omega \mid \forall \gamma \in \Gamma, d_\Omega(x,o) \leqslant d_\Omega(x,\gamma o)\}.$$

Recall that it is a closed subset of Ω, and that the translates of $\mathscr{D}$ by Γ cover Ω. Consider the closed subset $X = \mathscr{D} \cap \mathscr{C}(\Lambda_\Gamma) \smallsetminus \bigcup_p B_p$ of Ω. Let us show that X is bounded.

Assume it is not, then there exists a sequence $(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ such that $x_n \to p \in \Lambda_\Gamma$. Lemma 4.1 shows that p is not a conical limit point. Hence, p is a bounded parabolic point since $\Gamma \curvearrowright \partial\Omega$ is geometrically finite.

By our assumption, up to extracting a subsequence, there exists $\gamma_n \in \Gamma_p$ such that $\gamma_n(x_n) \to z \in \overline{\Omega} \smallsetminus \{p\}$, see Figure 4. In particular $\gamma_n \to +\infty$ and so $\gamma_n(o) \to p$, by Fact 5.2.

Pick any y in the interval (z,p), which is contained in Ω since it is strictly convex. Since $\gamma_n[x_n,o] \to [z,p]$, we can find $y_n \in \gamma_n(x_n,o) \subset \Omega$ that converge to y.

$$\begin{aligned}
d_\Omega(x_n,o) &= d_\Omega(\gamma_n(x_n),\gamma_n(o))) \\
&= d_\Omega(\gamma_n(x_n),y_n) + d_\Omega(y_n,\gamma_n(o)) \\
&\geqslant d_\Omega(\gamma_n(x_n),o) - d_\Omega(o,y_n) + d_\Omega(\gamma_n(o),o) - d_\Omega(o,y_n)
\end{aligned}$$

So:

$$\underbrace{d_\Omega(x_n,o) - d_\Omega(\gamma_n(x_n),o)}_{\leqslant 0 \text{ since } x_n \in \mathscr{D}} \geqslant -\underbrace{2d_\Omega(o,y_n)}_{\to d_\Omega(o,y)} + \underbrace{d_\Omega(\gamma_n(o),o)}_{\to +\infty}.$$

Absurd. □

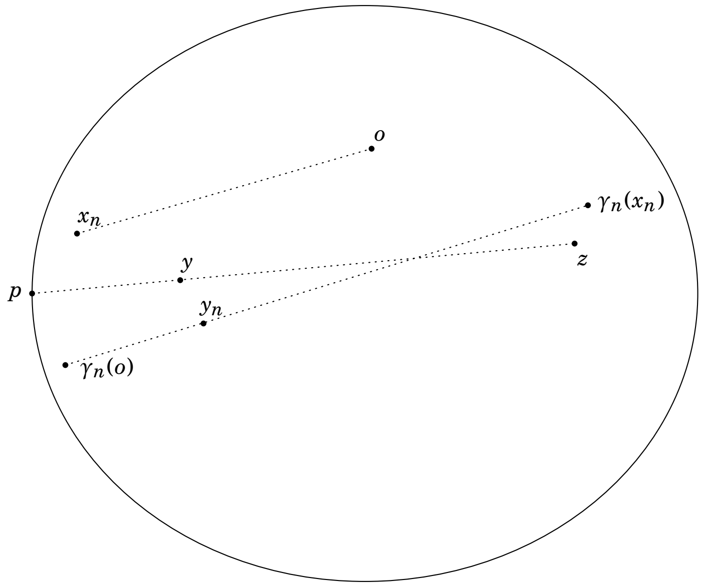

FIGURE 4. Illustration of the proof of Proposition 6.2

6.2. (gf) **as a consequence of cocompactness.** We now state the second main result of this section, which can be described as a converse to the first main result Proposition 6.2.

Proposition 6.3. *Let Ω be a round convex subset of $\mathbb{RP}^d$ and $\Gamma \subset Aut(\Omega)$ discrete non-elementary. Consider a Γ-invariant subset $\mathscr{P} \subset \Lambda_\Gamma$ and denote by $\Gamma_p \subset \Gamma$ the stabilizer of any $p \in \mathscr{P}$. Consider a (Γ, Γ_p)-precisely equivariant family $(B_p)_{p\in\mathscr{P}}$ of domains with B_p strongly star-shaped at p.*

Suppose that the action of Γ on $\mathscr{C}(\Lambda_\Gamma) \smallsetminus \bigcup_p B_p$ is cocompact.

Then Γ acts geometrically finitely on $\partial\Omega$ (Assumption (gf)), $\mathscr{P}$ is the set of bounded parabolic points in Λ_Γ, and Γ_p acts cocompactly on $\partial B_p \cap \mathscr{C}(\Lambda_\Gamma)$ for every $p \in \mathscr{P}$.

Proof. Let $q \in \mathscr{P}$. Since $(B_p)_{p\in\mathscr{P}}$ is $(\Gamma, (\Gamma_p)_p)$-precisely equivariant,

$$\big(\partial B_q \cap \mathscr{C}(\Lambda_\Gamma)\big)/\Gamma_q \hookrightarrow \left(\mathscr{C}(\Lambda_\Gamma) \smallsetminus \bigcup_p B_p\right)/\Gamma$$

is an embedding with closed image. In particular, $(\partial B_q \cap \mathscr{C}(\Lambda_\Gamma))/\Gamma_q$ is compact, in other words the action of Γ_q on $\partial B_q \cap \mathscr{C}(\Lambda_\Gamma)$ is cocompact.

Moreover, $\Lambda_\Gamma \smallsetminus \{q\}$ embeds Γ_q-equivariantly in $\partial B_q \cap \mathscr{C}(\Lambda_\Gamma)$ since B_q is strongly star-shaped at q (see **(iii)**), so the action of Γ_q on $\Lambda_\Gamma \smallsetminus \{q\}$ is proper and cocompact. This implies that q is bounded parabolic.

Let $q \in \Lambda_\Gamma \smallsetminus \mathscr{P}$. Consider $o \in \mathscr{C}(\Lambda_\Gamma)$. Note that the geodesic ray $[o,q[$ is not eventually contained in any B_p, for any $p \in \mathscr{P}$, since $\overline{B}_p \cap \partial\Omega = \{p\}$ as B_p is star-shaped at p (see **(ii)**). Hence there exists $x_n \in [o,q[$ such that $x_n \to q$ and $x_n \in \mathscr{C}(\Lambda_\Gamma) \smallsetminus \bigcup_p B_p$. So, by cocompactness of the action, up to extracting a subsequence, there exists $\gamma_n \in \Gamma$ such that $\gamma_n(x_n) \to z \in \Omega$, which implies that q is conical ($(\gamma_n^{-1} z)_n$ converge to q while remaining at bounded distance from $[o,q)$). □

7. (gf)$\Longleftrightarrow$ (CU)

Let Ω be a round convex subset of $\mathbb{RP}^d$. In this section we recall the definition of horoballs and some basic facts, e.g. that horoballs are the images of Ω under a projective transformation, and hence are round convex subsets of $\mathbb{RP}^d$. We then use Section 5 to prove (gf)$\Longleftrightarrow$ (CU).

7.1. **Horoballs.** We first give a very algebraic definition of horoballs, and then describe them more geometrically via Busemann functions, using a result of Benoist. See also [CM14a, §2.2] and [CLT15, p.16]

Definition 7.1. Let $p \in \partial\Omega$ and $x \in \Omega$. Let $q \in \partial\Omega$ be such that $x \in [p,q]$. Consider a basis $v_1,\dots,v_{d+1} \in \mathbb{R}^{d+1}$ such that $p = [v_1]$, $q = [v_2]$, $x = [v_1 + v_2]$, and $[v_i] \in T_p\partial\Omega$ for each $i \geqslant 3$. Then the horosphere $W \subset \Omega$ centered at p passing through x is the image $g\partial\Omega \smallsetminus \{p\}$ of $\partial\Omega \smallsetminus \{p\}$ under the projective transformation

$$g = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{I}_{d-1} \end{pmatrix}.$$

Note that g fixes any affine chart not containing $T_p\partial\Omega$, in which it acts as a translation in the direction of the line through p and q, sending q to x.

The open horoball H with boundary $W \cup \{p\}$ is $g\Omega$, which is a round convex subset of Ω; in particular it is strongly star-shaped at p. Note that $T_p\partial H = T_p\partial\Omega$.

This does not depend on the choice of $v_1,\dots,v_{d+1}$. Indeed, one can check that $\partial H \smallsetminus \{p\}$ is the set of $x' \in \Omega$ such that, if $x' \in [p,q']$ for $q' \in \partial\Omega$, then the two lines $\overline{qq'}$ and $\overline{xx'}$ intersect in the hyperplane $T_p\partial\Omega$.

Fact 7.2 ([Ben04, §3.2.3-4 & Fig.7])**.** *For all $p \in \partial\Omega$ and $x, y \in \Omega$,*

$$b_p(x,y) := \lim_{z\to p} d_\Omega(x,z) - d_\Omega(y,z)$$

is well defined.

Moreover, the horosphere centered at p through any given $x \in \Omega$ is $\{y \in \Omega : b_p(x,y) = 0\}$. The associated open horoball is $\{y \in \Omega : b_p(x,y) > 0\}$.

It is clear that projective transformations map horoballs to horoballs. The following states that parabolic groups preserve each horoball centered at the point they fix.

Fact 7.3 ([CM14a, Th. 3.3], [CLT15, Prop. 3.3])**.** *For all $p \in \partial\Omega$, $x \in \Omega$ and $\gamma \in Aut(\Omega)$ preserving p, the translation length of γ is exactly $|b_p(x,\gamma x)|$.*

In particular, if γ is parabolic (or elliptic) then it preserves each horoball centered at p.

Proof. Note that for every $y \in \Omega$ we have

$$b_p(y,\gamma y) = b_p(y,x) + b_p(x,\gamma x) + b_p(\gamma x,\gamma y) = b_p(x,\gamma x) + b_p(y,x) + b_p(x,y) = b_p(x,\gamma x).$$

Morever, by the triangle inequality $|b_p(y,\gamma y)| \leqslant d_\Omega(y,\gamma y)$ for every $y \in \Omega$.

As a consequence, $|b_p(x,\gamma x)|$ is bounded from above by the translation length.

If the translation length is zero then we are done. Otherwise, γ is hyperbolic and fixes exactly two points of $p, q \in \partial\Omega$ (see [CM14b, §3.1]). Then one can check that if $x \in [p,q]$ then $|b_p(x,\gamma x)|$ equals $d_\Omega(x,\gamma x)$, and hence equals the translation length of γ. $\square$

Finally, the following result says that geometrically finite groups always admit precisely equivariant families of horoballs. The result is not stated the same way in the reference, but the link is not hard to make.

Fact 7.4 ([BZ21, Lem. 8.11]). *Let Ω be a round convex subset of $\mathbb{RP}^d$. Let $\Gamma \subset Aut(\Omega)$ be discrete non-elementary and act geometrically finitely on $\partial\Omega$. Then there exists a $(\Gamma,(\Gamma_p)_p)$-precisely equivariant family of horoballs centered at the parabolic points of Λ_Γ.*

7.2. **Applications of Sections 5 and 6.**

Lemma 7.5. *Let Ω be a round convex subset of $\mathbb{RP}^d$. Let $\Gamma \subset Aut(\Omega)$ be discrete non-elementary. Let $p \in \Lambda_\Gamma$ be a bounded parabolic fixed point. For any open horoball H_p centered at p, the action of Γ_p on $\overline{\mathscr{C}(\Lambda_\Gamma)} \smallsetminus (H_p \cup \{p\})$ is cocompact.*

Proof. This is an immediate corollary of Lemmas 5.3 and 5.4, using the fact that horoballs are round convex with their center in their boundary (Definition 7.1) and invariant under the associated parabolic subgroups. □

Proof of (gf)⇔ (CU). This is an immediate corollary of Lemma 7.5, Propositions 6.2 and 6.3, and Fact 7.4. □

8. (gf) ⟺ (PNC)

In this section we recall the definition of the thin part of convex projective manifolds and some basic facts, e.g. that the components of the thin part in Ω are star-shaped. We then prove that they also satisfy the extra convexity condition of Lemma 5.3. We then use Sections 5 and 6 to prove (gf) ⟺ (PNC).

8.1. **Thick-thin decomposition.** Let Ω be a convex domain of $\mathbb{RP}^d$, $\Gamma \subset \mathrm{Aut}(\Omega)$ discrete and $\varepsilon > 0$. We use the following notation.

1. $S_\varepsilon(x) := \{\gamma \in \Gamma \mid d_\Omega(x,\gamma x) < \varepsilon\}$ for $x \in \Omega$;
2. $\Gamma_\varepsilon(x) := \langle S_\varepsilon(x)\rangle$ (the subgroup generated by $S_\varepsilon(x)$) for $x \in \Omega$;
3. $\Omega_\varepsilon(\Gamma) := \{x \in \Omega \mid \Gamma_\varepsilon(x)$ is infinite$\}$ is the ε-thin part of Ω, its complement is the ε-thick part;
4. $\mathscr{O}_\varepsilon(A) := \{x \in \Omega \mid d_\Omega(x,\gamma x) < \epsilon,\ \forall \gamma \in A\}$ for $A \subset \Gamma$;

Let us recall the Margulis lemma for convex projective geometry.

Fact 8.1 ([CM13] & [CLT15]). *There exists $\varepsilon_0 > 0$ which only depends on the dimension d, such that for every convex domain Ω of $\mathbb{RP}^d$, any $\Gamma \subset Aut(\Omega)$ discrete, any $0 < \varepsilon \leqslant \varepsilon_0$, any $x \in \Omega$, the group $\Gamma_\varepsilon(x)$ is virtually nilpotent.*

Assuming Ω is a round convex domain, one can use the previous Margulis lemma to obtain a thick-thin decomposition, and more precisely a nice decomposition of the thin part (see [CM14a, Lem. 6.2]).

If $\varepsilon < \varepsilon_0$, then the thin part $\Omega_\varepsilon(\Gamma)$ is the disjoint union of $(\Omega_\varepsilon(G))_G$ (in fact the *closures* are pairwise disjoint), where G runs over the maximal parabolic subgroups of Γ and the centralizers of hyperbolic elements of translation length less than ε.

The *ε-noncuspidal part* is the complement of the union of $(\Omega_\varepsilon(G))_G$, where G runs over the parabolic subgroups of Γ.

Fact 8.2 ([CM14a, Lem. 6.2.1 & Cor. 3.16])**.** *If $\varepsilon < \varepsilon_0$ and $\mathscr{P} \subset \Lambda_\Gamma$ is the set of parabolic points, then $(\Omega_\varepsilon(\Gamma_p))_{p\in\mathscr{P}}$ is $(\Gamma,(\Gamma_p)_p)$-precisely equivariant.*

8.2. **Star-shapedness and the weak convexity condition.** Here we check that the components of the thin part, as well as the domains of the form $\mathscr{O}_\varepsilon(A)$ defined in the previous section, are strongly star-shaped and satisfy the extra weak convexity condition in Lemma 5.3.

Fact 8.3. *Let Ω be a round convex subset of $\mathbb{RP}^d$ and $\Gamma_p \subset Aut(\Omega)$ a discrete infinite parabolic subgroup fixing $p \in \partial\Omega$.*

Then $\Omega_\varepsilon(\Gamma_p)$ is strongly star-shaped at p for any ε (see Definition 5.1).

Moreover, $\mathscr{O}_\varepsilon(A)$ is also strongly star-shaped at p for any finite $A \subset \Gamma_p$ that generates an infinite group.

Note also that $(\partial\Omega_\varepsilon(\Gamma_p))_\varepsilon$ and $(\partial\mathscr{O}_\varepsilon(A))_\varepsilon$ foliate Ω.

The above fact is a consequence of the following elementary result (which uses the fact that Ω is round).

Fact 8.4 ([Ben04, Lem. 3.4])**.** *Let Ω be a round convex subset of $\mathbb{RP}^d$ and $\gamma \in Aut(\Omega)$ a parabolic or elliptic transformation fixing $p \in \partial\Omega$. Consider a straight geodesic $(p_t)_{t\in\mathbb{R}} \subset \Omega$ going to p as $t \to \infty$.*

Then either γ fixes the geodesic or $t \mapsto d_\Omega(p_t, \gamma p_t)$ is decreasing from ∞ to 0.

Before we discuss the weak convexity condition needed in Lemma 5.3, let us discuss briefly the link between $\Omega_\varepsilon(\Gamma_p)$ and $\mathscr{O}_\varepsilon(A)$, where $\Omega \subset \mathbb{RP}^d$ is round convex, $\Gamma_p \subset \mathrm{Aut}(\Omega)$ is discrete infinite parabolic and fix $p \in \partial\Omega$, the subset $A \subset \Gamma_p$ is finite and generates an infinite group, and $\varepsilon > 0$.

1. $\mathscr{O}_\varepsilon(A) \subset \Omega_\varepsilon(\Gamma_p)$.
2. $\mathscr{O}_\varepsilon(A)$ is not necessarily Γ_p-invariant; it is if A is invariant under conjugacy.
3. Γ_p being virtually nilpotent, it admits a torsion-free nilpotent finite-index subgroup $G \subset \Gamma_p$, whose center has a nontrivial element $g \in G$; then $\mathscr{O}_\varepsilon(g)$ is G-invariant.

Let us now turn to the weak convexity condition needed in Lemma 5.3. We will need the following estimate on the Hilbert metric, which gives control on the distance between two segments via the distance between the endpoints.

Fact 8.5 ([Cra09, Lem. 8.3] & [Bla, Lem. 5.2])**.** *Let Ω be a convex domain. Consider two segments $[x,y],[x',y'] \subset \Omega$ and two points $z \in [x,y]$ and $z' \in [x',y']$ such that $\frac{d_\Omega(x,z)}{d_\Omega(x,y)} = \frac{d_\Omega(x',z')}{d_\Omega(x',y')}$. Then*

$$d_\Omega(z,z') \leqslant d_\Omega(x,x') + d_\Omega(y,y').$$

Corollary 8.6. *Let Ω be a convex domain and $y \in \partial\Omega$. Consider two segments $[x,y),[x',y) \subset \Omega$ and two points $z \in [x,y)$ and $z' \in [x',y)$ such that $d_\Omega(x,z) = d_\Omega(x',z')$. Then*

$$d_\Omega(z,z') \leqslant d_\Omega(x,x').$$

Proof. Let $(y_n)_n$ a sequence of points of the segment (z,y) converging to y. Let $(z'_n)_n$ be the sequence of points of (y_n,x') such that: $\frac{d_\Omega(x,z)}{d_\Omega(x,y_n)} = \frac{d_\Omega(x',z'_n)}{d_\Omega(x',y_n)}$. By Fact 8.5, $d_\Omega(z,z'_n) \leqslant d_\Omega(x,x')$. Hence, it is enough to show that $(z'_n)_n$ converges to z'.

The ratio $\frac{d_\Omega(x,y_n)}{d_\Omega(x',y_n)}$ converges to 1 since $|d_\Omega(x,y_n) - d_\Omega(x',y_n)| \leqslant d_\Omega(x,x')$. So, $d_\Omega(x',z'_n)$ converges to $d_\Omega(x,z) = d_\Omega(x',z')$, giving that $z'_n \to z'$ since z'_n is on the segment (y_n,x') which converges to the segment (y,x'). ☐

Corollary 8.7. *Let Ω be a convex domain of $\mathbb{RP}^d$. For all $\epsilon > 0$ and $A \subset Aut(\Omega)$, the convex hull of $\mathcal{O}_\varepsilon(A)$ is contained in $\mathcal{O}_{(d+1)\varepsilon}(A)$.*

Proof. It suffices to prove by induction on $k \geqslant 1$ that any convex combination of k points of $\mathcal{O}_\varepsilon(A)$ is in $\mathcal{O}_{k\varepsilon}(A)$.

If $k = 1$ then this is obvious.

Suppose $k \geqslant 2$ and the property we want to prove for convex combinations of fewer than k points. Let z be a convex combination of k points of $\mathcal{O}_\varepsilon(A)$. Then $z \in [x, y]$ where $x \in \mathcal{O}_\varepsilon(A)$ and y is a convex combination of $k-1$ points of $\mathcal{O}_\varepsilon(A)$. By the inductive hypothesis we have $y \in \mathcal{O}_{(k-1)\varepsilon}(A)$.

Consider $\gamma \in A$, and let us check that $d_\Omega(z,\gamma z) < k\varepsilon$. We have $\gamma z \in [\gamma x, \gamma y]$ and $\frac{d_\Omega(x,z)}{d_\Omega(x,y)} = \frac{d_\Omega(\gamma x,\gamma z)}{d_\Omega(\gamma x,\gamma y)}$, so by Fact 8.5

$$d_\Omega(z,\gamma z) \leqslant d_\Omega(x,\gamma x) + d_\Omega(y,\gamma y) < \varepsilon + (k-1)\varepsilon = k\varepsilon.$$

□

8.3. **Applications of Sections 5 and 6 bis.** We now apply the results from previous sections to establish (gf) ⇔ (PNC).

Lemma 8.8. *Let Ω be a round convex subset of $\mathbb{RP}^d$, $\Gamma \subset Aut(\Omega)$ discrete non-elementary, and $p \in \Lambda_\Gamma$ be a bounded parabolic fixed point with stabilizer Γ_p. Consider $G \subset \Gamma_p$ a finite index subgroup and $A \subset G$ finite, that generates an infinite subgroup, and invariant under conjugation by elements of G. Then the action of G on $\overline{\mathcal{C}(\Lambda_\Gamma)} \smallsetminus (\mathcal{O}_\epsilon(A) \cup \{p\})$ is cocompact for any ϵ.*

Note that G preserves $\mathcal{O}_\epsilon(A)$ because A is invariant under conjugation.

Proof. This is an immediate corollary of Fact 8.3, Corollary 8.7 and Lemmas 5.3 and 5.4. □

Corollary 8.9. *Let Ω be a round convex subset of $\mathbb{RP}^d$, $\Gamma \subset Aut(\Omega)$ discrete non-elementary, and $p \in \Lambda_\Gamma$ be a bounded parabolic fixed point with stabilizer Γ_p. Then the action of Γ_p on $\overline{\mathcal{C}(\Lambda_\Gamma)} \smallsetminus (\Omega_\epsilon(\Gamma_p) \cup \{p\})$ is cocompact for any ϵ.*

Proof. By Fact 8.1, we can find a finite-index torsion-free nilpotent subgroup $G \subset \Gamma_p$. Since G is nilpotent it has a nontrivial element g in the center. By definition, $\mathcal{O}_\varepsilon(g) \subset \Omega_\varepsilon(\Gamma_p)$ and hence

$$\overline{\mathcal{C}(\Lambda_\Gamma)} \smallsetminus (\Omega_\epsilon(\Gamma_p) \cup \{p\}) \subset \overline{\mathcal{C}(\Lambda_\Gamma)} \smallsetminus (\mathcal{O}_\epsilon(g) \cup \{p\}).$$

We conclude using Lemma 8.8. □

Proof of (gf) ⇔ (PNC). This is a corollary of Corollary 8.9, Fact 8.2 and Propositions 6.2 and 6.3. □

9. Counterexample to (GF) ⇒ (HC)

In this section we use a reducible representation of $\mathrm{SL}_2(\mathbb{R})$ to construct an example of group Γ satisfying (GF) but not (HC). Let $\tau : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}_3(\mathbb{R})$ be an irreducible representation. Consider the reducible semisimple representation $\rho : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}_5(\mathbb{R})$ such that for any $g \in \mathrm{SL}_2(\mathbb{R})$ we have

$$\rho(g) = \begin{pmatrix} \tau(g) & 0 \\ 0 & g \end{pmatrix}.$$

Note that by definition $\rho(\mathrm{SL}_2(\mathbb{R}))$ preserves the supplementary subspaces $\mathbb{R}^3 \times \{0\}$ and $\{0\} \times \mathbb{R}^2$ of $\mathbb{R}^5$. Moreover in $\mathbb{R}^3 \times \{0\}$ it preserves a properly convex (relatively) open cone $C = C \times \{0\}$, and of course it also preserves the open convex cone $C \times \mathbb{R}^2$ which is not properly convex.

The projectivisation $D = \mathbb{P}(C)$ is a 2-dimensional properly convex disc and $\Omega_{\max} = \mathbb{P}(C \times \mathbb{R}^2)$ is an open convex subset of $\mathbb{RP}^4$ which is contained in some affine chart, where it is $D \times \mathbb{R}^2$. Their relative boundaries are denoted by ∂D and $\partial\Omega_{\max}$.

The following result describes all $\rho(\mathrm{SL}_2(\mathbb{R}))$-invariant convex domains.

Fact 9.1. *We have the following.*

1. *The proximal limit set of $\rho(\mathrm{SL}_2(\mathbb{R}))$ is ∂D.*
2. *$\rho(\mathrm{SL}_2(\mathbb{R}))$ acts properly discontinuously on $\Omega_{\max}$; more precisely the orbit of any compact set accumulates on all of ∂D (and only there).*
3. *For any $x \in \Omega_{\max} \smallsetminus D$, the stabilizer is trivial.*
4. *For any $x \in \Omega_{\max} \smallsetminus D$, the disjoint union $\partial D \sqcup \rho(\mathrm{SL}_2(\mathbb{R})) \cdot x$ is the boundary of an invariant round convex domain $\Omega \subset \Omega_{\max}$. Moreover, every invariant convex domain Ω' is obtained in this way.*

Proof.

1. Let $g \in \mathrm{SL}_2(\mathbb{R})$ be proximal whose (real) eigenvalues have norm $\lambda > 1/\lambda$. Then the norms of the eigenvalues of $\tau(g)$ are $\lambda^2 > 1 > 1/\lambda^2$. Thus the biggest norm of eigenvalues of $\rho(g)$ is λ^2, and the corresponding eigenline is exactly the eigenline of $\tau(g)$, embedded in $\mathbb{R}^5$ via $\mathbb{R}^3 \to \mathbb{R}^3 \times \{0\}$. This concludes the proof since it is well known that the proximal limit set of $\tau(\mathrm{SL}_2(\mathbb{R}))$ in $\mathbb{RP}^2$ is ∂D.
2. Let $x \in \Omega_{\max}$ and $(g_n)_n$ a sequence of element of $\mathrm{SL}_2(\mathbb{R})$ such that $g_n \to \infty$. Let $\|\cdot\|$ be the induced norm on the space of $m \times m$ real matrix, by the canonical scalar product on $\mathbb{R}^m$, there exists $C > 1$ such that:
$$\forall g \in \mathrm{SL}_2(\mathbb{R}), \qquad C^{-1}\|g\|^2 \leqslant \|\tau(g)\| \leqslant C\|g\|^2$$
For another constant $C_2 > 1$, one has:
$$\forall g \in \mathrm{SL}_2(\mathbb{R}), \qquad C_2^{-1}\|g\|^2 \leqslant \|\rho(g)\| \leqslant C_2\|g\|^2$$
Hence, up to extraction, we may assume that $\tau(g_n)/\|\tau(g_n)\|$ converge to a rank-one matrix $T \in \mathscr{M}_3(\mathbb{R})$ such that $\mathrm{Im}(T) \subset \partial D$ and $\ker(T) \cap \partial D$ is a singleton. Thanks to the estimate, the matrix $\rho(g_n)/\|\rho(g_n)\|$ converges to the matrix:
$$\begin{pmatrix} T & \\ & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \end{pmatrix}$$
Hence, $\rho(g_n) \cdot x$ converges to $T(x) \in \partial D$.
3. Let $[(x,y)] \in \Omega_{\max} \smallsetminus D$ with $x \in \mathbb{R}^3$ and $y \in \mathbb{R}^2$ and consider $g \in \mathrm{SL}_2(\mathbb{R})$ such that $\rho(g) \cdot [(x,y)] = [(x,y)]$. We may assume that $\mathrm{Stab}_{\tau(\mathrm{SL}_2(\mathbb{R}))}([x]) = \tau(\mathrm{SO}_2(\mathbb{R}))$, hence we get that $g \in \mathrm{SO}_2(\mathbb{R})$. The only two rotations of $\mathbb{R}^2$ that preserves a line are Id and $-$Id, so we get that $g = \pm\mathrm{Id}$. But, the fixed point set of $\rho(-\mathrm{Id})$ is $\mathbb{P}(\mathbb{R}^3 \times \{0\}) \cup \mathbb{P}(\{0\} \times \mathbb{R}^2)$, hence $g = \mathrm{Id}$.
4. Take $x \in \Omega_{\max} \smallsetminus D$. The interior of the convex hull of $\rho(\mathrm{SL}_2(\mathbb{R})) \cdot x$ in $\Omega_{\max}$ is an invariant convex domain Ω. The orbit $\rho(\mathrm{SL}_2(\mathbb{R})) \cdot x$ of x accumulates on ∂D and only there (by (2.)), which implies $\rho(\mathrm{SL}_2(\mathbb{R})) \cdot x \sqcup \partial D$ is compact and $\overline{\Omega}$ is the convex hull of $\rho(\mathrm{SL}_2(\mathbb{R})) \cdot x \sqcup \partial D$ and is properly convex.

Thus the extremal points of $\overline{\Omega}$ are in $\rho(\mathrm{SL}_2(\mathbb{R}))\cdot x \sqcup \partial D$. They cannot be all in ∂D, otherwise $\overline{\Omega}\subset\overline{D}$ which contradicts $x\notin\overline{D}$. Thus at least one point of $\rho(\mathrm{SL}_2(\mathbb{R}))\cdot x$ must be extremal, and then all points of $\rho(\mathrm{SL}_2(\mathbb{R}))\cdot x$ are extremal since $\rho(\mathrm{SL}_2(\mathbb{R}))$ maps extremal points to extremal points. Moreover one can also check that any point $p\in\partial D$ is extremal. (Otherwise there would be $a,b\in\partial\Omega$ such that $p\in(a,b)$: if $a,b\in\partial D$ then $p\in D$, absurd, and if one of a,b is in $\rho(\mathrm{SL}_2(\mathbb{R}))\cdot x$ then $p\in\Omega_{\max}$, absurd too.)

We proved that the set of extremal points is exactly $\rho(\mathrm{SL}_2(\mathbb{R}))\cdot x\sqcup\partial D$, which is in particular contained in $\partial\Omega$. The orbit $\rho(\mathrm{SL}_2(\mathbb{R}))\cdot x$ of x is open in $\partial\Omega$ by Brouwer's invariance of the domain theorem, thanks to (3.). The orbit accumulates on ∂D and only there (by (2.)), hence is closed in $\partial\Omega\smallsetminus\partial D$. A classical result of topology shows that $\partial\Omega\smallsetminus\partial D$ is connected (see e.g. [Hat02, Prop. 2.B.1.b]). Hence, $\partial\Omega=\rho(\mathrm{SL}_2(\mathbb{R}))\cdot x\sqcup\partial D$, and Ω is strictly convex since all points of the boundary are extremal.

Let $\Omega'\subset\mathbb{RP}^4$ be an invariant properly convex open set. Then $\partial\Omega'$ contains the proximal limit set of $\rho(\mathrm{SL}_2(\mathbb{R}))$, i.e. ∂D. By convexity $\overline{\Omega}'$ must then contain D, and Ω' intersects $\Omega_{\max}$. However, $\overline{\Omega}'$ cannot intersect $\partial\Omega_{\max}\smallsetminus\partial D$. If it did then by applying powers of a suitable element of $\rho(\mathrm{SL}_2(\mathbb{R}))$ there would be a point of $\mathbb{P}(\{0\}\times\mathbb{R}^2)$ in $\overline{\Omega}'$, and hence there would in fact be the whole $\mathbb{P}(\{0\}\times\mathbb{R}^2)$ inside $\overline{\Omega}'$, which would compromise Ω''s proper convexity. As a consequence, $\partial\Omega'$ intersects $\Omega_{\max}$ at some point, say at the point x. Then $\Omega\subset\Omega'$, and by our results above we get $\partial\Omega=\rho(\mathrm{SL}_2(\mathbb{R}))\cdot x\sqcup\partial D\subset\partial\Omega'$, which implies at once that $\Omega'=\Omega$.

Finally, the dual representation ρ^* is conjugated to ρ, hence the dual convex of Ω is strictly convex too, hence Ω has $\mathscr{C}^1$-boundary. □

Next we prove that the image under ρ of a geometrically finite subgroup of $\mathrm{SL}_2(\mathbb{R})$ satisfies (GF) but not (HC).

Proposition 9.2. *For any $\rho(\mathrm{SL}_2(\mathbb{R}))$-invariant round convex domain $\Omega\subset\Omega_{\max}$, for any discrete subgroup $\Gamma\subset\mathrm{SL}_2(\mathbb{R})$, if Γ is finitely generated (which means geometrically finite in the classical sense), then $\rho(\Gamma)$ acts geometrically finitely on Ω, but no parabolic subgroup is conjugate into $\mathrm{O}_{4,1}(\mathbb{R})$ (even though all parabolic points are uniformly bounded).*

Proof. By Fact 9.1, the proximal limit set Λ_Γ of $\rho(\Gamma)$ is contained in ∂D, and the convex hull $\mathscr{C}(\Lambda_\Gamma)$ is contained in D, which is, we recall, isometric to the Poincaré disc. This implies Γ acts geometrically finitely on Ω.

Indeed every point p of Λ_Γ corresponds to a point q of the limit set of Γ acting on $\mathbb{H}^2$. If q is conical (there is $(\gamma_n)_n\subset\Gamma$ such that $(\gamma_n o)_n$ converges to q while remaining at bounded distance from $[o,q)$, for $o\in\mathbb{H}^2$) then p is conical too. If q is bounded parabolic for the action of Γ on $\mathbb{H}^2$ then the stabiliser Γ_q acts cocompactly on $\partial\mathbb{H}^2\smallsetminus\{q\}$, hence $\rho(\Gamma_q)$, which is the stabiliser of p, acts cocompactly on $\partial D\smallsetminus\{p\}$, which contains the stereographic projection of $\mathscr{C}(\Lambda_\Gamma)$, hence p is uniformly bounded parabolic.

Parabolic subgroups of $\rho(\Gamma)$ are virtually conjugate to the group generated by the following matrix, which is not conjugate into $\mathrm{O}_{4,1}(\mathbb{R})$

$$\begin{pmatrix} 1 & 1 & 1/2 & & \\ & 1 & 1 & & \\ & & 1 & & \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix}.$$ □

10. Under (Gen), uniformly bounded cusp groups are conjugate into $O_{d,1}(\mathbb{R})$

In this section, we prove that under the genericity assumption (Gen), the stabilisers of uniformly parabolic points are conjugate into $O_{d,1}(\mathbb{R})$. In particular, this establish the implication ((GF)&(Gen))⇒(HC).

Proposition 10.1. *Let Ω be a round convex subset of $\mathbb{RP}^d$, $\Gamma \subset Aut(\Omega)$ discrete non-elementary, and $p \in \Lambda_\Gamma$ be a uniformly bounded parabolic fixed point with stabilizer Γ_p.*

Suppose that the limit set Λ_Γ spans the whole $\mathbb{RP}^d$, or that its dual spans the dual of $\mathbb{RP}^d$.

Then Γ_p is conjugate to a parabolic subgroup of $O_{d,1}(\mathbb{R})$.

Moreover it preserves a projective subspace $\mathbb{RP}^{r+1} \subset \mathbb{RP}^d$ where r is the rank of Γ_p, that contains p and intersects Ω, and preserves ellipsoids $\mathscr{E}^{\rm int} \subset \mathscr{E}^{\rm ext}$ such that

$$\mathscr{E}^{\rm int} \cap \mathrm{Cone}(p, \mathscr{C}(\Lambda_\Gamma)) \subset \Omega \cap \mathrm{Cone}(p, \mathscr{C}(\Lambda_\Gamma)) \subset \mathscr{E}^{\rm ext} \cap \mathrm{Cone}(p, \mathscr{C}(\Lambda_\Gamma)),$$

where $\mathrm{Cone}(p, \mathscr{C}(\Lambda_\Gamma))$ is the union of the lines through p and a point of $\mathscr{C}(\Lambda_\Gamma)$.

Proof. We can assume that Λ_Γ spans $\mathbb{RP}^d$ since the other case is dual.

Let $\mathbb{A}^{d-1}$ be the affine chart of $\mathbb{P}(\mathbb{R}^{d+1}/p) \smallsetminus \mathbb{P}(T_p\partial\Omega/p)$, on which Γ_p acts properly discontinuously by affine transformation (see Fact 5.2), and preserves and acts cocompactly on the closed convex projection K of $\mathscr{C}(\Lambda_\Gamma)$ (since p is uniformly bounded).

Let $\mathbb{A}^r \subset \mathbb{A}^{d-1}$ be a maximal affine subspace contained in K and $K' \subset \mathbb{A}^{d-1}/\mathbb{A}^r$ the projection of K, which can be thought as the set of maximal affine subspaces contained in K. Note that $K' \subset \mathbb{A}^{d-1}/\mathbb{A}^r$ does not contain any line, and that K is isomorphic to $\mathbb{A}^r \times K'$.

Observe that Γ_p acting cocompactly on K implies that K' must be compact: indeed if it were not then K' would be homeomorphic to a halfspace, and so would be K, but no halfspace can be acted on properly discontinuously and cocompactly by a discrete group. (If a group G acts properly discontinuously and cocompactly on $\mathbb{R}^n \times \mathbb{R}_{\geqslant 0}$ then it acts properly discontinuously and cocompactly on both the boundary $\mathbb{R}^n \times \{0\}$ and the double $\mathbb{R}^{n+1}$, which is impossible.)

Moreover, K' has nonempty interior by our assumption that Λ_Γ spans $\mathbb{RP}^d$.

The group Γ_p acts on $\mathbb{A}^{d-1}/\mathbb{A}^r$ by affine transformations and preserves the compact convex subset K' with nonempty interior, so Γ_p must fix the barycenter of K' which is in its interior, and Γ_p must preserve some Euclidean structure on $\mathbb{A}^{d-1}/\mathbb{A}^r$.

This barycenter lifts to a Γ_p-invariant maximal affine subspace of K, which we assume to be $\mathbb{A}^r$ without loss of generality, and on which the action of Γ_p is cocompact. Then $\mathbb{A}^r$ lifts to a $(r+1)$-dimensional Γ_p-invariant subspace of $\mathbb{RP}^d$ which contains p and intersects Ω. Up to changing basis we assume that this subspace is $\mathbb{RP}^{r+1} = \mathbb{P}(\mathbb{R}^{r+2} \times \{0\}) \subset \mathbb{RP}^d = \mathbb{P}(\mathbb{R}^{d+1})$.

The intersection $\Omega' = \Omega \cap \mathbb{RP}^{r+1}$ is Γ_p-invariant, and Γ_p acts properly discontinuously and cocompactly on $\partial\Omega' \smallsetminus \{p\}$ since we have an equivariant identification with $\mathbb{A}^r$ via the stereographic projection.

By [CM14a, Th. 7.14] this implies that the restriction of Γ_p to $\mathbb{R}^{r+2} = \mathbb{R}^{r+2} \times \{0\} \subset \mathbb{R}^{d+1}$ is conjugate to a parabolic subgroup of $O_{r+1,1}(\mathbb{R})$ of rank r; up to conjugating everything we assume that this restriction is contained in $O_{r+1,1}(\mathbb{R})$. By Bieberbach's Theorem (see e.g. [Rat19, Th. 5.4.4]), up to changing the basis $\mathscr{B}$ of $\mathbb{R}^{r+2}$, the group Γ_p has a finite-index normal subgroup isomorphic to $\mathbb{Z}^r$ such that the restriction of any $k \in \mathbb{Z}^r$ to $\mathbb{R}^{r+2}$ acts by

$$\begin{pmatrix} 1 & k^t & \frac{||k||^2}{2} \\ 0 & I_{r-1} & k \\ 0 & 0 & 1 \end{pmatrix}.$$

The fact that Γ_p preserves an Euclidean structure on $\mathbb{A}^{d-1}/\mathbb{A}^r$ means that its action on $\mathbb{R}^{d-r-1} = \mathbb{R}^{d+1}/\mathbb{R}^{r+2}$ preserves an inner product, say the standard one.

To prove that Γ_p is conjugate to a parabolic subgroup of $\mathrm{O}_{d,1}(\mathbb{R})$ it suffices to find a Γ_p-invariant subspace of $\mathbb{R}^{d+1}$ which is supplementary to $\mathbb{R}^{r+2}$. The set E of subspaces supplementary to $\mathbb{R}^{r+2}$ is an affine space on which Γ_p acts by affine transformations. Thus it suffices to check that $\mathbb{Z}^r \subset \Gamma_p$ preserves such a subspace, i.e. fixes a point of E. Indeed, since $\mathbb{Z}^r$ is a normal subgroup of Γ_p, the subspace $E' \subset E$ of $\mathbb{Z}^r$-fixed points is Γ_p-invariant. As $\mathbb{Z}^r$ acts trivially on E', the Γ_p-action descends to an affine action of $\Gamma_p/\mathbb{Z}^r$ which is a finite group, and hence has a fixed point.

To write the matrices, we first choose $\mathscr{B}$ for the $(r+2)$ first elements of our basis. Then we choose the remaining $(d-r-1)$ elements of the basis of $\mathbb{R}^{d+1}$ in a lift of $T_p\partial\Omega$ in such a way that: an element $k \in \mathbb{Z}^r$ acts on $\mathbb{R}^{d+2}$ by

$$\begin{pmatrix} 1 & k^t & \frac{||k||^2}{2} & D_k \\ 0 & I_{r-1} & k & C_k \\ 0 & 0 & 1 & B_k \\ 0 & 0 & 0 & A_k \end{pmatrix},$$

where A_k is an orthogonal matrix. Since those last $(d-r-1)$ elements were chosen in a lift of $T_p\partial\Omega$, we get that $B_k = 0$.

Now if we put the $(r+2)$-th element of the basis of $\mathbb{R}^{d+1}$ into last position, then the first d vectors form a basis of the lift $\mathbb{R}^d$ of $T_p\partial\Omega$, and the new action of $k \in \mathbb{Z}^r$ will be given by

$$\begin{pmatrix} 1 & k^t & D_k & \frac{||k||^2}{2} \\ 0 & I_{r-1} & C_k & k \\ 0 & 0 & A_k & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and our goal is to arrange the elements of the basis from $(r+2)$-th to d-th so that $C_k = 0$ and $D_k = 0$ (and so that those vectors are still in $\mathbb{R}^d$, which is the lift of $T_p\partial\Omega$).

We can diagonalise simultaneously all A_k for $k \in \mathbb{Z}^r$, in the complex field. This gives us

- $v_1, \dots, v_\alpha \in \mathbb{R}^d/\mathbb{R}^{r+1}$ which are real-eigenvectors for all A_k's with eigenvalue 1,
- $w_1, \dots, w_\beta \in \mathbb{R}^d/\mathbb{R}^{r+1}$ eigenvectors such that the eigenvalue of A_k for w_j is $(-1)^{k\cdot\epsilon_j}$ for some $\epsilon_j \in \mathbb{Z}^r \smallsetminus 2\mathbb{Z}^r$,
- and $P_1, \dots, P_\gamma \subset \mathbb{R}^d/\mathbb{R}^{r+1}$ invariant planes such that each A_k acts as a rotation on P_j with angle $k \cdot \theta_j$ for some $\theta_j \in \mathbb{R}^r \smallsetminus \pi\mathbb{Z}^r$.

Using this basis, the new action of $k \in \mathbb{Z}^r$ is given by

$$\begin{pmatrix}
1 & k^t & d_k^1 & \cdots & d_k^\alpha & \bar{d}_k^1 & \cdots & \bar{d}_k^\beta & \tilde{d}_k^1 & \cdots & \tilde{d}_k^\gamma & \frac{||k||^2}{2} \\
 & I_{r-1} & c_k^1 & \cdots & c_k^\alpha & \bar{c}_k^1 & \cdots & \bar{c}_k^\beta & \tilde{c}_k^1 & \cdots & \tilde{c}_k^\gamma & k \\
 & & 1 & & & & & & & & & \\
 & & & \ddots & & & & & & & & \\
 & & & & 1 & & & & & & & \\
 & & & & & (-1)^{k\cdot\epsilon_1} & & & & & & \\
 & & & & & & \ddots & & & & & \\
 & & & & & & & (-1)^{k\cdot\epsilon_\beta} & & & & \\
 & & & & & & & & R_{k\cdot\theta_1} & & & \\
 & & & & & & & & & \ddots & & \\
 & & & & & & & & & & R_{k\cdot\theta_\gamma} & \\
 & & & & & & & & & & & 1
\end{pmatrix},$$

By dealing with each column (of width 1 or 2) independently, we can assume that we are in one of the three following elementary cases:

1. $A_k = 1$ for every k;
2. $A_k = (-1)^{k\cdot\epsilon}$ for every k, where $\epsilon \in \mathbb{Z}^r \smallsetminus 2\mathbb{Z}^r$;
3. $A_k = R_{k\cdot\theta}$ for every k, where $\theta \in \mathbb{R}^r \smallsetminus 2\pi\mathbb{Z}^r$.

Case 2. Since $\epsilon \notin 2\mathbb{Z}^r$ we can find k such that $(-1)^{k\cdot\epsilon} = -1$, and the action of k has a unique -1-eigenvector, which is invariant under the whole group $\mathbb{Z}^r$, and which makes C_ℓ and D_ℓ zero for every $\ell \in \mathbb{Z}^r$.

Case 3. Since $\theta \notin \pi\mathbb{Z}^r$ we can find k such that $R_{k\cdot\theta}$ is a nontrivial rotation, and the action of k has a unique invariant plane in $\mathbb{R}^{d+1}$ where it acts as this rotation, which is invariant under the whole group $\mathbb{Z}^r$, and which hence makes C_ℓ and D_ℓ zero for every $\ell \in \mathbb{Z}^r$.

Case 1. Let us show that C_k has to be zero for any k, no matter what choices for the basis have been made before.

The action of k is given by

$$\begin{pmatrix}
1 & k^t & D_k & \frac{||k||^2}{2} \\
0 & I_{r-1} & C_k & k \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.$$

The fact that we have a group action implies that for all $k, \ell \in \mathbb{Z}^r$ we have $C_{k+\ell} = C_k + C_\ell$ and $D_{k+\ell} = D_k + D_\ell + k^t \cdot C_\ell$.

Hence there is a matrix C such that $C_k = C \cdot k$ for every k.

Since $D_{k+\ell} = D_{\ell+k}$ we have $k^t C\ell = \ell^t Ck$ for all k, ℓ and hence C is symmetric. Set $D'_k := D_k - \frac{1}{2}k^t Ck$ and note that $D'_{k+\ell} = D'_k + D'_\ell$ for all k, ℓ, so there is a row vector D such that $D_k = Dk + \frac{1}{2}k^t Ck$ for any k.

The affine action of k on the affine horizontal hyperplane with height 1 is given by

$$v \mapsto k \cdot v = \begin{pmatrix}
1 & k^t & Dk + \frac{1}{2}k^t Ck \\
0 & I_{r-1} & Ck \\
0 & 0 & 1
\end{pmatrix} v + \begin{pmatrix}
\frac{||k||^2}{2} \\
k \\
0
\end{pmatrix}$$

We know that for every v the first entry of $k\cdot v$ must go to $+\infty$ as k leaves every compact set. This implies that C has to be zero. Indeed, if $x^tCx\neq 0$ for some unit vector x then we can find a diverging sequence $(k_n)_n$ with direction converging to x such that $\frac{1}{2}k_n^tCk_n\sim\frac{1}{2}x^tCx||k_n||^2$ and taking v with last entry $-2/x^tCx$ and all other entries zero, the first entry of $k_n\cdot v$ is

$$O(||k_n||)+\frac{1}{2}x^tCx||k_n||^2\cdot(-2/x^tCx)+\frac{1}{2}||k_n||^2=O(||k_n||)-\frac{1}{2}||k_n||^2\to-\infty,$$

which is absurd.

The action of k on $\mathbb{R}^{d+1}$ is now given by

$$\begin{pmatrix}1 & k^t & Dk & \frac{||k||^2}{2}\\ 0 & I_{r-1} & 0 & k\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

Replacing the $(r+2)$-th vector of the basis by

$$\begin{pmatrix}0\\ -D^t\\ 1\\ 0\end{pmatrix}$$

yields a new action of the form

$$\begin{pmatrix}1 & k^t & 0 & \frac{||k||^2}{2}\\ 0 & I_{r-1} & 0 & k\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix},$$

which is what we needed to conclude the proof. □

11. A correction on the proof of (GF) ⇒ (VF)$_1$

The implication (GF) ⇒ (VF)$_1$ is a consequence of (GF) ⇒(CU) and the following lemma.

We adopt here a slightly different strategy than in [CM14a]. There the crucial ingredient were estimates on the Hilbert volumes of cones in convex domains, with apex in the boundary, obtained in collaboration with Vernicos, using the Busemann volumes. We think it is possible to adapt this strategy to prove (VF)$_1$ (instead of just (VF)$_0$), but it would be much more complicated. Here upper estimates on volumes are obtained by covering our set with a well chosen collection of balls with the same radius.

Lemma 11.1 ([CM14a, p. 48])**.** *Let Γ be a discrete group preserving a round convex open subset Ω, and $p\in\Lambda_\Gamma$ a uniformly bounded parabolic point with stabiliser Γ_p. Fix a closed horoball $H\subset\Omega$ at p and let N be the 1-neighborhood of $H\cap\mathscr{C}(\Lambda_\Gamma)$ in Ω. Then N/Γ_p has finite Hilbert volume.*

Proof. Since p is uniformly bounded parabolic, Γ_p acts cocompactly on $\mathrm{Cone}(p,\mathscr{C}(\Lambda_\Gamma))\cap\partial H\smallsetminus\{p\}$. Fixing a point x_0 in this set, there exists $R>0$ such that all the other points are at Hilbert distance at most R from the Γ_p-orbit of x_0.

For each $t>0$ let x_t be the point of $[x_0,p)$ at distance t from x_0 and H_t the horoball with x_t in its boundary. Note that $\mathrm{Cone}(p,\mathscr{C}(\Lambda_\Gamma))\cap\partial H_t\smallsetminus\{p\}$ is contained in $\Gamma_p\cdot B_\Omega(x_t,R)$.

Indeed if $z_t \in \mathrm{Cone}(p,\mathscr{C}(\Lambda_\Gamma)) \cap \partial H_t \smallsetminus \{p\}$ then the line through p and z_t crosses ∂H at some point z_0. Then there is $\gamma \in \Gamma_p$ such that $\gamma z_0 \in B_\Omega(x_0,R)$, and $d_\Omega(\gamma z_t, x_t) \leqslant d_\Omega(\gamma z_0, x_0) < R$ by Corollary 8.6.

This implies that N is contained in

$$N \subset \bigcup_{n\in\mathbb{N}} \bigcup_{\gamma\in\Gamma_p} \gamma B_\Omega(x_n, R+2) = \bigcup_{n\in\mathbb{N}} B_\Omega(\Gamma_p \cdot x_n, R+2).$$

Let $\pi : \Omega \to \Omega/\Gamma_p$ be the projection map and Vol the quotient measure on Ω/Γ_p. Then

$$\mathrm{Vol}(\pi(N)) \leqslant \sum_{n\in\mathbb{N}} \mathrm{Vol}(\pi(B_\Omega(x_n, R+2)))$$

By definition of the quotient measure on Ω/Γ_p, to compute the volume of the quotient of $B_\Omega(x_n,R+2)$ one can either find a fundamental region for the action of Γ_p or one can consider all the points of $B_\Omega(x_n,R+2)$ and then divide by the number of orbit points in $B_\Omega(x_n,R+2)$:

$$\mathrm{Vol}(\pi(B_\Omega(x_n,R+2))) = \int_{x\in B_\Omega(x_n,R+2))} \frac{1}{\#\{\gamma\in\Gamma_p : \gamma x \in B_\Omega(x_n,R+2)\}} d\mathrm{Vol}_\Omega(x).$$

Here we use this second idea, except that we apply it to $V = B_\Omega(\Gamma_p\cdot x_n, R+2) \cap B_\Omega(x_n,R+3)$ instead of $B_\Omega(x_n,R+2)$, both have the same projection under π. Note that if $x \in V$ then $\gamma_0 x \in B_\Omega(x_n,R+2)$ for some γ_0, and then for each $\gamma' \in \Gamma$, if $\gamma' x_n \in B_\Omega(x_n,1)$ then $\gamma'\gamma_0 x \in B_\Omega(x_n,R+3)$. Using this we observe the following.

$$\begin{aligned}\mathrm{Vol}(\pi(B_\Omega(x_n,R+2))) = \mathrm{Vol}(\pi(V)) &= \int_{x\in V} \frac{1}{\#\{\gamma\in\Gamma_p : \gamma x \in V\}} d\mathrm{Vol}_\Omega(x)\\ &\leqslant \int_{x\in V} \frac{1}{\#\{\gamma\in\Gamma_p : \gamma x \in B_\Omega(x_n,R+3)\}} d\mathrm{Vol}_\Omega(x)\\ &\leqslant \frac{\mathrm{Vol}_\Omega(B_\Omega(x_n,R+3))}{\#\{\gamma'\in\Gamma_p : \gamma' x_n \in B_\Omega(x_n,1)\}} \leqslant \frac{C}{\#\{\gamma\in\Gamma_p : \gamma x_n \in B_\Omega(x_n,1)\}},\end{aligned}$$

where C is a constant that depends on R (see for instance [CV06, Th. 12]).

From Proposition 10.1 we know there is a Γ_p-invariant ellipsoid $\mathscr{E} \subset \mathscr{C}(\Lambda_\Gamma)$ of dimension 1 plus the rank of Γ_p tangent to p.

Up to shrinking H and making a different choice of x_0, we can assume that $x_0 \in \mathscr{E}$. Then the Hilbert distance in $\mathscr{E}$ from x_t to x_0 is t plus some constant. With this it is classical to deduce that that $\#\{\gamma\in\Gamma_p : \gamma x_n \in B_{\mathscr{E}}(x_n,1)\}$ increases exponentially fast with n, and hence so does the bigger number $\#\{\gamma\in\Gamma_p : \gamma x_n \in B_\Omega(x_n,1)\}$ (recall that distances in Ω are smaller than in $\mathscr{E}$ since $\mathscr{E} \subset \Omega$, see e.g. [CM14a, §2.1]). This makes $\sum_{n\in\mathbb{N}} \mathrm{Vol}(\pi(N \cap B_\Omega(\Gamma_p\cdot x_n, R+2)))$ summable and concludes the proof. □

12. A CORRECTION ON THE PROOF OF (GF) ⇒ ((gf)&(Hyp))

Lemma 12.1 ([CM14a, Lem. 9.3]). *Let Γ be a discrete group preserving a round convex open subset Ω. If Γ acts* (GF) *on Ω then the metric space $(\mathscr{C}(\Lambda_\Gamma), d_\Omega)$ is Gromov-hyperbolic.*

The lemma is correct but there is a mistake at the end of the proof in [CM14a, Lem. 9.3]. Let us reproduce the proof, with some minor modifications, to exhibit the mistake and explain how to fix it. One can assume that Λ_Γ spans $\mathbb{RP}^d$ (up to restricting to the span). The proof works by contradiction: one assumes there is a sequence of fatter and fatter triangles in the convex core $C = \mathscr{C}(\Lambda_\Gamma)$, with vertices x_n, y_n, z_n and a point u_n on the side $[x_n, y_n]$ whose Hilbert distance to the other sides goes to infinity.

If the projection of u_n in C/Γ stayed in a compact set, then up to translating the sequence of triangles and extracting a subsequence we could assume u_n converges to a point $u \in \Omega$ while x_n, y_n, z_n converge to points $x, y, z \in \partial\Omega$, but then by strict convexity of Ω, the point u would be at finite distance from one of the sides of the (possibly degenerate) triangle (x, y, z). Thus the projection of u_n in C/Γ does not stay in a compact set, and up to extraction we may assume it is contained in a single cusp and leaves every compact set (using that the action is geometrically finite).

Up to translating the triangles we can then assume that u_n lies in a fixed horoball H of Ω about a uniformly bounded parabolic point $u \in \partial\Omega$. Then $u_n \to u$. Up to translating again with elements of Γ_u, we may also assume that the intersection point $h_n \in \partial H \cap (u\, u_n)$ converges to a point $h \in \Omega$ (here $(u\, u_n)$ denotes the line spanned by u and u_n).

Letting Co be the cone at u spanned by C, by Proposition 10.1 there are Γ_u-invariant osculating ellipsoids $\mathscr{E}^{\text{int}} \subset \mathscr{E}^{\text{ext}}$ such that $\text{Co} \cap \mathscr{E}^{\text{int}} \subset \text{Co} \cap \Omega \subset \text{Co} \cap \mathscr{E}^{\text{ext}}$, and there is a Γ_u-invariant subspace $S \subset \mathbb{RP}^d$ of dimension one plus the rank of Γ_u, such that it intersects Ω and $S \cap \Omega \subset \text{Co}$ (see Proposition 10.1 and its proof). One can check that the Hilbert distance from u_n to $S \cap \Omega$ tends to zero: fix $o \in S \cap C$ and recall that, because u is a $\mathscr{C}^1$ point of $\partial\Omega$, the Hilbert distance between the rays $[h_n, u)$ and $[o, u)$ tends to zero as we get closer to u, so there is $o_n \in [o, u) \subset S \cap C$ whose distance to u_n tends to zero (see for instance [Ben04, Lem. 3.4]).

Now we fix a one-parameter subgroup $(\gamma^t)_t$ of hyperbolic automorphisms of $\mathscr{E}^{\text{ext}}$ that preserve S, u and the line $(o\, u)$ and use it to recenter the whole picture: we select times k_n such that $\gamma^{k_n} o_n = o$.

Now comes the small error: The sentence "Comme $\gamma^{k_n}\Omega \cap \mathscr{E}^{\text{ext}}$ est coincé entre $\gamma^{k_n}(\mathscr{E}^{\text{int}})$ et $\mathscr{E}^{\text{ext}}$" is incorrect (only when restricting to Co does the inclusion $\gamma^{k_n}(\text{Co} \cap \mathscr{E}^{\text{int}}) \subset \gamma^{k_n}(\Omega)$ become true), hence the conclusion "la suite de convexes $(\gamma^{k_n}(\Omega) \cap \mathscr{E}^{\text{ext}})$ tend, tout comme $(\gamma^{k_n}(\mathscr{E}^{\text{int}}))$, vers $\mathscr{E}^{\text{ext}}$" is incorrect too, and there are examples where $\gamma^{k_n}(\Omega)$ can converge to a convex with empty interior. To make the proof work we must recenter the picture via a sequence slightly different than $(\gamma^{k_n})_n$.

By [Ben03, Lem. 2.8] of Benoist (following Benzécri), and up to extracting a subsequence, there exists a sequence $(g_n)_n$ of projective transformations such that $\Omega_n := g_n(\Omega) \to \Omega_\infty$ and g_n equal to γ^{k_n} in restriction to S, and also such that Ω_∞ intersects S. Note that $g_n(\Omega \cap S)$ converges to $\mathscr{E}_{\text{ext}} \cap S$ and also to $\Omega_\infty \cap S$, which is therefore an ellipsoid. Note also that since $d_\Omega(u_n, o_n) \to 0$ and $g_n(o_n) = o$, we have $g_n(u_n) \to o$.

Suppose, up to extracting again, that the closures of the images of the convex core $C_n = g_n(C)$ converge to a closed convex set $\overline{C}_\infty$. To conclude, it suffices to show that $\overline{C}_\infty \subset S$. Indeed we can then conclude as earlier: we know $u'_n := g_n(u_n) \to o$ and up to extracting we also have $x'_n := g_n(x_n) \to x'$ and $y'_n := g_n(y_n) \to y'$ and $z'_n := g_n(z_n) \to z'$, all three contained in $\overline{C}_\infty \cap \partial\Omega_\infty$, hence $S \cap \partial\Omega_\infty$ which is an ellipsoid. By strict convexity of ellipsoids this means u' is at finite Hilbert distance from $[x', z']$ or $[y', z']$, which contradicts that the Hilbert distance from u'_n to $[x'_n, z'_n] \cup [y'_n, z'_n]$ goes to infinity.

Assume that there exists $(\mathrm{v}'_n)_n$ in C_n such that $\mathrm{v}'_n \to \mathrm{v}'_\infty \in \overline{C}_\infty$ but $\mathrm{v}'_\infty \notin S$. In particular, $d_{\Omega_n}(\mathrm{v}'_n, u'_n)$ is bounded from below by some constant $\epsilon > 0$ (since $u'_n \to o \in S$). Let v''_n be the intersection point of $[v'_n, u'_n]$ with the Hilbert sphere of radius ϵ around u'_n. Then $(v''_n)_n$ is a sequence in C_n converging to a point of $[o, v'_\infty] \subset \overline{C}_\infty$, which is not o, so this limit point is not in S (otherwise v'_∞ would be too). To simplify the notations we can assume that $v'_n = v''_n$, so that $d_{\Omega_n}(v'_n, u'_n) = \epsilon$ for all n.

Let $v_n = g_n^{-1}(v'_n)$, which remains at bounded Hilbert distance from u_n, and hence like u_n converges to u. Denote by a_n the intersection point of the line $(u\, v_n)$ with $\partial\Omega$, which is not u. Using that u is uniformly bounded parabolic we can find $\pi_n \in \Gamma_u$ such that $b_n := \pi_n(a_n) \to b_\infty \neq u$ after possibly extracting.

Let $w_n := \pi_n(v_n)$, and observe it also converges to u. Recall $o \in S \cap \Omega$. Since u is a $\mathscr{C}^1$-point of the boundary of Ω, there exists a $ww_n \in [o, u)$ such that $d_\Omega(w_n, ww_n) \to 0$ (see again [Ben04, Lem. 3.4]).

Hence, the Hilbert distance $d_{\Omega_n}(g_n \circ \pi_n^{-1}(ww_n), v'_n)$ tends to zero, whereas $g_n \circ \pi_n^{-1}(ww_n)$ is in S. Contradiction.

13. Counterexample to (VF)$_1$ ⇒ (GF) and ((gf)&(Hyp)) ⇒ (GF): an overview

The counterexamples to the implications (VF)$_1$ ⇒ (GF) and ((gf)&(Hyp)) ⇒ (GF) are similar to the example in Section 9, in the sense that they also come from a representation ρ of $\mathrm{SL}_2(\mathbb{R})$ into $\mathrm{SL}_5(\mathbb{R})$ that preserves round convex domains of $\mathbb{RP}^4$, except that this time we use an irreducible representation of $\mathrm{SL}_2(\mathbb{R})$, following [CM14a, §10.3].

As in Section 9, these $\rho(\mathrm{SL}_2(\mathbb{R}))$-invariant domains Ω can be described explicitly, as well as the convex hull $\mathscr{C}$ of the proximal limit set, and $\mathrm{SL}_2(\mathbb{R})$ acts cocompactly (but not transitively) on $\mathscr{C}$.

If $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ is a noncocompact lattice, then $\rho(\Gamma)$ does not act geometrically finitely on Ω, because the maximal parabolic subgroups are not conjugate into $\mathrm{O}_{4,1}(\mathbb{R})$ and the limit set spans the whole $\mathbb{RP}^4$ (see Proposition 10.1). However, that $\mathrm{SL}_2(\mathbb{R})$ acts cocompactly on $\mathscr{C}$ implies that $\mathscr{C}$ is quasi-isometric to $\mathrm{SL}_2(\mathbb{R})$, and hence is Gromov-hyperbolic. Moreover, using the ideas from the proof of Lemma 11.1, one can further prove that the quotient under $\rho(\Gamma)$ of the 1-neighborhood of $\mathscr{C}$ has finite volume.

All the above will be written in a forthcoming paper [BM]. We will also include other kinds of counterexamples, to (gf) ⇒ (VF)$_1$ and (gf) ⇒ (Hyp). In fact these examples will involve the same groups $\rho(\Gamma)$ (where ρ is the irreducible representation of $\mathrm{SL}_2(\mathbb{R})$ in $\mathrm{SL}_5(\mathbb{R})$ and Γ a noncocompact lattice of $\mathrm{SL}_2(\mathbb{R})$), but with different more subtle $\rho(\Gamma)$-invariant round domains, which are not invariant under the whole $\rho(\mathrm{SL}_2(\mathbb{R}))$.

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