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Abelian functional equations, planar web geometry and polylogarithms

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Abstract. We study some abelian functional equations (Afe). They are equations in the F_i's of the form $F_1(U_1) + \cdots + F_N(U_N) = 0$ where the U_i's are real rational functions in two variables. First we prove that the local measurable solutions are actually analytic and we characterize their components as solutions of linear differential equations constructed from the U_i's. Then we propose two methods for solving Afe. Next we apply these methods to the explicit resolution of generalized versions of classical (inhomogeneous) Afe satisfied by low order polylogarithms. Interpreted in the framework of web geometry, these results give us new nonlinearizable maximal rank planar webs. Then we observe that there is a relation between these webs and certain configurations of points in $\mathbb{CP}^2$, which leads us to define the notion of *web associated to a configuration*: these webs seem of high rank and could provide numerous new exceptional webs. Finally, we use the preceding results to show that, under weak regularity assumptions, the trilogarithm is the only function which satisfies the Spence–Kummer equation.

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1. Introduction and notation

1.1. Introduction

In this paper, we study the general solutions $(F_1, \ldots, F_N)$ of functional equations of the form below, in which the U_i's are fixed rational functions with real coefficients:

$$F_1(U_1(x, y)) + F_2(U_2(x, y)) + \cdots + F_N(U_N(x, y)) = 0. \qquad (\mathcal{E})$$

We will call them *abelian functional equations* with real rational inner functions.

One of the basic functional equations in mathematics has this form:

$$\mathbf{L}(x) + \mathbf{L}(y) - \mathbf{L}(xy) = 0. \qquad (\mathcal{C})$$

is satisfied for $x, y > 0$ by the classical logarithm. From a historical point of view, this equation is closely related to the definition of the logarithm itself and goes back to the 17th century.

It has been gradually discovered, from the early 19th century on, that low order polylogarithms $\mathbf{Li}_n$ satisfy some (inhomogeneous) functional equations of the type $(\mathcal{E})$ (see [Lew]). Abel, Kummer, Spence (and others) have established numerous versions of the following functional equation (now named *five terms Abel's functional equation*) satisfied by the bilogarithm $\mathbf{Li}_2$ for $0 < x < y < 1$:

$$\mathbf{L}(x) - \mathbf{L}(y) - \mathbf{L}\left(\frac{x}{y}\right) - \mathbf{L}\left(\frac{1-y}{1-x}\right) + \mathbf{L}\left(\frac{x(1-y)}{y(1-x)}\right) = -\frac{\pi^2}{6} + \log(y)\,\log\left(\frac{1-y}{1-x}\right) \qquad (\mathcal{A}\mathrm{b})$$

(this is Schaffer's form, see [Sch]). Spence and (mostly) Kummer have discovered many functional equations satisfied by polylogarithms of order less than 5, such as $\mathbf{Li}_3$, which satisfies the following *Spence–Kummer equation* for $0 < x < y < 1$:

$$\begin{aligned} 2\mathbf{L}(x) + 2\mathbf{L}(y) - \mathbf{L}\left(\frac{x}{y}\right) + 2\mathbf{L}\left(\frac{1-y}{1-x}\right) + 2\mathbf{L}\left(\frac{x(1-y)}{y(1-x)}\right) - \mathbf{L}(xy) \\ + 2\mathbf{L}\left(-\frac{x(1-y)}{1-x}\right) + 2\mathbf{L}\left(-\frac{1-y}{y(1-x)}\right) - \mathbf{L}\left(\frac{x(1-y)^2}{y(1-x)^2}\right) \\ = 2\mathbf{Li}_3(1) - \log^2(y)\log\left(\frac{1-y}{1-x}\right) + \frac{\pi^2}{3}\log(y) + \frac{1}{3}\log^3(y). \qquad (SK) \end{aligned}$$

By definition, $\mathbf{Li}_n(z) = \sum_{k=1}^{\infty} z^k/k^n$ for $|z| < 1$, so $\mathbf{Li}_1(z) = -\log(1-z)$. Then using the integral representation $\mathbf{Li}_{n+1}(z) = \int_0^z \mathbf{Li}_n(\xi)\xi^{-1}\,d\xi$ for $n > 0$, we see that the polylogarithms extend to multivalued holomorphic functions on $\mathbb{C}\backslash\{0,1\}$. For the last 30 years, the polylogarithms have appeared in many branches of mathematics (see for instance [Gon1], [Ost], [P], [Za]). In particular, Ramakrishnan, Wojtkowiak and Zagier have constructed some single-valued real versions $\mathcal{L}_n$ of the classical polylogarithms $\mathbf{Li}_n$ (see Section 3 in [Ost]). They are continuous functions defined on the whole $\mathbb{CP}^1$, analytic except at $0, 1$ and ∞.

These single-valued polylogarithms $\mathcal{L}_n$ globally satisfy "clean" versions of the inhomogeneous Afe satisfied by the classical polylogarithms (Théorème 2 in [Ost]): assume that $\mathbf{Li}_n$ satisfies locally an equation $\sum_{k=1}^{N} a_k \mathbf{Li}_n(U_k) = \mathbf{E}\mathrm{lem}_n$, where the a_k are integers, the U_k rational functions, and $\mathbf{E}\mathrm{lem}_n$ a polynomial evaluated on arguments of the form $\mathbf{Li}_k(V_k)$ with $k < n$ where the V_k are rational functions. Then the single-valued cousin $\mathcal{L}_n$ of $\mathbf{Li}_n$ will globally satisfy $\sum_{k=1}^{N} a_k \mathcal{L}_n(U_k) =$ const.

So the $\mathcal{L}_n$'s provide solutions to the homogeneous Afe $\sum_{k=1}^{N} F_k(U_k) = 0$ naturally associated to the functional equations satisfied by classical polylogarithms. This shows that the theory of functional equations in two variables of the polylogarithms can be considered a particular case of the theory of Afe.

The content of the paper is the following: Section 1 is the present introduction. Section 2 is devoted to the general study of local solutions $(F_1, \ldots, F_N)$ of the general equation $(\mathcal{E})$ at $\omega \in \mathbb{R}^2$. In the spirit of the second part of Hilbert's 5th problem (see [Acz]), we want to make minimal regularity assumptions on the F_i's "nice properties" for these solutions. We first prove (Proposition 1 of Section 2.2.1) that any local measurable solution of $(\mathcal{E})$ is actually analytic (modulo a genericity condition on ω): this allows us to complexify the problem and to restrict ourselves to the study of local holomorphic solutions.

As already noticed by Abel, one functional equation in several variables can determine several unknown functions which must be very specific. In our case, this philosophy works very well and gives us Theorem A. For $\mathsf{R} = (U_i) \in \mathbb{R}(x, y)^N$, we denote by $\mathcal{F}_i$ the foliation of $\mathbb{CP}^2$ given by $\{U_i = \text{const}\}$. We set $\mathcal{W}_{\mathsf{R}} = \{\mathcal{F}_i\}$. It is a web if the leaves of the $\mathcal{F}_i$'s are generically in general position. In this case its singular locus denoted Σ is a proper Zariski closed set in $\mathbb{CP}^2$ (see 1.2 for definitions).

Theorem A. *Assume that $\mathcal{W}_{\mathsf{R}}$ is a web and fix $\omega \in \mathbb{R}^2 \setminus \Sigma$. Then for each $i \in \{1, \ldots, N\}$ there exists a linear differential equation (Lde_i) whose coefficients are algebraic functions such that, if $F_1, \ldots, F_N$ are measurable germs satisfying $F_1(U_1) + \cdots + F_N(U_N) = 0$ in a neighbourhood of ω, then every F_i is analytic and generically satisfies (Lde_i). The germ F_i admits analytic continuation along any path in the Zariski open set $X_i = U_i(\mathbb{CP}^2 \setminus \Sigma) \subset \mathbb{CP}^1$.*

Our result is effective: given an N-uplet R, we can explicitly construct the equations (Lde_i) in terms of the U_i's. We prove this theorem by using elementary methods of complex analysis: from Proposition 1 (in Section 2.2.1), we know that the F_i's are analytic germs. Then by successive differentiations along the level curves of the functions U_i, we construct for each i the linear differential equation (Lde_i) from $(\mathcal{E})$. This is essentially an application to our case of Abel's method for solving functional equations in several variables, described in [Ab]. Finally, we prove the analytic continuation along any path in X_i by using a simple geometrical argument (see Proposition 3 in Section 2.2.1).

From Theorem A, we deduce two methods to find solutions to a given equation of the form $(\mathcal{E})$. The first, which we call *Abel's method* (see 2.3.1), is effective and can be implemented on a computer: it consists in solving the equation (Lde_i) given by Theorem A in order to reconstruct the solutions of $(\mathcal{E})$. The second method, presented in 2.3.2, is not so general. It is based on the fact that certain solutions of $(\mathcal{E})$ are characterized by their monodromy, which can be determined a priori from $(\mathcal{E})$.

As corollaries of Theorem A, we rediscover that there are no nonconstant solutions in the generic case and that the dimension of the space of local holomorphic solutions of $(\mathcal{E})$ is finite. A basic theorem of web geometry (see [Bol1]) says that this dimension is less than $N(N-1)/2$.

In Section 3, we explicitly solve three Afe for which this bound is reached. For instance, using monodromy arguments, we find again that the functional equation

$$F_1(x)+F_2(y)+F_3\left(\frac{x}{y}\right)+F_4\left(\frac{1-x}{1-y}\right)+F_5\left(\frac{x(1-y)}{y(1-x)}\right)=0, \qquad (\mathrm{L}_2)$$

naturally associated to ($\mathcal{A}$b), has six linearly independent holomorphic solutions, the most interesting of them involving Rogers' version of the bilogarithm. We next solve, in 3.2, the Afe similarly associated to (SK); then we solve in 3.3 an Afe denoted $(\mathcal{E}_a)$ depending on a parameter a.

In Section 4.1, we interpret the preceding results in the framework of planar web geometry. To equation $(\mathcal{E})$ we can associate a web $\mathcal{W}=\mathcal{W}\{U_i\}$. A nonconstant solution of $(\mathcal{E})$ is called an abelian relation for $\mathcal{W}$. The dimension of the space of abelian relations is a basic invariant for the geometrical study of $\mathcal{W}$ (i.e., its study up to local analytic equivalence). This dimension is named the rank of $\mathcal{W}$.

A class of maximal rank webs is given by considering the so-called algebraic webs associated by projective duality to algebraic plane curves. One can prove that maximal rank planar N-webs are algebraic for $N=3,4$. In 1936, Bol [Bol2] found that the 5-web associated to (L_2), now named Bol's web, is of maximal rank but is not algebraic. Chern called such webs exceptional or exotic. According to him, *the determination of such webs which are exotic is a problem of great interest and importance* (page 5 of [Che1]). Bol's web was until now the only known example of a nonalgebraic maximal rank planar web. Our results obtained in 3.2 and 3.3 yield new exceptional planar webs, such as the Spence–Kummer web $\mathcal{W}_{\mathcal{SK}}$:

Theorem B. *The web $\mathcal{W}_{\mathcal{SK}}$ associated to equation (SK) is an exceptional 9-web.*

The fact that we have found an explicit equivalent of the space of abelian relations for this web in 3.2 allows us to study its subwebs. Thus we discover an exceptional 7-web, two inequivalent exceptional 6-webs, and a new exceptional 5-web. Then we observe that, similar to Bol's web, modulo a suitable change of coordinates, $\mathcal{W}_{\mathcal{SK}}$ is related to a certain configuration of points in $\mathbb{CP}^2$. This remark leads us to define the notion of web associated to a configuration of points in the complex projective plane (see Definition 3). We next consider the most elementary case when the nature of the web associated to a configuration is not known. This case corresponds to the webs $\mathcal{W}_a$ associated to degenerate configurations C_a of five points depending on a parameter a (see Figure 3). The solutions of $(\mathcal{E}_a)$ obtained in 3.3 can now be interpreted as a basis of the space of abelian relations for $\mathcal{W}_a$, which therefore is exceptional. By studying the subwebs of the $\mathcal{W}_a$'s we obtain (in Propositions 9–11) several families of pairwise inequivalent exceptional k-webs for $k=6,8$. We can summarize some of the preceding results in the following

Theorem C. *For $n=3,4,5$, the web associated to any (degenerate if $n=5$) configuration of n points in $\mathbb{CP}^2$ is of maximal rank. Therefore it is exceptional if and only if it contains a subconfiguration of four points in general position.*

This suggests that webs associated to configurations of points could be of high rank. This could give numerous exceptional webs and therefore numerous equations of the form $(\mathcal{E})$. Since the equations in Section 3 (related to webs associated to configurations) are mostly constructed by using iterated integrals, it could give functional equations for higher order polylogarithms. Some important links between functional equations, polylogarithms and configurations of points already appear in several papers, including those by Gelfand–MacPherson [GeMcP] and Goncharov [Gon1]. Understanding the webs associated to configurations of points which we consider in this paper from their point of view could be very interesting.

In Section 4.2 we apply the preceding results to the problem of characterizing the measurable functions $\mathbf{L}$ which satisfy equations $(\mathcal{A}\mathrm{b})$ or (SK). We see again that Rogers' dilogarithm is characterized by the Afe it satisfies. We show that the same occurs with the trilogarithm relative to the Spence–Kummer equation:

Theorem D. *Let F be a real measurable function satisfying equation (SK) for $0 < x < y < 1$. If F is derivable at 0, then $F = \mathbf{Li}_3$.*

($\mathbf{Li}_3(x)$ is considered here as an analytic function on $]-\infty, 1[$ by using the integral representation formula $\mathbf{Li}_3(x) = \int_0^x \frac{1}{\xi} \int_0^\xi -\frac{1}{\sigma}\log(1-\sigma)\, d\sigma\, d\xi$ for $x < 1$.)

Remark. Theorems B and D first appeared in the preprint [Pi]. G. Robert has independently found that the Spence–Kummer web is of maximal rank by constructing an equivalent of the space of its abelian relations (see [Rob]).

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1.2. Notation

We introduce here some notations which we will use in the paper.

Throughout, N will be a fixed integer larger than 3. If not specified, U_i will denote (for $i = 1, \dots, N$) a nonconstant element of $\mathbb{R}(x, y)$ considered as a holomorphic map $\mathbb{CP}^2 \setminus S_i \to \mathbb{CP}^1$, where S_i denotes the indeterminacy locus of U_i; it is a finite set.

A functional equation of the form $F_1(U_1) + \cdots + F_N(U_N) = 0$ will be called an *abelian functional equation* (abbr. Afe) with real rational inner functions. The name comes from the notion of abelian relation in web geometry, itself related to the notion of abelian sum in algebraic geometry (see 4.1.1 or part 2 in [ChGr1]).

In the entire text, $(\mathcal{E})$ will denote a general Afe $\quad F_1(U_1) + \cdots + F_N(U_N) = 0$.

The foliation $\mathcal{F}_i = \mathcal{F}\{U_i\}$ will be the global singular foliation of $\mathbb{CP}^2$ whose leaves are the level curves of U_i. Let $\mathsf{R} = (U_1, \dots, U_N)$ be an N-uplet of nonconstant real rational functions. To the set $\mathcal{F}_{\mathsf{R}} = \{\mathcal{F}_i \mid i \leq N\}$ of foliations, we associate the following algebraic subset of $\mathbb{CP}^2$:

$$\Sigma[\mathsf{R}] := \bigcup_{i=1}^{N} \Sigma_i \cup \bigcup_{i \neq j} \{\omega \in \mathbb{CP}^2 \setminus (S_i \cup S_j) \mid (dU_i \wedge dU_j)(\omega) = 0\}$$

(where Σ_i denotes the singular locus of the foliation $\mathcal{F}_i$ for $i = 1, \dots, N$).

By definition, $\mathcal{F}_{\mathsf{R}}$ is a *web* if $\Sigma[\mathsf{R}]$ is a proper subset of $\mathbb{CP}^2$. In this case we write $\mathcal{W}\{U_i\}$ or $\mathcal{W}[\mathsf{R}]$ for $\mathcal{F}_{\mathsf{R}}$. The algebraic set $\Sigma[\mathsf{R}]$ will be called the *singular* *locus* of the web. Because it is the union of the singular loci of the foliations $\mathcal{F}_i$ and the locus in which the leaves of the foliations are not in general position, it depends only on the web and not on the functions U_i.

The web $\mathcal{W}\{U_i\}$ defined by the functions U_i appearing in the Afe $(\mathcal{E})$ will be denoted by $\mathcal{W}[\mathcal{E}]$. By definition, it is the *web associated to* $(\mathcal{E})$.

Let $\mathcal{W} = \{\mathcal{F}_i \mid i \leq N\}$ be a fixed web. For any subset $K \subset \{1, \ldots, N\}$ we write $\mathcal{W}_K$ for the subweb of $\mathcal{W}$ whose foliations are the $\mathcal{F}_k$ for $k \in K$. If $\kappa_1, \ldots, \kappa_p$ are distinct integers in $\{1, \ldots, N\}$, we set

$$\widehat{\kappa_1 \ldots \kappa_p} := \{1, \ldots, N\} \setminus \{\kappa_1, \ldots, \kappa_p\}.$$

In this text, X will denote a complex manifold. If $\mathcal{F}$ is a sheaf of function germs on X then for $\omega \in X$, $\underline{\mathcal{F}_\omega}$ will denote the function germs of this sheaf at ω. We will denote by $\underline{\mathcal{F}_\omega}(X)$ the image of the restriction morphism $\mathcal{F}(X) \to \underline{\mathcal{F}_\omega}$. We will mostly consider the sheaf $\mathcal{M}$ of measurable real-valued function germs and the sheaf $\mathcal{O}_X$ (abbr. $\mathcal{O}$) of holomorphic germs on X. Moreover, $\widetilde{X}$ will be the analytic simply connected covering of X. The sheaf $\mathcal{O}_{\widetilde{X}}$ of multivalued holomorphic functions on X will be denoted by $\widetilde{\mathcal{O}}_X$ or $\widetilde{\mathcal{O}}$. Most often, X will be a Zariski open set in $\mathbb{CP}^k$ with $k = 1, 2$. In Section 2.2.2, we will use the sheaf of multivalued holomorphic functions on X with logarithmic growth at infinity, denoted $\widetilde{\mathcal{O}}^{\log}$.

If γ is a path linking ω to ϖ in X and if $F \in \underline{\mathcal{O}_\omega}$ admits an analytic continuation along γ, then we write $F^{[\gamma]}$ for the holomorphic germ at ϖ obtained by this analytic continuation. If γ is a loop, we will write $\mathcal{M}_\gamma F$ for $F^{[\gamma]}$.

For $\omega \in \mathbb{CP}^2$, throughout the paper, we set $\omega_i := U_i(\omega) \in \mathbb{CP}^1$ when it is well defined. Then a *local solution of* $(\mathcal{E})$ *at* ω *in the class* $\mathcal{F}$ is an element of

$$\underline{\mathcal{S}^{\mathcal{F}}_\omega}(\mathcal{E}) = \Big\{ \mathbf{F} = (F_1, \ldots, F_N) \in \prod_{i=1}^{N} \underline{\mathcal{F}_{\omega_i}} \;\Big|\; \sum_{i=1}^{N} F_i(U_i) = 0 \text{ in } \underline{\mathcal{F}_\omega} \Big\}.$$

We remark that if $\mathcal{F} = \mathcal{O}$, then $\underline{\mathcal{S}^{\mathcal{O}}_\omega}(\mathcal{E})$ is the space of all local holomorphic solutions of the complex version of $(\mathcal{E})$ at ω. By definition a *solution with logarithmic growth* of $(\mathcal{E})$ will be an element of $\underline{\mathcal{S}^{\widetilde{\mathcal{O}}^{\log}}_\omega}(\mathcal{E})$.

In the entire paper, to any $\mathbf{H} = (H_1, \ldots, H_N) \in \prod_i \underline{\mathcal{F}_{\omega_i}}$ such that the sum $H_1(U_1) + \cdots + H_N(U_N)$ is constant and equal to c, we associate the element $(H_1 - c, \ldots, H_N)$ of $\underline{\mathcal{S}^{\mathcal{F}}_\omega}(\mathcal{E})$, again denoted $\mathbf{H}$.

If K is a subset of $\{1, \ldots, N\}$, we denote by $(\mathcal{E}_K)$ the equation $\sum_{k \in K} F_k(U_k) = 0$. For $\omega \notin \Sigma[\mathcal{E}]$ we have $\omega \notin \Sigma[\mathcal{E}_K]$ and there is a linear embedding $\underline{\mathcal{S}^{\mathcal{F}}_\omega}(\mathcal{E}_K) \hookrightarrow \underline{\mathcal{S}^{\mathcal{F}}_\omega}(\mathcal{E})$. So we will consider the local solutions of $(\mathcal{E}_K)$ as particular local solutions of $(\mathcal{E})$.

For $p \in \{3, \ldots, N\}$, we write $F^p\underline{\mathcal{S}^{\mathcal{F}}_\omega}(\mathcal{E})$ for the sum $\sum \underline{\mathcal{S}^{\mathcal{F}}_\omega}(\mathcal{E}_P)$ where P runs over all p-element subsets in $\{1, \ldots, N\}$. An element of $F^p\underline{\mathcal{S}^{\mathcal{F}}_\omega}(\mathcal{E}) \setminus F^q\underline{\mathcal{S}^{\mathcal{F}}_\omega}(\mathcal{E})$ (with $q = p - 1$) will be called a *solution of order* p *of equation* $(\mathcal{E})$. A solution of order $p < N$ (resp. N) will be called a *subsolution* (resp. a *genuine solution*) of $(\mathcal{E})$.

The components of most known solutions of Afe with rational inner functions are constructed from iterated integrals. This notion was introduced by K. T. Chen in the 60's. We now introduce the notation related to iterated integrals used in the paper.

Set $X = \mathbb{CP}^2 \setminus \Sigma[\mathcal{W}]$ and $Z = \mathbb{CP}^1 \setminus U_i(\Sigma[\mathcal{W}])$ where i is fixed $(1 \leq i \leq N)$. There exist a finite number of distinct points $a_0, \ldots, a_{M_i}$ in $\mathbb{CP}^1$ such that $Z = \mathbb{CP}^1 \setminus \{a_i\}$. We can always assume that $a_0 = \infty$. We inductively define the iterated integrals which are multivalued functions denoted $\mathbf{L}_{x_{i_1} \ldots x_{i_m}}$ with $i_k \in \{1, \ldots, M_i\}$: if $z \in Z$ and if γ is a path from ω_i to z in Z, then we set

$$\mathbf{L}_{x_{i_0} x_{i_1} \ldots x_{i_m}}(z, \gamma) := \int_\gamma \frac{\mathbf{L}_{x_{i_1} \ldots x_{i_m}}(\xi)}{a_{i_0} - \xi} \, d\xi.$$

These functions are holomorphic on the analytic universal covering of Z. We denote by $\mathcal{I}\{Z\}$ (or $\mathcal{I}\{a_i\}$) the subspace of $\widetilde{\mathcal{O}}(Z)$ spanned by the constants and the iterated integrals defined above (it is well defined).

In Section 3, we will use special notation for some elements of $\mathcal{I}\{0, 1, a\}$ where $a > 1$ is a parameter: let Ω_a be the domain $\mathbb{C} \setminus (\Delta_0 \cup \Delta_1 \cup \Delta_a)$, where Δ_0, Δ_1 and Δ_a are respectively the half-lines $i\mathbb{R}^-$, $1 + i\mathbb{R}^+$ and $a + i\mathbb{R}^+$ of $\mathbb{C}$. Now Ω_a is simply connected and does not contain 0, 1 and a, so for any $z \in \Omega_a$ the value of any function defined by the expression below is well defined if we integrate along any path in Ω_a; we set (with $\kappa \in \{1, a\}$, $\epsilon \in \{0, 1, a\}$ and with $\mathbf{L}$og standing for the principal branch of the logarithm on Ω_a)

$$\mathbf{L}_{x_0}(\bullet) = \mathbf{L}\mathrm{og}(\bullet), \qquad \mathbf{L}_{x_\kappa}(\bullet) = -\mathbf{L}\mathrm{og}(\kappa - \bullet), \qquad \mathbf{L}_{x_\epsilon^2}(\bullet) = \tfrac{1}{2}\mathbf{L}_{x_\epsilon}^2(\bullet),$$

$$\mathbf{L}_{x_0 x_1}(\bullet) = \mathbf{L}\mathrm{i}_2(\bullet), \qquad \mathbf{L}_{x_1 x_0}(\bullet) = \int_0^\bullet \tfrac{\mathbf{L}_{x_0}(\zeta)}{1-\zeta} \, d\zeta, \qquad \mathbf{L}_{x_0 x_a}(\bullet) = \int_1^\bullet \tfrac{\mathbf{L}_{x_a}(\zeta)}{\zeta} \, d\zeta,$$

$$\mathbf{L}_{x_a x_0}(\bullet) = \int_1^\bullet \tfrac{\mathbf{L}_{x_0}(\zeta)}{a-\zeta} \, d\zeta, \quad \mathbf{L}_{x_a x_1}(\bullet) = \int_0^\bullet \tfrac{\mathbf{L}_{x_1}(\zeta)}{a-\zeta} \, d\zeta, \qquad \mathbf{L}_{x_1 x_a}(\bullet) = \int_0^\bullet \tfrac{\mathbf{L}_{x_a}(\zeta)}{1-\zeta} \, d\zeta,$$

$$\mathbf{L}_{x_0^2 x_1}(\bullet) = \mathbf{L}\mathrm{i}_3(\bullet), \qquad \mathbf{L}_{x_0 x_1 x_0}(\bullet) = \int_0^\bullet \tfrac{\mathbf{L}_{x_1 x_0}(\zeta)}{\zeta} \, d\zeta, \quad \mathbf{L}_{x_1 x_0^2}(\bullet) = \int_0^\bullet \tfrac{\mathbf{L}_{x_0^2}(\zeta)}{1-\zeta} \, d\zeta.$$

A *polylogarithmic function* will be a function constructed from elements of $\mathcal{I}\{0, 1\}$.

2. General properties of solutions of $(\mathcal{E})$

2.1. Preliminary remarks

Our objective is to study the solutions of a general Afe $(\mathcal{E})$ with real rational inner functions: using the notation introduced in 1.2, we want to determine the space $\underline{\mathcal{S}_\omega^{\mathcal{F}}}(\mathcal{E})$ of local solutions of $(\mathcal{E})$ around ω in the class $\mathcal{F}$. What we want to prove is that, roughly speaking, the solutions of $(\mathcal{E})$ are analytic, admit analytic continuation onto a Zariski open set of $\mathbb{CP}^1$ and form a finite-dimensional linear space.

But we have to make some restrictions on $\mathcal{F}$ and ω to avoid pathological situations: we have to deal with at least measurable functions and we have to take

ω outside the singular locus Σ of the web $\mathcal{W}[\mathcal{E}]$. These two assumptions appear reasonable and quite natural if we consider the *generalized Cauchy equation*

$$F_1(x) + F_2(y) + F_3(xy) = 0. \tag{C}$$

It is well known that the space of multiplicative functions $F : \mathbb{R}_{>0} \to \mathbb{R}$ is infinite-dimensional. To any such function corresponds the solution $(F, F, -F)$ of (C). Such functions are generally not measurable: if the function is measurable, then it is constructed from the logarithm. So, if no restriction on the regularity of the F_i's is made, the space of solutions can be infinite-dimensional, which contrasts with the measurable setting in which we have $\dim_{\mathbb{R}} \underline{\mathcal{S}_\omega^{\mathcal{M}}}(C) = 3$. The assumption of measurability of solutions of the general equation $(\mathcal{E})$ appears natural.

Under this assumption we can expect the solutions to have some good regularity properties such as analyticity: in our case, the only nonconstant measurable solution of (C) is $(\log, \log, -\log)$, which is analytic indeed. But obtaining a precise local version of this statement requires another assumption, about ω: it is clear that (C) does not admit any nonconstant analytic (and even continuous) solution at the origin. This comes from the fact that 0 belongs to the singular locus of $\mathcal{W}\{x, y, xy\}$. Therefore the point ω must not belong to this set if we want to have nice properties of elements of $\underline{\mathcal{S}_\omega^{\mathcal{F}}}(C)$.

These elementary remarks show that both the hypotheses of measurability for the F_i's and of genericity for ω are quite natural and reasonable.

2.2. Properties of measurable solutions of $(\mathcal{E})$

2.2.1. Analyticity of measurable solutions. We now prove the analycity of a local measurable solution of $(\mathcal{E})$ at a generic ω:

Proposition 1. *Let $\omega \in \mathbb{R}^2 \setminus \Sigma$ and $\mathbf{F} = (F_1, \ldots, F_N) \in \underline{\mathcal{S}_\omega^{\mathcal{M}}}(\mathcal{E})$. Then each F_i is in fact an analytic germ at ω_i. Its complexification gives a holomorphic germ $F_i^{\mathbb{C}}$ such that $\mathbf{F}^{\mathbb{C}} = (F_1^{\mathbb{C}}, \ldots, F_N^{\mathbb{C}})$ is a holomorphic solution of $(\mathcal{E})$ at ω.*

Proof. By hypothesis we have $\omega \notin \Sigma$, so it follows from Theorem 3.3 of [Jár] that the F_i's are continuous germs at ω_i. By elementary tools of integration it follows next that they are C^∞ smooth germs. We obtain analyticity using the method of Joly and Rauch [JoRa], by formulating the equation $(\mathcal{E})$ in the form of a linear elliptic $(N + 2) \times N$ differential system. The analyticity of the F_i's follows from classical results on the regularity of solutions of elliptic systems. Finally, the unicity principle implies that $\mathbf{F}^{\mathbb{C}}$ is a holomorphic solution of $(\mathcal{E})$. □

So we have two $\mathbb{R}$-linear morphisms: the first (denoted ρ) is just the restriction to $\mathbb{R}^2$ of the real part of the holomorphic solutions, and the second (denoted ϱ) is the complexification of the solutions given by Proposition 1:

$$\begin{aligned} \rho : \underline{\mathcal{S}_\omega^{\mathcal{O}}}(\mathcal{E}) &\to \underline{\mathcal{S}_\omega^{\mathcal{M}}}(\mathcal{E}) \\ \mathbf{G} &\mapsto \Re e(\mathbf{G}|_{\mathbb{R}^2}) \end{aligned} \qquad\qquad \begin{aligned} \varrho : \underline{\mathcal{S}_\omega^{\mathcal{M}}}(\mathcal{E}) &\to \underline{\mathcal{S}_\omega^{\mathcal{O}}}(\mathcal{E}) \\ \mathbf{F} &\mapsto \mathbf{F}^{\mathbb{C}} \end{aligned}$$

It is clear that $\rho \circ \varrho = \mathbf{I}\mathrm{d}$, so the study of real measurable solutions of $(\mathcal{E})$ at ω amounts to the study of holomorphic ones.

2.2.2. Characterization of components of holomorphic solutions. It is known that, in the generic case, there is no nonconstant holomorphic solution of a general abelian functional equation. Consider now the very specific case when $(\mathcal{E})$ has a nontrivial local holomorphic solution $\mathbf{F} = (F_1, \ldots, F_N)$. Any nonconstant component germ F_i of $\mathbf{F}$ must be a function of a very specific kind. The point is that any germ F_i is just a local branch of a globally defined but ramified function which satisfies a linear differential equation with algebraic coefficients. Let $\mathsf{R} = (U_1, \ldots, U_N) \in \mathbb{R}(x, y)^N$ be such that $\Sigma = \Sigma[\mathsf{R}]$ is proper. Let $\omega \in \mathbb{R}^2 \setminus \Sigma$ be fixed. Then we have the following:

Theorem 1. *For every $i \in \{1, \ldots, N\}$ there exists a linear differential equation* (Lde_i) *whose coefficients are algebraic functions (meromorphic in a neighbourhood of ω_i) such that for all $(F_1, \ldots, F_N) \in \underline{\mathcal{S}}^{\mathcal{O}}_{\omega}(\mathcal{E})$, the germ F_i satisfies* (Lde_i) *in a neighbourhood of ω_i. The germ F_i is a local branch at ω_i of a globally defined multivalued function on $\mathbb{CP}^1$ whose ramification points belong to the finite set $U_i(\Sigma) \subset \mathbb{CP}^1$.*

Proof. Without any loss of generality, we can assume that $\omega = (0,0) \notin \Sigma$ and $U_i(\omega) = 0$ for $i = 1, \ldots, N$. Let $F_i \in \underline{\mathcal{O}}(\mathbb{C}, 0)$ be N germs such that $F_1(U_1) + \cdots + F_N(U_N) = 0$ in a neighbourhood of ω. For $\rho > 0$ set $\mathsf{D}_\rho = \{z \in \mathbb{C} \mid |z| < \rho\}$. If $i \neq j$, since $\omega \notin \Sigma$, (U_i, U_j) defines a system of holomorphic coordinates on a neighbourhood Ω_{ij} of ω. It is clear that we can find $\epsilon > 0$ such that each F_i is holomorphic on the whole D_ϵ, and $\Omega := \bigcap_k U_k^{-1}(\mathsf{D}_\epsilon) \subset \Omega_{ij}$ for all $i \neq j$.

We now want to deduce from the functional equation $(\mathcal{E})$ a linear differential equation (abbr. Lde) satisfied by F_N (or by any other F_i, the process remaining the same). To do this, it is useful to introduce a more general class of equations which will be a bridge between Afe and Lde:

Definition 1. Let $k > 0$, $(M_1, \ldots, M_k) \in \mathbb{N}^k$, and let V_i, $\mathcal{A}_i^j$ $(1 \leq i \leq k, 0 \leq j \leq M_i)$ be holomorphic functions on an open domain $\Theta \subset \mathbb{C}^2$. An *Abelian Differential Functional Equation* (abbr. Adfe) is an equation of the type

$$\sum_{i=1}^{k} \sum_{j=0}^{M_i} \mathcal{A}_i^j G_i^{(j)}(V_i) = 0 \qquad (\mathcal{A}\text{dfe})$$

where the unknowns are the function germs $G_1, \ldots, G_k$, supposed analytic ($G_i^{(\kappa)}$ denoting the κth derivative of G_i for $\kappa \in \mathbb{N}$).

The k-uplet $(M_1, \ldots, M_k)$ is called the *type* of equation $(\mathcal{A}\text{dfe})$, and its *true type* if $\mathcal{A}_i^{M_i} \not\equiv 0$ for all i.

The notion of Adfe generalizes Afe and Lde: Afe are Adfe of true type $(0, \ldots, 0)$, and Lde are Adfe of true type (M) with $M > 0$.

Assume that for all $i \neq j$, the couple (V_i, V_j) (as in Definition 1) defines holomorphic coordinates on Θ. We now describe a process of obtaining an Adfe of type $(M_1 - 1, M_2 + 1, \ldots, M_k + 1)$, or $(M_2 + 1, \ldots, M_k + 1)$ if $M_1 = 0$, from an Adfe

of true type $(M_1, \ldots, M_k)$. We give the details only for the case when $M_1 > 0$: by definition we have $\mathcal{A}_1^{M_1} \not\equiv 0$ on Θ; therefore equation ($\mathcal{A}$dfe) implies that on $\Theta' = \Theta \setminus \{\mathcal{A}_1^{M_1} = 0\}$, we have

$$G_1^{(M_1)}(V_1) + \sum_{j=0}^{M_1-1} \frac{\mathcal{A}_1^j}{\mathcal{A}_1^{M_1}} G_1^{(j)}(V_1) + \sum_{i=2}^{k}\sum_{j=0}^{M_i} \frac{\mathcal{A}_i^j}{\mathcal{A}_1^{M_1}} G_i^{(j)}(V_i).$$

Let ∂ be the vector field on Θ which corresponds to differentiation with respect to V_2 in the coordinate system (V_1, V_2). By applying ∂ to this last form of ($\mathcal{A}$dfe) we get a new Adfe on Θ':

$$\sum_{j=0}^{M_1-1} \partial\left(\frac{\mathcal{A}_1^j}{\mathcal{A}_1^{M_1}}\right) G_1^{(j)}(V_1) + \sum_{i=2}^{k}\sum_{j=0}^{M_i}\left(\partial\left(\frac{\mathcal{A}_i^j}{\mathcal{A}_1^{M_1}}\right) G_i^{(j)}(V_i) + \frac{\mathcal{A}_i^j}{\mathcal{A}_1^{M_1}} \partial(V_i) G_i^{(j+1)}(V_i)\right) = 0$$

which can be written as

$$\sum_{i=1}^{k}\sum_{j=0}^{\widehat{M}_i} \widehat{\mathcal{A}}_i^j G_i^{(j)}(V_i) = 0 \qquad (\mathcal{A}\text{dfe}')$$

where $\widehat{M}_i = M_i + 1$ (resp. $M_1 - 1$) if $i > 1$ (resp. $i = 1$) and

$$\widehat{\mathcal{A}}_i^j = \begin{cases} \partial\left(\dfrac{\mathcal{A}_1^j}{\mathcal{A}_1^{M_1}}\right) & \text{if } i = 1 \text{ and } 0 < j < M_1, \\ \partial\left(\dfrac{\mathcal{A}_i^0}{\mathcal{A}_1^{M_1}}\right) & \text{if } i > 1 \text{ and } j = 0, \\ \partial\left(\dfrac{\mathcal{A}_i^j}{\mathcal{A}_1^{M_1}}\right) + \dfrac{\mathcal{A}_i^{j-1}}{\mathcal{A}_1^{M_1}}\partial(V_i) & \text{if } i > 1 \text{ and } 0 < j \le M_i, \\ \dfrac{\mathcal{A}_i^{M_i}}{\mathcal{A}_1^{M_1}}\partial(V_i) & \text{if } i > 1 \text{ and } j = M_i + 1. \end{cases} \qquad (\star)$$

We remark that because $\partial(V_i) \neq 0$ for $i > 1$, no $\widehat{\mathcal{A}}_i^j$ for $j = M_i+1$ is identically zero, so the equation that we obtain is of true type $(M, M_2+1, \ldots, M_k+1)$, where M is an integer smaller than $M_1 - 1$. If $M_1 = 0$, by the same process, we obtain a new Adfe again, but of true type $(M_2 + 1, \ldots, M_k + 1)$. Thus, by appropriate eliminations, we can reduce our Adfe to another one, of strictly smaller true type (relative to a notion of order on the types which is clear from the context). This reduction process can always be performed, except when the equation is in fact a linear differential equation with coefficients depending on a single variable. Such an Adfe will be said to be *irreducible*.

Now let $(F_1, \ldots, F_N) \in (\mathcal{O}^2)^N$ be such that $F_1(U_1) + \cdots + F_N(U_N) = 0$ on Ω. By induction on the type, one shows that there exists a proper analytic subset $\Lambda \subset \Omega$ such that, on $\Omega' = \Omega \setminus \Lambda$, the germ F_N satisfies an equation of the form

$$\mathsf{A}_1(U_N) F_N^{(1)}(U_N) + \cdots + \mathsf{A}_K(U_N) F_N^{(K)}(U_N) = 0. \qquad (\mathsf{Lde}_N)$$

The A_i are holomorphic functions on Ω' depending only on U_N. The formulas in $(\star)$ show that the A_i can also be expressed as rational functions in the U_i's and their partial derivatives.

It is clear that (Lde_N) does not depend on the solution $(F_i)_{i=1}^n$ but only on the U_i's. Thus the N-th component of every solution $\mathbf{F} \in \underline{\mathcal{S}_\omega^{\mathcal{O}}}(\mathcal{E})$ will satisfy this equation, at least generically, in a neighbourhood of ω_N.

We now prove by induction on the type that the coefficients of (Lde_N) are algebraic functions of U_N. Let $\mathbb{C}\{U,V\}^{\mathrm{alg}}$ be the space of algebraic holomorphic germs:

$$\mathbb{C}\{U,V\}^{\mathrm{alg}} = \{h \in \mathbb{C}\{U,V\} \mid \exists Q \in \mathbb{C}[U,V,W] \text{ such that } Q(U,V,h(U,V)) = 0\}.$$

It is well known that this space has some strong closure properties:

Proposition 2 (see Section 5.4 in [BER])**.** *Let $F, G \in \mathbb{C}\{U,V\}^{\mathrm{alg}}$. Then $F+G$, $F \times G$, $\partial_U F$, $\partial_V F$, $1/F$ (if $F(0,0) \neq 0$) and $G \circ F$ (if $F(0,0) = 0$) are still elements of $\mathbb{C}\{U,V\}^{\mathrm{alg}}$. If $\phi = (F,G)$ defines a germ of diffeomorphism of $\mathbb{C}^2$ fixing the origin, then the components of the local inverse ϕ^{-1} are also algebraic functions.*

Denote by ∂_k^{kl} the derivation with respect to U_k on Ω in the coordinate system (U_k, U_l). One can easily prove that $\partial_k^{kl} = \mathsf{U}_k^{kl}\partial_U + \mathsf{V}_k^{kl}\partial_V$ with $\mathsf{U}_k^{kl}, \mathsf{V}_k^{kl} \in \mathbb{C}\{U_k, U_l\}^{\mathrm{alg}}$. Thus by Proposition 2, the last set is closed under the action of the ∂_k^{kl}'s. By Proposition 2 again and from the above relations $(\star)$ it follows that if the $(\mathcal{A}_i^j)$ of $\mathcal{A}$dfe are algebraic functions, then so are the $\widehat{\mathcal{A}}_i^j$ of $(\mathcal{A}\mathrm{dfe}')$. Because all coefficients of Afe $(\mathcal{E})$ are equal to 1 and thus are algebraic, we deduce by induction that the A_i's of (Lde_N) are algebraic functions of U_N.

Because the A_i's are algebraic, they are globally defined but ramified. A classical result of the theory of linear differential equations of a complex variable implies that the germ F_N can be analytically extended along any curve in $\mathbb{CP}^1\backslash \mathrm{R}$, where R consists of the poles and ramification points of the A_i's. But this argument does not allow us to prove that F_N admits an analytic continuation along any path in the whole $U_N(\mathbb{CP}^2 \setminus \Sigma)$ because, though it is not hard to see that the ramification points of the A_i's are in $U_N(\Sigma)$, it is not so for their possible poles, which can generate some ramification for any solution of (Lde_N).

The last part of Theorem 1 comes from Proposition 3 below.

Proposition 3. *Let X be a connected (paracompact) complex manifold of dimension 2 and let $U_i : X \to \mathbb{C}$ $(i = 1, \ldots, N)$ be holomorphic functions such that, if $i \neq j$, we have $dU_i \wedge dU_j \neq 0$ on X. If for $\omega \in X$ we have N holomorphic germs F_i such that $\psi = F_1 \circ U_1 + \cdots + F_N \circ U_N$ is a holomorphic germ at ω which can be analytically continued along any path in the whole X, then every F_i can be analytically continued along any path in $U_i(X)$.*

Proof. We will prove this proposition under the assumption that $\psi \equiv 0$. The proof in the general case is similar.

For $i = 1, \dots, N$, set $\psi_i = F_i \circ U_i \in \underline{\mathcal{O}_\omega}$. Because X is paracompact, it is metrizable. So we can fix a metric on X, compatible with its topology. Then there exists $\epsilon > 0$ such that each ψ_i is defined on $B(\omega, \epsilon) \subset\subset X$.

Lemma 1. *Let $X = B_\rho := B(\omega, \rho)$ be the open ball in $\mathbb{C}^2$ centred at ω, of radius $\rho \geq \epsilon$. Then each ψ_i can be analytically extended to X.*

Proof of Lemma 1. For $\delta > 0$, we write $B_\delta = B(\omega, \delta)$. Let

$$\tau := \sup\{\delta \in [\epsilon, \rho] \mid \text{each } \psi_i \text{ extends to } B_\delta\}.$$

We want to prove that $\tau = \rho$. Suppose that $\tau < \rho$; by definition each ψ_i extends analytically to B_τ. We call this extension ψ_i again. Choose arbitrarily $\varpi \in \partial B_\tau$. We are going to prove that each ψ_i has a holomorphic extension to a neighbourhood of ϖ. By compactness, this will imply that each ψ_i extends to a neighbourhood of the closure of B_τ, which will contradict the definition of τ.

Let x and y denote the standard complex coordinates on $\mathbb{C}^2$. We introduce the holomorphic vector fields $\mathcal{X}_i := (\partial_y U_i)\partial_x - (\partial_x U_i)\partial_y$ of differentiation along the level curves of the U_i's. According to the definition of ψ_i, we have $\mathcal{X}_i \psi_i \equiv 0$ on B_ϵ and therefore on B_τ by the unicity theorem: ψ_i is constant along the level curves of U_i in B_τ. But these level curves are globally defined on X and in particular in a neighbourhood of ϖ. This fact combined with the general position assumption on these level curves at ϖ (formulated by $dU_i \wedge dU_j(\varpi) \neq 0$ according to the hypothesis of the theorem) will allow us to extend each ψ_i near ϖ.

We denote by $\mathrm{T}^{\mathbb{R}}_\varpi \partial B_\tau$ the real tangent space to ∂B_τ at ϖ. It is a real subspace of real dimension 3 of the complex tangent space to $\mathbb{C}^2$ at ϖ, denoted $\mathrm{T}_\varpi \mathbb{C}^2$. It contains a unique complex line denoted $\mathrm{T}_\varpi \partial B_\tau$. Let us extend ψ_j to a neighbourhood of ϖ for $j \in \{1, \dots, N\}$. Let $C^j(\varpi)$ be the level curve of U_j through ϖ. Since $dU_j(\varpi) \neq 0$, there exists a neighbourhood $\mathcal{V}$ of ϖ such that $C^j(\varpi) \cap \mathcal{V}$ is a complex 1-dimensional manifold. Let $\mathrm{T}_\varpi C^j$ be its holomorphic tangent space in ϖ.

Assume first that $\mathrm{T}_\varpi \partial B_\tau$ and $\mathrm{T}_\varpi C^j$ are transverse (i.e., their intersection in $\mathrm{T}_\varpi \mathbb{C}^2$ is zero). We name this condition of transversality *condition $\mathcal{T}$*. The fact that $\mathrm{T}_\varpi \partial B_\tau$ contains a unique complex line implies (for dimensional reasons) that $\mathrm{T}_\varpi C^j \not\subset \mathrm{T}_\varpi \partial B_\tau$. So $C^j(\varpi) \cap B_\tau \neq \emptyset$. Because all the geometrical objects considered here are analytic, therefore smooth, condition $\mathcal{T}$ is open: there exists an open connected neighbourhood $V \subset \mathcal{V}$ of ϖ such that for all $\zeta \in V$ we have $C^j(\zeta) \cap B_\tau \neq \emptyset$. So consider $\zeta' \in C^j_\zeta \cap B_\tau$ and $\psi_j(\zeta) := \psi_j(\zeta')$. Because ψ_j is constant along the level curves of U_j in $V \cap B_\tau$, it is evident that $\psi_j(\zeta)$ does not depend on ζ'. We may suppose V to be small enough to make ψ_j well defined on it. We have $\mathcal{X}_j \psi_j \equiv 0$ near ϖ again for this extension, so we have holomorphically extended ψ_j to $B_\tau \cup V$.

Suppose now that condition $\mathcal{T}$ is not satisfied by $C^j(\varpi)$. This means that $\mathrm{T}_\varpi \partial B_\tau = \mathrm{T}_\varpi C^j$. But the hypothesis $dU_j \wedge dU_k(\varpi) \neq 0$ for $j \neq k$ has the geometrical interpretation that the curves $C^j(\varpi)$ and $C^k(\varpi)$ are transverse at ϖ. Therefore all the level curves $C^k(\varpi)$ (for $k \neq j$) satisfy the transversality condition $\mathcal{T}$ at ϖ. By the same argument as above, we can extend analytically each ψ_k (for $k \neq j$) to a neighbourhood V of ϖ. To extend ψ_j close to ϖ we set $\psi_j := -\sum_{k\neq j}\psi_k$ on V, which will do. This finishes the proof of Lemma 1. □

Now let us prove Proposition 3. Let $\gamma : [0,1] \to X_1 = U_1(X)$ be a path starting from ω_1. We want to extend ψ_1 along γ. Because $dU_1 \neq 0$ on X, we can find a lift $\widetilde{\gamma}$ of γ to X through U_1 with ω as origin. The support $|\widetilde{\gamma}|$ of $\widetilde{\gamma}$ is compact. So there is a subdivision $\alpha_{-1} < 0 = \alpha_0 < \alpha_1 < \cdots < \alpha_M = 1$ of $[0,1]$ such that, for every $\kappa \in \{0,\dots,M-1\}$, there exists a holomorphic chart $(\Theta_\kappa, \Gamma_\kappa)$ centred at $\widetilde{\gamma}_\kappa(\alpha_\kappa)$ such that $\widetilde{\gamma}([\alpha_{\kappa-1},\alpha_{\kappa+1}]) \subset \Theta_\kappa$, and $\Gamma_\kappa(\Theta_\kappa)$ is a ball in $\mathbb{C}^2$ with centre $\Gamma_\kappa(\alpha_\kappa)$. We can apply the preceding lemma taking $X = \Gamma_0(\Theta_0)$ and considering the functions $U_i \circ \Gamma_0^{-1}$ instead of U_i. So the ψ_i can be extended along $\widetilde{\gamma}|_{[\alpha_0,\alpha_1]}$. By iterating this process $M-1$ times, we finally get an extension of each ψ_i along $\widetilde{\gamma}$, again denoted ψ_i.

This gives us an analytic extension of ψ_1 along γ: for each chart $(\Theta_\kappa, \Gamma_\kappa)$ we have on Θ_κ some holomorphic vector fields $\mathcal{X}_i^\kappa$ of differentiation along the level curves of U_i. By the construction of the lemma, we have $\mathcal{X}_1^\kappa \phi_1 \equiv 0$ on each Θ_κ. The holomorphic inverse function theorem implies that we can write $\psi_1 = F_1^\kappa(U_1)$ on each Θ_κ, F_1^κ being holomorphic in a neighbourhood of $(U_1 \circ \widetilde{\gamma})([\alpha_{\kappa-1},\alpha_{\kappa+1}]) = \gamma([\alpha_{\kappa-1},\alpha_{\kappa+1}])$. We see that $F_1^\kappa = F_1^{\kappa+1}$ on $\gamma([\alpha_\kappa,\alpha_{\kappa+1}])$ (for $0 < \kappa < M-2$). Now F_1^0 is an extension of the original germ F_1 to $U_1(\Theta_0)$. By setting $F_1 = F_1^\kappa$ on $U_1(\Theta_\kappa)$ for $\kappa = 1,\dots,M-1$, we get an analytic extension of F_1 along γ. In the same way we can construct such an analytic continuation for every F_j.

Theorem 1 is now totally proved. □

Remarks. **1.** Theorem 1 implies that $\underline{\mathcal{S}_\omega^{\mathcal{O}}}(\mathcal{E})$ is finite-dimensional. The following proposition gives an effective bound on its dimension:

Proposition 4. *For $\omega \notin \Sigma$, we have* $\dim_{\mathbb{C}} \underline{\mathcal{S}_\omega^{\mathcal{O}}}(\mathcal{E}) \leq N(N-1)/2$.

This bound is optimal and will be used in Sections 3 and 4. It is a particular case (for rational inner functions) of a basic result of planar web geometry (see 4.1).

2. From the proof of Theorem 1, we get, using the same notation:

Corollary 1. *In the generic case, there are no nonconstant local holomorphic solutions of $(\mathcal{E})$ at any $\omega \notin \Sigma$.*

Proof. The Afe $(\mathcal{E})$ can always be reduced to an irreducible Adfe in F_N, of type (K) with $K \geq 1$:

$$A_1 F_N^{(1)} + \cdots + A_K F_N^{(K)} = 0. \qquad (*)$$

The coefficients of $(*)$ are differential expressions in the U_i's. This equation is irreducible if and only if $\mathcal{X}_N(A_k) = 0$ for $k = 1,\dots,K$. Then the fact that equation $(*)$ is reducible to an Adfe of type (1) corresponds to the nonvanishing of a finite number of differential expressions in the U_i's. This implies that the N-th components of solutions of $(\mathcal{E})$ are all constant. Then the nonvanishing of a finite number of differential expressions in the U_i's is a sufficient condition for any solution of $(\mathcal{E})$ to be constant, and this condition will be generically satisfied by $(U_1,\dots,U_N)$. □

We have not found any reference for such a result.

3. Some of the preceding results remain valid in a more general situation. For instance, if instead of taking the U_i's rational, we consider some analytic germs, then the H_i's that are solutions of $H_1(U_1)+\cdots+H_N(U_N)=0$ generically satisfy a linear differential equation which can be constructed from the U_i's by the same process.

4. The method used here to obtain a linear differential equation from a functional equation is the one described by Abel in his first publication [Ab]. We will call it *Abel's method.*

5. The point is that this method is effective: for a given N-uplet of rational functions, we can explicitly find a linear differential equation satisfied by any component of any local solution. We can even do this in an algorithmic way (see 2.3.1). Similarly, for a fixed N, we can explicitly find sufficient conditions on the U_i's so that there is no nonconstant solution of $(\mathcal{E})$.

2.3. Two methods to solve Afe with rational inner functions

In this section, we use the setting and notation of Theorem 1.

The proofs of the preceding results contain some useful tools for two methods to solve Afe of type $(\mathcal{E})$. The first is essentially based on Abel's method. It is well formalized and appears very general: its only defect is that it is computational. The second is just a remark and is not well established as a general method. However, this remark allows us to solve the two Afe $(\mathcal{R})$ and $(\mathcal{SK})$, associated respectively to equations (L_2) and (SK). It is based on the fact that certain solutions of $(\mathcal{E})$ are determined by their monodromy.

2.3.1. Abel's method of resolution of Afe with rational inner functions. Let $\mathsf{R}=(U_1,\ldots,U_N)\in\mathbb{R}(x,y)^N$ be such that there exist nonconstant holomorphic solutions of the Afe defined by the U_i's. Let $\omega\notin\Sigma$. Assume that, for each $i\in\{1,\ldots,N\}$, we know a basis $\{G_i^\nu \mid \nu\leq\nu_i\}$ of the space of solutions at ω_i of the linear differential equation (Lde_i) given by Theorem 1. Then we have

$$\underline{\mathcal{S}_\omega^{\mathcal{O}}}(\mathcal{E})=\Big\{\Big(\sum_{\nu=1}^{\nu_1}a_1^\nu G_1^\nu,\ldots,\sum_{\nu=1}^{\nu_N}a_N^\nu G_N^\nu\Big)\ \Big|\ \sum_{i=1}^{N}\sum_{\nu=1}^{\nu_i}a_i^\nu G_i^\nu(U_i)=0\text{ in }\underline{\mathcal{O}_\omega^2}\Big\}.$$

Thus, modulo the resolution of the Lde_i's, the explicit resolution of $(\mathcal{E})$ at ω amounts to some linear algebra in a finite-dimensional space.

In the standard coordinates (x,y) on $\mathbb{C}^2$, the vector fields $\partial_p^{k\ell}$ (where $p\in\{k,\ell\}$) are elements of $\mathbb{C}(x,y)\partial_x+\mathbb{C}(x,y)\partial_y$. Thus the coefficients $\mathcal{A}_i^\kappa$ of any Adfe obtained through the application of several steps of Abel's method to $(\mathcal{E})$ belong to $\mathbb{C}(x,y)$. Therefore the process of obtaining (Lde_i) from the Afe $(\mathcal{E})$ can be performed within $\mathbb{C}(x,y)$. This fact allows us to easily implement a computer algebra algorithm which constructs Lde_N from $(U_1,\ldots,U_N)$. The author has used this method to solve equations $(\mathcal{E}_a)$ of Section 3.3.

2.3.2. Method of monodromy a priori. In contrast to the preceding method, the one described here does not seem to be valid in the general case, but its interest lies in the fact that it works for at least the two general Afe associated to classical functional equations of polylogarithms.

It is a method to find solutions with logarithmic growth of an Afe when the U_i's satisfy a certain condition called *condition* (C), which is defined below. Roughly speaking, it is based on the fact that solutions with logarithmic growth are determined by their monodromy, which can be determined a priori when the solutions of some subequations of $(\mathcal{E})$ are known.

For $i \in \{1, \ldots, N\}$, there exist a positive integer m_i and distinct points a_i^κ of $\mathbb{CP}^1$ (for $\kappa = 1, \ldots, m_i$) such that $X_i := U_i(X) = \mathbb{CP}^1 \setminus \{a_i^\kappa \mid 1 \leq \kappa \leq m_i\}$. For $\kappa = 1, \ldots, m_i$, let γ_i^κ be a loop in X_i with basepoint ω_i, and of index with respect to a_i^σ equal to 1 if $\sigma = \kappa$, and 0 otherwise. Then the homotopy classes $[\gamma_i^\kappa]$ and their inverses generate the fundamental group $\Pi_1(X_i, \omega_i)$.

Let δ_i denote the degree of a generic level curve $\{U_i = \text{const}\}$ as a subscheme of $\mathbb{C}^2$. For all $k \neq i$, the map $U_{ik} = (U_i, U_k) : X \to X_i \times X_k$ is a finite δ_{ik}-sheeted covering, with $0 < \delta_{ik} \leq \delta_i \delta_k$.

Let us now consider the following *condition* (C):

for every i, there exists $\ell = \ell(i)$ such that the covering $U_{i\ell}$ is 1-1.

Throughout this section, we will assume that this strong condition is satisfied. Then for $\kappa = 1, \ldots, m_i$, there exists a loop $\overline{\gamma}_{i\ell}^\kappa$ in X, with basepoint ω, such that

$$[U_i \circ \overline{\gamma}_{i\ell}^\kappa] = [\gamma_i^\kappa] \quad \text{in } \Pi_1(X_i, \omega_i), \qquad [U_\ell \circ \overline{\gamma}_{i\ell}^\kappa] = [1] \quad \text{in } \Pi_1(X_\ell, \omega_\ell).$$

Now let $\mathbf{F} = (F_1, \ldots, F_N)$ be a genuine holomorphic solution of $(\mathcal{E})$ at ω: we have

$$F_1(U_1) + \cdots + F_N(U_N) = 0 \quad \text{in } \underline{\mathcal{O}_\omega}. \tag{$\mathcal{E}_\mathbf{F}$}$$

By Proposition 3, we know that each F_j can be analytically extended along any path in X_j. Then by analytic continuation of $(\mathcal{E}_\mathbf{F})$ along $\overline{\gamma}_{i\ell}^\kappa$ in X, we get

$$F_1^{[U_1 \circ \overline{\gamma}_i^\kappa]}(U_1) + \cdots + F_i^{[\gamma_i^\kappa]}(U_i) + \cdots + F_\ell^{[1]}(U_\ell) + \cdots + F_N^{[U_N \circ \overline{\gamma}_i^\kappa]}(U_N) = 0 \quad \text{in } \underline{\mathcal{O}_\omega}.$$

This can be summarized by writing $\mathbf{F}^{[\overline{\gamma}_i^\kappa]} \in \underline{\mathcal{S}_\omega^\mathcal{O}}(\mathcal{E})$ where $\mathbf{F}^{[\overline{\gamma}_i^\kappa]} := (F_s^{[U_s \circ \overline{\gamma}_i^\kappa]})$.

Now, by subtraction, we get $\mathbf{F}^{[\overline{\gamma}_i^\kappa]} - \mathbf{F} \in \underline{\mathcal{S}_\omega^\mathcal{O}}(\mathcal{E})$ or, more explicitly,

$$\begin{aligned}(F_1^{[U_1 \circ \overline{\gamma}_i^\kappa]}(U_1) - F_1(U_1)) + \cdots + (F_i^{[\gamma_i^\lambda]}(U_i) - F_i(U_i))\\ + \cdots + (F_\ell^{[1]}(U_\ell) - F_\ell(U_\ell)) + \cdots + (F_N^{[U_N \circ \overline{\gamma}_i^\kappa]}(U_N) - F_N(U_N)) = 0.\end{aligned}$$

But $F_\ell^{[1]} - F_\ell \equiv 0$, therefore $\mathbf{F}^{[\overline{\gamma}_i^\kappa]} - \mathbf{F}$ is now a subsolution of $(\mathcal{E})$, so we have

$$\mathbf{F}^{[\overline{\gamma}_i^\kappa]} - \mathbf{F} \in F^q \underline{\mathcal{S}_\omega^\mathcal{O}}(\mathcal{E}) \quad \text{with } q < N.$$

Assume that we know a basis $\{\mathbf{B}^\nu = (B_1^\nu, \ldots, B_N^\nu) \mid 1 \leq \nu \leq M\}$ of the space of subsolutions of $(\mathcal{E})$. Then for all $\kappa = 1, \ldots, m_i$, we have a linear relation

$$\mathbf{F}^{[\overline{\gamma}_i^\kappa]} = \mathbf{F} + \sum_{\nu=1}^{M} \beta_i^{\kappa,\nu} \mathbf{B}^\nu$$

where the $\beta_i^{\kappa,\nu}$ are complex constants (depending linearly on $\mathbf{F}$). The latter relations are concise forms of the following:

$$\mathcal{M}_{[\gamma_i^\lambda]} F_i = F_i + \sum_{\nu=1}^{M} \beta_i^{\kappa,\nu} B_i^\nu \qquad (\star)_i^\kappa$$

$$\mathcal{M}_{[U_s \circ \overline{\gamma}_i^\lambda]} F_s = F_s + \sum_{\nu=1}^{M} \beta_i^{\kappa,\nu} B_s^\nu \quad (s = 1, \ldots, N,\, s \neq i). \qquad (\star\star)_{i,s}^\kappa$$

If Y is a complex manifold, knowing the monodromy of $G \in \widetilde{\mathcal{O}}(Y)$ means knowing a monodromy representation

$$\Pi_1(Y, y) \to \mathrm{Aut}(\mathcal{G}_y), \qquad [\gamma] \mapsto (\mathcal{M}_{[\gamma]} : g \mapsto g^{[\gamma]}),$$

for (at least) one $y \in Y$, where $\mathcal{G}_y$ denotes the linear space spanned by the branches of G at y.

Because the γ_i^κ are such that the family $\{[\gamma_i^\kappa], [\gamma_i^\kappa]^{-1} \mid 1 \leq \kappa \leq m_i\}$ spans the group $\Pi_1(X_i, \omega_i)$, the relations $(\star)_i^\kappa$ give us a priori the monodromy of each component F_i, as a function of the i-th components of the subsolutions $\mathbf{B}^\nu$ of $(\mathcal{E})$.

Proposition 5. *Under condition* (C)*, the monodromy of each component F_i of a genuine solution of $(\mathcal{E})$ can be expressed in terms of the i-th components of some subsolutions of $(\mathcal{E})$.*

This transforms our point of view on equation $(\mathcal{E})$: although considering it in functional form, we will now see relations $(\star)_i^\kappa$ as *monodromy equations* for the components of solutions, and relations $(\star\star)_{i,s}^\kappa$ as *compatibility relations* between those equations of monodromy. We now want to find some genuine solution of $(\mathcal{E})$ by solving the monodromy equations $(\star)_i^\kappa$.

Assume that there exists a genuine solution $\mathbf{F} = (F_1, \ldots, F_N)$ of $(\mathcal{E})$ at ω. From the above, there are complex constants $\beta_i^{\kappa,\nu}$ satisfying both $(\star)_i^\kappa$ and $(\star\star)_{i,s}^\kappa$. Let $\mathbf{H} = (H_1, \ldots, H_N)$ be such that each $H_i \in \widetilde{\mathcal{O}}(X_i)$ has a branch, still denoted H_i, at ω_i satisfying all the equations $(\star)_i^\kappa$. Then the germ $H_i - F_i$ can be extended analytically to X_i without ramifications. This implies that the germ $\mathcal{H} = \sum_{i=1}^N (H_i - F_i) \circ U_i$ is the restriction to a neighbourhood of ω of a global holomorphic function on X; we write $\mathcal{H} \in \mathcal{O}(X)$.

Now suppose that we can choose H_i with logarithmic growth. Then $\mathcal{H}$ is a holomorphic function on $\mathbb{CP}^2 \setminus \Sigma$ with logarithmic growth at infinity. By a Liouville type theorem, this implies that $\mathcal{H}$ is constant, so $(H_1, \ldots, H_N) \in \underline{\mathcal{S}_\omega^{\mathcal{O}}}(\mathcal{E})$.

Thus the problem of finding genuine solutions in $\underline{\mathcal{S}_\omega^{\mathcal{O}}}(\mathcal{E})$ with logarithmic growth amounts to solving the equations $(\star)_i^\kappa$ in $\prod_i \underline{\mathcal{O}_{\omega_i}^{\log}}(X_i)$. What makes this conceptually interesting is that the problem is now reduced to a linear form.

Let $\mathbf{F} \in \underline{\mathcal{S}_\omega^{\mathcal{O}}}(\mathcal{E})$ with logarithmic growth. The subsolutions of the form $\mathbf{F} - \mathbf{F}^{[\overline{\gamma}_i^\kappa]}$ which appeared in the preceding discussion also have logarithmic growth and are now elements of $\underline{\mathcal{S}_\omega^{\mathcal{O}}}(\mathcal{E}_{K_i^\kappa})$ where K_i^κ is a proper subset of $\{1, \ldots, N\}$. Under suitable conditions on the U_i's, the $\{U_\sigma\}_{\sigma \in K_i^\kappa}$ satisfy (C) again. In this case it could be possible to inductively determine the solutions with logarithmic growth of equation $(\mathcal{E})$ by their monodromy.

Remarks. **1.** The components of numerous solutions of Afe with rational inner functions are constructed from iterated integrals (see Section 3). Therefore it is interesting and useful to solve the monodromy equations considered above in the proper subspace $\prod_i \underline{\mathcal{I}_i}$ of $\prod_i \underline{\widetilde{\mathcal{O}}_{\omega_i}^{\log}}(X_i)$ where $\underline{\mathcal{I}_i}$ denotes the space of branches at ω_i of elements of $\mathcal{I}\{X_i\}$.

2. However, not all components of solutions with logarithmic growth are constructed from iterated integrals. For instance, the function $x \mapsto \mathrm{Arctanh}\sqrt{x}$ is a component of a solution of the Afe $(\mathcal{SK})$ with real rational inner functions considered in 3.2. This function cannot be expressed through iterated integrals although it is ramified with logarithmic growth on $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$.

3. Explicit resolution of three Afe

In this section we apply the methods sketched above to the explicit resolution of three Afe: to begin with, using monodromy arguments, we solve the Afe naturally associated to equation (L_2). Then we give an explicit basis of solutions of the *generalized Spence–Kummer equation* associated to (SK). We finish with another Afe, depending on a real parameter $a > 1$.

3.1. Rogers' dilogarithm equation revisited

In [Rog], Rogers established a clean version of the equation $(\mathcal{A}\mathrm{b})$ satisfied by the Rogers dilogarithm $\mathbf{D}$, for $0 < x < y < 1$:

$$\mathbf{D}(x) - \mathbf{D}(y) - \mathbf{D}\left(\frac{x}{y}\right) - \mathbf{D}\left(\frac{1-y}{1-x}\right) + \mathbf{D}\left(\frac{x(1-y)}{y(1-x)}\right) = 0. \tag{R}$$

Here $\mathbf{D}(x) := \mathbf{Li}_2(x) - \frac{1}{2}\mathbf{Log}(x)\mathbf{Li}_1(x) - \pi^2/6$ is a normalized version (by addition of $-\pi^2/6$) of the original Rogers dilogarithm, in order to have 0 for the rhs of (R).

We consider the more general equation in five unknowns (denoted (L_2) in Section 1):

$$F_1(x) + F_2(y) + F_3\left(\frac{x}{y}\right) + F_4\left(\frac{1-y}{1-x}\right) + F_5\left(\frac{x(1-y)}{y(1-x)}\right) = 0. \tag{$\mathcal{R}$}$$

We let $\mathcal{W}_{\mathcal{R}}$ be the singular web on $\mathbb{CP}^2$ associated to the inner functions $U_1, U_2, \ldots, U_5$ of $(\mathcal{R})$, where $U_1 = x$, $U_2 = y$, $U_3 = \frac{x}{y}$, $U_4 = \frac{1-y}{1-x}$ and $U_5 = \frac{x(1-y)}{y(1-x)}$.

After computation we find that its singular locus is the union of five lines:

$$\Sigma[\mathcal{R}] := \{(z, \zeta) \in \mathbb{C}^2 \mid z\zeta(1-z)(1-\zeta)(z-\zeta) = 0\}.$$

In the framework of web geometry (see Section 4.1), G. Bol in [Bol2] determines a basis of the space of *abelian relations* of $\mathcal{W}[\mathcal{R}]$, which is equivalent to solving $(\mathcal{R})$. We want to rediscover Bol's results by using the two methods described in 2.3.

3.1.1. Resolution of $(\mathcal{R})$ by the monodromy a priori method. We choose $\omega = (1/3, 1/2) \in X := \mathbb{CP}^2 \setminus \Sigma[\mathcal{R}]$ as a basepoint. For $i = 1, \ldots, 5$, we have $U_i(X) = \mathbb{C}\setminus\{0, 1\}$. So, by Proposition 3, if $(F_1, \ldots, F_5) \in \underline{\mathcal{S}_\omega^{\mathcal{O}}}(\mathcal{R})$, then each F_i is a branch at $\omega_i = U_i(\omega)$ of a global multivalued function on $\mathbb{C}\setminus\{0, 1\}$.

We want to determine those solutions of $(\mathcal{R})$ whose components are iterated integrals of elements of $\mathcal{I} = \mathcal{I}\{0, 1\}$. We first search for order-3 solutions of this type, i.e., we want to determine $F^3\underline{\mathcal{S}_\omega^{\mathcal{I}}}(\mathcal{R})$. The monodromy a priori method works without difficulties and without too many computations. It gives us the following five nonconstant independent elements of $F^3\underline{\mathcal{S}_\omega^{\mathcal{I}}}(\mathcal{R})$:

$$\mathbf{\Delta}_1 := (\mathbf{L}_{x_0}, -\mathbf{L}_{x_0}, -\mathbf{L}_{x_0}, 0, 0), \qquad \mathbf{\Delta}_2 := (0, 0, \mathbf{L}_{x_0}, \mathbf{L}_{x_0}, -\mathbf{L}_{x_0}),$$
$$\mathbf{\Delta}_3 := (\mathbf{L}_{x_1}, -\mathbf{L}_{x_1}, 0, -\mathbf{L}_{x_0}, 0), \qquad \mathbf{\Delta}_4 := (\mathbf{L}_{x_1}, 0, -\mathbf{L}_{x_1}, 0, \mathbf{L}_{x_1}),$$
$$\mathbf{\Delta}_5 := (\mathbf{L}_{x_1+x_0}, 0, -\mathbf{L}_{x_1+x_0}, \mathbf{L}_{x_1}, 0).$$

Now we have to determine the last nonconstant solution of $(\mathcal{R})$. We will use our method again. We will detail the computation to be well understood.

For $\kappa = 1, \ldots, 5$, we have $\omega_\kappa \in]0, 1[$, and we let c_0^κ and c_1^κ be the loops $\sigma \mapsto \omega_\kappa \exp(2i\pi\sigma)$ and $\sigma \mapsto 1-(1-\omega_\kappa)\exp(2i\pi\sigma)$ respectively. Let $\mathbf{F} = (F_1, \ldots, F_5)$ be a genuine solution of $(\mathcal{R})$ at ω. Consider the loop $\gamma : [0, 1] \ni \sigma \mapsto (\exp(2i\pi\sigma)/3, 1/2) \in \mathbb{C}^2 \setminus \Sigma[\mathcal{R}]$. Computations give

$$[U_1 \circ \gamma] = [c_0^1], \quad [U_2 \circ \gamma] = [1], \quad [U_3 \circ \gamma] = [c_0^3], \quad [U_4 \circ \gamma] = [1], \quad [U_5 \circ \gamma] = [c_0^5].$$

These are equalities (respectively) in $\Pi_1(\mathbb{C}\setminus\{0, 1\}, \omega_i)$ for $i = 1, \ldots, 5$. By analytic continuation along γ and then subtracting $\mathbf{F}$, we get a new functional equation

$$(F_1^{[c_0^1]}(U_1) - F_1(U_1)) + (F_3^{[c_0^3]}(U_3) - F_3(U_3)) + (F_5^{[c_0^5]}(U_5) - F_5(U_5)) = 0,$$

which corresponds to an element of $F^3\underline{\mathcal{S}_\omega^{\mathcal{O}}}(\mathcal{R})$. But we know this space explicitly, and in this case we obtain the following relations of monodromy for the components of any solution $\mathbf{F}$ (a, a_1, a_2 are complex constants depending linearly on $\mathbf{F}$):

$$\mathcal{M}_0 F_1 = F_1 + a\mathbf{L}_{x_1} + a_1,$$
$$\mathcal{M}_0 F_3 = F_3 - a\mathbf{L}_{x_1} + a_2,$$
$$\mathcal{M}_0 F_5 = F_5 + a\mathbf{L}_{x_1} - (a_1 + a_2).$$

Now considering the loop $[0,1] \ni \sigma \mapsto (\frac{1}{3}, 1 - \frac{1}{2}\exp(2i\pi\sigma))$ in $\mathbb{C}^2 \setminus \Sigma[\mathcal{R}]$, we show in the same way that there exist $a', a'_1, a'_2 \in \mathbb{C}$ such that

$$\begin{aligned}\mathcal{M}_1 F_2 &= F_2 + a'\mathbf{L}_{x_0} + a'_1,\\ \mathcal{M}_0 F_4 &= F_4 + a'\mathbf{L}_{x_1} + a'_2,\\ \mathcal{M}_0 F_5 &= F_5 - a'\mathbf{L}_{x_1} - (a'_1 + a'_2).\end{aligned}$$

From these relations it is evident that $a = -a'$ and $a_1 + a_2 = a'_1 + a'_2$.

We can continue this kind of computation to conclude that a priori, the monodromy of the components of holomorphic solutions of $(\mathcal{R})$ is of the form

$$\begin{aligned}\mathcal{M}_0 F_k &= F_k - \epsilon_k a\mathbf{L}_{x_1} + a_k,\\ \mathcal{M}_1 F_k &= F_k + \epsilon_k a\mathbf{L}_{x_0} + b_k,\end{aligned}$$

for $k = 1, \ldots, 5$, with $\epsilon_k = 1$ for $k = 1, 5$ and $\epsilon_k = -1$ otherwise (a_k, b_k being complex constants satisfying certain linear relations). It can be proved that the a_k's and b_k's are such that there exists a linear combination $\mathbf{H} = \alpha_1\mathbf{\Delta}_1 + \cdots + \alpha_5\mathbf{\Delta}_5$ such that the monodromy of the components of $\widehat{\mathbf{F}} = (\widehat{F}_k) = \mathbf{F} + \mathbf{H}$ now satisfies

$$\begin{aligned}\mathcal{M}_0 \widehat{F}_k &= \widehat{F}_k + \epsilon_k a\mathbf{L}_{x_1},\\ \mathcal{M}_1 \widehat{F}_k &= \widehat{F}_k + \epsilon_k a\mathbf{L}_{x_0}.\end{aligned}$$

For $\alpha = -a/2i\pi$, the function $\mathbf{f} = \alpha\mathbf{D} \in \mathcal{I}\{0,1\}$ satisfies the following monodromy equations:

$$\mathcal{M}_0\mathbf{f} = \mathbf{f} - a\mathbf{L}_{x_1} \quad \text{and} \quad \mathcal{M}_1\mathbf{f} = \mathbf{f} + a\mathbf{L}_{x_0}.$$

We deduce that $\mathbf{\Delta}_6 := (\mathbf{D}, -\mathbf{D}, -\mathbf{D}, -\mathbf{D}, \mathbf{D}) \in \underline{\mathcal{S}_\omega^{\mathcal{O}}}(\mathcal{R})$ because $\mathbf{D}(U_1) + \mathbf{D}(U_2) + \cdots + \mathbf{D}(U_5)$ is not ramified on $\mathbb{CP}^2$ and has logarithmic growth at infinity, so must be a constant (in fact, it is 0).

We can easily construct a basis $\{\mathbf{\Delta}_i \mid i = -3, \ldots, 0\}$ of the constant solutions of $(\mathcal{R})$. It is not difficult to see that the ten elements $\mathbf{\Delta}_k$ ($k = -3, \ldots, 6$) described above are linearly independent. Thus we have

$$10 = \dim_{\mathbb{C}}\langle\{\mathbf{\Delta}_k\}\rangle \leq \dim_{\mathbb{C}} \underline{\mathcal{S}_\omega^{\mathcal{O}}}(\mathcal{R}) \leq 5(5-1)/2 = 10$$

where the right bound is given by Proposition 4. Hence we deduce that

$$\underline{\mathcal{S}_\omega^{\mathcal{O}}}(\mathcal{R}) = \langle\{\mathbf{\Delta}_k \mid k = -3, \ldots, 6\}\rangle.$$

This solves $(\mathcal{R})$ at ω in the holomorphic class. We get the local holomorphic solutions around $\varpi \notin \Sigma[\mathcal{R}]$ by analytic continuation of the $\mathbf{\Delta}_k$'s along any path joining ω to ϖ in X.

3.1.2. Resolution of $(\mathcal{R})$ by Abel's method. A simple application of Abel's method implies that on the whole Ω_1 (see 1.2 for definition), the first component F_1 of every solution of $\mathcal{R}$ must satisfy the following linear differential equation (where ∂ denotes $\frac{d}{dx}$):

$$\left(\partial^4 + \frac{4(2x^3-3x^2+x)}{x^2(1-x)^2}\,\partial^3 + \frac{2(1-7x+7x^2)}{x^2(1-x)^2}\,\partial^2 + \frac{2(2x-1)}{x^2(1-x)^2}\,\partial\right) g = 0.$$

The general solution of this equation is of the form $c_1\mathbf{D} + c_2\mathbf{L}_{x_0} + c_3\mathbf{L}_{x_1} + c_4$, which is the form that any first component F_1 of any solution of $(\mathcal{R})$ can have.

3.2. The generalized Spence–Kummer equation

To the Spence–Kummer equation (SK) satisfied by $\mathbf{Li}_3$ we can associate the following abelian functional equation in nine unknown functions:

$$F_1(U_1) + F_2(U_2) + F_3(U_3) + \cdots + F_9(U_9) = 0 \qquad (\mathcal{SK})$$

where the U_i's are the rational inner functions which appear in (SK): $U_1, \dots, U_5$ have been defined in 3.1 and we set $U_6 = xy$, $U_7 = x(1-y)/(x-1)$, $U_8 = (1-y)/(y(x-1))$ and $U_9 = x(1-y)^2/(y(1-x)^2)$. The 9-web on $\mathbb{CP}^2$ defined by these nine rational functions is the *Spence–Kummer web*, and will be denoted by $\mathcal{W}_{\mathcal{SK}}$. Its singular locus is (with $\mathbb{CP}^1_\infty = \{[z:\zeta:0] \in \mathbb{CP}^2\}$)

$$\begin{aligned}\Sigma[\mathcal{SK}] = \mathbb{CP}^1_\infty \cup \{[z:\zeta:1] \in \mathbb{CP}^2 \mid\ & z\zeta(1-z)(1-\zeta)(z-\zeta)\\ & \times (1+\zeta)(1+z)(1-z\zeta)(2-z-\zeta)\\ & \times (z\zeta-2\zeta+1)(z\zeta-2z+1)(2z\zeta-\zeta-z) = 0\}.\end{aligned}$$

We set $X = \mathbb{CP}^2 \setminus \Sigma[\mathcal{SK}]$. We choose again $\omega = (1/3, 1/2) \in \mathbb{R}^2 \setminus \Sigma[\mathcal{SK}]$ as basepoint. We want to find the local holomorphic solutions of $(\mathcal{SK})$ at ω. Simple computations give $U_i(X) = \mathbb{C} \setminus \Sigma_i$ where $\Sigma_i = \{-1, 0, 1, \infty\}$ if $i = 1, 2$ and $\Sigma_i = \{0, 1, \infty\}$ for $i \geq 3$. One can prove that the first and second components of any solution of $\mathcal{SK}$ cannot be ramified at -1. So if F_k is the k-th component of a solution of $(\mathcal{SK})$ at ω, then F_k is the branch at ω_k of a multivalued function on $\mathbb{CP}^1$, ramified at 0, 1 and ∞, i.e., $F_k \in \underline{\widetilde{\mathcal{O}}_\omega}(\mathbb{C}\setminus\{0,1\})$.

Equation $(\mathcal{SK})$ can be solved by using the two methods stated in 2.3: the method of monodromy a priori can be applied to find all the elements of $\underline{\mathcal{S}^{\mathcal{I}}_\omega}(\mathcal{SK})$. Then by applying Abel's method, we get the missing solutions whose components are not iterated integrals (denoted $\mathbf{F}_8, \mathbf{F}_{10}, \mathbf{F}_{15}, \mathbf{F}_{16}$ and $\mathbf{F}_{17}$ below).

We first obtain 17 subequations of order 3: we set $\mathbf{F}_i = \mathbf{\Delta}_i$ for $i = 1, \dots, 5$ (we consider solutions of $(\mathcal{R})$ as subsolutions of $(\mathcal{SK})$) and

$$\begin{aligned}\mathbf{F}_6 &= (\mathbf{L}_{x_0}, \mathbf{L}_{x_0}, 0, 0, 0, -\mathbf{L}_{x_0}, 0, 0, 0),\\ \mathbf{F}_7 &= (\mathbf{L}_{x_0}, 0, 0, \mathbf{L}_{x_0}, 0, 0, -\mathbf{L}_{x_0} + i\pi, 0, 0),\\ \mathbf{F}_8 &= (\mathbf{I}\mathrm{v}, 0, 0, 0, \mathbf{I}\mathrm{v}, 0, \mathbf{I}\mathrm{v} - 1, 0, 0),\end{aligned}$$

$$\mathbf{F}_9 = (\mathbf{L}_{x_1}, 0, 0, 0, 0, -\mathbf{L}_{x_1}, \mathbf{L}_{x_1}, 0, 0),$$
$$\mathbf{F}_{10} = (0, \mathbf{I}\mathrm{d}, 0, \mathbf{I}\mathrm{d}, 0, 0, \mathbf{I}\mathrm{d} - 1, 0, 0),$$
$$\mathbf{F}_{11} = (0, 0, 0, 0, 0, \mathbf{L}_{x_0}, -\mathbf{L}_{x_0}, \mathbf{L}_{x_0}, 0),$$
$$\mathbf{F}_{12} = (0, \mathbf{L}_{x_0}, 0, 0, 0, 0, \mathbf{L}_{x_1}, -\mathbf{L}_{x_1}, 0),$$
$$\mathbf{F}_{13} = (-2i\pi, 0, 0, 0, 0, 0, \mathbf{L}_{x_0}, \mathbf{L}_{x_0}, -\mathbf{L}_{x_0}),$$
$$\mathbf{F}_{14} = (0, 0, 0, 0, \mathbf{L}_{x_1}, 0, \mathbf{L}_{x_1}, 0, -\mathbf{L}_{x_1}),$$
$$\mathbf{F}_{15} = (0, \mathbf{I}\mathrm{v},\ 0,\ 0, \mathbf{I}\mathrm{d}, 0, 0, \mathbf{I}\mathrm{d} - 1, 0),$$
$$\mathbf{F}_{16} = (\mathbf{I}\mathrm{d}, 0, 0, \mathbf{I}\mathrm{v}, 0, 0, 0, \mathbf{I}\mathrm{v}, -1),$$
$$\mathbf{F}_{17} = (0, 0, \mathbf{A}, 0, 0, -\mathbf{A}, 0, 0, -\mathbf{A}),$$
$$\mathbf{F}_{18} = (2\mathbf{L}_{x_0^2}, 2\mathbf{L}_{x_0^2}, -\mathbf{L}_{x_0^2}, 0, 0, -\mathbf{L}_{x_0^2}, 0, 0, 0),$$
$$\mathbf{F}_{19} = (0, 0, 0, 0, 0, \mathbf{L}_{x_0^2}, -2\mathbf{L}_{x_0^2}, -2\mathbf{L}_{x_0^2}, \mathbf{L}_{x_0^2} + 2i\pi\mathbf{L}_{x_0} - 2\pi^2),$$
$$\mathbf{F}_{20} = (0, 0, \mathbf{L}_{x_0^2}, -2\mathbf{L}_{x_0^2}, -2\mathbf{L}_{x_0^2}, 0, 0, 0, \mathbf{L}_{x_0^2}),$$
$$\mathbf{F}_{21} = (\mathbf{D}, -\mathbf{D}, -\mathbf{D}, -\mathbf{D}, \mathbf{D}, 0, 0, 0, 0),$$
$$\mathbf{F}_{22} = \left(\mathbf{D}, \mathbf{D} - \frac{i\pi}{2}\,\mathbf{L}_{x_0}, 0, 0, 0, -\mathbf{D}, \mathbf{D}, -\mathbf{D}, 0\right),$$
$$\mathbf{F}_{23} = \left(\pi^2, 0, 0, \mathbf{D} - \frac{i\pi}{2}\mathbf{L}_{x_0}, \mathbf{D}, 0, \mathbf{D}, \mathbf{D} + \frac{i\pi}{2}\mathbf{L}_{x_0} + i\pi\mathbf{L}_{x_1}, -\mathbf{D}\right),$$
$$\mathbf{F}_{24} = \left(\mathbf{L}_{x_0x_1}, \mathbf{L}_{x_0x_1}, 0, \mathbf{L}_{x_0^2}, 0, -\mathbf{L}_{x_0x_1}, \mathbf{L}_{x_0x_1}, -\mathbf{L}_{x_0x_1} - \mathbf{L}_{x_0^2} + i\pi\mathbf{L}_{x_0}, \frac{\pi^2}{3}\right),$$
$$\mathbf{F}_{25} = (0, \mathbf{L}_{x_0x_0}, 0, \mathbf{L}_{x_0x_1}, \mathbf{L}_{x_0x_1}, 0, \mathbf{L}_{x_0x_1}, \mathbf{L}_{x_0x_1}, -\mathbf{L}_{x_0x_1}),$$
$$\mathbf{F}_{26} = (2\mathbf{L}_{x_0x_1}, 0, -\mathbf{L}_{x_0x_1}, 0, 2\mathbf{L}_{x_0x_1}, -\mathbf{L}_{x_0x_1}, 2\mathbf{L}_{x_0x_1}, 0, -\mathbf{L}_{x_0x_1}),$$
$$\mathbf{F}_{27} = (2\ell_3, 2\ell_3, -\ell_3, 2\ell_3, 2\ell_3, -\ell_3, 2\widehat{\ell_3}, 2\widehat{\ell_3}, -\ell_3),$$
$$\mathbf{F}_{28} = (2L_3, 2L_3, -L_3, 2L_3, 2L_3, -L_3, 2\widehat{L}_3, 2\widehat{L}_3, -L_3),$$

where $\mathbf{I}\mathrm{d} : z \mapsto z$, $\mathbf{I}\mathrm{v} : z \mapsto z^{-1}$ and (with $\zeta(3) = \mathbf{L}\mathrm{i}_3(1) = \sum_{k>0} 1/k^3$)

$$\mathbf{A} = \mathrm{Arctanh}(\sqrt{\bullet}), \qquad \mathbf{D} = \frac{1}{2}(\mathbf{L}_{x_0x_1} - \mathbf{L}_{x_1x_0}) - \frac{\pi^2}{6},$$
$$\ell_3 = \mathbf{L}_{x_0^2x_1} + \mathbf{L}_{x_0x_1x_0} - 2\mathbf{L}_{x_1x_0^2}, \qquad \widehat{\ell_3} = \ell_3 - i\pi\mathbf{L}_{x_0x_1} + 2i\pi\mathbf{L}_{x_1x_0} + \pi^2\mathbf{L}_{x_1} + \frac{3}{2}\zeta(3),$$
$$L_3 = \mathbf{L}_{x_0^2x_1} - \mathbf{L}_{x_1x_0^2} - \frac{\pi^2}{6}\mathbf{L}_{x_0}, \qquad \widehat{L}_3 = L_3 + i\pi\mathbf{L}_{x_1x_0} + \frac{\pi^2}{2}\mathbf{L}_{x_1} + \frac{\pi^2}{12}\mathbf{L}_{x_0}.$$

Let $\{\mathbf{F}_k \mid k = -7, \ldots, 0\}$ be a basis of the space of constant solutions of $\mathcal{SK}$. One can verify that the $\mathbf{F}_i$'s (for $i = -7, \ldots, 28$) are linearly independent. Then using the majorization given by Proposition 4, we get

$$36 = \dim_{\mathbb{C}}\langle\{\mathbf{F}_i\}\rangle \leq \dim_{\mathbb{C}} \underline{\mathcal{S}^{\mathcal{O}}_{\omega}}(\mathcal{SK}) \leq 9(9-1)/2 = 36.$$

So we have

$$\underline{\mathcal{S}_\omega^{\mathcal{O}}}(\mathcal{SK}) = \langle \{ \mathbf{F}_i \mid -7 \leq i \leq 28 \} \rangle.$$

This solves $(\mathcal{SK})$ at ω in the holomorphic (in fact measurable) class. We get the holomorphic solutions around $\varpi \notin \Sigma[\mathcal{SK}]$ by analytic continuation of the $\mathbf{F}_i$'s.

3.3. An Afe depending on a parameter

For a fixed real parameter $a > 1$, consider the following Afe:

$$G_1(x) + G_2(y) + G_3\left(\frac{x}{y}\right) + G_4\left(\frac{1-y}{1-x}\right) + G_5\left(\frac{x(1-y)}{y(1-x)}\right) + G_6\left(\frac{a-y}{a-x}\right) + G_7\left(\frac{x(a-y)}{y(a-x)}\right) + G_8\left(\frac{(1-y)(a-x)}{(1-x)(a-y)}\right) = 0. \qquad (\mathcal{E}_a)$$

We set $U_6^a = (a-y)/(a-x)$, $U_7^a = x(1+y)/(y(1+x))$, $U_8^a = (1-y)(a+x)/((1-x)(a+y))$ and $U_i^a = U_i$ for $i = 1, \ldots, 5$. Let $\mathcal{W}_a$ the 8-web associated to the U_i^a's. Its singular locus is

$$\Sigma_a = \mathbb{CP}^1_\infty \cup \{ [z:\zeta:1] \in \mathbb{CP}^2 \mid z\zeta(1-z)(1-\zeta)(z-\zeta) \times (a-\zeta)(a-z)(a-z-\zeta+z\zeta) \times (a-z\zeta)(z\zeta - a\zeta - az + a) = 0 \}.$$

We take again $\omega = (1/3, 1/2) \in \mathbb{R} \setminus \Sigma_a$ as basepoint and we want to find all holomorphic solutions of $(\mathcal{E}_a)$ at ω. We can consider solutions of $(\mathcal{R})$ as subsolutions of $(\mathcal{E}_a)$. For $i = 1, \ldots, 5$, we define $\mathbf{G}_i^a = \mathbf{\Delta}_i$. By applying Abel's method, we construct the following 21 elements $\mathbf{G}_k^a$ of $\underline{\mathcal{S}_\omega^{\mathcal{O}}}(\mathcal{E}_a)$ (for $k = 6, \ldots, 21$):

$$\begin{aligned}
\mathbf{G}_6^a &= (\mathbf{L}_{x_a}, -\mathbf{L}_{x_a}, 0, 0, 0, -\mathbf{L}_{x_0}, 0, 0),\\
\mathbf{G}_7^a &= (\mathbf{L}_{x_0+x_a}, 0, -\mathbf{L}_{x_0+x_1}, 0, 0, \mathbf{L}_{x_1}, 0, 0),\\
\mathbf{G}_8^a &= \left(-1, 0, \mathbf{J}, \frac{a}{a-1}\mathbf{J}, 0, \frac{-1}{a-1}\mathbf{J}, 0, 0\right),\\
\mathbf{G}_9^a &= (0, 0, \mathbf{L}_{x_0}, 0, 0, \mathbf{L}_{x_0}, -\mathbf{L}_{x_0}, 0),\\
\mathbf{G}_{10}^a &= (\mathbf{L}_{x_a}, 0, -\mathbf{L}_{x_1}, 0, 0, 0, \mathbf{L}_{x_1}, \nu_a),\\
\mathbf{G}_{11}^a &= \left(0, 0, \mathbf{J}, 0, \frac{1}{a-1}\mathbf{J}, 0, \frac{-a}{a-1}\mathbf{J}, 0\right),\\
\mathbf{G}_{12}^a &= (0, 0, 0, \mathbf{L}_{x_0}, 0, -\mathbf{L}_{x_0}, 0, -\mathbf{L}_{x_0}),\\
\mathbf{G}_{13}^a &= (\kappa_a, \mathbf{L}_{x_a}, 0, -\mathbf{L}_{x_1}, 0, 0, 0, \mathbf{L}_{x_1}),\\
\mathbf{G}_{14}^a &= \left(1, 0, 0, 0, 0, \frac{1}{a-1}\mathbf{J}, \frac{-a}{a-1}\mathbf{J}, -\mathbf{J}\right),\\
\mathbf{G}_{15}^a &= (\mathbf{D}, -\mathbf{D}, -\mathbf{D}, -\mathbf{D}, \mathbf{D}, 0, 0, 0),\\
\mathbf{G}_{16}^a &= \left(\mathbf{D}_a + \frac{\nu_a}{2}\mathbf{L}_{x_0+x_a}, -\mathbf{D}_a - \frac{\nu_a}{2}\mathbf{L}_{x_0+x_a}, -\mathbf{D}, 0, 0, -\mathbf{D}, \mathbf{D}, 0\right),\\
\mathbf{G}_{17}^a &= (\mathbf{L}_{x_a x_1 - x_1 x_a}, -\mathbf{L}_{x_a x_1 - x_1 x_a}, 0, 2\mathbf{D} - \kappa_a \mathbf{L}_{x_0}, 0, -2\mathbf{D} + \kappa_a \mathbf{L}_{x_0}, 0, -2\mathbf{D}),
\end{aligned}$$

$$\mathbf{G}^a_{18} = \left(0, 0, \mathbf{B}, \frac{-a}{a-1}\mathbf{B}, \frac{1}{a-1}\mathbf{B}, \frac{1}{a-1}\mathbf{B} + \mathbf{L}_{x_0}, \frac{-a}{a-1}\mathbf{B}, \mathbf{B}\right),$$

$$\mathbf{G}^a_{19} = \Big(\mathbf{L}_{x_0x_1 - x_0x_a} - \nu_a \mathbf{L}_{x_0+x_a}, \mathbf{L}_{x_1x_0 - x_ax_0} + \nu_a \mathbf{L}_{x_a}, 0, -\mathbf{L}_{x_1^2+x_0x_1},$$
$$\mathbf{L}_{x_1^2+x_0x_1}, \mathbf{L}_{x_1^2+x_0x_1} - \ell_a \mathbf{L}_{x_1}, -\mathbf{L}_{x_1^2+x_0x_1}, \frac{\nu_a^2}{2}\Big),$$

$$\mathbf{G}^a_{20} = \Big(\mathbf{L}_{x_ax_1+x_0x_1}, \mathbf{L}_{x_1x_a+x_1x_0}, -\mathbf{L}_{x_1^2+x_0x_1} + \kappa_a \mathbf{L}_{x_0+x_1}, \frac{\kappa_a^2}{2} + \frac{\pi^2}{3},$$
$$\mathbf{L}_{x_1^2+x_0x_1} - \kappa_a \mathbf{L}_{x_0+x_1}, \mathbf{L}_{x_1^2+x_1x_0} - \kappa_a \mathbf{L}_{x_1}, 0, -\mathbf{L}_{x_1^2+x_0x_1}\Big),$$

$$\mathbf{G}^a_{21} = (0, 0, (a-1)^3\mathbf{C} + (a-1)^2\mathbf{J}, -a^3\mathbf{C} + a(2a^2 - a - 1)\mathbf{J}, \mathbf{C},$$
$$\mathbf{C} + (a-1)\mathbf{J}, -a^3\mathbf{C}, (a-1)^3\mathbf{C} - 2a(a-1)^2\mathbf{J})$$

with $\nu_a = \log(a)$, $\kappa_a = \log(a-1)$, $\mathbf{D}_a = \frac{1}{2}(\mathbf{L}_{x_0x_a} - \mathbf{L}_{x_ax_0})$ and

$$\mathbf{J} = \frac{1}{1 - \mathbf{I}\mathrm{d}}, \qquad \mathbf{C} = \frac{1}{(1 - \mathbf{I}\mathrm{d})^2}, \qquad \mathbf{B} = \frac{\mathbf{L}_{x_0}}{1 - \mathbf{I}\mathrm{d}}.$$

Let $\{\mathbf{G}^a_\ell \mid \ell = -6, \dots, 0\}$ be a basis of the space of constant solutions of $(\mathcal{E}_a)$. Then it is just an exercise in linear algebra to satisfy that the $\mathbf{G}^a_i$'s (for $i = -6, \dots, 21$) are 28 linearly independent elements of $\underline{\mathcal{S}^{\mathcal{O}}_{\omega}}(\mathcal{E}_a)$. Thus it is clear that

$$28 = \dim_{\mathbb{C}} \langle \{\mathbf{G}^a_i\} \rangle \leq \dim_{\mathbb{C}} \underline{\mathcal{S}^{\mathcal{O}}_{\omega}}(\mathcal{E}_a) \leq 8(8-1)/2 = 28.$$

So we have

$$\underline{\mathcal{S}^{\mathcal{O}}_{\omega}}(\mathcal{E}_a) = \langle \{\mathbf{G}^a_i \mid i = -6, \dots, 21\} \rangle.$$

4. Application to web theory and to the characterization of low order polylogarithms by their functional equation

In the introduction we noticed that abelian functional equations arise in many areas of mathematics. We now give some applications of the material presented in the preceding parts to two of these areas: planar web geometry and theory of polylogarithmic functional equations.

4.1. Application to web theory

4.1.1. A brief introduction to planar web geometry. We now briefly recall, in the complex analytic setting, the basic notions of planar web geometry (the standard reference is [BB]; see [AkGo], [Che2], [ChGr1] or [Hé] for more modern points of view).

A *planar N-web $\mathcal{W}$* on a simply connected domain Ω in $\mathbb{C}^2$ is a set $\{\mathcal{F}_i\}$ of N foliations of Ω such that their leaves are in general position on Ω. Such a web will be called *linear* if these leaves are segments of straight lines.

A classical example of N-web is the *algebraic web $\mathcal{W}_C$ associated to an algebraic reduced curve $C \subset \mathbb{CP}^2$* of degree N. Let L_0 be a generic line in $\mathbb{CP}^2$

which transversally intersects the regular part of C in N distinct points; we have $C.L_0 = P_1(L_0) + \cdots + P_N(L_0)$ as 0-cycles on C. There exists an open neighbourhood Ω_0 in the dual projective space $(\mathbb{CP}^2)^\star$ and there are N holomorphic maps $P_i : \Omega_0 \to C$ such that all $L \in \Omega_0$ transversally intersect C and $C.L = P_1(L) + \cdots + P_N(L)$. Let $\mathcal{F}_i^C$ be the foliation of Ω_0 whose leaf at L is the line segment $\{L' \mid P_i(L) \in L'\}$. Then $\mathcal{W}_C = \{\mathcal{F}_i^C\}$ is an N-web on Ω_0. Because the leaves of the foliations are segments of straight lines, it is a *linear* web.

We are interested in the local geometric study of webs. Two (germs of) webs $\mathcal{W}$ and W will be called *equivalent* if there exists a local biholomorphism sending $\mathcal{W}$ to W. The objective is to classify the germs of webs modulo this notion of equivalence. A natural way to undertake a classification of webs will be to attach invariants to them, relative to this notion of equivalence.

We now define an interesting invariant for webs: the *rank*, defined as the dimension of the space of certain functional equations which can be associated to a web. Let $\mathcal{W} = \{\mathcal{F}_i\}$ be a (germ of) N-web at the origin of $\mathbb{C}^2$. Then each $\mathcal{F}_i$ admits a holomorphic first integral $U_i \in \mathcal{O}(\mathbb{C}^2, 0)$. By definition, an *abelian relation* for $\mathcal{W}$ (relative to $\mathcal{U} = (U_1, \ldots, U_N)$) is an equation in the space of germs of holomorphic 1-forms at the origin of $\mathbb{C}^2$, of the form $G_1(U_1)dU_1 + \cdots + G_N(U_N)dU_N = 0$. The space of such abelian relations will be denoted by $\mathcal{A}[\mathcal{W},\mathcal{U}]$ or less rigorously $\mathcal{A}[\mathcal{W}]$. It is a linear space. By definition, the rank of $\mathcal{W}$ is

$$r_k[\mathcal{W}] := \dim_{\mathbb{C}} \mathcal{A}[\mathcal{W},\mathcal{U}].$$

The rank is well defined: it does not depend on the choice of the U_i's. It is an invariant of webs relative to the notion of equivalence defined above. A basic result of planar web geometry is that the rank is always finite and satisfies the majorization $r_k[\mathcal{W}] \leq (N-1)(N-2)/2$. This was first proved by Bol in [Bol1].

Using the notations of this paper, we see that if the first integrals U_i are fixed, then, modulo the constant solutions, there is an isomorphism between $\underline{\mathcal{S}_\omega^{\mathcal{O}}}(\mathcal{E}\{\mathcal{U}\})$ and $\mathcal{A}[\mathcal{W},\mathcal{U}]$ where $\mathcal{E}\{\mathcal{U}\}$ is the Afe $F_1(U_1) + \cdots + F_n(U_N) = 0$. Thus, the results of Section 2 may now be seen as tools to study the abelian relations of webs whose foliations are the level curves of real rational functions. We will see that such webs are of particular interest.

We are led to consider those functional equations which can be attached to a web by the interpretation of Abel's addition theorem for algebraic webs. Let us consider our algebraic webs $\mathcal{W}_C$ again. Let ω be a differential of the first kind on C (i.e. an element of $H^0(C, \omega_C^1)$). A particular case of Abel's theorem says that the abelian sum relative to ω is identically zero: we have $P_1^*(\omega) + \cdots + P_N^*(\omega) = 0$ on Ω_0. This relation can be interpreted as an abelian relation for $\mathcal{W}_C$. So we have a linear morphism, denoted Tr (for trace), which is injective:

$$\mathrm{Tr} : H^0(C, \omega_C^1) \to \mathcal{A}[\mathcal{W}_C], \qquad \omega \mapsto P_1^*(\omega) + \cdots + P_N^*(\omega) = 0.$$

But because $\dim_{\mathbb{C}} H^0(C, \omega_C^1) = (N-1)(N-2)/2 \geq r_k[\mathcal{W}_C]$ we deduce that Tr is an isomorphism, therefore $\mathcal{W}_C$ is of maximal rank. Thus Abel's theorem gives us a wide class of maximal rank webs, coming from algebraic geometry.

The Abel-inverse theorem implies that a planar N-web of maximal rank is algebraic if and only if it is linearizable. A 3-web of maximal rank 1 will be called *hexagonal* (see [BB] for the terminology). One can prove that hexagonal 3-webs are linearizable, therefore algebraic. According to the work of Lie on translation surfaces [Lie], this is still true for planar 4-webs. A naive idea would be that all maximal rank planar webs are linearizable and therefore algebraic. But this is no longer true for N-webs with $N \geq 5$: in [Bol2], Bol gave an example of a maximal rank planar 5-web which cannot be linearized, and therefore is not algebraic; this web (now known as *Bol's web* and denoted $\mathcal{B}$) is the web whose foliations are the level curves of the U_i's of equation $(\mathcal{R})$ (see Section 3.1). From the elements $\boldsymbol{\Delta}_1, \boldsymbol{\Delta}_2, \boldsymbol{\Delta}_3, \boldsymbol{\Delta}_4, \boldsymbol{\Delta}_5$ and $\boldsymbol{\Delta}_6$ of $\underline{\mathcal{S}^{\mathcal{O}}_{\omega}}(\mathcal{R})$, we can construct a basis of the space $\mathcal{A}[\mathcal{B}]$ of all abelian relations of $\mathcal{B}$. So we have $\dim_{\mathbb{C}} \mathcal{A}[\mathcal{B}] = 6$ and the web $\mathcal{B}$ is of maximal rank 6, although it is not linearizable.

Such maximal rank but nonalgebraic webs are called *exceptional*. Since its discovery by Bol in the 30's, Bol's web $\mathcal{B}$ has been the only known example of an exceptional planar web.[1]

4.1.2. Exceptional planar webs and configurations of points. According to Chern and Griffiths (see [ChGr2, p. 83]), classifying the nonlinearizable maximal rank webs is a fundamental problem in web geometry. Since Bol's web is related to the functional equation of the dilogarithm, the Spence–Kummer web $\mathcal{W}_{\mathcal{SK}}$ associated to the Spence–Kummer equation of the trilogarithm *seems to be a good candidate as an exceptional* 9-*web* (see Section 3.3 of [Hé]). We now prove that this web is actually exceptional. The explicit resolution of equations $(\mathcal{SK})$ and $(\mathcal{E}_a)$ in Sections 3.2 and 3.3 respectively allows us to find numerous other new examples of such exceptional webs.

We first study the Spence–Kummer web $\mathcal{W}_{\mathcal{SK}}$ defined in 3.2 and its subwebs. If K is a subset of $\{1, \ldots, 9\}$, we write $\mathcal{W}_K$ for $\mathcal{W}_{\mathcal{SK},K}$. Then we have the following

Theorem 2. *The Spence–Kummer web is an exceptional* 9-*web. Its* 7-*subweb* $\mathcal{W}_{\widehat{69}}$ *is also exceptional. The two* 6-*webs* $\mathcal{W}_{\widehat{679}}$ *and* $\mathcal{W}_{\widehat{248}}$ *are equivalent exceptional subwebs of* $\mathcal{W}_{\mathcal{SK}}$ *while* $\mathcal{W}_{\widehat{369}}$ *is hexagonal (i.e., all its* 3-*subwebs are hexagonal). Finally, the web* $\mathcal{W}_{12369}$ *is an exceptional* 5-*web inequivalent to Bol's.*

Proof. For each of these webs, we have to prove two distinct things: first, that the rank is maximal, second, that the web is nonlinearizable.

From the explicit basis $\{\mathbf{F}_i\}$ of $\underline{\mathcal{S}^{\mathcal{O}}_{\omega}}(\mathcal{W}_{\mathcal{SK}})$ given in 3.2, we can easily construct 28 linearly independent abelian equations for $\mathcal{W}_{\mathcal{SK}}$, which therefore is of maximal rank. For ω generic and for any subset $K \subset \{1, \ldots, N\}$, we have a natural linear inclusion $\underline{\mathcal{S}^{\mathcal{O}}_{\omega}}(\mathcal{W}_K) \hookrightarrow \underline{\mathcal{S}^{\mathcal{O}}_{\omega}}(\mathcal{W}_{\mathcal{SK}})$. From this we can easily deduce an explicit basis of the space $\underline{\mathcal{S}^{\mathcal{O}}_{\omega}}(\mathcal{W}_K)$, and consequently of $\mathcal{A}[\mathcal{W}_K]$. So we can calculate the rank

[1] There were however some examples of exceptional 2-codimensional 4-webs in 4-dimensional spaces (see [Lit] and Section 8.4 in [Gol]) and some generalizations of Bol's web in exceptional curvilinear $(n+3)$-webs in $\mathbb{R}^n$ (see [Dam]).

of any subweb of $\mathcal{W}_{\mathcal{SK}}$: all the webs in Theorem 2 have maximal rank and all 3-subwebs of $\mathcal{W}_{\widehat{369}}$ have rank 1, so $\mathcal{W}_{\widehat{369}}$ is hexagonal.

For $k \geq 6$, let $\mathcal{W}$ be a k-web of Theorem 2, distinct from $\mathcal{W}_{\widehat{248}}$. We remark that $\mathcal{W}$ contains $\mathcal{B}$ as a 5-subweb. Because $\mathcal{B}$ cannot be linearized, the same is true for $\mathcal{W}$ which is thus exceptional.

We cannot use this argument to prove that $\mathcal{W}_{\widehat{248}}$ is not linearizable: all its 5-subwebs have rank 5, so are not equivalent to Bol's web (this already shows that $\mathcal{W}_{\widehat{248}}$ and $\mathcal{W}_{\widehat{679}}$ are not equivalent). Suppose that $\mathcal{W}_{\widehat{248}}$ is linearizable. Because it is of maximal rank, the Abel-inverse theorem implies that it is equivalent to a web $\mathcal{W}_C$ associated to an algebraic curve C of degree 6. One can verify that exactly eight of its 3-subwebs are hexagonal. This implies that C contains exactly eight distinct algebraic curves of degree 3. Consequently, C must be the union of four lines and of a nondegenerate conic. Then C contains exactly four curves of degree five, so exactly four 5-subwebs of $\mathcal{W}_C$ are of maximal rank (by Abel's theorem). But all subwebs of $\mathcal{W}_{\widehat{248}}$ have rank 5, so this web is not algebraic.

By the change of variables $(\mathsf{x}, \mathsf{y}) = (\log(x), \log(y))$, the web $W = \mathcal{W}_{12369}$ transforms into the web $\mathsf{W} = \mathcal{W}\{\mathsf{x}, \mathsf{y}, \mathsf{x}+\mathsf{y}, \mathsf{x}-\mathsf{y}, \sinh(\mathsf{x})/\sinh(\mathsf{y})\}$. A theorem by Mayrhofer and Reidemeister (see §10 in [BB]) says that a linearization of a web formed by four pencils of lines is projective. Hence we deduce that W (and so W) is nonlinearizable, because it is impossible to transform a transcendental leave $\{\sinh(\mathsf{x}) = \lambda \sinh(\mathsf{y})\}$ into a straight line by a projective map. Thus W is exceptional. Only eight 3-subwebs of W are of rank 1. Thus W is not hexagonal and so cannot be equivalent to $\mathcal{B}$. Theorem 2 is now fully proved. □

Remarks. 1. The subwebs $\mathcal{W}_{\widehat{36}}$ and $\mathcal{W}_{\widehat{39}}$ are exceptional but equivalent to $\mathcal{W}_{\widehat{69}}$.
2. The subwebs $\mathcal{W}_{\widehat{147}}$, $\mathcal{W}_{\widehat{257}}$, and $\mathcal{W}_{\widehat{158}}$ are exceptional and equivalent to $\mathcal{W}_{\widehat{248}}$.
3. The subwebs $\mathcal{W}_{\widehat{689}}$, $\mathcal{W}_{\widehat{349}}$, $\mathcal{W}_{\widehat{236}}$, $\mathcal{W}_{\widehat{359}}$, and $\mathcal{W}_{\widehat{136}}$ are equivalent to $\mathcal{W}_{\widehat{679}}$ and therefore exceptional.
4. The subwebs $\mathcal{W}_{34569}$ and $\mathcal{W}_{36789}$ are exceptional but equivalent to $\mathcal{W}_{12369}$.
5. All the exceptional subwebs of $\mathcal{W}_{\mathcal{SK}}$ are those which are described in Theorem 2 and in the above remarks **1**, **2**, **3** and **4**.
6. One can prove that exactly 8 (resp. 16) 3-subwebs of $\mathcal{W}_{\widehat{248}}$ (resp. $\mathcal{W}_{\widehat{679}}$) are hexagonal. We will use these facts later.

According to a theorem by Bol (see [BB, p. 108]), the fact that $\mathcal{W}_{\widehat{369}}$ is hexagonal implies that it is linearizable into a web formed by six pencils of lines. Hence, by duality, it is associated to the configuration formed by the six basepoints of these pencils. A linearization of this web is given by the quadratic Cremona transform $\mathcal{C} : (x, y) \mapsto (1/(x-1), 1/(y-1))$. It is natural to ask what is the action of $\mathcal{C}$ on the whole web $\mathcal{W}_{\mathcal{SK}}$. We introduce some definitions. For $d > 0$, let δ_d be the dimension of the space of algebraic curves of degree d in $\mathbb{CP}^2$.

Definition 2. Let $d \geq 1$. If K is a set of $\delta_d - 1$ points in general position in the complex projective plane, then the family of curves of degree d through these $\delta_d - 1$ points is denoted by $\mathcal{F}_K$. It is a singular foliation of $\mathbb{CP}^2$.

For $n \geq 3$, we set $\Delta_n = \bigcup_{i<j}\{(p_1,\ldots,p_n) \in (\mathbb{CP}^2)^n \mid p_i = p_j\}$. We define the space of configurations of n points in $\mathbb{CP}^2$ as the set $C_n^2 = (\mathbb{CP}^2)^n - \Delta_n$. If three distinct points p_i, p_j, p_k of a configuration $(p_1,\ldots,p_n)$ lie on the same line, the configuration is said to be *degenerate*. The natural $\mathrm{PGL}_3(\mathbb{C})$-action on $\mathbb{CP}^2$ induces projective group actions on the spaces of configurations C_n^2.

Definition 3. Let $n \geq 3$. The web $\mathcal{W}_{\mathbf{c}}$ associated to a configuration $\mathbf{c} = (p_1,\ldots,p_n)$ of n points in $\mathbb{CP}^2$ is the singular web defined on the whole projective plane, the foliations of which are the $\mathcal{F}_K$'s where K runs over the set of subsets of $\delta_k - 1$ points in $\{p_1,\ldots,p_n\}$, in general position, with k such that $1 \leq \delta_k \leq n$.

It was first remarked by Bol that $\mathcal{B}$ is associated to a configuration of four points in generic position in $\mathbb{CP}^2$, in the sense of the preceding definition. According to our notations, the web given by the level curves of the functions $U_1,\ldots,U_5$ is associated to the configuration $\mathbf{b}$ shown in Figure 1 below.

For the Spence–Kummer web $\mathcal{W}_{\mathcal{SK}}$, we have the following.

Proposition 6. *The image of* $\mathcal{W}_{\mathcal{SK}}$ *under* $\mathcal{C}$ *is the web associated to the degenerate configuration* $\mathsf{q}_{\mathcal{SK}}$ *of six points in* $\mathbb{CP}^2$ *shown in Figure* 2.

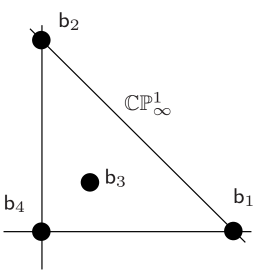

Fig. 1. Configuration b with $\mathsf{b}_1 = [1:0:0]$, $\mathsf{b}_2 = [0:1:0]$, $\mathsf{b}_3 = [1:1:1]$, $\mathsf{b}_4 = [0:0:1]$.

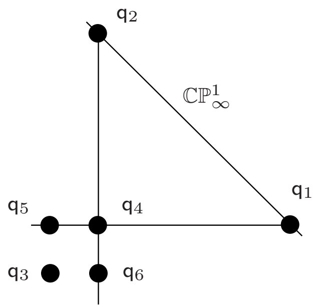

Fig. 2. Configuration $\mathsf{q}_{\mathcal{SK}}$: $\mathsf{q}_1 = \mathsf{b}_1$, $\mathsf{q}_2 = \mathsf{b}_2$, $\mathsf{q}_4 = \mathsf{b}_4$, $\mathsf{q}_3 = [-1:-1:1]$, $\mathsf{q}_5 = [-1:0:1]$, $\mathsf{q}_6 = [0:-1:1]$.

The 6-subweb $\mathcal{W}_{\widehat{679}}$ of $\mathcal{W}_{\mathcal{SK}}$ is associated to a subconfiguration of $\mathsf{q}_{\mathcal{SK}}$:

Proposition 7. *The image of the exceptional web* $\mathcal{W}_{\widehat{679}}$ *under* $\mathcal{C}$ *is the web associated to the subconfiguration* $(\mathsf{q}_1,\mathsf{q}_2,\mathsf{q}_3,\mathsf{q}_4,\mathsf{q}_6)$ *of* $\mathsf{q}_{\mathcal{SK}}$.

The other exceptional subwebs of $\mathcal{W}_{\mathcal{SK}}$ must be associated to configurations of points as well, but in a more complicated way than in Definition 4.

In Section 3.3 we have considered some webs $\mathcal{W}_a$ depending on a real parameter $a > 1$. We can now define these webs as the webs associated to configuration $\mathsf{C}_a = (\mathsf{c}_1,\mathsf{c}_2,\mathsf{c}_3,\mathsf{c}_4,\mathsf{c}_a)$ shown in Figure 3.

More generally, we will denote $\mathcal{W}_a$ by the 8-web associated to C_a for $a \in \mathbb{CP}^1 \setminus \{0,1,\infty\}$. Configuration b is a subconfiguration of C_a, so Bol's web is a subweb of $\mathcal{W}_a$. This implies that the $\mathcal{W}_a$'s are nonlinearizable. It is natural to study their rank.

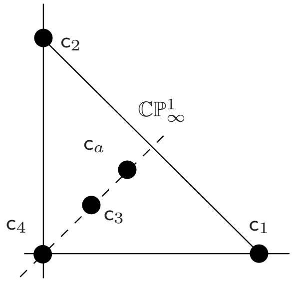

Fig. 3. Configuration C_a which is associated to the web $\mathcal{W}_a$ of 3.3 with $\mathsf{c}_1 = \mathsf{b}_1$, $\mathsf{c}_2 = \mathsf{b}_2$, $\mathsf{c}_3 = \mathsf{b}_3$, $\mathsf{c}_4 = \mathsf{b}_4$ and $\mathsf{c}_a = [a : a : 1]$.

Proposition 8. *The web $\mathcal{W}_a$ associated to C_a is an exceptional planar 8-web.*

Proof. For $a > 1$, the family $\{\mathbf{G}_i^a \mid i = 1, \ldots, 21\}$ forms a basis of $\underline{\mathcal{S}_\omega^{\mathcal{O}}}(\mathcal{W}_a)$ modulo the constant solutions. Thus $\mathcal{W}_a$ is of maximal rank 21. We observe that the germs $\mathbf{G}_i^a$ analytically depend on the parameter a and can be analytically extended (with respect to a) along any path in $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$. So we can construct 21 nonconstant solutions of $(\mathcal{E}_a)$ for any $a \notin \{0, 1, \infty\}$. One can verify that these solutions are linearly independent again. This implies that the $\mathcal{W}_a$'s are of maximal rank. Because they are not linearizable, they are exceptional. □

Thus we have a family of exceptional 8-webs, parametrized by the set $A := \mathbb{CP}^1 \setminus \{0, 1, \infty\}$. It is natural to study this family relative to the notion of (local analytic or projective) equivalence of webs.

Consider the group $\mathfrak{G} = \{g \in \mathrm{PGL}_2(\mathbb{C}) \mid g(\{0, 1, \infty\}) = \{0, 1, \infty\}\}$ which acts on A in a natural way. One can easily prove that if the two webs $\mathcal{W}_a$ and $\mathcal{W}_\alpha$ are projectively equivalent (for $a, \alpha \in A$) then a and α are $\mathfrak{G}$-equivalent.

We write $\mathfrak{A}$ for the quotient of A by $\mathfrak{G}$ and $[a]$ for the class of $a \in A$ in $\mathfrak{A}$. The classical map $j : a \mapsto (a^2 - a + 1)^3 (a(1 - a))^{-2}$ is $\mathfrak{G}$-invariant and gives an isomorphism $\mathfrak{A} \simeq \mathbb{C}$. For $z \in \mathbb{C}$ we choose $a_z \in A$ such that $j(a_z) = z$. Then the $\mathcal{W}_{a_z}$ form a family of pairwise nonprojectively equivalent webs. It is natural to ask if these webs are equivalent (up to local analytic diffeomorphism).

Proposition 9. *For $a, \alpha \in A$, the webs $\mathcal{W}_a$ and $\mathcal{W}_\alpha$ are equivalent (up to local biholomorphism) if and only if a and α are $\mathfrak{G}$-equivalent. Thus the webs $\mathcal{W}_{a_z}$ form a family of pairwise inequivalent exceptional 8-webs parametrized by $\mathbb{C}$.*

Proof (sketch). Let $a, \widehat{a} \in A$. Assume that $\mathcal{W}_a$ and $\mathcal{W}_{\widehat{a}}$ are equivalent. This means that there exist two connected open neighbourhoods $\Omega_a \subset \mathbb{CP}^2 \setminus \Sigma_a$, $\Omega_{\widehat{a}} \subset \mathbb{CP}^2 \setminus \Sigma_{\widehat{a}}$ of two points ω and $\widehat{\omega}$, and an analytic diffeomorphism $\Phi : \Omega_a \to \Omega_{\widehat{a}}$ such that $\mathcal{W}_{\widehat{a}}|_{\Omega_{\widehat{a}}}$ is the image of $\mathcal{W}_a|_{\Omega_a}$ under Φ. Hence, there are a permutation $\sigma \in \mathfrak{S}_8$ and invertible holomorphic germs h_i such that for all $i \in \{1, \ldots, 8\}$ we have

$$U_i^a = \mathsf{h}_{\sigma i} \circ U_{\sigma i}^{\widehat{a}} \circ \Phi \quad \text{in a neighbourhood of } \omega, \tag{$\star$}$$

which we may suppose to be Ω_a (we use the notations of Section 3.3 for the functions U_i^a).

For $k=1,\dots 8$, $\mathsf{a}\in A$ and $w\notin\Sigma_{\mathsf{a}}$, we denote by $[\mathcal{E}_{\mathsf{a}}]_k^w$ the space spanned by the k-th components G_k of all solutions $(G_1,\dots,G_8)$ of $(\mathcal{E}_{\mathsf{a}})$ at w. The dimension of this space does not depend on w and will be denoted $\dim[\mathcal{E}_{\mathsf{a}}]_k$. The relations $(\star)$ give us linear isomorphisms between $[\mathcal{E}_a]_i^\omega$ and $[\mathcal{E}_{\widehat{a}}]_{\sigma i}^{\widehat{\omega}}$ $(i=1,\dots,8)$. For $\mathsf{a}\in A$, the results of Section 3.3 imply that $\dim[\mathcal{E}_{\mathsf{a}}]_i=9$ if $i=1,2$ while $\dim[\mathcal{E}_{\mathsf{a}}]_i=8$ for $i\geq 3$. Thus we have $\sigma\{1,2\}=\{1,2\}$. Modulo composition with the involution $(x,y)\mapsto(y,x)$ which leaves the $\mathcal{W}_{\mathsf{a}}$'s unchanged, we may assume that $\sigma 1{=}1$ and $\sigma 2{=}2$.

Let $\mathfrak{E}$ be the subgroup of $\mathfrak{S}_8$ generated by $(34)(78),(36)(58),(46)(57)$ and $(45)(67)$. For $\tau\in\mathfrak{E}$, one can find $a_\tau\in[a]$ and a projective (or birational) map g_τ such that $g_\tau(\mathsf{C}_{a_\tau})=\mathsf{C}_a$ and $U_i^{a_\tau}=\mathsf{k}_{\tau i}\circ U_{\tau i}^a\circ g_\tau$ where the k_i's are invertible rational functions. Then by considering $\mathcal{W}_{a_\tau}$ and $\Phi\circ g_\tau$ instead of $\mathcal{W}_a$ and Φ, we may assume that $\sigma 3=3$ and $\sigma 4\in\{4,8\}$. But the case when $\sigma 3=3$, $\sigma 4=8$ cannot happen because for all $\mathsf{a}\in A$, the web $\mathcal{W}_{\mathsf{a},\{1,3,4\}}$ is hexagonal, while $\mathcal{W}_{\mathsf{a},\{1,3,8\}}$ is not. So we may assume that $\sigma j=j$ for $j=1,2,3,4$: Φ is a symmetry for $\mathcal{W}_{a,\{1,2,3,4\}}$.

A theorem by Mayrhofer and Reidemeister (see [BB, p. 93]) implies that a local equivalence between two 4-webs formed by pencils of lines is a projective transformation. Because Φ is a symmetry for $\mathcal{W}_{a,\{1,2,3,4\}}$ which is formed by four pencils of lines, we infer that it is a projective map. Because Φ leaves the foliation $\{U_i=\text{const}\}$ invariant for $i=1,2,3,4$, it follows that Φ is the identity. This proves the proposition.

From the results of 3.3, we explicitly know the space of abelian relations of the webs $\mathcal{W}_a$. This allows us to study their subwebs. We get new exceptional webs.

Proposition 10. *Let $a,\alpha\in A$. The two webs $\mathcal{W}_{a,\widehat{12}}$ and $\mathcal{W}_{\alpha,\widehat{38}}$ are exceptional 6-webs which are not equivalent. Moreover none of them is equivalent to the exceptional 6-subwebs of $\mathcal{W}_{\mathcal{SK}}$.*

Proof (sketch). The webs of the proposition have maximal rank: since they are subwebs of the $\mathcal{W}_a$'s, which are webs whose abelian equations are explicitly known, this point can be checked easily.

One can prove that among the 3-subwebs of $\mathcal{W}_{a,\widehat{12}}$ (resp. $\mathcal{W}_{\alpha,\widehat{38}}$), exactly 8 (resp. 12) are hexagonal. Because a plane algebraic curve of degree 6 cannot admit exactly twelve cubics as components, a simple application of Abel's theorem shows that $\mathcal{W}_{\alpha,\widehat{38}}$ is not algebraic and so is exceptional. We show that $\mathcal{W}_{a,\widehat{12}}$ is exceptional by an argument of the same type.

Two equivalent d-webs have the same number of maximal rank k-subwebs. Then Remark 5 following Theorem 2 shows that among the exceptional 6-webs of Theorem 2 and Proposition 10, only $\mathcal{W}_{\mathcal{SK},\widehat{248}}$ and $\mathcal{W}_{a,\widehat{12}}$ may be equivalent: they are the only ones with the same number of hexagonal 3-subwebs. But among the 4-subwebs of $\mathcal{W}_{\mathcal{SK},\widehat{248}}$ exactly six are of maximal rank while $\mathcal{W}_{a,\widehat{12}}$ has only three subwebs of this kind. So these webs are not equivalent.

It is now natural to study the (local) equivalence of the $\mathcal{W}_{a_z,\widehat{\kappa\varsigma}}$ with $\{\kappa,\varsigma\}\in\{\{1,2\},\{3,8\}\}$ and $z\in\mathbb{C}$ (we recall that $\widehat{\kappa\varsigma}=\{1,\dots,8\}\setminus\{\kappa,\varsigma\}$).

Proposition 11. *Let $\{\kappa,\varsigma\}=\{1,2\}$ or $\{\kappa,\varsigma\}=\{3,8\}$. Then, for $a,\alpha\in A$, the webs $\mathcal{W}_{a,\widehat{\kappa\varsigma}}$ and $\mathcal{W}_{\alpha,\widehat{\kappa\varsigma}}$ are equivalent if and only if a and α are $\mathfrak{G}$-equivalent. Thus the $\mathcal{W}_{a_z,\widehat{\kappa\varsigma}}$ form a family of pairwise inequivalent exceptional 6-webs.*

Proof (sketch). Assume that $\mathcal{W}_{a,\widehat{\kappa\varsigma}}$ and $\mathcal{W}_{\alpha,\widehat{\kappa\varsigma}}$ are equivalent. We use the same notation as in Proposition 9: there is a permutation $\sigma\in\mathfrak{S}_{\widehat{\kappa\varsigma}}$ such that for $i\in\widehat{\kappa\varsigma}$ the relations

$$U_i^a=\mathsf{h}_{\sigma i}\circ U_{\sigma i}^{\widehat{a}}\circ\Phi \tag{$\star$}$$

are satisfied on Ω_a.

When $\{\kappa,\varsigma\}=\{3,8\}$, the proof is similar to that of Proposition 9: the explicit knowledge of the abelian relations of $\mathcal{W}_{\mathsf{a},\widehat{38}}$ allows us to notice that $\mathcal{W}_{\mathsf{a},\{4,5,6,7\}}$ is the single maximal rank 4-subweb of $\mathcal{W}_{\mathsf{a},\widehat{38}}$ whose 3-subwebs are all nonhexagonal (for all $\mathsf{a}\in A$). This implies that $\mathcal{W}_{\alpha,\{4,5,6,7\}}$ is the image of $\mathcal{W}_{a,\{4,5,6,7\}}$ under Φ. So $\sigma\{4,5,6,7\}=\{4,5,6,7\}$. Thus $\sigma\{1,2\}=\{1,2\}$, hence we may assume that $\Phi(x,y)=(\phi_1(x),\phi_2(y))$. Among the 3-subwebs $\mathcal{W}_{\mathsf{a},14k}$ of $\mathcal{W}_{\mathsf{a},\widehat{38}}$ (with $k=2,5,6,7$), only $\mathcal{W}_{\mathsf{a},147}$ is not of rank one. This implies that $\sigma\{4,7\}=\{4,7\}$. Because we are working up to equivalence, we may assume that $\sigma4=4$ and $\sigma7=7$. The fact that $U_4\circ\Phi$ is constant along the level curves $\{U_4=\text{const}\}$ implies that ϕ_1 and ϕ_2 satisfy the following differential equation with separated variables:

$$\left(\partial_x+\frac{1-y}{1-x}\partial_y\right)\left[\frac{1-\phi_2(y)}{1-\phi_1(x)}\right]=0.$$

So there is a constant $c\neq0$ such that

$$\phi_i'(z)=c(1-\phi_i(z))/(1-z)\qquad\text{for } i=1,2. \tag{1}$$

We know that $\sigma\{5,6\}=\{5,6\}$. Assume that $\sigma6=5$. Then $U_5\circ\Phi$ is constant along the level curves $\{U_6^a=\text{const}\}$. This implies that we have the relation

$$((a-x)\partial_x+(a-y)\partial_y)\left[\frac{\phi_1(x)}{\phi_2(y)}\cdot\frac{1-\phi_2(y)}{1-\phi_1(x)}\right]=0.$$

So there exists $\lambda\neq0$ such that

$$\phi_i'(z)=\lambda\phi_i(z)(1-\phi_i(z))/(a-z)\qquad\text{for } i=1,2. \tag{2}$$

Then (1) and (2) imply that $\phi_i(z)=(c/\lambda)(a-z)/(1-z)$, which is impossible. Thus we have $\sigma6=6$ and Φ transforms $\mathcal{W}_{\widehat{a},1246}$ into $\mathcal{W}_{a,1246}$, and because these webs are formed by pencils of lines, we can conclude as in the proof of Proposition 9.

We now study the case when $\{\kappa,\varsigma\}=\{1,2\}$. We may assume that $\sigma3=3$ and $\sigma4\in\{4,8\}$. The case when $\sigma4=8$ cannot happen because, for $\mathsf{a}\in A$, none of the 3-webs $\mathcal{W}_{\mathsf{a},3k8}$ $(k=4,5,6,7)$ is hexagonal, while $\mathcal{W}_{\mathsf{a},345}$ is. We cannot deduce a particular form for $\Phi=(\phi_1,\phi_2)$ by considering the action of σ on the maximal rank subwebs of $\mathcal{W}_{\mathsf{a},\widehat{12}}$. Therefore equations ($\star$) do not give differential equations with separate variables but a system of partial differential equations on ϕ_1 and ϕ_2 which is not easily solvable. So we will use another argument.

The diffeomorphism Φ induces an isomorphism $\underline{\Phi}\colon\underline{\mathcal{S}}_{\widehat{\omega}}^{\mathcal{O}}(\mathcal{W}_{\widehat{a},\widehat{12}})\to\underline{\mathcal{S}}_{\omega}^{\mathcal{O}}(\mathcal{W}_{a,\widehat{12}})$: if $F_3(U_3^{\widehat{a}})+\cdots+F_8(U_8^{\widehat{a}})=0$ on $\Omega_{\widehat{a}}$, then composition with Φ shows that we have

$(F_{\sigma 3} \circ \mathsf{h}_{\sigma 3}^{-1})(U_3^a) + \cdots + (F_{\sigma 8} \circ \mathsf{h}_{\sigma 8}^{-1})(U_8^a) = 0$ on Ω_a. The isomorphism $\underline{\Phi}$ is clearly compatible with the filtration by order, so $\underline{\Phi}(F^3 \underline{\mathcal{S}}_{\omega}^{\mathcal{O}}(\mathcal{W}_{\widehat{a},\widehat{12}})) = F^3 \underline{\mathcal{S}}_{\omega}^{\mathcal{O}}(\mathcal{W}_{a,\widehat{12}})$.

The components of order-3 solutions of $(\mathcal{E}_{\mathsf{a}})$ are linear combinations of suitable branches of $\mathbf{L}_{x_0}$, $\mathbf{L}_{x_1}$ and $\mathbf{J}$ (up to constants). So for $i \in \{3, \dots, 8\}$ and $\mathbf{f} \in \{\mathbf{J}, \mathbf{L}_{x_0}, \mathbf{L}_{x_1}\}$, we have $\mathbf{f} \circ \mathsf{h}_i^{-1} = \kappa_0^{\mathbf{f}} \mathbf{L}_{x_0} + \kappa_1^{\mathbf{f}} \mathbf{L}_{x_1} + \kappa_2^{\mathbf{f}} \mathbf{J} + \kappa_3^{\mathbf{f}}$ where the $\kappa_s^{\mathbf{f}}$ are complex constants. From these relations we deduce that $\mathsf{h}_i = \mathbf{I}\mathrm{d}$ or $\mathbf{I}\mathrm{v}$ (for $i \geq 3$).

Modulo the change of coordinates $(x, y) \mapsto (y, x)$, we may assume that $\mathsf{h}_3 = \mathbf{I}\mathrm{d}$. In this case, if $\mathsf{h}_4 = \mathbf{I}\mathrm{d}$, then the relations $(\star)$ yield $\Phi(x, y) = (x, y)$, so we are done. If we assume that $\mathsf{h}_4 = \mathbf{I}\mathrm{v}$, the relations $(\star)$ imply that $\Phi = (\phi, \phi)$ with $\phi(x, y) = x(y - x)(y(1 - y) - x(1 - x))^{-1}$. It is not difficult to see that with this definition, Φ cannot be an equivalence between $\mathcal{W}_{a,\widehat{12}}$ and $\mathcal{W}_{\widehat{a},\widehat{12}}$.

Remark. Let K be one of the sets $\widehat{12}$, $\widehat{38}$ or $\{1, \dots, 8\}$. Let Φ be a local analytic equivalence between two exceptional webs $\mathcal{W}_{a,K}$ and $\mathcal{W}_{\widehat{a},K}$. Then the proofs of the preceding propositions show that Φ is the restriction of a global (projective or birational) equivalence. From these proofs, it is also possible to determine the symmetry groups of the exceptional webs considered above.

The fact that the only known exceptional planar webs described above are related to configurations of points may be an important fact which deserves to be studied. In [Dam], Damiano considers some webs of curves in $\mathbb{R}^{N+3}$ ($N \geq 2$) similarly associated to configurations of points. He shows, using the work of Gelfand and MacPherson [GeMcP], that those webs are exceptional curvilinear webs.

All these results suggest that there could be a link between configurations of points and exceptional webs. In this spirit we can reformulate the preceding results in the following general form:

Theorem 3. *The web associated to any configuration of four points in $\mathbb{CP}^2$ is of maximal rank. It is nonlinearizable if and only if the configuration is nondegenerate. In that case it is (projectively) equivalent to Bol's web $\mathcal{B}$.*

For configurations of five points in the plane, we have:

Theorem 4. *The web associated to any degenerate configuration C of five points in $\mathbb{CP}^2$ is of maximal rank. It is exceptional if and only if C has a subconfiguration of four points in general position.*

Proof. We consider the stratification of C_5^2 described by Figure 4 below:

- $\mathbf{S}_0$ is the open subset of generic configurations;
- $\mathbf{S}_1$ is the analytic stratum of degenerate configurations such that exactly three points lie on the same line;
- $\mathbf{S}_2$ is the analytic stratum of degenerate configurations such that exactly four points lie on the same line;
- $\mathbf{S}_3$ is the analytic stratum of degenerate configurations $(p_1, \dots, p_5)$ outside $\mathbf{S}_2$ with a unique p_j such that for all $i \neq j$ there exists k distinct from i and j and such that the three points p_i, p_j and p_k are collinear;

- $\mathbf{S}_4$ is the analytic stratum of degenerate configurations such that the five points are collinear.

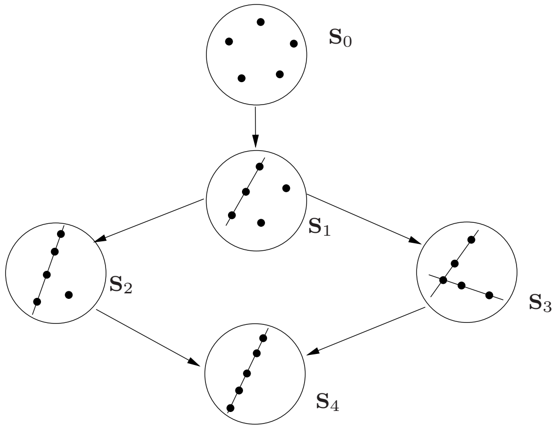

Fig. 4. Stratification of C_5^2 by degenerate configurations. (An arrow $\mathcal{S} \to \widehat{\mathcal{S}}$ between two strata means that $\widehat{\mathcal{S}} \subset \partial\mathcal{S}$ in C_5^2.)

We set $N_0 = 10, N_1 = 8, N_2 = 5, N_3 = 6,$ and $N_4 = 5$. For each $i \in \{0, \ldots, 4\}$, the web associated to a configuration $\mathsf{C} \in \mathbf{S}_i$ (denoted $\mathcal{W}_\mathsf{C}$) is an N_i-web.

- If $\mathsf{C} \in \mathbf{S}_2 \cup \mathbf{S}_4$, then the foliations of $\mathcal{W}_\mathsf{C}$ are pencils of lines: this web is associated by duality to an algebraic curve of degree 5 which is the union of five lines. So $\mathcal{W}_\mathsf{C}$ is an algebraic web, thus it is of maximal rank.
- Assume that $\mathsf{C} \in \mathbf{S}_3$. Because the natural $\mathrm{PGL}_3(\mathbb{C})$-action on C_4^2 is transitive when restricted to the generic configurations, there exists a projective map g such that $g(\mathsf{C})$ is the configuration $\widehat{\mathsf{q}}$ of Proposition 7. Therefore the image of $\mathcal{W}_\mathsf{C}$ under g is the web $\mathcal{W}_{\widehat{\mathsf{q}}}$ itself (birationally) equivalent to the subweb $\mathcal{W}_{\widehat{679}}$ of $\mathcal{W}_{\mathcal{SK}}$ which is of maximal rank by Theorem 2. Then the same is true for $\mathcal{W}_\mathsf{C}$.
- Let $\mathsf{C} = (p_1, \ldots, p_5) \in \mathbf{S}_1$ be such that p_1, p_2 and p_3 are on the same line $\mathsf{L} \subset \mathbb{CP}^2$. We write p_6^c for the intersection point of L and $\overline{p_4p_5}$. We write $c_r[\mathsf{C}]$ for the class of the cross-ratio $c_r(p_1, p_2, p_3, p_6^\mathsf{c})$ in A. For $a \in c_r[\mathsf{C}]$, the webs $\mathcal{W}_\mathsf{C}$ and $\mathcal{W}_a$ are projectively equivalent, so they have the same rank. By Proposition 9 this implies that $\mathcal{W}_\mathsf{C}$ is of maximal rank.

Thus Theorem 4 is proved. □

To a generic configuration of five points in $\mathbb{CP}^2$ is associated a 10-web formed by five pencils of lines and five pencils of conics. While we know that some of these webs are of maximal rank, this has not been proved in general yet, although we think it is true. Some webs associated to degenerate configurations of six points are also of maximal rank. But it is known that some webs associated to configurations of $n \geq 6$ points are not of maximal rank, although they carry numerous abelian relations. Therefore we find the following problem interesting:

Problem. *Find the rank and all abelian relations of webs associated to configurations of $n \geq 3$ points in the projective plane.*

It is because of this problem that the author has studied the Afe's $(\mathcal{E}_a)$ which characterize the abelian relations of the webs $\mathcal{W}_a$ with $a \in A$.

A web associated to a configuration C is nonlinearizable as soon as C contains a subconfiguration of four points in general position. So if all the webs associated to generic configurations of five points in $\mathbb{CP}^2$ are of maximal rank, we will get a 2-dimensional family of exceptional planar 10-webs.

The results of Section 3 show that numerous abelian functional equations of webs associated to configurations studied in Section 4.1 are constructed from iterated integral functions. Then if, as expected, webs associated to configurations carry numerous Afe constructed from iterated integrals, it could be a way to find new Afe satisfied by higher order polylogarithms.

Another interesting question is whether the webs associated to Kummer's inhomogeneous Afe for $\mathbf{Li}_4$ and $\mathbf{Li}_5$ (see [Lew]) are exceptional and linked to configurations of points.

4.2. Application to the problem of characterizing polylogarithmic functions by their functional equation

Our objective here is to study the functions which satisfy equation $(\mathcal{A}b)$ or equation (SK). For the Cauchy equation (C) (see 2.1), it has been known for a long time that any nonconstant measurable local solution of (C) is constructed from the logarithm. The explicit resolution of equations $(\mathcal{R})$ and $(\mathcal{SK})$ in Section 3 allows us to get the same kind of results for the dilogarithm and the trilogarithm: these functions are characterized (in the measurable class) by their functional equation $(\mathcal{A}b)$ and (SK) respectively.

4.2.1. Characterization of the dilogarithm by the Rogers equation $(\mathcal{R})$. We first have this characterization of $\mathbf{D}$ which follows easily from a result established by Rogers in the early 20th century (see [Rog, Section 4]): let $\mathbf{F}$ be a real-valued function of class C^3 on $]0,1[$ which satisfies, for $0 < x < y < 1$,

$$\mathbf{F}(x) - \mathbf{F}(y) - \mathbf{F}\left(\frac{x}{y}\right) - \mathbf{F}\left(\frac{1-y}{1-x}\right) + \mathbf{F}\left(\frac{x(1-y)}{y(1-x)}\right) = 0. \tag{R}$$

Then $\mathbf{F}$ is a real multiple of $\mathbf{D}$. The proof is essentially an application of Abel's method to this case.

The classical bilogarithm $\mathbf{Li}_2$ has another well known cousin: the Bloch–Wigner function defined by $\mathcal{L}_2(z) = \Im m(\mathbf{Li}_2(z) + \log(1-z)\log|z|)$ for z in $\mathbb{C}\setminus\{0,1\}$ and extended to $\mathbb{CP}^1$ by continuity. This function satisfies the following relation for all $z_0, \ldots, z_4 \in \mathbb{CP}^1$ (where c_r denotes the cross ratio):

$$\sum_{i=0}^{4} (-1)^i \mathcal{L}_2(c_r(z_0, \ldots, \widehat{z_i}, \ldots, z_4)) = 0. \tag{$*$}$$

When we specialize $(z_0, \ldots, z_4)$ to $(\infty, 0, 1, x, y)$, equation $(*)$ simplifies. Now using the inversion relation $\mathcal{L}_2(1/z) = -\mathcal{L}_2(z)$ (for $z \in \mathbb{CP}^1$) we conclude that $\mathcal{L}_2$ is a global continuous solution of equation (R).

In [Blo], Bloch characterizes $\mathcal{L}_2$ as the unique measurable solution of $(*)$:

Proposition 12. *Let* $\mathbf{F} : \mathbb{CP}^1 \to \mathbb{R}$ *be measurable and satisfy* $(*)$*. Then* $\mathbf{F}$ *is a real multiple of* $\mathcal{L}_2$*.*

Using Proposition 1 and Bol's description of the space $\mathcal{A}[\mathcal{B}]$ (see 3.1), we get a semi-local characterization of Rogers' dilogarithm by its functional equation (R), for two unknown functions, in the class of measurable functions:

Proposition 13. *If* $\mathbf{F}, \mathbf{G}$ *are real measurable functions on* $]0,1[$ *satisfying*

$$\mathbf{F}(x) - \mathbf{F}(y) - \mathbf{F}\left(\frac{x}{y}\right) - \mathbf{F}\left(\frac{1-y}{1-x}\right) + \mathbf{G}\left(\frac{x(1-y)}{y(1-x)}\right) = 0$$

for $0 < x < y < 1$*, then* $\mathbf{F} = \mathbf{G} = \alpha\mathbf{D}$ *for some* $\alpha \in \mathbb{R}$*.*

The explicit knowledge of $\underline{\mathcal{S}}^{\mathcal{O}}_{\omega}(\mathcal{R})$ allows us to state numerous variants of Proposition 13. Those results can be formulated in an inhomogeneous form to obtain some characterization of $\mathbf{Li}_2$ by functional equations inspired from $(\mathcal{A}b)$.

4.2.2. Characterization of the trilogarithm by equation (SK)**.** The fact that the logarithm and dilogarithm are characterized by the Afe with rational inner functions which they satisfy naturally leads us to ask if the same is true for any trilogarithmic function.

In [Gon2], Goncharov obtains some results of this kind: he considers the real single-valued cousin of $\mathbf{Li}_3$ introduced by Ramakrishnan and Zagier:

$$\mathcal{L}_3 : z \mapsto \Re e\left(\mathbf{Li}_3(z) - \mathbf{Li}_2(z)\log|z| + \frac{1}{3}\mathbf{Li}_1(z)\log^2|z|\right)$$

defined on the whole $\mathbb{CP}^1$ and extended to $\mathbb{R}[\mathbb{CP}^1]$ by linearity.

When it is well defined, he considers the following element of $\mathbb{Q}[\mathbb{CP}^1]$:

$$\begin{aligned} R_3(\alpha_1,\alpha_2,\alpha_3) := \sum_{i=1}^{3}\Bigg(&\{\alpha_{i+2}\alpha_i - \alpha_i + 1\} + \left\{\frac{\alpha_{i+2}\alpha_i - \alpha_i + 1}{\alpha_{i+2}\alpha_i}\right\} - \{1\} \\ &+ \left\{\frac{\alpha_{i+2}\alpha_{i+1} - \alpha_{i+2} + 1}{(\alpha_{i+2}\alpha_i - \alpha_i + 1)\alpha_{i+1}}\right\} - \left\{\frac{\alpha_{i+2}\alpha_i - \alpha_i + 1}{\alpha_{i+2}}\right\} \\ &- \left\{\frac{\alpha_{i+2}\alpha_{i+1} - \alpha_{i+1} + 1}{(\alpha_{i+2}\alpha_i - \alpha_i + 1)\alpha_{i+1}\alpha_{i+2}}\right\} + \{\alpha_{i+2}\} \\ &+ \left\{-\frac{(\alpha_{i+2}\alpha_{i+1} - \alpha_{i+1} + 1)\alpha_i}{\alpha_{i+2}\alpha_i - \alpha_i + 1}\right\}\Bigg) + \{-\alpha_1\alpha_2\alpha_3\} \end{aligned}$$

(with $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{CP}^1$, the indices i being taken modulo 3).

Next he proves that we have the functional equation with 22 terms

$$\mathcal{L}_3(R_3(a,b,c)) = 0 \qquad (a,b,c \in \mathbb{C}). \tag{$\mathcal{G}$}$$

Then in part (a) of Theorem 1.10 in [Gon2], he shows that *the space of real* *continuous functions on* $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$ *that satisfy the functional equation* $(\mathcal{G})$ *is generated by the functions* $z \mapsto \mathcal{L}_3(z)$ *and* $z \mapsto \mathcal{L}_2(z) \log |z|$ (in fact, what he proves implies that this theorem is valid for measurable functions). He had previously remarked that if we specialize this equation by setting $a = 1, b = x$, and $c = (1-y)/(1-x)$, equation $(\mathcal{G})$ simplifies and by using the inversion relation $\mathcal{L}_3(1/z) = \mathcal{L}_3(z)$ (holding for all $z \in \mathbb{CP}^1$), we deduce that $\mathcal{L}_3$ satisfies the homogeneous version (i.e., without the right hand side) of equation (SK). We denote this homogeneous equation by (SKh). This leads Goncharov to ask ([Gon2, p. 209]) if this specialization characterizes the solutions of $(\mathcal{G})$.

The explicit determination of a basis of $\underline{\mathcal{S}_\omega^\mathcal{O}}(\mathcal{SK})$ in Section 3.2 allows us to give a positive answer to this question: Proposition 14 is a real semi-local characterization of the measurable solutions of (SKh). We need the following continuous real version of Rogers' dilogarithm, defined on $]-\infty, 1[$:

$$\delta_2 : x \mapsto \mathbf{Li}_2(x) + \frac{1}{2} \log |x| \log(1-x) - \frac{\pi^2}{6}.$$

Then we can consider the real-valued function L_3 defined by

$$\mathrm{L}_3(x) := \begin{cases} \delta_2(x) \log |x| & \text{for } 0 < x < 1, \\ \delta_2(x) \log |x| + \dfrac{\pi^2}{4} \log |x| & \text{for } x < 0. \end{cases}$$

Using the notation of 3.2 for the U_i's, we have the following:

Proposition 14. *Let* $\mathbf{L} :]-\infty, 1[\to \mathbb{R}$ *be measurable and satisfy the equation*

$$2\mathbf{L}(U_1) + 2\mathbf{L}(U_2) - \mathbf{L}(U_3) + 2\mathbf{L}(U_4) + 2\mathbf{L}(U_5) \\ - \mathbf{L}(U_6) + 2\mathbf{L}(U_7) + 2\mathbf{L}(U_8) - \mathbf{L}(U_9) = \text{const.}$$

when $0 < x < y < 1$. *Then there are* $\alpha, \beta, \gamma \in \mathbb{R}$ *such that* $\mathbf{L} = \alpha \mathcal{L}_3 + \beta \mathrm{L}_3 + \gamma$.

Remark. The function L_3 here seems to play the role of the function $z \mapsto \mathcal{L}_2(z) \log |z|$ which is a solution of $(\mathcal{G})$ but vanishes on the real axis and therefore gives a trivial equation under the specialization (SKh) when $x, y \in \mathbb{R}$. But the relation between these two functions is not clear although their definitions are similar.

In a certain sense, Proposition 14 gives a solution to the problem raised by Goncharov in [Gon2, p. 209]. It implies this result for the trilogarithm $\mathbf{Li}_3$:

Theorem 5. *Let* $\mathbf{L} :]-\infty, 1[\to \mathbb{R}$ *be a measurable function which satisfies*

$$2\mathbf{L}(x) + 2\mathbf{L}(y) - \mathbf{L}\left(\frac{x}{y}\right) + 2\mathbf{L}\left(\frac{1-y}{1-x}\right) + 2\mathbf{L}\left(\frac{x(1-y)}{y(1-x)}\right) - \mathbf{L}(xy) \\ + 2\mathbf{L}\left(\frac{x(1-y)}{x-1}\right) + 2\mathbf{L}\left(\frac{y-1}{y(1-x)}\right) - \mathbf{L}\left(\frac{x(1-y)^2}{y(1-x)^2}\right) \\ = 2\mathbf{Li}_3(1) - \log^2(y) \log\left(\frac{1-y}{1-x}\right) + \frac{\pi^2}{3} \log(y) + \frac{1}{3} \log^3(y)$$

when $0 < x < y < 1$. *If* $\mathbf{L}$ *is derivable at* 0, *then* $\mathbf{L} = \mathbf{Li}_3$.

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