The Classification of Exceptional CDQL Webs on Compact Complex Surfaces

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Codimension one webs are configurations of finitely many codimension one foliations in general position. Much of the classical theory evolved around the concept of abelian relation: a functional equation among the first integrals of the foliations defining the web reminiscent of Abel’s Addition Theorem. The abelian relations of a given web form a finite-dimensional vector space with dimension (the rank of the web) bounded by Castelnuovo number \( \pi(n, k) \) where \( n \) is the dimension of the ambient space and \( k \) is the number of foliations defining the web. A fundamental problem in web geometry is the classification of exceptional webs, that is, webs of maximal rank not equivalent to the dual of a projective curve. Recently, Trépreau proved that there are no exceptional \( k \)-webs for \( n \geq 3 \) and \( k \geq 2n \). In dimension two, there are examples for arbitrary \( k \geq 5 \) and the classification of exceptional webs is wide open. In this paper, we classify the exceptional completely decomposable quasi-linear (CDQL) webs globally defined on compact complex surfaces. By definition, these are the exceptional \( (k + 1) \)-webs on compact complex surfaces that are formed by the superposition of \( k \) “linear” and one non-linear foliations. For instance, we show that up to projective transformations there are exactly four countable families and thirteen sporadic examples of exceptional CDQL webs on the projective plane.
1 Introduction and Statement of the Main Results

1.1 Codimension one webs of maximal rank

A germ of smooth $k$-web $W = \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_k$ of codimension one on $(\mathbb{C}^n, 0)$ is a collection of $k$ germs of smooth holomorphic foliations $\mathcal{F}_i$ with tangent spaces in general position at the origin. By definition, the $\mathcal{F}_i$'s are the defining foliations of $W$. If they are respectively induced by differentials 1-forms $\omega_1, \ldots, \omega_k$, then the space of abelian relations of $W$ is the vector space

$$\mathcal{A}(W) = \left\{ (\eta_i)_{i=1}^k \in \Omega^1(\mathbb{C}^n, 0)^k \mid \forall i \quad d\eta_i = \eta_i \wedge \omega_i = 0 \quad \eta_1 + \cdots + \eta_k = 0 \right\}.$$ 

If $u_i : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ are local submersions defining the foliations $\mathcal{F}_i$ then, after integration, the abelian relations can be read as functional equations of the form

$$F_1(u_1) + \cdots + F_k(u_k) \equiv 0$$

for suitable germs of holomorphic functions $F_1, \ldots, F_k : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$.

The dimension of $\mathcal{A}(W)$ is commonly called the rank of $W$ and denoted by $rk(W)$. It is a theorem of Bol (for $n = 2$) and Chern (for $n \geq 3$) that

$$rk(W) \leq \pi(n, k) = \sum_{j=1}^{\infty} \max \left( 0, k - j(n-1) - 1 \right). \quad (1)$$

A $k$-web $W$ on $(\mathbb{C}^n, 0)$ is of maximal rank if $rk(W) = \pi(n, k)$. The integer $\pi(n, k)$ is the well-known Castelnuovo’s bound for the arithmetic genus of irreducible and non-degenerated algebraic curves of degree $k$ on $\mathbb{P}^n$.

One of the main topics of the theory concerns the characterization of webs of maximal rank. It follows from Abel’s Addition Theorem that all the webs $W_C$ obtained from Castelnuovo curves (non-degenerate irreducible algebraic curve with maximal arithmetical genus) $C$ by projective duality are of maximal rank (see [35] for instance). The webs analytically equivalent to $W_C$ for some non-degenerated projective curve $C$ are the so-called algebraizable webs.

It can be traced back to Lie the proof that all 4-webs on $(\mathbb{C}^2, 0)$ of maximal rank are algebraizable. In [7], Bol proved that a maximal rank $k$-web on $(\mathbb{C}^3, 0)$ is algebraizable when $k \geq 6$. Recently, building up on previous work by Chern and Griffiths [16], Trépureau
extended Bol’s result and established in [46] that $k$-webs of maximal rank on $(\mathbb{C}^n, 0)$ are algebraizable whenever $n \geq 3$ and $k \geq 2n$.

The non-algebraizable webs of maximal rank on $(\mathbb{C}^2, 0)$ are nowadays called exceptional webs. For almost 70 years there was just one example, due to Bol [8], of exceptional planar web in the literature. Recently, a number of new examples have appeared, see [31, 39, 41, 43]. Despite these new examples, the classification problem for exceptional planar webs is wide open.

1.2 Characterization of planar webs of maximal rank

Although a classification seems out of reach, there are methods to decide if a given web has maximal rank. The first result in this direction is due to Pantazi [34]. It was published during the second world war and remained unknown to the practitioners of web theory until recently, see [39]. Unaware of this classical result, Hénaut [24] worked out an alternative approach to determine if a given web has maximal rank. Both approaches share in common the use of prolongations of linear differential systems to express the maximality of the rank by the vanishing of certain differential expressions determined by the defining equations of the web.

It has to be noted that these results are wide generalizations of the classical criterion of Blaschke–Dubourdieu for the maximality of the rank of 3-webs. If $\mathcal{W} = F_1 \boxtimes F_2 \boxtimes F_3$ is a planar 3-web and if the foliations $F_i$ are defined by 1-forms $\omega_i$ satisfying $\omega_1 + \omega_2 + \omega_3 = 0$ then a simple computation ensures the existence of a unique 1-form $\gamma$ such that $d\omega_i = \gamma \wedge \omega_i$ for $i = 1, 2, 3$. Although $\gamma$ does depend on the choice of the 1-forms $\omega_i$, its differential $d\gamma$ is intrinsically attached to $\mathcal{W}$. It is the so-called curvature $K(\mathcal{W})$ of $\mathcal{W}$. In [6], it is proved that a 3-web $\mathcal{W}$ has maximal rank if and only if $K(\mathcal{W}) = 0$.

Building on Pantazi’s result, Mihăileanu gave in [32] a necessary condition for a planar $k$-web be of maximal rank, for $k \geq 3$ arbitrary: if $\mathcal{W}$ has maximal rank then $K(\mathcal{W}) = 0$. Now, the curvature $K(\mathcal{W})$ is the sum of the curvatures of all 3-subwebs of $\mathcal{W}$. Recently, Hénaut, Ripoll, and Robert (see [25, page 281],[42]) have rediscovered Mihăileanu’s necessary condition using Hénaut’s approach.

As in the case of 3-webs, the curvature $K(\mathcal{W})$ is a holomorphic 2-form intrinsically attached to $\mathcal{W}$: it does not depend on the choice of the defining equations of $\mathcal{W}$. Another nice feature of the curvature is that it still makes sense, as a meromorphic 2-form, for global webs. More precisely, if $S$ is a complex surface then a global $k$-web on $S$ can be defined as an element $\mathcal{W} = [\omega]$ of $\mathbb{P}H^0(S, \text{Sym}^k\Omega^1_S \otimes \mathcal{N})$—where $\mathcal{N}$ denotes a line bundle and $\text{Sym}^k\Omega^1_S$ the sheaf of $k$-symmetric powers of holomorphic differential
1-forms on $S$—subjected to the following two conditions: (i) the zero locus of $\omega$ has codimension at least two; (ii) $\omega(p)$ factors as the product of pairwise linearly independent 1-forms at some point $p \in S$. For $k = 1$, the condition (ii) is vacuous and we recover one of the usual definitions of foliations. When $k \geq 2$, the set where the condition (ii) does not hold is the discriminant of $W$ and will be denoted by $\Delta(W)$. For $k \geq 3$, the curvature $K(W)$ is a global meromorphic 2-form on $S$ with polar set contained in $\Delta(W)$.

Elementary arguments (see [39, Theorem 1.2.2] for instance) imply that the space of abelian relations of $W$, in this global setup, is a local system over $S \setminus \Delta(W)$. The rank of $W$ appears now as the rank of the local system $A(W)$.

One has to be careful when talking about defining foliations of a global web since these will make sense only in sufficiently small analytic open subsets of $S$. When it is possible to write globally $W = F_1 \boxtimes \cdots \boxtimes F_k$ we will say that $W$ is completely decomposable.

Recall that a complex manifold of dimension $n$ is called pseudo-parallelizable if it carries $n$ global meromorphic 1-forms $\omega_1, \ldots, \omega_n$ with exterior product $\omega_1 \wedge \cdots \wedge \omega_n$ not identically zero. When $S$ is a pseudo-parallelizable surface, a global $k$-web on $S$ can be alternatively defined as an element $W = [\omega]$ of the projective space $\mathbb{P}_{\mathbb{C}(S)}(\text{Sym}^k \Omega^1_S)$—where $\mathbb{C}(S)$ is the field of meromorphic functions on $S$ and $\text{Sym}^k \Omega^1_S$ denotes now the $\mathbb{C}(S)$-vector space of meromorphic $k$-symmetric powers of differential 1-forms on $S$—subjected to the condition that $\omega$ factors as the product of pairwise linearly independent 1-forms at some point of $S$.

1.3 Mihăileanu necessary condition and $F$-barycenters

The present work stems from an attempt to understand geometrically Mihăileanu’s necessary condition for the maximality of the rank. More precisely, we try to understand the conditions imposed by the vanishing of the curvature on the behavior of $W$ over its discriminant. It has to be mentioned that the idea of analyzing webs through their discriminants is not new, see [13] and [30]. More recently, [25] advocates the study of webs (decomposable or not) in neighborhoods of their discriminants.

Our result in this direction is stated in terms of $\beta_F(W)$—the $F$-barycenter of a web $W$. Suppose that $S$ is a pseudo-parallelizable surface and $F \in \mathbb{P}_{\mathbb{C}(S)}(\Omega^1_S)$ is a foliation on it. There is a naturally defined affine structure on $\mathbb{A}^1_F = \mathbb{P}_{\mathbb{C}(S)}(\Omega^1_S) \setminus F$. If $W \in \mathbb{P}_{\mathbb{C}(S)}(\text{Sym}^k \Omega^1_S)$ is a $k$-web not containing $F$ as one of its defining foliations, then it can be loosely interpreted as $k$ points in $\mathbb{A}^1_F$. The $F$-barycenter of $W$ is then the foliation
\( \beta_{F}(W) \) defined by the barycenter of these \( k \) points in \( \mathbb{A}_{\mathbb{F}}^{1} \). For a precise definition and some properties of \( \beta_{F}(W) \), see Sections 5 and 6.

**Theorem 1.1.** Let \( F \) be a foliation and \( W = F_{1} \boxtimes F_{2} \boxtimes \cdots \boxtimes F_{k} \) be a \( k \)-web, \( k \geq 2 \), both defined on the same domain \( U \subset \mathbb{C}^{2} \). Suppose that \( C \) is an irreducible component of \( \text{tang}(F, F_{1}) \) that is not contained in \( \Delta(W) \). The curvature \( K(F \boxtimes W) \) is holomorphic over a generic point of \( C \) if and only if the curve \( C \) is \( F \) invariant or \( \beta_{F}(W') \) invariant, where \( W' = F_{2} \boxtimes \cdots \boxtimes F_{k} \).

Theorem 1.1 is the cornerstone of our approach to the classification of exceptional completely decomposable quasi-linear webs (CDQL webs for short) on compact complex surfaces.

### 1.4 Linear webs and CDQL webs

Linear webs are classically defined as the ones for which all the leaves are open subsets of lines. Here we will adopt the following global definition. A web \( W \) on a compact complex surface \( S \) is linear if (a) the universal covering of \( S \) is an open subset \( \tilde{S} \) of \( \mathbb{P}^{2} \); (b) the group of deck transformations acts on \( \tilde{S} \) by automorphisms of \( \mathbb{P}^{2} \), and; (c) the pullback of \( W \) to \( \tilde{S} \) is linear in the classical sense.

A CDQL \((k + 1)\)-web on a compact complex surface \( S \) is, by definition, the superposition of \( k \) linear foliations and one non-linear foliation.

It follows from [26, 28] that the only compact complex surfaces satisfying (a) and (b) are: the projective plane; surfaces covered by the unit ball; Kodaira primary surfaces; complex tori; Inoue surfaces; Hopf surfaces and principal elliptic bundles over hyperbolic curves with odd first Betti number.

If \( S \) is not \( \mathbb{P}^{2} \) then the group of deck transformations is infinite. Because it acts on \( \tilde{S} \) without fixed points, every linear foliation on \( S \) is a smooth foliation. An inspection of Brunella’s classification of smooth foliations [10] reveals that the only compact complex surfaces admitting at least two distinct linear foliations are the projective plane, the complex tori and the Hopf surfaces. Moreover, the only Hopf surfaces admitting four distinct linear foliations are the primary Hopf surfaces \( H_{\alpha} \) for \( |\alpha| > 1 \), obtained by taking the quotient of \( \mathbb{C}^{2} \setminus \{0\} \) by the map \((x, y) \mapsto (ax, \alpha y)\).

The linear foliations on complex tori are pencils of parallel lines on their universal coverings, in the sense that they are dual to product of lines. The ones on Hopf surfaces are either pencils of parallels lines or the pencil of lines through the origin of
C\(^2\). In particular, all completely decomposable linear webs on compact complex surfaces are algebraic on their universal coverings, in the sense that they are dual to product of lines.

1.5 Classification of exceptional CDQL webs on the projective plane

On \(\mathbb{P}^2\) the CDQL webs can be written as \(W \boxtimes F\) where \(W\) is a product of pencils of lines and \(F\) is a non-linear foliation. These webs are determined by the pair \((P, F)\) where \(P \subset \mathbb{P}^2\) is the set of singularities of the linear foliations defining \(W\). One key example is the already mentioned Bol's 5-web. It is the exceptional CDQL 5-web on \(\mathbb{P}^2\) with \(F\) equal to the pencil generated by two smooth conics intersecting transversely and \(P\) equal to the set of four base points of this pencil. Other examples of exceptional CDQL webs on the plane have appeared in [39, 43].

We will deduce from Theorem 1.1 a complete classification of exceptional CDQL webs on the projective plane. In succinct terms it can be stated as follows:

**Theorem 1.2.** Up to projective automorphisms, there are exactly four countable families and thirteen sporadic exceptional CDQL webs on \(\mathbb{P}^2\). \(\square\)

In suitable affine coordinates \((x, y) \in \mathbb{C}^2 \subset \mathbb{P}^2\), the four infinite families are

- \(A^k_I = [(dx^k - dy^k)] \boxtimes [d(xy)]\) where \(k \geq 4\);
- \(A^k_{II} = [(dx^k - dy^k)(xdy - ydx)] \boxtimes [d(xy)]\) where \(k \geq 3\);
- \(A^k_{III} = [(dx^k - dy^k) dx dy] \boxtimes [d(xy)]\) where \(k \geq 2\);
- \(A^k_{IV} = [(dx^k - dy^k) dx dy (xdy - ydx)] \boxtimes [d(xy)]\) where \(k \geq 1\).

The diagram below shows how these webs relate to each other in terms of inclusions for a fixed \(k\). Moreover, if \(k\) divides \(k'\) then \(A^k_I, A^k_{II}, A^k_{III},\) and \(A^k_{IV}\) are subwebs of \(A^k_I, A^k_{II}, A^k_{III},\) and \(A^k_{IV}\), respectively.

All the webs above are invariant by the \(\mathbb{C}^*\)-action \(t \cdot (x, y) = (tx, ty)\) on \(\mathbb{P}^2\). Among the thirteen sporadic examples of exceptional CDQL webs on the projective plane, seven
The web $\mathcal{H}_{10}$ shares a number of features with Bol’s web $B_5$. They both have a huge group of birational automorphisms (the symmetric group $S_5$ for $B_5$ and Hesse’s
Fig. 1. A sample of real models for exceptional CDQL webs on \( \mathbb{P}^2 \). In the first and second rows, the first three members of the infinite family are \( \mathcal{A}_k^I \) and \( \mathcal{A}_k^{II} \), respectively. In the third row, from left to right, \( \mathcal{A}_3^{III}, \mathcal{A}_4^{IV} \) and \( \mathcal{A}_5^{IV} \). In the fourth row: \( \mathcal{A}_5^a, \mathcal{A}_5^b \) and \( \mathcal{A}_5^c \).
group $G_{216}$ for $\mathcal{H}_{10}$, both are naturally associated to nets in the sense of Section 3.1, and their abelian relations can be expressed in terms of logarithms and dilogarithms.

Because they have parallel 4-subwebs whose slopes have non-real cross-ratio, the webs $A^k_{III}, A^k_{IV}$ for $k \geq 3$, $A^6_5, A^6_6$ and $A_7$ do not admit real models. The web $\mathcal{H}_{10}$ also does not admit a real model. To verify this fact, one possibility is to observe that the lines passing through two of the nine base points always contain a third and notice that this contradicts Sylvester–Gallai Theorem [17]: for every finite set of non-collinear points in $\mathbb{P}^2_R$ there exists a line containing exactly two points of the set. All the other exceptional CDQL webs admit real models. Some of them are pictured in Figure 1.

1.6 Exceptional CDQL webs on Hopf surfaces

The classification of CDQL webs on $\mathbb{P}^2$ admits as a corollary the classification of exceptional CDQL webs on Hopf surfaces.

**Corollary 1.1.** Up to automorphism, the only exceptional CDQL webs on Hopf surfaces are quotients of the restrictions of the webs $A^*_\alpha$ to $\mathbb{C}^2 \setminus \{0\}$ by the group of deck transformations.

The proof is automatic. One has just to remark that a foliation on a Hopf surface of type $H_\alpha$ when lifted to $\mathbb{C}^2 \setminus \{0\}$ gives rise to an algebraic foliation on $\mathbb{C}^2$ invariant by the $\mathbb{C}^*$-action $t \cdot (x, y) = (tx, ty)$.

1.7 From global to local...

Although based on global methods, the classification of exceptional CDQL webs on $\mathbb{P}^2$ also yields information about the singularities of local exceptional webs.

**Corollary 1.2.** Assume that $k \geq 4$. Let $\mathcal{W}$ be a smooth $k$-web and $\mathcal{F}$ be a foliation, both defined on $(\mathbb{C}^2, 0)$. If $\mathcal{W} \boxtimes \mathcal{F}$ is a (possibly singular) germ of $(k + 1)$-web with maximal rank, then one of the following situations holds:

1. the foliation $\mathcal{F}$ is of the form $[H(x, y)(\alpha dx + \beta dy) + h.o.t.]$ where $H$ is a non-zero homogeneous polynomial and $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{0\}$;

2. the foliation $\mathcal{F}$ is of the form $[H(x, y)(ydx - xdy) + h.o.t.]$ where $H$ is a non-zero homogeneous polynomial;
(3) $W \otimes F$ is exceptional and its first non-zero jet defines, up to linear automorphisms, one of the following webs

$$A^k_1, A^{k-2}_{III}, A^d_5 \text{ (only when } k = 4) \text{ or } A^b_6 \text{ (only when } k = 5).$$

\[\Box\]

In fact, as it will be clear from its proof, it is possible to state a slightly more general result in the same vein. Nevertheless, the result above suffices for the classification of exceptional CDQL webs on complex tori.

1.8 ... and back: classification of exceptional CDQL webs on tori

A CDQL web on a torus of dimension two is the superposition of a non-linear foliation with a product of foliations induced by global holomorphic 1-forms. Since étale coverings between complex tori abound and because the pullbacks of exceptional CDQL webs under these are still exceptional CDQL webs, we are naturally led to extend the notion of isogenies between complex tori. Two webs $W_1, W_2$ on complex tori $T_1, T_2$ are isogenous if there exist a two-dimensional complex torus $T$ and two étale morphisms $\pi_i : T \to T_i$ for $i = 1, 2$, such that $\pi_1^*(W_1) = \pi_2^*(W_2)$.

**Theorem 1.3.** Up to isogenies, there are exactly three sporadic (one for each $k \in \{5, 6, 7\}$) and one continuous family (with $k = 5$) of exceptional CDQL $k$-webs on complex tori. \[\Box\]

The elements of the continuous family are the 5-webs

$$E_\tau = \left[ \right. d\tau d\tau (d\tau^2 - d\tau^2) \left. \right] \otimes \left[ d\left( \frac{\vartheta_1(x, \tau) \vartheta_1(y, \tau)}{\vartheta_4(x, \tau) \vartheta_4(y, \tau)} \right)^2 \right]$$

defined, respectively, on the torus $E^2_\tau$ for arbitrary $\tau \in \mathbb{H} = \{ z \in \mathbb{C} \mid \Im(z) > 0 \}$ where $E_\tau = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z} \tau)$. The functions $\vartheta_i$ involved in the above definition are the classical Jacobi theta functions, see Example 4.1.

These webs first appeared in Buzano’s work [11] but their rank was not determined at that time. They were later rediscovered in [41] where it is proved that they are all exceptional and that $E_\tau$ is isogenous to $E_{\tau'}$ if and only if $\tau$ and $\tau'$ belong to the same orbit under the natural action on $\mathbb{H}$ of the $\mathbb{Z}/2\mathbb{Z}$ extension of $\Gamma_0(2) \subset \text{PSL}(2, \mathbb{Z})$ generated by $\tau \mapsto -2\tau^{-1}$. Thus, the continuous family of exceptional CDQL webs on tori is parameterized by a $\mathbb{Z}/2\mathbb{Z}$-quotient of the modular curve $X_0(2)$. 
The sporadic CDQL 7-web $\mathcal{E}_7$ is strictly related to a particular element of the previous family. Indeed, $\mathcal{E}_7$ is the 7-web on $E^2_{1+i}$

$$
\mathcal{E}_7 = [dx^2 + dy^2] \otimes E_{1+i}.
$$

The sporadic CDQL 5-web $\mathcal{E}_5$ lives naturally in $E^2_{\xi_3}$ and can be described as

$$
[dx dy(dx - dy)(dx + \xi_3^2 dy)] \otimes \left[ d\left( \frac{\partial_1(x, \xi_3)\partial_1(y, \xi_3)\partial_1(x - y, \xi_3)\partial_1(x + \xi_3^2 y, \xi_3)}{\partial_2(x, \xi_3)\partial_3(y, \xi_3)\partial_4(x - y, \xi_3)\partial_3(x + \xi_3^2 y, \xi_3)} \right) \right].
$$

The sporadic CDQL 6-web $\mathcal{E}_6$ also lives in $E^2_{\xi_3}$ and is best described in terms of Weierstrass $\wp$-function.

$$
\mathcal{E}_6 = [dx dy(dx^3 + dy^3)] \otimes \left[ \wp(x, \xi_3)^{-1} dx + \wp(y, \xi_3)^{-1} dy \right].
$$

Although not completely evident from the above presentation, it turns out that the non-linear foliation of $\mathcal{E}_6$ admits a rational first integral, see Proposition 4.2.

A more geometric description of these exceptional elliptic webs will be given in Section 4 together with the proof that they are indeed exceptional.

The proof of Theorem 1.3 follows the same lines of the proof of Theorem 1.2 but with some twists. The key extra ingredients are Corollary 1.2 and the following (considerably easier) analog for two-dimensional complex tori of [38, Theorem 1].

**Theorem 1.4.** If $T$ is a two-dimensional complex torus and if $f : T \rightarrow \mathbb{P}^1$ is a meromorphic map, then the number of linear fibers of $f$, when finite, is at most six. $\square$

For us, the linear fibers of a rational map from a two-dimensional complex torus to a curve are the ones that are set-theoretically equal to a union of subtori.

1.9 **Plan of the paper**

The remaining of the paper can be roughly divided in five parts. The first goes from Section 2 to Section 4 and is devoted to prove that all the webs presented in the Introduction are exceptional. The highlights are Theorems 3.1 and 4.1 that show that the webs $B_5, \mathcal{H}_{10}, \mathcal{E}_r, \mathcal{E}_5, \mathcal{E}_6$ and $\mathcal{E}_7$ are exceptional thanks to essentially the same reason. Their abelian relations are expressed in terms of logarithms, dilogarithms and their elliptic counterparts.
Sections 5, 6 and 7 form the second part of the paper which is mainly devoted to the study of the $\mathcal{F}$-barycenter of a web. Besides the proof of Theorem 1.1 of the Introduction, it also contains a very precise description of the barycenters of decomposable linear webs centered at linear foliations on $\mathbb{P}^2$. This description lies at the heart of our approach to the classification of exceptional CDQL webs on $\mathbb{P}^2$.

The third part of the paper goes from Section 8 to Section 9 and contains the classification of flat CDQL webs on the projective plane. It is then not difficult to obtain the classification of exceptional CDQL webs on $\mathbb{P}^2$.

The fourth part is contained in the last two sections and deals with the classification of exceptional CDQL webs on two-dimensional complex tori. Besides this classification, it also contains the proofs of Corollary 1.2 and of Theorem 1.4.

Finally, in the Appendix we give some details concerning the proof of Theorem 5.1 (see also Table 1) that is a projective classification of non-constant rational maps on $\mathbb{P}^1$ with a special behavior relative to barycenters.

2 Abelian Relations for CDQL Webs Invariant by $\mathbb{C}^*$-actions

We start things off with the following well-known proposition.

**Proposition 2.1.** Let $\mathcal{W}$ be a linear $k$-web of maximal rank and $\mathcal{F}$ be a non-linear foliation on $(\mathbb{C}^2, 0)$. The $(k+1)$-web $\mathcal{W} \boxtimes \mathcal{F}$ is exceptional if and only if it has maximal rank and $k \geq 4$. $\square$

**Proof.** For $k \leq 3$, all $(k+1)$-webs of maximal rank are algebraizable thanks to Lie’s Theorem. Suppose that $k \geq 4$ and let $\varphi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be a biholomorphism algebraizing $\mathcal{W} \boxtimes \mathcal{F}$. Since $\mathcal{W}$ has maximal rank, $\varphi^*(\mathcal{W})$ must be algebraic. According to [23] (see also [5, page 247]), the biholomorphism $\varphi$ must be the restriction of a projective transformation. It follows that the foliation $\varphi^*(\mathcal{F})$ is non-linear and consequently it cannot exist an algebraization of $\mathcal{W} \boxtimes \mathcal{F}$. $\blacksquare$

As a corollary, one sees that in order to prove that a CDQL $k$-web $\mathcal{W}$ is exceptional, when $k \geq 5$, it suffices to verify that it has maximal rank. The most obvious way to accomplish this task is to exhibit a basis of the space of its abelian relations. In general, the explicit determination of $\mathcal{A}(\mathcal{W})$ is a fairly difficult problem. To our knowledge, the only general method available is Abel’s method for solving functional equations (see...
[1] and [39, Chapter 2]). It assumes the knowledge of first integrals for the defining foliations of \( W \) and it tends to involve rather lengthy computations.

In particular cases, there are more efficient ways to determine the space of abelian relations. For instance, if the web admits an infinitesimal automorphism then the results of [31] reduce the problem to plain linear algebra. In Section 2.1, we recall the results of [31] and use them in Sections 2.2 and 2.3 to deal with the CDQL webs invariant by \( \mathbb{C}^* \)-actions described in the Introduction.

2.1 Webs with infinitesimal automorphisms

Let \( F \) be a regular foliation on \((\mathbb{C}^2, 0)\) induced by a 1-form \( \omega \). We say that a vector field \( X \) is an infinitesimal automorphism of \( F \) if \( L_X \omega \wedge \omega = 0 \). When such infinitesimal automorphism \( X \) is transverse to \( F \), that is when \( \omega(X) \neq 0 \), the 1-form \( \eta = (i_X \omega)^{-1} \omega \) is closed and satisfies \( L_X(\eta) = 0 \). By definition, the integral

\[
u(z) = \int_0^z \eta\]

is the canonical first integral of \( F \) (with respect to \( X \)).

Assume now that \( W \) is a regular \( k \)-web on \((\mathbb{C}^2, 0)\) admitting an infinitesimal automorphism \( X \). Let \( u_1, \ldots, u_k \) be the associated canonical first integrals of \( W \).

The Lie derivative \( L_X \) induces a \( \mathbb{C} \)-linear map

\[
L_X : A(W) \longrightarrow A(W) \quad (\eta_1, \ldots, \eta_k) \longmapsto (L_X(\eta_1), \ldots, L_X(\eta_k)).
\]

The study of this linear map leads to the following proposition.

**Proposition 2.2.** Let \( \lambda_1, \ldots, \lambda_\tau \in \mathbb{C} \) be the eigenvalues of the map \( L_X \) acting on \( A(W) \) corresponding to minimal Jordan blocks with dimensions \( \sigma_1, \ldots, \sigma_\tau \). The abelian relations of \( W \) are of the form

\[
P_1(u_1) e^{\lambda_1 u_1} du_1 + \cdots + P_k(u_k) e^{\lambda_k u_k} du_k = 0
\]

where \( P_1, \ldots, P_k \) are polynomials of degree less or equal to \( \sigma_i - 1 \). Moreover, the abelian relations corresponding to eigenvectors are the ones for which the \( P_i \)'s are constant. \( \square \)
Proposition 2.2 suggests an effective method to determine $\mathcal{A}(\mathcal{W})$ from the study of the linear map (3). For details see [31]. It also follows from the study of (3) the main result of [31].

**Theorem 2.1.** Let $\mathcal{W}$ be a $k$-web which admits a transverse infinitesimal automorphism $X$. If $\mathcal{F}_X$ stands for the foliation induced by $X$, then

$$\text{rk}(\mathcal{W} \boxtimes \mathcal{F}_X) = \text{rk}(\mathcal{W}) + (k - 1).$$

In particular, $\mathcal{W}$ is of maximal rank if and only if $\mathcal{W} \boxtimes \mathcal{F}_X$ is also of maximal rank. □

Below, we will make use of Theorem 2.1 to prove that certain webs have maximal rank without giving a complete list of their abelian relations. Nevertheless, the proof of Theorem 2.1 (see [31]) is constructive and the interested reader can easily determine a complete list of the abelian relations.

### 2.2 Four infinite families

Recall the definition of the webs $\mathcal{A}_I^k$, $\mathcal{A}_{II}^k$, $\mathcal{A}_{III}^k$, and $\mathcal{A}_{IV}^k$:

- $\mathcal{A}_I^k = [(dx^k - dy^k) \boxtimes [d(xy)]]$ where $k \geq 4$;
- $\mathcal{A}_{II}^k = [(dx^k - dy^k)(x dy - y dx)] \boxtimes [d(xy)]$ where $k \geq 3$;
- $\mathcal{A}_{III}^k = [(dx^k - dy^k) dx dy] \boxtimes [d(xy)]$ where $k \geq 2$;
- $\mathcal{A}_{IV}^k = [(dx^k - dy^k) dx dy (xy - y dx)] \boxtimes [d(xy)]$ where $k \geq 1$.

The exceptionality of these webs follows from the next proposition.

**Proposition 2.3.** For every $k \geq 1$, the webs $\mathcal{A}_I^k$, $\mathcal{A}_{II}^k$, $\mathcal{A}_{III}^k$, and $\mathcal{A}_{IV}^k$ have maximal rank. □

**Proof.** Let $R = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$ be the radial vector field. Note that it is an infinitesimal automorphism of all the webs above. Moreover,

$$\mathcal{A}_{II}^k = \mathcal{A}_I^k \boxtimes \mathcal{F}_R \quad \text{and} \quad \mathcal{A}_{IV}^k = \mathcal{A}_{III}^k \boxtimes \mathcal{F}_R.$$

It follows from Theorem 2.1 that $\mathcal{A}_{II}^k$ (resp. $\mathcal{A}_{IV}^k$) has maximal rank if and only if $\mathcal{A}_I^k$ (resp. $\mathcal{A}_{III}^k$) also does.
To prove that $A_k^I$ has maximal rank, consider the linear automorphism of $\mathbb{C}^2$, $\varphi(x, y) = (x, \xi_k y)$. Consider also the induced automorphism of the vector space $\mathbb{C}_{2k-2}[x, y]$ of homogeneous polynomials of degree $2k-2$:

$$\varphi^*: \mathbb{C}_{2k-2}[x, y] \longrightarrow \mathbb{C}_{2k-2}[x, y]$$

$$p \mapsto p \circ \varphi.$$

For $k = 1$, there is nothing to prove: every 2-web has maximal rank. Assume that $k \geq 2$. If $\xi_k = \exp(2\pi i/k)$, then the $(\xi_k^{k-1})$-eigenspace of $\varphi^*$ has dimension one and is generated by $(xy)^{k-1}$.

If $V \subset \mathbb{C}_{2k-2}[x, y]$ denotes the vector subspace generated by the homogeneous polynomials $(x - \xi_k^i y)^{2k-2}$ with $i$ ranging from 0 to $k - 1$, then $\varphi^*$ preserves $V$ and the characteristic polynomial of $\varphi^*|_V$ is equal to $t^k - 1$. It follows that there exists $p \in V \setminus \{0\}$ such that $\varphi^* p = (\xi_k^{k-1}) p$. Since the eigenspace of $\varphi^*$ associated to the eigenvalue $\xi_k^{k-1}$ has dimension one, $p$ must be a complex multiple of $(xy)^{k-1}$. Therefore, there exist complex constants $\mu_1, \ldots, \mu_k$ such that

$$(xy)^{k-1} = \sum_{i=1}^{k} \mu_i \left(x - \xi_k^i y\right)^{2k-2}.$$

This identity can be interpreted as an abelian relation of $A_k^I$. If we apply the second-order differential operator $\frac{\partial^2}{\partial x \partial y}$ to it, we obtain another abelian relation

$$(k - 1)^2(xy)^{k-2} = \sum_{i=1}^{k} \mu_i(2k - 2)(2k - 1)\xi_k^i \left(x - \xi_k^i y\right)^{2(k-1)-2}.$$

When $k \geq 3$, this abelian relation is clearly linearly independent from the previous one. Iteration of this procedure shows that

$$\dim \frac{\mathcal{A}(A_k^I)}{\mathcal{A}([dx^k - dy^k])} \geq k - 1.$$

Since $[dx^k - dy^k]$ is an algebraic $k$-web, its rank is $(k - 1)(k - 2)/2$. Thus, $\dim \mathcal{A}(A_k^I) = k(k - 1)/2$ and $A_k^I$ are indeed of maximal rank. Theorem 2.1 implies that the $(k+2)$-web $A_k^{II}$ is also of maximal rank.

The proof that $A_k^{III}$ and $A_k^{IV}$ are of maximal rank is analogous. As before, it suffices to show that the $(k+3)$-web $A_k^{III}$ has maximal rank.
Consider now the induced automorphism \( \varphi^* \) on the space \( \mathbb{C}^{2k}[x, y] \) of homogeneous polynomials of degree 2\( k \). The 1-eigenspace of \( \varphi^* \) has dimension three and is generated by \( x^{2k}, y^{2k}, \) and \((xy)^k\). If \( V \subset \mathbb{C}^{2k}[x, y] \) denotes now the vector subspace generated by the polynomials \((x - \xi^i_k y)^{2k}\) with \( i = 0, \ldots, k - 1 \), then the characteristic polynomial of \( \varphi^*|_V \) is also equal to \( t^k - 1 \). Thus, there exists an abelian relation of \( A^k_{\text{III}} \) of the form

\[
(xy)^k = \sum_{i=1}^{k} \mu_i \left( x - \xi^i_k y \right)^{2k} + \mu_{k+1} x^{2k} + \mu_{k+2} y^{2k}.
\]

Applying the operator \( \frac{\partial^2}{\partial x \partial y} \) and iterating as above, one deduces that

\[
\dim \frac{A(A^k_{\text{III}})}{A([dx \, dy(dx^k - dy^k)])} \geq k.
\]

Taking into account the logarithmic abelian relation

\[
\log(xy) = \log x + \log y
\]

we conclude that \( \dim \frac{A(A^k_{\text{III}})}{A([dx \, dy(dx^k - dy^k)])} \geq k + 1 \). Since \([dx \, dy(dx^k - dy^k)]\) has rank \( k(k + 1)/2 \), it follows that the \((k + 3)\)-web \( A^k_{\text{III}} \) also has maximal rank.

2.3 The seven sporadic exceptional CDQL webs invariant by \( \mathbb{C}^* \)-actions

We now consider the seven exceptional CDQL webs (2) that are also invariant by the \( \mathbb{C}^* \)-action \( t \cdot (x, y) \mapsto (tx, ty) \). Of course, they all share the same infinitesimal automorphism: the radial vector field \( R = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \). Because

\[
A^6_6 = A^d_5 \boxtimes \mathcal{F}_R \quad \text{and} \quad A_7 = A^b_6 \boxtimes \mathcal{F}_R.
\]

Theorem 2.1 implies that the maximality of the rank of \( A^6_6 \) (resp. \( A_7 \)) is equivalent to the maximality of the rank of \( A^d_5 \) (resp. \( A^b_6 \)). Thus, to prove that all the seven webs (2) are exceptional, it suffices to prove that \( A^d_6, A^b_6, A^d_5, A^b_5, A^d_3, \) and \( A^b_0 \) have maximal rank. For this sake, we list below a basis for a subspace of the space of abelian relations of these webs that is transverse to the space of abelian relations of the maximal linear subweb contained in each of them.
2.3.1 Abelian relations for $A_5^a$

If $g_0 = xy(x + y)$, $g_1 = x$, $g_2 = y$, $g_3 = x + y$, and $g_4 = \frac{x}{y}$, then the sought abelian relations for $A_5^a$ are

\[
\ln g_0 = \ln g_1 + \ln g_2 + \ln g_3
\]
\[
\ln^2 g_0 = 3 \ln^2 g_1 + 3 \ln^2 g_2 + 3 \ln^2 g_3 - \varphi(g_4)
\]
\[
g_0^{-1} = g_1^{-1} - g_2^{-1} + g_3^{-1}
\]

where $\varphi(t) = \ln^2 t + \ln^2(t + 1) + \ln^2(t^{-1} + 1)$.

2.3.2 Abelian relations for $A_5^b$

If $g_0 = xy/(x + y)$ and $g_1, g_2, g_3, g_4$ are as above, then

\[
\ln g_0 = \ln g_1 + \ln g_2 - \ln g_3
\]
\[
\ln^2 g_0 = \ln^2 g_1 + \ln^2 g_2 - 3 \ln^2 g_3 - \varphi(g_4)
\]
\[
g_0^{-1} = g_1^{-1} + g_2^{-1}
\]

where $\varphi(t) = \ln^2 t - \ln^2(t + 1) - \ln^2(t^{-1} + 1)$.

2.3.3 Abelian relations for $A_5^c$

If $g_0 = (x^2 + xy + y^2)/(xy(x + y))$ and $g_1, g_2, g_3, g_4$ are as above, then

\[
\ln g_0 = \ln g_1 + \ln g_2 + \ln(g_4 + g_4^{-1} + 1)
\]
\[
g_0 = g_1^{-1} + g_2^{-1} - g_3^{-1}
\]
\[
g_0^2 = g_1^{-2} + g_2^{-2} - g_3^{-2}
\]

2.3.4 Abelian relations for $A_5^d$

Notice that $A_5^d$ is equivalent to

\[
[dx dy (dx + dy) (dx - \xi_3 dy)] \boxtimes [d(xy(x + y)(x - \xi_3 y))]
\]
under a linear change of coordinates. If \( g_0 = xy(x + y)(x - \xi_3 y) \), \( g_4 = x - \xi_3 y \), and \( g_1, g_2, g_3 \) are as above, then

\[
\ln g_0 = \ln g_1 + \ln g_2 + \ln g_3 + \ln g_4 \\
12 g_0 = (-2 - \xi_3) g_1^4 + (1 + 2 \xi_3) g_2^4 + (1 - \xi_3) g_3^4 + (1 + 2 \xi_3) g_4^4 \\
28 g_0^2 = (1 + \xi_3) g_1^8 - g_2^8 - \xi_3 g_3^8 - g_4^8.
\]

### 2.3.5 Abelian relations for \( A^b_6 \)

If \( g_0 = x^3 + y^3 \), \( g_4 = x + \xi_3 y \), \( g_5 = x + \xi_3^2 y \), and \( g_1, g_2, g_3 \) are as above, then

\[
g_0 = g_1^3 + g_2^3 \\
\ln g_0 = \ln g_3 + \ln g_4 + \ln g_5 \\
30 g_0^2 = 27 g_1^6 + 27 g_2^6 + g_3^6 + g_4^6 + g_5^6 \\
84 g_0^3 = 81 g_1^9 + 81 g_2^9 + g_3^9 + g_4^9 + g_5^9.
\]

### 3 Abelian Relations for Planar Webs Associated to Nets

The determination of \( A(B_5) \) is due to Bol, see [7]. The determination of \( A(B_6), A(B_7) \), and \( A(B_8) \) is treated in [43] (see also [39, 40] for the determination of \( A(B_6) \) and \( A(B_7) \) through Abel’s method). In this section, we will prove that \( \mathcal{H}_5 \) and \( \mathcal{H}_{10} \)—the two remaining exceptional CDQL webs on \( \mathbb{P}^2 \) presented in the Introduction—have maximal rank. We adopt here an approach similar to the one used by Robert in [43] and that can be traced back to [22]. We look for the abelian relations among \( k \)-uples of Chen’s iterated integrals of logarithmic 1-forms with poles on certain hyperplane arrangements. It turns out that this particular class of webs carries logarithmic and dilogarithmic abelian relations thanks to purely combinatorial reasons.

#### 3.1 Webs associated to nets

Let \( r \geq 3 \) be an integer. Recall from [47] that a \( r \)-net in \( \mathbb{P}^2 \) is a pair \((\mathcal{L}, \mathcal{P})\) where \( \mathcal{L} \) is a finite set of lines partitioned into \( r \) disjoint subsets \( \mathcal{L} = \bigsqcup_{i=1}^r \mathcal{L}_i \) and \( \mathcal{P} \) is a finite set of points subjected to the two conditions:

1. for every \( i \neq j \) and every \( \ell \in \mathcal{L}_i, \ell' \in \mathcal{L}_j \), we have that \( \ell \cap \ell' \in \mathcal{P} \);
(2) for every $p \in \mathcal{P}$ and every $i = 1, 2, \ldots, r$, there exists a unique $\ell \in \mathcal{L}_i$ passing through $p$.

The definition implies first that the cardinalities of the sets $\mathcal{L}_i$ do not depend on $i$ and are all equal to $m = \text{Card}(\ell \cap \mathcal{P})$ for any $\ell \in \mathcal{L}$. This fact implies in its turn that $\mathcal{P}$ has cardinality $m^2$. We say that $\mathcal{L}$ is a $(r, m)$-net.

For every pair $(\alpha, \beta) \in \{1, \ldots, r - 1\}^2$, there is a function

$$
n^\beta_\alpha : \mathcal{L}_\alpha \times \mathcal{L}_r \to \mathcal{L}_\beta
$$

that assigns to $(\ell, \ell') \in \mathcal{L}_\alpha \times \mathcal{L}_r$ the line in $\mathcal{L}_\beta$ passing through $\ell \cap \ell'$. Notice that for a fixed $\ell \in \mathcal{L}_r$ the functions $n^\beta_\alpha(\cdot, \ell) : \mathcal{L}_\alpha \to \mathcal{L}_\beta$ are bijective.

It follows from the definition of a $r$-net (cf. [47]) that there exists a rational function $F : \mathbb{P}^2 \to \mathbb{P}^1$ of degree $m$ with $r$ distinct values $c_1, \ldots, c_r \in \mathbb{P}^1$ for which $F^{-1}(c_i)$ can be identified with $\mathcal{L}_i$. Although there is some ambiguity in the definition of $F$ (we can compose it with any automorphism of $\mathbb{P}^1$) the induced foliation is uniquely determined and will be denoted by $F(\mathcal{L})$. Similarly, if $(\mathcal{L}, \mathcal{P})$ is a $(r, m)$-net then we will denote by $\mathcal{W}(\mathcal{L})$ the CDQL $(m^2 + 1)$-web $\mathcal{W}(\mathcal{P}) \boxtimes F(\mathcal{L})$, where $\mathcal{W}(\mathcal{P})$ is the completely decomposable linear $m^2$-web formed by the superposition of the pencils of lines through the points of $\mathcal{P}$.

Among the thirteen sporadic examples of exceptional CDQL webs on $\mathbb{P}^2$ presented in the Introduction, two are webs associated to nets. The first one is Bol’s web $B_5$ which is associated to a $(3, 2)$-net with $\mathcal{P}$ equal to four points in general position and $\mathcal{L}$ equal to the set of lines joining any two of them. In this case, $F(\mathcal{L})$ is the pencil of conics through the four points. The other example is $\mathcal{H}_{10}$ that is the CDQL 10-web associated to a $(4, 3)$-net with $\mathcal{P}$ equal to the set of base points of the Hesse pencil, $\mathcal{L}$ equal to the set of lines through any two of them and $F(\mathcal{L})$ equal to the Hesse pencil.

The result below implies that both $B_5$ and $\mathcal{H}_{10}$ are exceptional.

**Theorem 3.1.** If $\mathcal{L}$ is a $(r, m)$-net then

$$
\text{rk}(\mathcal{W}(\mathcal{L})) \geq \frac{(m^2 - 1)(m^2 - 2)}{2} + (r - 1)^2 - 1.
$$

In particular, if $\mathcal{L}$ is a $(3, 2)$-net or a $(4, 3)$-net then $\mathcal{W}(\mathcal{L})$ has maximal rank.
Proof of Theorem 3.1. Since the \(m^2\)-subweb \(W(P)\) is linear, it has maximal rank. To prove the theorem, it suffices to show that

\[
\dim_{\mathbb{C}} \frac{\mathcal{A}(W(L))}{\mathcal{A}(W(P))} \geq (r - 1)^2 - 1.
\]

Set \(L_i = \{\ell_1^{(i)}, \ldots, \ell_m^{(i)}\}\) and let \(L_j^{(i)}\) be a linear homogenous polynomial in \(\mathbb{C}[x, y, z]\) defining \(\ell_j^{(i)}\). For \(i, j = 1, \ldots, m\), let \(p_{ij} = \ell_i^{(1)} \cap \ell_j^{(r)}\) and let by \(L_{ij}\) be the subset of \(L\) formed by the lines passing through \(p_{ij}\). Notice that \(P = \cup_{i, j} \{p_{ij}\}\). Let also \(L_{p_{ij}}\) be the linear foliation induced by the pencil of lines through \(p_{ij}\). It will be useful to introduce the vector space \(V = H^0(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(\log L))\) (resp. \(V_{ij} = H^0(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(\log L_{ij}))\)) of logarithmic 1-forms with poles in \(L\) (resp. in \(L_{ij}\)).

Lemma 3.1.

1. The union of the subspaces \(V_{ij} \subset V\) spans \(V\).
2. Every element in \(V_{ij}\) is closed and vanishes when restricted to the leaves of \(L_{p_{ij}}\). □

Proof of Lemma 3.1. We first prove (1). The space \(V\) is generated by elements of the form \(\omega = \frac{dL}{L} - \frac{dL'}{L'}\) where \(L\) and \(L'\) are linear forms cutting out \(\ell, \ell' \in L\). If \(\ell \cap \ell' = p_{ij} \in P\), then \(\omega \in V_{ij}\). Otherwise, \(\ell\) and \(\ell'\) belong to the same set \(L_i\). Then if \(\ell''\) is an element of \(L_j\) for \(j \neq i\) cut out by a linear form \(L''\), one can write

\[
\omega = \left(\frac{dL}{L} - \frac{dL''}{L''}\right) - \left(\frac{dL'}{L'} - \frac{dL''}{L''}\right).
\]

If \(\ell_{n_{\ell}(i,j)}\) denotes the line \(n_{\ell}(i,j) = (\ell_i^{(a)}, \ell_j^{(b)})\), then the logarithmic 1-forms

\[
\frac{dL_{n_{\ell}(i,j)}^{(a)}}{L_{n_{\ell}(i,j)}^{(a)}} - \frac{dL_j^{(r)}}{L_j^{(r)}}, \quad \alpha = 1, \ldots, r - 1,
\]

can be taken as a basis of \(V_{ij}\). This immediately implies point (2) of the lemma. □
For a suitable choice of the linear forms \( L_j^{(i)} \), the rational function \( F : \mathbb{P}^2 \to \mathbb{P}^1 \) associated to the net satisfies (for every \( \alpha = 1, \ldots, r-1 \))

\[
F - c_\alpha = \frac{\prod_{i=1}^{m} L_i^{(\alpha)}}{\prod_{i=1}^{m} L_i^{(r)}}.
\]

Taking the logarithmic derivatives of these relations, it follows that \( \frac{dF}{F - c_\alpha} \in V \) for every \( \alpha \) ranging from 1 to \( r - 1 \). Therefore, by Lemma 3.1 (1), there exist some logarithmic 1-forms \( \omega_{ij}^{(\alpha)} \in V_{ij} \) (for \( i, j = 1, \ldots, m \)) such that

\[
\frac{dF}{F - c_\alpha} + \sum_{i,j=1}^{m} \omega_{ij}^{(\alpha)} = 0.
\]

Point (2) of Lemma 3.1 implies that the preceding relations can be interpreted as elements of \( \mathcal{A}(\mathcal{W}(\mathcal{L})) \). Since the 1-forms \( \frac{dF}{F - c_\alpha} \) are linearly independent, the classes of these equations span a \((r-1)\)-dimensional subspace \( A_0 \subset \mathcal{A}(\mathcal{W}(\mathcal{L})) \).

For distinct \( \alpha, \beta \in \{1, \ldots, r-1\} \) and \( i \in \{1, \ldots, m\} \) fixed, one has \( \cup_{j=1}^{m} \ell_{\mathcal{L}(\beta)}(i,j) = \mathcal{L}_\beta \).

Using this fact, one obtains that

\[
\frac{dF}{F - c_\alpha} \otimes \frac{dF}{F - c_\beta} = \sum_{i,j=1}^{m} \left( \frac{dL_i^{(\alpha)}}{L_i^{(\alpha)}} - \frac{dL_j^{(r)}}{L_j^{(r)}} \right) \otimes \left( \frac{dL_j^{(\beta)}}{L_j^{(\beta)}} - \frac{dL_j^{(r)}}{L_j^{(r)}} \right) + K
\]

where \( K \) is given by

\[
K = \sum_{i \neq j} \frac{dL_i^{(r)}}{L_i^{(r)}} \otimes \frac{dL_j^{(r)}}{L_j^{(r)}} - (m - 1) \sum_{i=1}^{m} \left( \frac{dL_i^{(r)}}{L_i^{(r)}} \right)^{\otimes 2}.
\]

Notice that \( K \) does not depend on the indices \( \alpha \) and \( \beta \). Thus for ordered pairs \((\alpha, \beta)\) and \((\gamma, \delta)\) with distinct entries in \( \{1, \ldots, r-1\} \) and for suitable \( \omega_{ij}^{(\alpha\beta\gamma\delta)} \in V_{ij} \otimes V_{ij} \), the following identity holds true

\[
\left( \frac{dF}{F - c_\alpha} \otimes \frac{dF}{F - c_\beta} - \frac{dF}{F - c_\gamma} \otimes \frac{dF}{F - c_\delta} \right) + \sum_{i,j=1}^{m} \omega_{ij}^{(\alpha\beta\gamma\delta)} = 0. \tag{4}
\]
It follows from Chen’s theory of iterated integrals (see [15, Theorem 4.1.1] and [43, Theorem 2.1]) that after integration these identities can be interpreted as elements of \( A(L) \). This can be explained more concretely as follows: let \( U \subset \mathbb{P}^2 \) be a small open ball centered at a generic point \( x_0 \in \mathbb{P}^2 \). If \( \omega = \varphi \otimes \psi \) with \( \varphi, \psi \in \Omega^1(U) \) closed, one sets \( I(\omega)(x) = (\int_{\gamma} \varphi) \psi(x) \) for every \( x \in U \), where \( \gamma \) is a smooth path joining \( x_0 \) to \( x \) in \( U \). Clearly, it does not depend on the path \( \gamma \) but only on the end-point \( x \). Hence, it defines a holomorphic 1-form \( I(\omega) \) on \( U \). The proof of the following fact is straightforward:

**Fact 3.1.** If \( \varphi, \psi \in \Omega^1(U) \) are closed and vanish along the leaves of a smooth foliation \( F \) on \( U \) then \( I(\varphi \otimes \psi) \) is closed and also vanishes along the leaves of \( F \). □

Assume now that \( U \) does not meet the discriminant of \( W(L) \). Applying the \( \mathbb{C} \)-linear operator \( I : V \otimes^2 \rightarrow \Omega^1(U) \) to (4), one obtains

\[
(\log(F - c_\alpha) - \hat{c}_\alpha) \frac{dF}{F - c_\beta} - (\log(F - c_\gamma) - \hat{c}_\gamma) \frac{dF}{F - c_\delta} + \sum_{i,j=1}^m I(\omega_{ij}^{(\alpha\beta\gamma\delta)}) = 0 \tag{5}
\]

for some constant \( \hat{c}_\alpha, \hat{c}_\gamma \in \mathbb{C} \). Since \( \omega_{ij}^{(\alpha\beta\gamma\delta)} \in V_{ij} \otimes V_{ij} \) for every \( i, j = 1, \ldots, m \), Fact 3.1 above and assertion (2) in Lemma 3.1 imply that the iterated integrals \( I(\omega_{ij}^{(\alpha\beta\gamma\delta)}) \) can be interpreted as closed 1-forms on \( U \) vanishing when restricted to the leaves of the foliations \( L_{p_{ij}} \). Hence the relations (5) can be interpreted as abelian relations for \( W(L) \) as asserted above. Moreover, their classes modulo \( A(W(P)) \) span a subspace \( A_1 \subset A(W(P)) \) of dimension \((r - 1)(r - 2) - 1\). Since \( A_0 \cap A_1 = 0 \), the theorem follows. □

It has to be noted that Theorem 3.1 has a rather limited scope. Indeed, the Hesse net is the only \( r \)-net in \( \mathbb{P}^2 \) known with \( r \geq 4 \) and recently J. Stipins has proved that there is no \( r \)-net in \( \mathbb{P}^2 \) if \( r \geq 5 \) see [48]. Nevertheless, Theorem 3.1 might give some clues on how to approach the problem about the abelian relations of webs associated to hyperplane arrangements proposed in [38]. We refer to this paper and the references therein for further examples of nets.

The maximality of the rank of \( \mathcal{H}_5 \) follows from similar reasons. If \( \mathcal{L} \) is the Hesse arrangement of lines then an argument similar to the one used in the proof of Theorem 3.1 shows that \( V = H^0(\mathbb{P}^2, \Omega^1(\log \mathcal{L})) \) can be generated by logarithmic 1-forms
inducing the defining foliations of the maximal linear subweb of $\mathcal{H}_5$. Since the Hesse pencil has four linear fibers, it follows that

$$\dim \frac{\mathcal{A}(\mathcal{H}_5)}{\mathcal{A}([xdy - ydx)(dx^3 + dy^3)])} \geq 3.$$ 

Consequently, $\mathcal{H}_5$ has maximal rank.

### 3.2 Explicit abelian relations for $\mathcal{H}_5$

Alternatively, one can also establish directly that the rank of $\mathcal{H}_5$ is maximal. Indeed, the functions $g_0 = (x^3 + y^2 + 1)/(xy)$, $g_1 = \xi_3 x + y$, $g_2 = x + y$, $g_3 = x + \xi_3 y$, and $g_4 = x/y + y/x$ are first integrals of $\mathcal{H}_5$ and they verify the abelian relations:

$$\ln \left( \frac{g_0 - 3 \xi_3}{g_0 - 3 (\xi_3)^2} \right) = \ln \left( \frac{g_1 + (\xi_3)^2}{g_1 + 1} \right) + \ln \left( \frac{g_2 + 1}{g_2 + \xi_3} \right) + \ln \left( \frac{g_3 + (\xi_3)^2}{g_3 + 1} \right).$$

$$\ln \left( \frac{g_0 - 3 \xi_3}{g_0 - 3 (\xi_3)^2} \right) = \ln \left( \frac{g_1 + (\xi_3)^2}{g_1 + 1} \right) + \ln \left( \frac{g_2 + \xi_3}{g_2 + (\xi_3)^2} \right) + \ln \left( \frac{g_3 + 1}{g_3 + \xi_3} \right).$$

$$\ln (g_0 - 3) = \ln \left( \frac{g_1 + (\xi_3)^2}{g_1} \right) - \ln (g_2 + 1) - \ln \left( \frac{g_3 + (\xi_3)^2}{g_3} \right) - \ln (1 - g_4).$$

These abelian relations span a three-dimensional vector space $\mathcal{A}_1$ such that

$$\mathcal{A}(\mathcal{H}_5) = \mathcal{A}([xdy - ydx)(dx^3 + dy^3)]) \oplus \mathcal{A}_1.$$ 

### 4 Abelian Relations for the Elliptic CDQL Webs

In this section, we will prove that the elliptic CDQL webs presented in the Introduction are exceptional. Their abelian relations not coming from the maximal linear subweb that they contain are all captured by Theorem 4.1 below.

The analogy with Theorem 3.1 is evident and probably not very surprising for the specialists in polylogarithms since, according to the terminology of Beilinson and Levine [4], the integrals $\int dz$ and $\int d \log \vartheta$ ($\vartheta$ being a theta function) can be considered as elliptic analogs of the classical logarithm and dilogarithm.

### 4.1 Rational maps on complex tori with many linear fibers

Let $T$ be a two-dimensional complex torus and $F : T \to \mathbb{P}^1$ be a meromorphic map. We will say that a fiber $F^{-1}(\lambda)$ is linear if it is supported on a union of subtori.
Notice that each subtorus $E$ of $T$ determines a unique linear foliation with $E$ and its translates being the leaves. We will say that a linear web $\mathcal{W}$ on $T$ supports a fiber $F^{-1}(\lambda)$ if it contains all the linear foliations determined by the irreducible components of $F^{-1}(\lambda)$.

**Theorem 4.1.** Let $\mathcal{F}$ be the foliation induced by a meromorphic map $F : T \rightarrow \mathbb{P}^1$. If $\mathcal{W}$ is a linear $k$-web with $k \geq 3$ that supports $m$ distinct linear fibers of $F$, then

$$\dim \frac{A(\mathcal{W} \boxtimes \mathcal{F})}{A(\mathcal{W})} \geq m - 1.$$ 

□

Before proving Theorem 4.1, let us briefly review some basic facts about theta functions. For details see for instance [18, Chapter IV]. If $V$ is a complex vector space and $\Gamma \subset V$ is a lattice, then a theta function associated to $\Gamma$ is any entire function $\vartheta$ on $V$ such that for each $\gamma \in \Gamma$ there exists a linear form $a_\gamma$ and a constant $b_\gamma$ such that

$$\vartheta(x + \gamma) = \exp \left( 2i\pi (a_\gamma (x) + b_\gamma) \right) \vartheta(x) \quad \text{for every } x \in V.$$

Any effective divisor on the complex torus $T = V/\Gamma$ is the zero divisor of some theta function. Moreover, if the divisors of two theta functions, say $\vartheta$ and $\tilde{\vartheta}$, coincide then their quotient is a trivial theta function, that is

$$\frac{\tilde{\vartheta}(x)}{\vartheta(x)} = \exp \left( P(x) \right)$$

where $P : V \rightarrow \mathbb{C}$ is a polynomial of degree at most two.

**Example 4.1.** If $(\mu, \nu) \in \{0, 1\}^2$ and $\tau \in \mathbb{H}$, then the entire functions on $\mathbb{C}$

$$\vartheta_{\mu, \nu}(x, \tau) = \sum_{n = -\infty}^{+\infty} (-1)^n \exp \left( i\pi \left( n + \frac{\mu}{2} \right)^2 \tau + 2i\pi \left( n + \frac{\mu}{2} \right) x \right)$$

satisfy the following relations

$$\vartheta_{\mu, \nu}(x + 1, \tau) = (-1)^\mu \vartheta_{\mu, \nu}(x, \tau) \quad \text{(6)}$$
$$\vartheta_{\mu, \nu}(x + \tau, \tau) = (-1)^\nu \exp \left( -i\pi (2x + \tau) \right) \vartheta_{\mu, \nu}(x, \tau).$$
It is then clear that they are examples of theta functions with respect to the lattice \( \mathbb{Z} \oplus \mathbb{Z} \tau \subset \mathbb{C} \). The theta functions \( \vartheta_i \) that appeared in the Introduction are nothing more than
\[
\begin{align*}
\vartheta_1 &= -i \vartheta_{1,1}, \\
\vartheta_2 &= \vartheta_{1,0}, \\
\vartheta_3 &= \vartheta_{0,0}, \quad \text{and} \quad \\
\vartheta_4 &= \vartheta_{0,1}.
\end{align*}
\]

If \( E_\tau \) denotes the elliptic curve \( \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z} \tau) \), then the zero divisors of the functions \( \vartheta_i = \vartheta_i(\cdot, \tau) \) are
\[
(\vartheta_1)_0 = 0, \quad (\vartheta_2)_0 = \frac{1}{2}, \quad (\vartheta_3)_0 = \frac{1 + \tau}{2}, \quad \text{and} \quad (\vartheta_4)_0 = \frac{\tau}{2}.
\]

**Proof of Theorem 4.1.** With notation as above, suppose that \( T = V/\Gamma \). If \( F^{-1}(\lambda) \) is a linear fiber, then one can write
\[
F^{-1}(\lambda) = D_1^\lambda + \cdots + D_{r(\lambda)}^\lambda
\]
where each divisor \( D_i^\lambda \) (for \( i = 1, \ldots, r(\lambda) \)) is supported on a union of translates of a subtori \( E_i^\lambda \). Therefore, there exist complex vector spaces \( V_i^\lambda \) of dimension one, linear maps \( p_i^\lambda : V \to V_i^\lambda \) and lattices \( \Gamma_i^\lambda \subset V_i^\lambda \) such that
\[
\begin{align*}
(1) & \quad p_i^\lambda(\Gamma) \subset \Gamma_i^\lambda; \\
(2) & \quad D_i^\lambda \text{ is the pullback by the map } [p_i] : T \to V_i^\lambda/\Gamma_i^\lambda \text{ of a divisor on } V_i^\lambda/\Gamma_i^\lambda.
\end{align*}
\]
Notice that \( p_i^\lambda \) can be interpreted as a linear form on \( V \) and its differential \( dp_i^\lambda \) as a 1-form defining the linear foliation determined by \( E_i^\lambda \).

Composing \( F \) with an automorphism of \( \mathbb{P}^1 \) we can assume that the linear fibers are \( F^{-1}(\lambda_1), F^{-1}(\lambda_2), \ldots, F^{-1}(\lambda_{m-1}) \) and \( F^{-1}(\infty) \). Thus, for \( j \) ranging from 1 to \( m - 1 \), we can write
\[
F - \lambda_j = \exp(P_j(z)) \prod_i [p_i^{\lambda_j}]^* \vartheta_i^{\lambda_j} \prod_i [p_i^{\infty}]^* \vartheta_i^{\infty}
\]
(7)
where the \( P_j \)'s are polynomials of degree at most two and \( \vartheta_i^{\lambda_j} \) are theta functions on \( V_i^{\lambda_j} \) associated to the lattices \( \Gamma_i^{\lambda_j} \). Taking the logarithmic derivative of (7), we obtain
\[
\frac{dF}{F - \lambda_j} = dP_j(z) + \sum_i [p_i^{\lambda_j}]^* d \log \vartheta_i^{\lambda_j} - \sum_i [p_i^{\infty}]^* d \log \vartheta_i^{\infty}.
\]
(8)
Since \( W \) is a \( k \)-web with \( k \geq 3 \), there exist three pairwise linearly independent linear forms \( p_1, p_2, \) and \( p_3 \) among the \( p_i^j \) such that \( dP_j \) can be written as a linear combination of \( dp_1, dp_2, p_1 dp_1, p_2 dp_2, \) and \( p_3 dp_3 \). It follows that (8) is an abelian relation for \( W \boxtimes F \). Since the logarithmic 1-forms \( dF \) \( F - \lambda_1 \), \( \ldots \), \( F - \lambda_m \) are linearly independent over \( \mathbb{C} \), the abelian relations described in (8) are also linearly independent and generate a subspace of \( \mathcal{A}(W \boxtimes F) \) of dimension \( m - 1 \) intersecting \( \mathcal{A}(W) \) trivially. The theorem follows.

In the next three subsections, we will derive the exceptionality of the CDQL webs \( E_5^1, E_5, E_6, \) and \( E_7 \) from Theorem 4.1. Along the way, a more geometric description of these webs will emerge.

4.2 The harmonic 5-webs \( E_5^1 \) and the superharmonic 7-web \( E_7 \)

For \( \tau \in \mathbb{H} \), let \( E_\tau \) be the elliptic curve \( \mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z}) \) and \( T_\tau \) be the complex torus \( E_\tau^2 \). For every \( \tau \in \mathbb{H} \), the 5-web \( E_5^1 = [dx dy (dx^2 - dy^2) dF_\tau] \) is naturally defined on \( T_\tau \) where

\[
F_\tau(x, y) = \left( \frac{\vartheta_1(x, \tau) \vartheta_1(y, \tau)}{\vartheta_4(x, \tau) \vartheta_4(y, \tau)} \right)^2.
\]

(9)

The 7-web \( E_7 = [dx dy (dx^2 - dy^2)(dx^2 + dy^2) dF_{1+i}] \) in its turn is naturally defined on \( T_{1+i} \).

For every \( \alpha, \beta \in \text{End}(E_\tau) \), denote by \( E_{\alpha,\beta} \) the elliptic curve described by the image of the morphism

\[
\varphi_{\alpha,\beta} : E_\tau \longrightarrow T_\tau,
\]

\[
x \longmapsto (\alpha \cdot x, \beta \cdot x).
\]

(10)

For example \( E_{1,0} \) is the horizontal elliptic curve through \( 0 \in T_\tau \), \( E_{0,1} \) is the vertical one, \( E_{1,1} \) is the diagonal, and \( E_{1,-1} \) is the anti-diagonal. The translation of \( E_{\alpha,\beta} \) by an element \( (a, b) \in T_\tau \) will be denoted by \( L(a,b)E_{\alpha,\beta} \).

Let \( D_1 = E_{1,0} + E_{0,1} \) and \( D_2 = L(0,\tau/2)E_{1,0} + L(\tau/2,0)E_{0,1} \) be divisors in \( T_\tau \). Notice that the rational function \( F_\tau \) is such that \( \text{div}(F_\tau) = 2D_1 - 2D_2 \). Thus, the indeterminacy set of \( F_\tau \) is

\[
\text{Indet}(F_\tau) = \{ (\tau/2, 0), (0, \tau/2) \}.
\]
Blowing up the two indeterminacy points of $F_{\tau}$, we obtain a surface $\tilde{T}_{\tau}$ containing two pairwise disjoint divisors $\tilde{D}_1$ and $\tilde{D}_2$: the strict transforms of $D_1$ and $D_2$, respectively. Let $D_3 = L_{(\tau/2, 0)}E_{1,1} + L_{(0, \tau/2)}E_{1,-1}$. The pairwise intersection of the supports of the divisors $D_1, D_2$, and $D_3$ are all equal, that is

$$\mid D_1 \cap \mid D_3 \mid = \mid D_2 \cap \mid D_3 \mid = \mid D_1 \cap \mid D_2 \mid = \text{Indet}(F_{\tau}).$$

Therefore, $\tilde{D}_3$, the strict transform of $D_3$, is a divisor in $\tilde{T}_{\tau}$ with support disjoint from the supports of $\tilde{D}_1$ and $\tilde{D}_2$. The lifting of $F_{\tau}$ to $\tilde{T}_{\tau}$ must map the support of $\tilde{D}_3$ to $\tilde{T}_{\tau}(\tilde{T}_{\tau} - (\mid \tilde{D}_1 \cup \mid \tilde{D}_2 \mid)) = \mathbb{P}^1 \setminus \{0, \infty\} = \mathbb{C}^*$. The maximal principle implies that the image must be a point. Since $\tilde{D}_3 \cdot \tilde{D}_3 = 0$, $D_3$ must be a connected component of a fiber of $F_{\tau}$. Since $D_3$ is numerically equivalent to $2D_1$ and $2D_2$, it turns out that $\tilde{D}_3$ is indeed a fiber of $F_{\tau}$. Moreover, because $\tilde{D}_3$ is connected and reduced, the generic fiber of $F_{\tau}$ is irreducible. In particular, the linear equivalence class of the divisor $\frac{1}{2}\text{div}(F_{\tau}) = D_1 - D_2$ is a non-trivial two-torsion point in $\text{Pic}_0(T_{\tau})$.

So far, we have proved that $F_{\tau}$ has at least three linear fibers. For generic $\tau$, it can be verified that three is the exact number of linear fibers of $F_{\tau}$. But if $\tau = 1 + i$, then

$$D_4 = L_{((1+i)/2, 0)}E_{1,i} + L_{(0,(1+i)/2)}E_{1,-i}$$

is such that $\mid D_1 \cap \mid D_4 \mid = \mid D_2 \cap \mid D_4 \mid = \mid D_3 \cap \mid D_4 \mid = \mid D_1 \cap \mid D_2 \mid = \text{Indet}(F_{1+i})$. The arguments above imply that $F_{1+i}$ has at least four linear fibers.

Theorem 4.1 can be applied to the 5-webs $E_5^\tau = \{dx dy(dx^2 - dy^2) dF_{\tau}\}$ (resp. to the 7-web $E_7 = \{dx dy(dx^2 - dy^2)(dx^2 + dy^2) dF_{1+i}\}$) to ensure that $rk(E_5^\tau) \geq 3 + 2 = 5$ (resp. $rk(E_7) \geq 10 + 3 = 13$). To prove that $E_5^\tau$ and $E_7$ are exceptional, it remains to find two extra abelian relations for the latter web and one for the former.

The missing abelian relations are also captured by Theorem 4.1. The point is that the torsion of $D_1 - D_2$ is hiding two extra linear fibers of $F_{1+i}$ and one extra linear fiber of $F_{\tau}$. More precisely, since $D_1 - D_2$ is a non-trivial two-torsion element of $\text{Pic}_0(T_{\tau})$, there exists a complex torus $X_{\tau}$, an étale covering $\rho : X_{\tau} \rightarrow T_{\tau}$, and a rational function $G_{\tau} : X_{\tau} \rightarrow \mathbb{P}^1$, with irreducible generic fiber, fitting in the commutative diagram:

$$\xymatrix{ X_{\tau} \ar[r]^{\rho} & T_{\tau} \\
\mathbb{P}^1 \ar[r]^-{\psi} \ar[u]_{G_{\tau}} & \mathbb{P}^1 \ar[u]_-{F_{\tau}}.}$$
The étale covering above can be assumed to lift to the identity over the universal covering of $X_r$ and $T_r$. In other words, if $T_r = \mathbb{C}^2 / \Gamma_r$ for some lattice $\Gamma_r \subset \mathbb{C}^2$ then $X_r$ is induced by a sublattice of $\Gamma_r$. In particular

$$\rho^*[dxdy(d^k - dy^k)] = [dxdy(d^k - dy^k)]$$

for every $k \geq 1$.

Clearly, $G_r$ has at least four linear fibers supported by the linear web $[dxdy(dx^2 - dy^2)]$ and $G_{1+i}$ has at least six linear fibers supported by the linear web $[dxdy(dx^4 - dy^4)]$. Theorem 4.1 implies that $\rho^*\mathcal{E}_5^T$ and $\rho^*\mathcal{E}_7$ are webs of maximal rank. Since the rank is locally determined, the same holds for $\mathcal{E}_5^T$ and $\mathcal{E}_7$.

### 4.3 The equianharmonic 5-web $\mathcal{E}_5$

Let $F : T_{\xi_3} \to \mathbb{P}^1$ be the rational function

$$F = \frac{\vartheta_1(x, \xi_3) \vartheta_1(y, \xi_3) \vartheta_1(x - y, \xi_3) \vartheta_1(x + \xi_3^2 y, \xi_3)}{\vartheta_2(x, \xi_3) \vartheta_3(y, \xi_3) \vartheta_4(x - y, \xi_3) \vartheta_3(x + \xi_3^2 y, \xi_3)} .$$

(11)

**Proposition 4.1.** The function $F$ has four linear fibers on $T_{\xi_3}$. Moreover, each of these fibers is supported on the linear 4-web $\mathcal{W} = [dxdy(dx - dy)(dx + \xi_3^2 dy)]$.

**Proof.** Consider the divisor $D_1 = E_{1,0} + E_{0,1} + E_{1,1} + E_{1,-\xi_3}$. Notice that $D_1$ can be given by the vanishing of

$$f_1(x, y) = \vartheta_1(x, \xi_3) \vartheta_1(y, \xi_3) \vartheta_1(x - y, \xi_3) \vartheta_1(x + \xi_3^2 y, \xi_3).$$

The complex torus $T_{\xi_3}$ has sixteen two-torsion points and the support of $D_1$ contains thirteen of them. The two-torsion points that are not contained in $|D_1|$ are

$$p_2 = \left(\frac{1}{2}, \frac{1 + \xi_3}{2}\right), \quad p_3 = \left(\frac{\xi_3}{2}, \frac{1}{2}\right), \quad \text{and} \quad p_4 = \left(\frac{1 + \xi_3}{2}, \frac{\xi_3}{2}\right).$$

If we set $D_i = L_{p_i} D_1$ (the translation of $D_1$ by $p_i$) for $i = 2, 3, 4$, then the support of $D_i \cap D_j$ (with $j \neq i$) does not depend on $(i, j)$ and is the set of 12 non-trivial two-torsion points of $T_{\xi_3}$ contained in $D_1$. Notice that $D_2$ can be given by the vanishing of

$$f_2(x, y) = \vartheta_2(x, \xi_3) \vartheta_3(y, \xi_3) \vartheta_4(x - y, \xi_3) \vartheta_3(x + \xi_3^2 y, \xi_3).$$

The quotient $F = f_1(x, y)/f_2(x, y)$ is the rational function we are interested in.
Blowing up the twelve indeterminacy points of $F$, one sees that the strict transforms of the divisors $D_i$ are connected and pairwise disjoint divisors of self-intersection zero. This is sufficient to prove that each of the divisors $D_i$ is a linear fiber of $F$ and that $F$ has generic fiber irreducible as in the analysis of the webs $E_5^7$ and $E_7$. Clearly, each one of these fibers is supported on the linear web $\mathcal{W}$.

The proposition above combined with Theorem 4.1 implies at once that the web

$$E_5 = [dx dy (dx - dy) (dx + \xi_2^2 dy)] \boxtimes [dF]$$

is exceptional.

4.4 The equianharmonic 6-web $E_6$

It remains to analyze the 6-web

$$E_6 = [dx dy (dx + dy) (dx + \xi_3 dy) (dx + \xi_3^2 dy)] \boxtimes [dx/\wp(x) + dy/\wp(y)]$$

on $T_{\xi_3} = E_6^{\xi_3}$. We will proceed exactly as in the previous cases.

**Proposition 4.2.** The foliation $F = [dx/\wp(x) + dy/\wp(y)]$ on $T_{\xi_3}$ admits a rational first integral $F : T_{\xi_3} \to \mathbb{P}^1$ with generic fiber irreducible and with three linear fibers, one reduced and two of multiplicity three. Moreover, these three linear fibers are supported on the linear web $\mathcal{W} = [dx dy (dx + dy) (dx + \xi_3 dy) (dx + \xi_3^2 dy)]$. □

**Proof.** Recall that if $\Gamma \subset \mathbb{C}$ is a lattice, then the Weierstrass $\wp$-function associated to $\Gamma$ is defined as

$$\wp(z, \Gamma) = \frac{1}{z^2} + \sum_{\gamma \in \Gamma \setminus \{0\}} \left( \frac{1}{(z - \gamma)^2} - \frac{1}{\gamma^2} \right). \quad (12)$$

It is an entire meromorphic function with poles of order two on $\Gamma$ and for a fixed $\Gamma$, the function $\wp(\cdot, \Gamma)$ is $\Gamma$-periodic, that is $\wp(\cdot, \Gamma)$ descends to a meromorphic function on the elliptic curve $E(\Gamma) = \mathbb{C}/\Gamma$ with a unique pole of order two at zero.

Recall also that $\wp$ is homogeneous of degree $-2$, that is, for any $\mu \in \mathbb{C}^*$

$$\wp(\mu z, \mu \Gamma) = \mu^{-2} \wp(z, \Gamma). \quad (13)$$
Set $\Gamma = \mathbb{Z} \oplus \mathbb{Z} \xi_3$ in what follows. Because $\xi_3 \Gamma = \Gamma$, multiplication by $\xi_3$ induces an automorphism of $E = E(\Gamma)$, of order three with two fixed points besides the origin:

$$p_+ = \frac{2 + 3 \xi_3}{3} + \Gamma \quad \text{and} \quad p_- = \frac{1 + 3 \xi_3}{3} + \Gamma.$$

The relation (13) implies that

$$\wp(p_{\pm}, \Gamma) = \tau^{-2} \wp(p_{\pm}, \Gamma).$$

It follows that $p_+$ and $p_-$ are two zeroes of $\wp(\cdot, \Gamma)$. Since $\wp(\cdot, \Gamma)$ has a unique pole of order two, there are no other zeroes. The points $0$, $p_+$, and $p_-$ form a subgroup $T$ of the three-torsion group $E(3)$ of $E$.

The 1-form $\omega = \frac{dx}{\wp(x)} + \frac{dy}{\wp(y)}$ is a logarithmic 1-form with polar set at $E_{\pm} = \{p_{\pm}\} \times E$ and $E_{\pm} = E \times \{p_{\pm}\}$. The residues of $\omega$ along $E_-$ and $E_-$ are equal and so are those along $E_+$ and $E_+$. Moreover, the residue of $\omega$ along $E_-$ is the opposite of its residue along $E_+$. The singular set of the foliation $F = [\omega]$ consists of five points: $p_{00} = (0, 0)$ and

$$p_{--} = (p_-, p_-), \quad p_{+-} = (p_-, p_+), \quad p_{++} = (p_+, p_+).$$

The inspection of the first non-zero jet of the closed 1-form $\omega$ at the singularity $p_{00}$ reveals that $F$ admits a local first integral analytically equivalent to $x^3 + y^3$ at this point. The two singularities $p_{--}$, $p_{+-}$ are radial ones with local meromorphic first integrals analytically equivalent to $x/y$. Finally, the last two singularities $p_{-+}$ and $p_{++}$ have local holomorphic first integrals analytically equivalent to $xy$.

If $\alpha \Gamma = \Gamma$, then (13) implies that

$$\varphi_1^* \omega = \phi_1^* \frac{\alpha \, dx}{\wp(\alpha x, \Gamma)} + \frac{dx}{\wp(x, \Gamma)} = \frac{\alpha^3 + 1}{\wp(x, \Gamma)} \, dx.$$

A simple consequence is that the three separatrices of $F$ through $p_{00}$ are the elliptic curves $E_{(1, -1)}$, $E_{(1, -\xi_3)}$, and $E_{(1, -\xi_3^2)}$.

Let $\pi : \mathcal{T}_{\xi_3} \to \mathcal{T}_{\xi_3}$ be the blowup of $\mathcal{T}_{\xi_3}$ at the radial singularities of $F$ and denote by $\mathcal{F}$ the transformed foliation. If

$$D_1 = E_+ + E^+, \quad D_2 = E_- + E^-, \quad D_3 = E_{(1, -1)} + E_{(1, -\xi_3)} + E_{(1, -\xi_3^2)}$$
and \( \widetilde{D}_1, \widetilde{D}_2, \widetilde{D}_3 \) denote their respective strict transforms then

\[
\widetilde{D}_1^2 = \widetilde{D}_2^2 = \widetilde{D}_3^2 = 0.
\]

The polar set of \( \pi^*\omega \) has two connected components, one supported on \( |\widetilde{D}_1| \) and the other on \( |\widetilde{D}_2| \). The divisor \( \widetilde{D}_3 \) has connected support and is disjoint from the polar set of \( \pi^*\omega \). It follows from the Hodge index Theorem that this divisor is numerically equivalent to multiples of \( \widetilde{D}_1 \) and \( \widetilde{D}_2 \). Of course, this can be verified by a direct computation. Indeed,

\[
3 (E_- + E^-) \equiv 3 (E_+ + E^+) \equiv E_{(1,-1)} + E_{(1,-\xi_3)} + E_{(1,-\xi_3^2)}
\]

where \( \equiv \) denotes numerical equivalence.

We can apply [45, Theorem 2.1] (see also [37, Theorem 2]) to conclude that the divisors \( \widetilde{D}_1, \widetilde{D}_2, \) and \( \widetilde{D}_3 \) are fibers of a fibration on \( \widetilde{T} \). Consequently, \( F \) admits a first integral \( F : T_{\xi_3} \rightarrow \mathbb{P}^1 \) satisfying \( F^{-1}(0) = 3D_1, F^{-1}(\infty) = 3D_2, \) and \( F^{-1}(1) = D_3 \). As before, \( F \) has generic fiber irreducible because \( \widetilde{D}_3 \) is connected and reduced.

To conclude, we proceed as in Section 4.2. The proof of Proposition 4.2 shows that the linear equivalence class of \( D_1 - D_2 \) is a non-trivial three-torsion point of \( \text{Pic}_0(T_{\xi_3}) \). Therefore, there exists a complex torus \( X \), an étale covering \( \rho : X \rightarrow T_{\xi_3} \) and a rational function \( G : X \rightarrow \mathbb{P}^1 \) with generic irreducible fiber fitting into the commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\rho} & T_{\xi_3} \\
\downarrow G & & \downarrow F \\
\mathbb{P}^1 & \xrightarrow{z \mapsto z^3} & \mathbb{P}^1 \\
\end{array}
\]

(14)

Notice that \( G \) has five linear fibers (three of them over \( F^{-1}(1) \)) and, as in Section 4.2, Theorem 4.1 implies that \( E_0 \) has maximal rank and therefore is exceptional.

4.5 Explicit abelian relations for elliptic exceptional CDQL webs

The results of the four preceding subsections give geometrical descriptions of the abelian relations of the webs \( E_5, E_6, \) and \( E_\tau \). Closed explicit forms for the abelian relations of these elliptic exceptional CDQL webs can be deduced from their proofs.
4.5.1 Explicit abelian relations for $E^\tau_5$

We recall the description of $A(E^\tau_5)$ that have been obtained in [41]. We fix $\tau \in \mathbb{H}$ and set $G = F^{1/2}_\tau$ (see (9)), $g_1 = x$, $g_2 = y$, $g_3 = x + y$, and $g_4 = x - y$. Then the following multiplicative abelian relations hold:

\[
G = \frac{\vartheta_1(g_1, \tau) \vartheta_1(g_2, \tau)}{\vartheta_4(g_1, \tau) \vartheta_4(g_2, \tau)},
\]

\[
1 - G = \frac{\vartheta_3(\frac{g_3}{2}, \frac{\tau}{2}) \vartheta_4(\frac{g_4}{2}, \frac{\tau}{2})}{\vartheta_4(g_1, \tau) \vartheta_4(g_2, \tau)},
\]

\[
1 + G = \frac{\vartheta_4(\frac{g_3}{2}, \frac{\tau}{2}) \vartheta_3(\frac{g_4}{2}, \frac{\tau}{2})}{\vartheta_4(g_1, \tau) \vartheta_4(g_2, \tau)}.
\]

4.5.2 Explicit abelian relations for $E_7$

We fix $\tau = 1 + i$ in this section and we note $H = F^{1/2}_\tau = F^{1/2}_{1+i}$. Let $g_1, \ldots, g_4$ be the same functions than above and set $g_5 = ix + y$, $g_6 = x + iy$. The relations (4.5.2) of the subweb $E^\tau_5$ are of course three abelian relations for $E_7$. To obtain the last two, we just substitute $ix$ to $x$ in (4.5.2) and use the transformation formulas for theta functions admitting complex multiplication (see Section §8 of [14, Chapter V] for instance) to get:

\[
1 - iH = \frac{\vartheta_3(\frac{g_5}{2}, \frac{\tau}{2}) \vartheta_4(\frac{g_6}{2}, \frac{\tau}{2})}{\vartheta_4(i g_1, \tau) \vartheta_4(g_2, \tau)},
\]

\[
1 + iH = \frac{\vartheta_4(\frac{g_5}{2}, \frac{\tau}{2}) \vartheta_3(\frac{g_6}{2}, \frac{\tau}{2})}{\vartheta_4(i g_1, \tau) \vartheta_4(g_2, \tau)}.
\]

4.5.3 Explicit abelian relations for $E_5$

To simplify the formulas, we shall abbreviate $\xi_3$ by $\xi$, will write $\vartheta_i(z) = \vartheta_i(z, \xi)$ ($i = 1, \ldots, 4$) and will set $q = e^{i\pi \xi}$ in this subsection. We will also use the notations introduced in the proof of Proposition 4.1.

Let $F$ be the rational function (11), that is $F = f_1/f_2$ with

\[
f_1(x, y) = \vartheta_1(x) \vartheta_1(y) \vartheta_1(x - y) \vartheta_1(x + \xi^2 y)
\]

and

\[
f_2(x, y) = \vartheta_2(x) \vartheta_3(y) \vartheta_4(x - y) \vartheta_3(x + \xi^2 y).
\]
Since $f_1(x + \frac{\xi}{2}, y + \frac{1}{2}) = i q^{-1/2} e^{i(x-x)^2} \partial_4(x) \partial_2(y) \partial_3(x-y) \partial_2(x+\xi^2) (y)$ (see [14, pages 63–64]), the linear divisor $D_3 = L_{p_3}(D_1)$ on $T_\xi$ is cut out by

$$f_3(x, y) = \partial_4(x) \partial_2(y) \partial_3(x-y) \partial_2(x+\xi^2) y.$$ 

One verifies that $f_3 \equiv a_3 f_1 + b_3 f_2$ where $a_3 = i \frac{\partial_2(0)}{\partial_3(0)}$ and $b_3 = \frac{\partial_2(0)}{\partial_3(0)}$. Consequently, $D_3$ is the linear fiber $F^{-1}(c_3)$ where $c_3 = -b_3/a_3 = i/\partial_4(0)$. According to (the proof of) Theorem 4.1, there is an associated logarithmic abelian relation. Explicitly, it is (in multiplicative form)

$$a_3 F + b_3 = \frac{\partial_4(x) \partial_2(y) \partial_3(x-y) \partial_2(x+\xi^2) y}{\partial_2(x) \partial_3(y) \partial_4(x-y) \partial_3(x+\xi^2) y}.$$

In the same way, one proves that the linear divisor $D_4 = L_{p_4}(D_1)$ is cut out by

$$f_4(x, y) = \partial_3(x) \partial_4(y) \partial_2(x-y) \partial_4(x+\xi^2) y.$$

One verifies that $f_4 \equiv a_4 f_1 + b_4 f_2$ where $a_4 = i \frac{\partial_3(0)}{\partial_4(0)}$ and $b_4 = \frac{\partial_3(0)}{\partial_4(0)}$. So $D_4 = F^{-1}(c_4)$ where $c_4 = i \partial_4(0)/\partial_2(0)$. The associated logarithmic abelian relation is

$$a_4 F + b_4 = \frac{\partial_3(x) \partial_4(y) \partial_2(x-y) \partial_4(x+\xi^2) y}{\partial_2(x) \partial_3(y) \partial_4(x-y) \partial_3(x+\xi^2) y}.$$

### 4.5.4 Explicit abelian relations for $\mathcal{E}_6$

We shall also abbreviate $\xi_3$ by $\xi$ in this subsection and use the notations introduced in the proof of Proposition 4.2. Let $\wp(z)$ be the Weierstrass $\wp$-function (12) associated to the lattice $\Gamma = \mathbb{Z} \oplus \mathbb{Z} \xi$. It satisfies the differential equation

$$\wp'(z)^2 = 4 \wp(z)^3 - \left( \frac{\Gamma(1/3)^3}{2\pi} \right)^6$$

and $(\wp) = (p_+ + (p_-) - 2(0)$ as divisors on the elliptic curve $E = \mathbb{C}/\Gamma$.

We want to make explicit the abelian relations of

$$\mathcal{E}_6 = [dx \, dy \, (dx + dy) \, (dx + \xi \, dy) \, (dx + \xi^2 \, dy)] \otimes [dx/\wp(x) + dy/\wp(y)].$$
defined on $E^2$. Let $f$ be the elliptic function defined by
\[ f(x) = \frac{\wp'(x) - \wp'(p_\perp)}{\wp'(x) - \wp'(p_-)} . \]

Using (16), one verifies by a straightforward computation that $F = f(x)f(y)$ is a first integral for the foliation $[dx/\wp(x) + dy/\wp(y)]$. We claim that this rational function corresponds exactly to the first integral deduced in the proof of Proposition 4.2 (also denoted by $F$ there). One verifies that $(f) = 3(p_\perp) - 3(p_-)$.

Recall (from [14, Chapter IV] for instance) the definition of the Weierstrass sigma function associated to a lattice $\Lambda \subset \mathbb{C}$:
\[ \sigma(z, \Lambda) = z \prod_{\lambda \in \Lambda \setminus \{0\}} \left(1 - \frac{z}{\lambda}\right) e^{\frac{z}{\lambda} + \frac{z^2}{2\lambda^2}}. \]

**Lemma 4.1.** Let $\sigma_1$ be the Weierstrass sigma function associated to the lattice
\[ \Gamma_1 = (2 + \xi) \Gamma = (2 + \xi) \mathbb{Z} \oplus (1 + 2 \xi) \mathbb{Z}. \]

If $E_1 = \mathbb{C}/\Gamma_1$ and
\[ g(x) = -\frac{\sigma_1(x - p_\perp)\sigma_1(x - \xi p_\perp)\sigma_1(x - \xi^2 p_\perp)}{\sigma_1(x - p_-)\sigma_1(x - \xi p_-)\sigma_1(x - \xi^2 p_-)} . \]

then the product $G = g(x)g(y)$ is a function that makes commutative the diagram (14). More precisely, let $X = E_1^2$ and set $\rho = (\mu, \mu) : X \to E^2$ where $\mu : E_1 \to E$ denotes the isogeny of degree three induced by the natural inclusion $\Gamma_1 \subset \Gamma$. Then

1. the functions $g$ and $G$ are rational functions on $E_1$ and $X$, respectively;
2. they satisfy $g^3 = f \circ \mu$ and $G^3 = F \circ \rho$ on $E_1$ and $X$, respectively. \(\square\)

**Proof.** Item (1) follows at once from formula (6). To establish item (2), one proceeds as usual by comparing the zeroes and the poles of $g^3$ and $f \circ \mu$ on $E_1$. \(\blacksquare\)

Using the function $G$, one can give closed explicit formulas for the non-elementary abelian relations of
\[ [dx dy (dx + \xi dy) (dx + \xi^2 dy)] \otimes [dG]. \]
The simplest is certainly (in multiplicative form) $G = g(x)g(y)$. If we set $g_3 = x + y$, $g_4 = x + \xi y$, and $g_5 = x + \xi^2 y$, then the other three are

\[
1 - G = \epsilon_0 \frac{\sigma_1(g_3)\sigma_1(g_4)\sigma_1(g_5)}{\prod_{\ell=0}^{2} \sigma_1(x - \xi^\ell p_-)\sigma_1(y - \xi^\ell p_-)},
\]

\[
1 - \xi G = \epsilon_1 \frac{\sigma_1(g_3 + \xi^2)\sigma_1(g_4 + \xi)\sigma_1(g_5 + 1)}{\prod_{\ell=0}^{2} \sigma_1(x - \xi^\ell p_-)\sigma_1(y - \xi^\ell p_-)},
\]

\[
1 - \xi^2 G = \epsilon_2 \frac{\sigma_1(g_3 - \xi^2)\sigma_1(g_4 - \xi)\sigma_1(g_5 - 1)}{\prod_{\ell=0}^{2} \sigma_1(x - \xi^\ell p_-)\sigma_1(y - \xi^\ell p_-)}.
\]

where $\epsilon_0$, $\epsilon_1$, and $\epsilon_2$ are complex constants. Notice that (18) and (19) can be obtained from (17) by using the relations $g(x + 1) = g(x + \xi) = \xi g(x)$.

**Remark 4.1.** Since $1 - F = (1 - G)(1 - \xi G)(1 - \xi^2 G)$, multiplying (17), (18), and (19) one gets a multiplicative abelian relation of $E_6$ involving $1 - F$. After several simplifications (left to the reader), we find the relation

\[
1 - F = -\wp'(p_+)\sigma(p_-)^6 \frac{\sigma(g_3)\sigma(g_4)\sigma(g_5)}{\prod_{\ell=0}^{2} \sigma(x - \xi^\ell p_-)\sigma(y - \xi^\ell p_-)},
\]

where $\sigma$ denotes the Weierstrass sigma function associated to the lattice $\Gamma$.

Since $\wp'(x) - \wp'(p_+) = \frac{2}{\sigma(p_-)^2} \frac{\sigma(x-p_+\sigma(x-\xi^\ell p_-)\sigma(x-\xi^2 p_-)}{\sigma(x)^3}$ on $E$, one has also

\[
1 - F = \wp'(p_+)\sigma(p_-)^6 \frac{\sigma(x)^3\sigma(y)^3 (\wp'(x) + \wp'(y))}{2 \prod_{\ell=0}^{2} \sigma(x - \xi^\ell p_-)\sigma(y - \xi^\ell p_-)}.
\]

Comparing (20) and (21) yields the relation

\[-\frac{1}{2} (\wp'(x) + \wp'(y)) = \frac{\sigma(x+y)\sigma(x+\xi y)\sigma(x+\xi^2 y)}{\sigma(x)^3\sigma(y)^3}.
\]

This is the recently discovered addition formula (6.6) of [20].
5 The Barycenter Transform

5.1 The \([v]\)-barycenter of a configuration

Let \(V\) be a two-dimensional vector space over \(\mathbb{C}\) equipped with a non-zero alternating 2-form \(\sigma \in \bigwedge^2 V^*\). For a fixed \(k \geq 1\) and \(v \in V\) distinct from zero, consider the map

\[
\alpha_v : V^k \rightarrow V \\
(v_1, \ldots, v_k) \mapsto \sum_{i=1}^k \left( \prod_{j \neq i} \sigma(v, v_j) \right) v_i.
\]  

These maps have the following properties:

1. \(\alpha'_{v} = \lambda^{k-1} \alpha_v\) if \(\sigma' = \lambda \sigma\) with \(\lambda \in \mathbb{C}^*\);
2. \(\alpha_{\lambda v} = \lambda^{k-1} \alpha_v\) for every \(\lambda \in \mathbb{C}^*\);
3. \(\alpha_v\) is symmetric;
4. \(\alpha_v(v_1, \ldots, v_k) = 0\) if and only if there exist \(i\) and \(j\) distinct such that \(v_i, v_j\) and \(v\) are multiples of each other or if one of the \(v_i\)'s is zero.

The projectivization of \(\alpha_v\) is a rational map \(\beta_{[v]} : \mathbb{P}(V)^k \rightarrow \mathbb{P}(V)\) that admits a nice geometric interpretation: if \([v_i] \neq [v]\) for every \(i \in \{1, \ldots, k\}\) then \(\beta_{[v]}([v_1], \ldots, [v_k])\) is nothing but the barycenter of \([v_1], \ldots, [v_k]\) seen as points of the affine line \(\mathbb{C} \simeq \mathbb{P}(V) \setminus \{[v]\}\). Unlike \(\alpha_v\), \(\beta_{[v]}\) does not depend on the choice of \(\sigma\). The point \(\beta_{[v]}([v_1], \ldots, [v_k])\) will be referred as the \([v]-\)barycenter of \([v_1], \ldots, [v_k]\).

The naturalness of \(\beta_{[v]}\) is testified by its \(\text{PSL}(V)\)-equivariance, that is, for every \(g \in \text{PSL}(V)\), \(\beta_{g[v]}(gv_1, \ldots, gv_k) = g\beta_v(v_1, \ldots, v_k)\).

5.2 Symmetric versions

Since \(\beta_{[v]}\) is a symmetric function, it factors through the natural map \(\mathbb{P}(V)^k \rightarrow \mathbb{P}(\text{Sym}^k V)\). Still denoting by \(\beta_{[v]}\) the resulting rational map from \(\mathbb{P}(\text{Sym}^k V)\) to \(\mathbb{P}(V)\), it has been observed in [21] (see also [19]) that \(\beta_{[v]}\) admits the affine expression

\[
\beta_x(p(t)) = x - k \frac{p(x)}{p'(x)}
\]

where \(x \in \mathbb{C}\) and the roots of the degree \(k\) polynomial \(p \in \mathbb{C}[t]\) correspond to \(k\) points in an affine chart \(\mathbb{C} = \mathbb{P}(V) \setminus \{\infty\}\).
There are also symmetrized versions of the above maps. Namely, we can define

\[ \alpha : \text{Sym}^k V \rightarrow \text{Sym}^k V \]
\[ v_1 \cdot v_2 \cdots v_k \mapsto \prod_{i=1}^{k} \alpha_{v_i}(v_1, \ldots, \hat{v_i}, \ldots, v_k). \]

Its projectivization

\[ \beta : \mathbb{P}(\text{Sym}^k V) \rightarrow \mathbb{P}(\text{Sym}^k V) \]

is a PSL(V)-equivariant rational map.

A concise affine expression for \( \beta \) is presented in [21]. If all the \( k \) points belong to the same affine chart \( \mathbb{C} \subset \mathbb{P}(V) \), then

\[ \beta \left( p(t) \right) = \text{Resultant}_z(p(z), (t - z)p'(z) + 2(k - 1)p'(z)) \]  

(24)

where \( p \in \mathbb{C}[t] \) is a degree \( k \) polynomial whose roots correspond to \( k \) points in \( \mathbb{C} \).

**Remark 5.1.** For \( k = 2 \), the rational map \( \beta : \mathbb{P}(\text{Sym}^k V) \rightarrow \mathbb{P}(\text{Sym}^k V) \) is nothing more than the identity map. For \( k = 3 \), it is still rather simple: it is a birational involution of \( \mathbb{P}^3 \) with indeterminacy locus equal to a cubic rational normal curve. For \( k = 4 \), it is already more interesting from the dynamical point of view. Recall that for four unordered points of \( \mathbb{P}^1 \) there is a unique invariant, the so-called \( j \) invariant. It can be interpreted as a rational map \( j : \mathbb{P}(\text{Sym}^4 V) \rightarrow \mathbb{P}^1 \) whose generic fiber contains an orbit of the natural PSL(V)-action of \( \mathbb{P}(\text{Sym}^4 V) \) as an open and dense subset. Therefore, there exists a rational map \( \beta_* : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \) that fits in the commutative diagram below:

\[ \begin{array}{c}
\mathbb{P}(\text{Sym}^4 V) \\
\downarrow j \\
\mathbb{P}^1 \\
\downarrow \beta_* \\
\mathbb{P}^1
\end{array} \]

\[ \begin{array}{c}
\mathbb{P}(\text{Sym}^4 V) \\
\downarrow j \\
\mathbb{P}^1 \\
\downarrow \beta_* \\
\mathbb{P}^1
\end{array} \]

We learned from David Marín that there exists a choice of coordinates in \( \mathbb{P}^1 \) where

\[ \beta_*(z) = \frac{z^2(z + 540)^3}{(5z - 216)^4}. \]
It can be immediately verified that \( \beta^* \) is a post-critically finite map. We do not know if a similar property holds for the map \( \beta \) when \( k \geq 5 \). For a more comprehensive discussion about the dynamic of \( \beta \), see [21].

5.3 A technical lemma

Let \( q_1, \ldots, q_k \in \mathbb{P}(V) \) be \( k \geq 3 \) pairwise distinct points. Then for \( i = 1, \ldots, k \), one sets

\[
\hat{q}_i = \beta_{q_i}(q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_k).
\]

**Lemma 5.1.** For every \( i = 1, \ldots, k \), one has

\[
\bigcap_{j \neq i} \{q_j, \hat{q}_j\} = \emptyset.
\]

**Proof.** One can assume that \( i = k \) and one sets \( q_k = \infty \). For \( j = 1, \ldots, k - 1 \), let \( t_j \) (resp. \( \hat{t}_j \)) be the value corresponding to \( q_i \) (resp. to \( \hat{q}_j \)) relative to a fixed affine coordinate \( t \) on \( C = \mathbb{P}(V) \setminus \{\infty\} \). If we set \( p(t) = \prod_{j=1}^{k-1}(t - t_j) \) and define the polynomial \( p_j(t) \) by the relation \( p(t) = p_j(t)(t - t_j) \), then \( p_j(t_j) = p'(t_j) \) and \( p_j'(t_j) = \frac{1}{2} p''(t_j) \) for every \( j = 1, \ldots, k - 1 \). Thus, from (23), one can deduce that

\[
\hat{t}_i = t_j + 2(1 - k) \frac{p'(t_j)}{p''(t_j)}.
\]

Assume that \( \bigcap_{j=1}^{k-1} \{q_j, \hat{q}_j\} \) is not empty. One distinguishes two cases:

1. there exists \( \tau \in C \) such that \( \tau = \hat{t}_1 = \ldots = \hat{t}_{k-1} \);
2. there exists \( l \) such that \( t_j = \hat{t}_l \) for all \( j \in \{1, \ldots, k - 1\} \setminus \{l\} \).

In case (1), one has

\[
(\tau - t) p''(t) + 2(k - 1) p'(t) = 0 \tag{25}
\]

for every \( t \in \{t_1, \ldots, t_{k-1}\} \). Hence the same holds true for every \( t \) since \( p'(t) \) is of degree \( k - 2 \). Considering the coefficient of \( t^{k-2} \) in (25), it follows that \(- (k - 1)(k - 2) + 2(k - 1)(k - 1) = 0 \). This contradicts the assumption since \( k > 2 \).
In case (2), one can assume that $t_i = 0$. Then there exists $\rho(t) \in \mathbb{C}[t]$ such that 
$p(t) = t\rho(t)$. Then for every $t$ such that $\rho(t) = 0$, one has $t(t\rho''(t) + 2\rho'(t)) - 2(k-1)(t\rho'(t) + \rho(t)) = 0$ that is

$$t\rho''(t) + 2(2-k)\rho'(t) = 0. \quad (26)$$

Since $\rho(t)$ is of degree $k - 2$, (26) holds for every $t$, that is again impossible. ■

Since $\mathbb{P}(\text{Sym}^k V)$ can be naturally identified with the set of degree $k$ effective divisors on $\mathbb{P}(V)$, it makes sense to talk about the support of an element in $\mathbb{P}(\text{Sym}^k V)$.

**Corollary 5.1.** If $Q \subset \mathbb{P}(V)$ has cardinality $k \geq 3$, then every point in the support of $\beta(Q)$ appear with multiplicity at most $k - 2$. □

### 5.4 Some special rational maps relative to the barycenter transform

We state here a result that will be crucial in Section 8.5 in order to classify the homogeneous flat CDQL webs on $\mathbb{P}^2$.

We consider pair $(f, Q)$ where $f : \mathbb{P}^1 \to \mathbb{P}^1$ is a non-constant rational map and $Q = \{q_1, \ldots, q_k\} \subset \mathbb{P}^1$ a finite set of cardinal $k \geq 3$. Two such pairs $(f_1, Q_1)$ and $(f_2, Q_2)$ are said to be *projectively equivalent* if there exists $g \in \text{Aut}(\mathbb{P}^1)$ such that

$$f_2 = g \circ f_1 \circ g^{-1} \quad \text{and} \quad Q_2 = g(Q_1).$$

We want to classify (up to projective equivalence) pairs $(f, Q)$ such that

$$\forall q \in Q, \quad f^{-1}(q) \subset \{q, \hat{q}\} \quad (27)$$

(where $\hat{q}$ stands for the barycenter of $Q \setminus \{q\}$ on $\mathbb{C} \simeq \mathbb{P}^1 \setminus \{q\}$ for every $q \in Q$).

By definition, the *action* of a pair $(f, Q)$ satisfying (27) with $f$ of topological degree $d$ is the $k$-uplet $(e_i)_{i=1}^k \in \{0, \ldots, d\}^k$ such that as divisors on $\mathbb{P}^1$, one has

$$f^{-1}(q_i) = e_i q_i + (d - e_i)\hat{q}_i \quad \text{for } i = 1, \ldots, k. \quad (28)$$
Let us consider the following subsets of $\mathbb{P}^1$:

\[ Q_3 = \left\{ [0 : 1], [0 : 1], [1 : -1] \right\} \]
\[ Q_4 = Q_3 \cup \left\{ e^{\frac{2\pi i}{3}} : 1 \right\} \]
\[ Q_5 = Q_3 \cup \left\{ [e^{\frac{2\pi i}{3}}] : 1 \right\} \]

and \[ Q(k) = \left\{ e^{\frac{2\pi i \ell}{k}} : 1 \right\} | \ell = 0, \ldots, k \], \quad k \geq 3.

**Theorem 5.1.** Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a non-constant map and $Q = \{q_1, \ldots, q_k\}$ a finite subset of $\mathbb{P}^1$ with $k \geq 3$ elements such that (27) holds. If $f \neq \text{Id}$, then $(f, Q)$ is projectively equivalent to one of the pairs appearing in Table 1. \hfill \Box

The proof of this result involves elementary but rather cumbersome considerations. We postpone it to the Appendix at the end of this paper.

6 The $F$-barycenter of a Web

If $V$ is a two-dimensional vector space over an arbitrary field $F$ of characteristic zero, then it is still possible to define the $[v]$-barycenter of an element $\mathbb{P}(\text{Sym}^k V)$. This can be inferred directly from Equation (24) in Section 5.1.

More explicitly, we can specialize (23) to the $F$-barycenter of a $k$-web $W$ when there are at our disposal global rational coordinates $x, y$ on $S$. Assume that $F = [dy - a \, dx]$ with $a \in \mathbb{C}(S)$. If $W$ is defined by an implicit differential equation $F(x, y, dy/dx) = 0$ where $F(x, y, p)$ is a polynomial of degree $k$ in $p$ with coefficients in $\mathbb{C}(S)$, then

\[ \beta_F(W) = \left[ dy - \left( a - k \frac{F(a)}{\partial p(a)} \right) dx \right]. \tag{29} \]

Note also that the PSL$(V)$-equivariance of the barycenter transform yields

\[ \beta_{\varphi^*}F(\varphi^*W) = \varphi^*(\beta_F(W)) \]

for any $\varphi \in \text{Diff}(S)$. Therefore, the $F$-barycenter of $W$ is a foliation geometrically attached to the pair $(F, W)$ and, as such, can be defined on an arbitrary surface by patching together over local coordinate charts the construction presented above.
Table 1  Normal forms for pairs \((f, Q)\) satisfying (27) with \(Q\) normalized such that \(q_1 = [0 : 1], q_2[1 : 0]\) and \(q_3 = [-1 : 1]\).

<table>
<thead>
<tr>
<th>(k)</th>
<th>(d)</th>
<th>action</th>
<th>normal form for (f(x : y))</th>
<th>set (Q)</th>
<th>label</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2</td>
<td>(f^{-1}(q_1) = q_1 + \hat{q}_i)</td>
<td>((x(2y + x) : -y(2x + y)))</td>
<td>(Q_3)</td>
<td>(a.1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(f^{-1}(q_1) = 2q_1)</td>
<td>((x^2 : -y^2))</td>
<td>(Q_3)</td>
<td>(a.2)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(f^{-1}(q_2) = 2q_2)</td>
<td>((x \pm y)^2 : -(2x + y)^2)</td>
<td>(Q_3)</td>
<td>(a.3)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(f^{-1}(q_3) = q_3 + \hat{q}_3)</td>
<td>((x(3y + x) : -y(3x + y)))</td>
<td>(Q_3)</td>
<td>(c.1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(f^{-1}(q_1) = q_1 + 3\hat{q}_i)</td>
<td>((3x(2y + x)^3 : -y(3x + y)^3))</td>
<td>(Q_3)</td>
<td>(c.2)</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>(f^{-1}(q_i) = q_i + 2\hat{q}_i)</td>
<td>((3x(x + y(1 - \xi^2_3))^2 : -y(3x + y(1 - \xi^2_3))^2))</td>
<td>(Q_4)</td>
<td>(b.1)</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>(f^{-1}(q_1) = 2\hat{q}_1)</td>
<td>((y^2 : -x^2))</td>
<td>(Q_5)</td>
<td>(a.4)</td>
</tr>
<tr>
<td>(k \geq 3)</td>
<td>1</td>
<td>(f^{-1}(q_1) = \hat{q}_i)</td>
<td>((-x : y))</td>
<td>(Q(k))</td>
<td>(d.1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(f^{-1}(q_1) = q_1)</td>
<td>((\frac{2-k}{k})x : y))</td>
<td>(Q(k-1) \cup {\infty})</td>
<td>(d.2)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(f^{-1}(q_j) = \hat{q}_j, \ j \geq 2)</td>
<td>((-x : y))</td>
<td>(Q(k-2) \cup {0, \infty})</td>
<td>(d.3)</td>
</tr>
</tbody>
</table>
Remark 6.1. In [33], Nakai defines the dual three-line configuration of a configuration $L = L_1 \cup L_2 \cup L_3$ of three concurrent lines in the plane: it is “the unique invariant three-line configuration distinct from $L$ invariant by the group generated by three involutions respecting the line $L_i$ and $L$.” The dual 3-web $W^*$ of a 3-web $W$ is then defined as the one obtained by integrating the dual three-line configuration of the tangent three-line fields of $W$.

It turns out that $W^*$ is nothing more than the barycenter transform of $W$ in our terminology. Since $\beta$ is an involution in the case of three points, it follows that $(W^*)^* = W$, a fact already noted by Nakai. Moreover, he also observed that $K(W) = K(\beta(W))$, see [33, Theorem 4.1]. In particular, a 3-web $W$ is flat if and only if $\beta(W)$ is flat [33, Corollary 4.1].

We have verified, with the help of a computer algebra system, that the identity $K(W) = K(\beta(W))$ also holds for 4-webs as soon as the four defining foliations of $\beta(W)$ are distinct. For 5-webs, the situation is different: the barycenter transforms of most algebraic 5-webs do not have zero curvature.

These blind constatations are crying for geometric interpretations. □

6.1 Barycenters of completely decomposable linear webs

Let $p_0, \ldots, p_k$ be $(k + 1)$ pairwise distinct points in $\mathbb{P}^2$. For any $i = 0, \ldots, k$, let $L_i$ denote the foliation of $\mathbb{P}^2$ tangent to the pencil of lines through $p_i$. In what follows, we give a description of the foliation $\beta_{\mathcal{F}}(W)$ when $\mathcal{F} = L_0$ and $W = L_1 \boxplus \cdots \boxplus L_k$.

In the simplest case, the points $p_0, \ldots, p_k$ are aligned. If one chooses an affine coordinate system where all the $p_i$’s belong to the line at infinity then the foliations $L_i$ are induced by constant 1-forms and so is the $\mathcal{F}$-barycenter of $W$. The corresponding foliation $\beta_{\mathcal{F}}(W)$ is tangent to the pencil of lines through the point $\beta_{\mathcal{F}}(p_0, p_1, \ldots, p_k)$ on the line at infinity.

If we think of $\mathbb{C}^2$ as the universal covering of a two-dimensional complex torus $T$ then, if $p_0, \ldots, p_k$ are at the line at infinity, the foliations $L_i$ are pullbacks of linear foliations on $T$ under the covering map. In this geometric picture, the line at infinity is identified with $\mathbb{P}H^0(T, \Omega^1_T)$ and the the linear foliations $L_i$ are defined by points $p_i$ in $\mathbb{P}H^0(T, \Omega^1_T)$. The $\mathcal{F}$-barycenter of $W$ is the linear foliation on $T$ determined by $\beta_{\mathcal{F}}(p_0, p_1, \ldots, p_k) \in \mathbb{P}H^0(T, \Omega^1_T)$.

In the next simplest case, $p_1, \ldots, p_k$ are on the same line while $p_0$ is not. In an affine coordinate system where $p_1, \ldots, p_k$ are on the line at infinity and $p_0$ is at the
origin, the $\mathcal{F}$-barycenter will be induced by the 1-form $\sum_{i=1}^{k} d \log L_i$, where $L_i$ is a linear polynomial vanishing on the line $\overline{p_0 p_i}$. In particular, the product

$$
\prod_{i=1}^{k} L_i
$$

is a first integral of the foliation $\beta_{\mathcal{F}}(\mathcal{W})$.

In order to describe the $\mathcal{F}$-barycenter of $\mathcal{W}$ without further restrictions on the points $p_0, \ldots, p_k$, let $\Pi : (S, E) \to (\mathbb{P}^2, p_0)$ be the blowup of $p_0$; $\pi : S \to \mathbb{P}^1$ be the fibration on $S$ induced by the lines through $p_0$; $\mathcal{G}$ be the foliation $\Pi^* \beta_{\mathcal{F}}(\mathcal{W})$; and $\ell_i$ be the strict transform of the line $\overline{p_0 p_i}$ under $\Pi$ for $i = 1, \ldots, k$.

**Lemma 6.1.** If the points $\{p_0, \ldots, p_k\}$ are not aligned, then the foliation $\mathcal{G}$ is a Riccati foliation with respect to $\pi$, that is, $\mathcal{G}$ has no tangencies with the generic fiber of $\pi$. Moreover, $\mathcal{G}$ has the following properties:

1. the exceptional divisor $E$ of $\Pi$ is $\mathcal{G}$ invariant;
2. the only fibers of $\pi$ that are $\mathcal{G}$ invariant are the lines $\ell_i$, for $i = 1, \ldots, k$;
3. the singular set of $\mathcal{G}$ is contained in the lines $\ell_i$, for $i = 1, \ldots, k$;
4. over each line $\ell_i$ there are two singularities of $\mathcal{G}$. One is a complex saddle at the intersection of $\ell_i$ with $E$, the other is a complex node at the $p_0$-barycenter of $\{p_1, \ldots, p_k\} \cap \ell_i$. Moreover, if $r_i$ is the cardinality of $\{p_1, \ldots, p_k\} \cap \ell_i$ then

![Diagram](image-url)

**Fig. 2.** The $\mathcal{L}_{p_0}$-barycenter of the linear web $\mathcal{L}_{p_1} \times \cdots \times \mathcal{L}_{p_k}$. 
the quotient of eigenvalues of the saddle (resp. node) over $\ell_i$ is $-r_i/k$ (resp. $r_i/k$);
(5) the monodromy of $G$ around $\ell_i$ is finite of order $k/\gcd(k, r_i)$;
(6) the only separatrices of $\beta_{\mathcal{F}}(\mathcal{W})$ through $p_0$ are the lines $p_0p_i$, $i = 1, \ldots, k$. □

**Proof.** Let $(x, y) \in \mathbb{C}^2 \subset \mathbb{P}^2$ be an affine coordinate system where $\mathcal{F} = \mathcal{L}_0 = [dx]$ (that is $p_0 = [0:1:0]$) and $\mathcal{L}_i = [(x - x_i)dy - (y - y_i)dx]$ (that is $p_i = [x_i : y_i : 1]$) for $i = 1, \ldots, k$. It is convenient to assume also that $y_i \neq 0$ for $i = 1, \ldots, k$.

By definition, $\beta_{\mathcal{F}}(\mathcal{W})$ is

$$\beta_{\mathcal{F}}(\mathcal{W}) = \left[ k \, dy - \left( \sum_{i=1}^{k} \frac{y - y_i}{x - x_i} \right) \, dx \right].$$

(31)

Since $\Pi : (S, E) \to (\mathbb{P}^2, p_0)$ is the blowup of a point at infinity, the coordinates $(x, y)$ still define affine coordinates on an affine chart of $S$. Notice that the fibration $\pi : S \to \mathbb{P}^1$ is nothing more than $\pi(x, y) = x$ in these new coordinates.

If we set $z = 1/y$, then $(x, z) \in \mathbb{C}^2$ is another affine chart of $S$. The intersection of the exceptional divisor $E = \pi^{-1}(p_0)$ with this chart is equal to $\{z = 0\}$. Notice that in the new coordinates $(x, z)$ we have

$$G = \left[ k \, dz + z \left( \sum_{i=1}^{k} \frac{1 - zy_i}{x - x_i} \right) \, dx \right].$$

(32)

It is clear from Equations (31) and (32) that: $G$ has no tangencies with the generic fiber of $\pi$, that is, $G$ is a Ricatti foliation; (1) the exceptional divisor $E$ is $G$ invariant; (2) the only $G$-invariant fibers of $\pi$ are the lines $\ell_i$; and (3) the singularities of $G$ are contained in the lines $\ell_i$.

To prove items (4) and (5) suppose, without loss of generality, that $\ell_1 \cap \{p_1, \ldots, p_k\} = \{p_1, \ldots, p_r\}$ and that $x_1 = 0$. In particular, $x_i \neq 0$ for $i > r_1$. Therefore, in the open set $U = \{(x, z) \in \mathbb{C}^2 \mid |x| \ll 1\}$ we can write

$$G = \left[ k \, x \, u(x) \, dz + \left( z \left( r - \sum_{i=1}^{r} z y_i \right) + z \, x \, v(x, z) \right) \, dx \right]$$
where \( u \) is a unity in \( \mathcal{O}_U \) that does not depend on \( z \) and \( v \in \mathcal{O}_U \) is a regular function. It follows that the singularities of \( G \) over \( \ell_1 = \{x_1 = 0\} \) are \((0, 0)\) and

\[
\left( 0, \frac{r}{\sum_{i=1}^{r_1} y_i} \right).
\]

Notice that this last point is the \( p_0 \)-barycenter of \( \{p_1, \ldots, p_{r_1}\} \) on \( \ell_1 \).

The local expression for \( G \) over \( U \) also shows that \( G \) is induced by a vector field \( X \) with linear part at \((0, 0)\) equal to

\[
kx \frac{\partial}{\partial x} - r_1 z \frac{\partial}{\partial z}.
\]

Clearly, the quotient of eigenvalues in the direction of \( \ell_1 \) is \(-r_1/k\). Since the points \( \{p_0, \ldots, p_k\} \) are not aligned \( r_1 < k \) and, consequently, \(-r_1/k \in \mathbb{Q} \setminus \mathbb{Z} \). Since \( \ell_1 \) has zero self-intersection, it follows from Camacho–Sad index Theorem that the quotient of eigenvalues (in the direction of the fiber of \( \pi \)) of the other singularity of \( G \) on \( \ell_1 \) is \( r_1/k \). Since this number is not an integer, it follows (see [9, page 52]) that this singularity is a complex node. Moreover, the monodromy around \( \ell_1 \) is analytically conjugated to \( z \mapsto \exp(2\pi i r_1/k)z \). This concludes the proof of (4) and (5).

Finally, to settle (6) notice that the singular points of \( G \) contained in \( E \) are complex saddles. A classical result by Briot and Bouquet says that these singularities admit exactly two separatrices. In our setup, one separatrix corresponds to \( E \) and the other corresponds to one of the lines \( \ell_i \). Thus, (6) follows and so does the lemma.

It is interesting to notice that the generic leaf of \( \beta_f(W) \) is transcendental in general. Indeed, the cases when there are more algebraic leaves than the obvious ones (the lines \( p_0 \overline{p_i} \)) are characterized by the following proposition.

**Proposition 6.1.** The foliation \( \beta_f(W) \) has an algebraic leaf distinct from the lines \( p_0 \overline{p_i} \) if and only if all its singularities distinct from \( p_0 \) are aligned. Moreover, in this case all its leaves are algebraic.

**Proof.** Since the Riccati foliation \( G \) leaves the exceptional divisor \( E \) invariant, it has affine monodromy. It follows from Lemma 6.1 item (5) that its monodromy group is generated by elements of finite order.

Suppose that \( G \) has an algebraic leaf \( L \) distinct from \( E \) and the lines \( \ell_i \). The existence of such leaf implies that the monodromy group \( G \subset \text{Aut}(\mathbb{P}^1) \) of \( G \) must have a periodic point corresponding to the intersections of \( L \) with a generic fiber of \( \pi \). Since \( G \)
already has a fixed point (thanks to the \( G \) invariance of \( E \)), it follows from Lemma 6.1 item (5) that \( G \) is conjugated to a finite subgroup of \( \mathbb{C}^* \subset \text{Aff}(\mathbb{C}) \subset \text{Aut}(\mathbb{P}^1) \). This is sufficient to show that \( G \) admits a holomorphic first integral defined on the complement of the \( G \)-invariant fibers of \( \pi \). Lemma 6.1 item (4) implies that \( G \) is conjugated to \([r_1y\,dx - k\,xdy]\) in a neighborhood of \( \ell_i \) and, consequently, the restriction of \( G \) to this neighborhood has a local meromorphic first integral. Putting all together, it follows that \( G \) has a global rational first integral.

Notice that \( G \) admits two distinguished leaves that correspond to the two fixed points of the monodromy. One of these is the exceptional divisor \( E \) and the other is an algebraic curve \( C \) invariant by \( G \) such that \( \pi|_C : C \to \mathbb{P}^1 \) is a one to one covering.

For every \( i = 1, \ldots, k \), the distinguished leaf \( C \) must intersect the line \( \ell_i \) at a singularity of \( G \) away from \( E \) (by Lemma 6.1 item (6)). In a neighborhood of these singularities, \( G \) has a meromorphic first integral of the form \( y^kx^{-r_i} \) where \( r_i \) is the cardinality of \( \{p_1, \ldots, p_k\} \cap \ell_i \) and the local coordinates \((x, y)\) are such that \( [\,dx\,] \) defines the reference fibration. The restriction of the projection \((x, y) \mapsto x\) to any local leaf not contained in \( \{xy = 0\} \) is a \( \frac{k}{\text{gcd}(k, r_i)} \) to 1 covering of \( \mathbb{D}^* \). Therefore, in these local coordinates around \( \ell_i \), the distinguished leaf \( C \) must be contained in \( \{y = 0\} \). Notice that the Camacho–Sad index of the leaf \( \{y = 0\} \) is \( \frac{r_i}{k} \). Summing over the lines \( \ell_i \), we obtain from the Camacho–Sad index Theorem that \( C^2 = 1 \). Since \( C \) does not intersect \( E \) (Lemma 6.1 item (6)), it follows that \( \Pi(C) \) has self-intersection one. Thus, \( \Pi(C) \) is a line containing all the singularities of \( \beta_F(W) \) different from \( p_0 \). The proposition follows.

**Corollary 6.1.** If the foliation \( \beta_F(W) \) has an irreducible algebraic leaf \( C \) distinct from the lines \( \overline{p_0p_i} \), then \( C \) is a line or

\[
\deg C = \frac{\sum_{i=1}^m r_i}{\text{gcd}(r_1, \ldots, r_m)} \quad (33)
\]

where \( \ell_1, \ldots, \ell_m = \bigcup_{i=1}^k \overline{p_0p_i} \) and \( r_i \) is the cardinality of \( \ell_i \cap \{p_1, \ldots, p_k\} \) for \( i = 1, \ldots, m \). In particular, the degree of \( C \) is bounded from below by \( m \).

**Proof.** It follows from Proposition 6.1 that the singularities of \( \beta_F(W) \) distinct from \( p_0 \) are all contained in an invariant line \( \ell \). We can assume that \( \ell \) is the line at infinity in an affine chart \((x, y) \in \mathbb{C}^2 \subset \mathbb{P}^2 \). We can also assume that \( p_0 = (0, 0) \) and, as a by product, that the \( m \) lines \( \overline{p_0p_i} \) are cut out by homogeneous linear polynomials \( L_1, \ldots, L_m \). It can easily be verified that the polynomial \( P = L_1^{r_1} \cdots L_m^{r_m} \) is a first integral
for $\beta F(W)$. Of course, if $s_i = r_i / \gcd(r_1, ..., r_m)$ then $(L_1^{s_1} \cdots L_m^{s_m})^{\gcd(r_1, ..., r_m)} = P$ and therefore $Q = L_1^{s_1} \cdots L_m^{s_m}$ is also a polynomial first integral for $\beta F(W)$.

To conclude, one has just to prove that the plane curve $C = \{Q - c = 0\} \subset \mathbb{P}^2$ is irreducible when $c \neq 0$. To this end, consider the blowup $\pi : X \to \mathbb{P}^2$ of the projective plane at the origin of $C^2 \subset \mathbb{P}^2$ and denote by $z_1, \ldots, z_m$ the points on the exceptional divisor $E = \pi^{-1}(0, 0) \simeq \mathbb{P}^1$ that correspond to the $\ell_i$’s. If $C'$ is the strict transform of $C$, then one has a finite covering $\mu : C' \mapsto E$ of degree $s = s_1 + \cdots + s_m$ that is ramified at the $z_i$’s. Clearly $C'$ thus $C$ is irreducible if and only if the monodromy of $\mu$ is transitive. It can be shown that this monodromy is isomorphic to the subgroup $\langle s_1, \ldots, s_m \rangle$ of $\mathbb{Z}/s\mathbb{Z}$. But $\gcd(s_1, ..., s_m) = 1$ so $\langle s_i \rangle = \mathbb{Z}/s\mathbb{Z}$ hence $C$ is irreducible. The corollary follows. ■

7 Curvature

To settle the notation, we recall the definition of curvature for a completely decomposable $(k + 1)$-web $W = F_0 \boxtimes F_1 \boxtimes \cdots \boxtimes F_{k}$. We start by considering 1-forms $\omega_i$ with isolated singularities such that $F_i = [\omega_i]$. For every triple $(r, s, t)$ with $0 \leq r < s < t \leq k$, we define

$$\eta_{rst} = \eta(F_r \boxtimes F_s \boxtimes F_t)$$

as the unique meromorphic 1-form such that

$$\begin{align*}
d(\delta_{st} \omega_r) &= \eta_{rst} \wedge \delta_{st} \omega_r \\
d(\delta_{tr} \omega_s) &= \eta_{rst} \wedge \delta_{tr} \omega_s \\
d(\delta_{rs} \omega_t) &= \eta_{rst} \wedge \delta_{rs} \omega_t
\end{align*}$$

where $\delta_{ij} = \sigma(\omega_i, \omega_j)$ and $\sigma$ is the alternating 2-form characterized by

$$\omega_i \wedge \omega_j = \sigma(\omega_i, \omega_j) \, dx \wedge dy.$$

Notice that the 1-forms $\omega_i$ are not uniquely defined but any two differ by an invertible function. Therefore, although dependent on the choice of the $\omega_i$’s, the 1-forms $\eta_{rst}$ are well defined modulo the addition of a closed holomorphic 1-form. The curvature of the web $W = F_0 \boxtimes F_1 \boxtimes \cdots \boxtimes F_{k}$ is thus defined by the formula

$$K(W) = K(F_0 \boxtimes F_1 \boxtimes \cdots \boxtimes F_{k}) = d\eta(W)$$
where

$$\eta(\mathcal{W}) = \eta(\mathcal{F}_0 \boxtimes \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_k) = \sum_{0 \leq r < s < t \leq k} \eta_{rst}.$$  

Clearly, $K(\mathcal{W})$ is a meromorphic 2-form intrinsically attached to $\mathcal{W}$. More precisely, for any invertible holomorphic map $\varphi$, one has

$$K(\varphi^*\mathcal{W}) = \varphi^*(K(\mathcal{W})).$$

We will say that a $k$-web $\mathcal{W}$ is flat if its curvature $K(\mathcal{W})$ vanishes identically. This extends to every $k \geq 3$ a classical terminology used for 3-webs.

7.1 On the regularity of the curvature

Our main motivation to introduce the $\mathcal{F}$-barycenter of a web $\mathcal{W}$ stems from an attempt to characterize the absence of poles of $K(\mathcal{W})$ at a generic point of an irreducible component of $\Delta(\mathcal{W})$.

In order to state our result in this direction, we introduce the following notation. If $\mathcal{F}$ is one of the defining foliations of a $(k+1)$-web $\mathcal{W}$, then we define the $k$-web $\mathcal{W}/\mathcal{F}$ by the relation

$$\mathcal{W} = (\mathcal{W}/\mathcal{F}) \boxtimes \mathcal{F}.$$  

Recall the usual definition of the tangency between two foliations: if $\mathcal{F}_1$ and $\mathcal{F}_2$ are two distinct holomorphic foliations then $\text{tang}(\mathcal{F}_1, \mathcal{F}_2)$ is the divisor locally defined by the vanishing of

$$\omega_1 \wedge \omega_2 = 0$$

where $\omega_i$ are holomorphic 1-forms with isolated zeros locally defining $\mathcal{F}_i$ for $i = 1, 2$.

**Theorem 7.1.** Let $\mathcal{F}$ be a foliation and $\mathcal{W} = \mathcal{F}_1 \boxtimes \mathcal{F}_2 \boxtimes \cdots \boxtimes \mathcal{F}_k$ be a completely decomposable $k$-web, $k \geq 2$, both defined on the same domain $U \subset \mathbb{C}^2$. Suppose that $C$ is an irreducible component of $\text{tang}(\mathcal{F}, \mathcal{F}_1)$ that is not contained in $\Delta(\mathcal{W})$. The curvature $K(\mathcal{F} \boxtimes \mathcal{W})$ is holomorphic over a generic point of $C$ if and only if the curve $C$ is invariant by $\mathcal{F}_1$ or by $\beta_{\mathcal{F}_1}(\mathcal{W}/\mathcal{F}_1)$.  \(\square\)
We will need the following lemma.

**Lemma 7.1.** If $C$ is an irreducible component of $\text{tang}(\mathcal{F}, \mathcal{F}_1)$ that is not contained in $\Delta(\mathcal{W})$, then $\eta(\mathcal{F} \boxtimes \mathcal{W})$ is holomorphic over the generic point of $C$ if and only if $C$ is $\beta_{\mathcal{F}_1}(\mathcal{W}/\mathcal{F}_1)$ invariant. \hfill \Box

**Proof.** From the hypothesis, we can choose a local coordinate system over a generic point of $C$ such that

\[
\mathcal{F} = \{\omega_0 = dx + b \, dy = 0\},
\]
\[
\mathcal{F}_1 = \{\omega_1 = dx = 0\}
\]
and \[
\mathcal{F}_i = \{\omega_i = a_i \, dx + dy = 0\} \quad \text{for} \quad i = 2, \ldots, k.
\]

A straightforward computation shows that for every $i$:

\[
\eta_{01i} = \left[ \frac{\partial b}{\partial x} - a_i \frac{\partial b}{\partial y} - b \left( a_i \frac{\partial b}{\partial x} + \frac{\partial a_i}{\partial y} \right) \right] dx - \left[ \frac{a_i b \frac{\partial b}{\partial y} + \frac{\partial a_i}{\partial y}}{a_i b - 1} \right] dy.
\]

Over a generic point of $C$, we have that $C$ coincides with the zero locus of $b$. Thus, $C$ is not contained in the polar set of $\sum_{i=2}^k \eta_{01i}$ if and only if the expression

\[
\sum_{i=2}^k \frac{\partial b}{\partial x} - a_i \frac{\partial b}{\partial y} \quad (a_i b - 1)
\]

is divisible by $b$. But

\[
b \text{ divides } \sum_{i=2}^k \frac{\partial b}{\partial x} - a_i \frac{\partial b}{\partial y} \quad \iff \quad b \text{ divides } \sum_{i=2}^k \left( \frac{\partial b}{\partial x} - a_i \frac{\partial b}{\partial y} \right) \cdot
\]

The right-hand side above is equivalent to

\[
b \text{ divides } \left( \sum_{i=2}^k a_i \right) dx + (k - 1) \, dy \wedge db.
\]
From the very definition of the barycenter (see Equation (22)) it follows that

$$\beta_{\mathcal{F}_1}(\mathcal{W}/\mathcal{F}_1) = \left[ \sum_{i=2}^{k} \left( \prod_{j=2}^{k} \delta_{1i} \right) \omega_i \right] = \left[ \left( \sum_{i=2}^{k} a_i \right) dx + (k - 1) dy \right].$$

Notice that the 1-form $$\left( \sum_{i=2}^{k} a_i \right) dx + (k - 1) dy$$ has no singularities. Thus, $$\sum_{i=2}^{k} \eta_{01i}$$ is holomorphic on $$C$$ if and only if $$C$$ is invariant by $$\beta_{\mathcal{F}_1}(\mathcal{W}/\mathcal{F}_1)$$.

Since $$C$$ is not contained in $$\Delta(\mathcal{F}_r \boxtimes \mathcal{F}_s \boxtimes \mathcal{F}_t)$$ for every set $$\{r, s, t\}$$ that does not contain $$\{0, 1\}$$, it follows that $$\eta_{rst}$$ is holomorphic on $$C$$. The lemma follows. $\blacksquare$

**Proof of Theorem 7.1.** In the notation of the proof of Lemma 7.1

$$d\omega_0 = \frac{1}{k-1} \left( \sum_{i=2}^{k} (\eta_{01i} - d\log \delta_{1i}) \right) \wedge \omega_0. \quad (34)$$

The definition of $$\eta(\mathcal{W})$$ laid down in the beginning of this section implies that

$$\sum_{i=2}^{k} \eta_{01i} = \eta(\mathcal{F} \boxtimes \mathcal{W}) - \eta(\mathcal{W}).$$

Because the curve $$C$$ is not contained in $$\Delta(\mathcal{W})$$, the 1-form $$\eta(\mathcal{W})$$ is holomorphic at the generic point of $$C$$.

Suppose first that $$K(\mathcal{F} \boxtimes \mathcal{W})$$ is holomorphic over the generic point of $$C$$. If $$C$$ is $$\mathcal{F}$$ invariant then there is nothing to prove. Thus, assume that $$C$$ is not $$\mathcal{F}$$ invariant. If $$p$$ is a generic point of $$C$$ and $$\alpha$$ is a holomorphic primitive of $$d\eta(\mathcal{F} \boxtimes \mathcal{W})$$ on a neighborhood of $$p$$ then

$$\eta(\mathcal{F} \boxtimes \mathcal{W}) - \alpha = \frac{df(b)}{b^n} + dg$$

where $$f$$ and $$g$$ are holomorphic functions on a neighborhood of $$p$$ and $$n$$ is a positive integer. Therefore, (34) implies

$$d\omega_0 = \frac{1}{k-1} \left( \frac{df(b)}{b^k} + \alpha' \right) \wedge \omega_0.$$
for some holomorphic 1-form $\alpha'$. Since $d\omega_0$ is holomorphic and, by assumption, $\{b = 0\}$
is not $\mathcal{F}$ invariant, the only possibility is that $f \equiv 0$. Therefore, $\eta(\mathcal{F} \boxtimes \mathcal{W})$
is holomorphic along $C$. It follows from Lemma 7.1 that $C$ is $\beta_{\mathcal{F}_1}(\mathcal{W}/\mathcal{F}_1)$ invariant.

Suppose now that $C$ is left invariant by $\mathcal{F}$ or $\beta_{\mathcal{F}_1}(\mathcal{W}/\mathcal{F}_1)$. In the latter case, the re-
sult follows from Lemma 7.1. In the former case we can assume, for a fixed $i \in \{2, \ldots, k\}$, that $C = \{x = 0\}$, $\omega_0 = dx + x^nu dy$, $\omega_1 = dx$ and $\omega_i = dy$ where $u$ does not vanish identi-
cally on $C$. A straightforward computation shows that

$$d\eta_{01i} = \frac{u^2}{u^2} \left( \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial u}{\partial y} \frac{\partial u}{\partial x} \right).$$

Thus, the 2-forms $d\eta_{01i}$ are holomorphic for every $i = 2, \ldots, k$. Because

$$K(\mathcal{F} \boxtimes \mathcal{W}) = \sum_{i=2}^{k} d\eta_{01i} + d\eta(\mathcal{W}),$$

and the right-hand side is a sum of holomorphic 2-forms, the curvature $K(\mathcal{W} \boxtimes \mathcal{F})$ is also holomorphic and the theorem follows.

\section*{7.2 Specialization to CDQL webs on complex tori}

Theorem 7.1 completely characterizes in geometric terms the flat CDQL webs on complex tori.

\textbf{Theorem 7.2.} Let $\mathcal{W} = \mathcal{L}_1 \boxtimes \cdots \boxtimes \mathcal{L}_k$ be a linear $k$-web and $\mathcal{F}$ be a non-linear foliation
on a two-dimensional complex torus $T$. If $k \geq 2$, then $K(\mathcal{W} \boxtimes \mathcal{F}) = 0$ if and only if any irreducible component of $\text{tang}(\mathcal{F}, \mathcal{L}_i)$ is invariant by $\mathcal{F}$ or by $\beta_{\mathcal{L}_i}(\mathcal{W}/\mathcal{L}_i)$ for every $i = 1, \ldots, k$.

\textbf{Proof.} Notice that the discriminant of $\mathcal{W}$ is empty. Therefore, the hypotheses of
Theorem 7.1 are all satisfied.

If every irreducible component of $\text{tang}(\mathcal{F}, \mathcal{L}_i)$ is invariant by $\mathcal{F}$ or $\beta_{\mathcal{L}_i}(\mathcal{W}/\mathcal{L}_i)$ for
every $i = 1, \ldots, k$, then Theorem 7.1 implies that $K(\mathcal{W})$ is a holomorphic 2-form. Since
every foliation on $T$ is induced by a global meromorphic 1-form, one can proceed as in
the beginning of this section to define a global meromorphic 1-form $\eta$ on $T$ such that $K(\mathcal{W}) = d\eta$. The result follows from the next proposition. ■

**Proposition 7.1.** Let $\omega$ be a meromorphic 1-form on a compact Kähler manifold $M$. If $d\omega$ is holomorphic, then $\omega$ is closed. □

**Proof.** We learned the following proof from Marco Brunella. Notice that although $\omega$ is not closed a priori, the holomorphicity of $\Omega = d\omega$ ensures that its residues along the irreducible components $Z_i$ of its polar set are well-defined complex numbers. If $S$ is a real subvariety of $M$ of real dimension two, then Stoke’s Theorem implies that

$$\int_{S} \omega = \sum_{i=1}^{m} \text{res}_{Z_i}(\omega) \cdot (S \cdot Z_i)$$

where $S \cdot Z_i$ stands for the topological intersection number of $S$ with $Z_i$. It follows that the class of $\Omega$, seen as a current, lies in $H^{1,1}(M, \mathbb{C})$.

On the other hand, $\Omega$ being a closed holomorphic 2-form, its class lies also in $H^{2,0}(M, \mathbb{C})$. But $H^{1,1}(M, \mathbb{C}) \cap H^{2,0}(M, \mathbb{C}) = 0$ since $M$ is Kähler. This implies that $\Omega$ is zero and consequently $\omega$ is closed. ■

Theorem 7.2 admits the following consequence.

**Corollary 7.1.** Let $\mathcal{W}$ be a linear $k$-web and $\mathcal{F}$ be a foliation both defined on the same complex torus $T$. Suppose that $\mathcal{W}$ decomposes as $\mathcal{W}_1 \boxtimes \mathcal{W}_2$ in such a way that $\mathcal{W}_1$ and $\mathcal{W}_2$ are not foliations. Suppose also that for every defining foliation $\mathcal{L}$ of $\mathcal{W}_i$, $i = 1, 2$, we have

$$\beta_{\mathcal{L}}(\mathcal{W}_i / \mathcal{L}) = \beta_{\mathcal{L}}(\mathcal{W} / \mathcal{L}) .$$

Then $K(\mathcal{W} \boxtimes \mathcal{F}) = 0$ if and only if $K(\mathcal{W}_i \boxtimes \mathcal{F}) = 0$ for $i = 1, 2$. □

**Example 7.1.** Consider the linear 4-web

$$\mathcal{W} = \left[ \begin{array}{c} dx \\ dy \end{array} \right] \boxtimes \left[ \begin{array}{c} (dx - dy)(dx + dy) \\ \mathcal{W}_1 \\ \mathcal{W}_2 \end{array} \right]$$
on a two-dimensional complex torus $T$. Notice that
\[
\beta_{[dx]}(W/[dx]) = [dy] = \beta_{[dx]}(W_1) \quad \text{and} \quad \beta_{[dy]}(W/[dy]) = [dx] = \beta_{[dy]}(W_1).
\]
Similarly, $\beta_{[dx\pm dy]}(W) = [dx \mp dy] = \beta_{[dx\pm dy]}(W_2)$.

In [41], germs of exceptional CDQL 5-webs on $(\mathbb{C}^2, 0)$ of the form
\[
[dxdy(dx - dy)(dx + dy)] \boxtimes F
\]
are classified under the additional assumption that $K([dxdy] \boxtimes F) = 0$. Mihăileanu’s criterion combined with the Corollary 7.1 above yields that the additional assumption is superfluous if $F$ is supposed to be globally defined on a complex torus $T$. Translating the classification of [41] to our setup, we obtain that every flat and global 5-web on complex tori of the form $[dxdy(dx - dy)(dx + dy)] \boxtimes F$ is isogenous to one of the 5-webs $E_\tau$ (with $\tau \in \mathbb{H}$) presented in the Introduction. In particular, the torus $T$ has to be isogenous to the square of an elliptic curve. □

7.3 Specialization to CDQL webs on the projective plane

It would be interesting to extend Theorem 7.1 in order to deal with more degenerated discriminants. We do not know how to do it in general. Nevertheless, under the assumption that $W$ is a product of linear foliations on the projective plane we have the following weaker result.

**Theorem 7.3.** Let $F$ be a foliation and $W = L_1 \boxtimes L_2 \boxtimes \cdots \boxtimes L_k$ be a totally decomposable linear $k$-web, $k \geq 2$, both globally defined on $\mathbb{P}^2$. Suppose that $C$ is an irreducible component of $\text{tang}(F, L_1)$. If $K(W \boxtimes F)$ is holomorphic over a generic point of $C$ then the curve $C$ is invariant by $L_1$ or by $\beta_{L_1}(W/L_1)$.

**Proof.** If $C$ is not contained in $\Delta(W)$, then the result follows from Theorem 7.1. Thus, assume that $C \subset \Delta(W)$. The tangency of two linear foliations on $\mathbb{P}^2$ is a line invariant by both and, therefore, $C$ must be a line invariant by at least two of the defining foliations of $W$.

If $C$ is $L_1$ invariant, then there is nothing to prove. Thus, assume that this is not the case. Because $C \subset \text{tang}(F, L_1)$, we are also assuming that $C$ is not $F$ invariant.

First remark that Theorem 7.1 implies that $K(F \boxtimes L_i \boxtimes L_j)$ is holomorphic over the generic point of $C$ for every choice of distinct $i, j \in \{2, \ldots, k\}$. Indeed, on the one hand,
if \( C \subset \text{tang}(\mathcal{F}, \mathcal{L}_i) \) then \( \mathcal{L}_i \) and \( \mathcal{L}_1 \) have to be tangent along \( C \). Thus, \( C \) is \( \mathcal{L}_1 \) invariant contrary to our assumptions. On the other hand, if \( C \subset \text{tang}(\mathcal{L}_i, \mathcal{L}_j) \) then \( C \) is invariant by both \( \mathcal{L}_i \) and \( \mathcal{L}_j \) and the triple \((\mathcal{F}, \mathcal{F}_1, \mathcal{W}) = (\mathcal{L}_i, \mathcal{L}_j, \mathcal{F} \boxtimes \mathcal{L}_j)\) satisfies the hypotheses of Theorem 7.1. Thus, \( K(\mathcal{F} \boxtimes \mathcal{L}_i \boxtimes \mathcal{L}_j) \) is indeed holomorphic over the generic point of \( C \).

Similarly, Theorem 7.1 implies that \( K(\mathcal{F} \boxtimes \mathcal{L}_1 \boxtimes \mathcal{L}_i) \) is holomorphic along \( C \) whenever \( C \) is \( \mathcal{L}_i \) invariant.

If we write \( \mathcal{W} = \mathcal{L}_1 \boxtimes \mathcal{W}_0 \boxtimes \mathcal{W}_1 \) with \( \mathcal{W}_1 \) being the product of foliations in \( \mathcal{W} \) leaving \( C \) invariant and \( \mathcal{W}_0 \) being the product of foliations in \( \mathcal{W}/\mathcal{L}_1 \) not leaving \( C \) invariant, then \( K(\mathcal{F} \boxtimes \mathcal{L}_1 \boxtimes \mathcal{W}_0) \) is holomorphic over the generic point of \( C \).

Because \( C \) is not contained in \( \Delta(\mathcal{L}_1 \boxtimes \mathcal{W}_0) \), Theorem 7.1 implies that \( C \) is \( \beta_{\mathcal{L}_1}(\mathcal{W}_0) \) invariant. From the definition of the \( \mathcal{L}_1 \)-barycenter, it follows that \( C \) is also invariant by \( \beta_{\mathcal{L}_1}(\mathcal{W}_0 \boxtimes \mathcal{W}_1) = \beta_{\mathcal{L}_1}(\mathcal{W}/\mathcal{L}_1) \).

Notice that in Theorem 7.3, unlike in Theorem 7.1, the invariance condition imposed on \( C \subset \text{tang}(\mathcal{F}, \mathcal{L}_1) \) is no longer a necessary and sufficient condition for the regularity of the curvature: it is just necessary. In fact, the converse to Theorem 7.3 does not hold in general. For instance, if \( \mathcal{F} = [ydx + dy], \mathcal{L}_1 = [dy], \) and \( \mathcal{L}_2 = [ydx - xdy] \), then the line \( L = \{ y = 0 \} \) is invariant by \( \mathcal{F}, \mathcal{L}_1, \) and \( \beta_{\mathcal{L}_1}(\mathcal{L}_2) = \mathcal{L}_2 \) but \( K(\mathcal{F} \boxtimes \mathcal{L}_1 \boxtimes \mathcal{L}_2) \) is not holomorphic over \( L \) since

\[
K(\mathcal{F} \boxtimes \mathcal{L}_1 \boxtimes \mathcal{L}_2) = \frac{dx \wedge dy}{y(x + 1)^2}.
\]

8 Constraints on Flat CDQL Webs

In this section, we start the classification of flat CDQL webs on the projective plane. As already mentioned in the Introduction the starting point is Mihăileanu criterion: If \( \mathcal{W} \) is a web of maximal rank then \( K(\mathcal{W}) = 0 \).

We will combine this criterion with Theorem 7.3 in order to restrict the possibilities for the pairs \((\mathcal{F}, \mathcal{P})\). For instance, Theorem 8.1 below shows that the degree of \( \mathcal{F} \) is bounded by four when \( \mathcal{F} \boxtimes \mathcal{W}(\mathcal{P}) \) is flat.

Here, as usual, the degree of a holomorphic foliation \( \mathcal{F} \) on \( \mathbb{P}^2 \) is the number of tangencies with a generic line \( \ell \subset \mathbb{P}^2 \). Concretely, in affine coordinates \((x, y) \in \mathbb{C}^2 \subset \mathbb{P}^2 \), a foliation \( \mathcal{F} \) has degree \( d \) if and only if \( \mathcal{F} \) is defined by a polynomial 1-form \( \omega \) with isolated zeros that can be written in the form

\[
\omega = a(x, y)dx + b(x, y)dy + h(x, y)(xdy - ydx)
\]
where \( h \) is a homogeneous polynomial of degree \( d \); \( a \) and \( b \) are polynomials of degree at most \( d \) and; when \( h \) is the zero polynomial, the polynomial \( xa + yb \) has degree exactly \( d + 1 \).

We point out that \( h \) vanishes identically if and only if the line at infinity is \( \mathcal{F} \) invariant. In this case, the zeros of the homogenous component of degree \( d + 1 \) of the polynomial \( xa + yb \) correspond to the singularities of \( \mathcal{F} \) on the line at infinity. If \( h \) is non-zero, then the points at infinity determined by \( h \) are in one to one correspondence with the tangencies of \( \mathcal{F} \) with the line at infinity.

8.1 Notations

The notations below will be used in the proof of the classification of flat CDQL webs on the projective plane.

- \( \mathcal{P} \): finite set of points in \( \mathbb{P}^2 \);
- \( k \): the cardinality of \( \mathcal{P} \);
- \( p_1, \ldots, p_k \): the points of \( \mathcal{P} \);
- \( \mathcal{P}_i \): \( \mathcal{P} \setminus \{p_i\} \);
- \( \mathcal{L}_i \): the linear foliation determined by \( p_i \);
- \( \mathcal{W}(\mathcal{P}) \): \( \mathcal{L}_1 \boxtimes \cdots \boxtimes \mathcal{L}_k \);
- \( \mathcal{W}(\mathcal{P}_i) \): \( \mathcal{W}(\mathcal{P})/\mathcal{L}_i = \mathcal{L}_1 \boxtimes \cdots \boxtimes \mathcal{L}_{i-1} \boxtimes \mathcal{L}_{i+1} \boxtimes \cdots \boxtimes \mathcal{L}_k \);
- \( \hat{\mathcal{L}}_i \): the \( \mathcal{L}_i \)-barycenter of \( \mathcal{W}(\mathcal{P}_i) \) that is, \( \beta_{\mathcal{L}_i}(\mathcal{W}(\mathcal{P}_i)) \);
- \( \mathcal{L}_p \): the pencil of lines through a point \( p \in \mathbb{P}^2 \);
- \( \hat{\mathcal{L}}_p \): in case \( p \in \mathcal{P} \), the \( \mathcal{L}_p \)-barycenter of \( \mathcal{W}(\mathcal{P} \setminus \{p\}) \);
- \( \ell \): a line on \( \mathbb{P}^2 \);
- \( \mathcal{P}_\ell \): \( \mathcal{P} \cap \ell \);
- \( k_\ell \): the cardinality of \( \mathcal{P}_\ell \);
- \( q_1, \ldots, q_{k_\ell} \): the points of \( \mathcal{P}_\ell \);
- \( \hat{q}_i \): the \( q_i \)-barycenter of \( \mathcal{P}_\ell \setminus \{q_i\} \) in \( \ell \).

A set \( \mathcal{P} \) is in \( p_{i\text{-barycentric general position}} \) if the only algebraic leaves of \( \hat{\mathcal{L}}_i \) are the lines \( \overline{p_ip_j} \) (compare with Proposition 6.1). We will write \( b(\mathcal{P}) \) for the cardinality of

\[
B(\mathcal{P}) = \{ p \in \mathcal{P} \mid \mathcal{P} \text{ is in } p_{i\text{-barycentric general position}} \}.
\]
8.2 Configurations of points in barycentric general position

As an immediate consequence of Theorem 7.1, it follows that a completely decomposable 3-web $W = \mathcal{F} \otimes L_1 \otimes L_2$ on $\mathbb{P}^2$ induced by two pencils of lines and a foliation has curvature zero if and only if it is projectively equivalent to a web of the form

$$[a(y)dx + b(x)dy] \otimes [dx] \otimes [dy]$$

where $a$ and $b$ are rational functions.

In the same vein, the next result combines Proposition 6.1 with Theorem 7.3 to show how generic configurations of points impose strong restrictions on a foliation $\mathcal{F}$ when $\mathcal{F} \otimes \mathcal{W}(\mathcal{P})$ has curvature zero.

**Proposition 8.1.** Let $\mathcal{W}(\mathcal{P})$ be the linear web associated to a collection $\mathcal{P}$ of $k \geq 2$ distinct points in $\mathbb{P}^2$. If $\mathcal{F}$ is a non-linear foliation on $\mathbb{P}^2$ such that $K(\mathcal{W}(\mathcal{P}) \otimes \mathcal{F}) = 0$, then $b(\mathcal{P})$ is at most 4. Moreover, there exist affine coordinates $x, y$ such that

(a) if $b(\mathcal{P}) = 1$, then $\mathcal{F} = [a(y)dx + b(x, y)dy]$ for some $a \in \mathbb{C}[y], b \in \mathbb{C}[x, y]$;
(b) if $b(\mathcal{P}) = 2$, then $\mathcal{F} = [a(y)dx + b(x)dy]$ for some $a, b \in \mathbb{C}[t]$;
(c) if $b(\mathcal{P}) = 3$, then the points in $B(\mathcal{P})$ are not aligned and

$$\mathcal{F} = [y(y^{d-1} - \epsilon_1)dx - x(x^{d-1} - \epsilon_2)dy]$$

for some integer $d \geq 2$ and $\epsilon_1, \epsilon_2 \in \{0, 1\}$ or

$$\mathcal{F} = [ydx - \lambda xdy]$$

for some constant $\lambda \in \mathbb{C} \setminus \{0, 1\}$;
(d) if $b(\mathcal{P}) = 4$, then the points in $B(\mathcal{P})$ are in general position and $\mathcal{F}$ is the pencil of conics through them. □

**Proof.** Suppose that $\mathcal{P}$ is in $p_1$-barycentric general position and assume that $p_1 = [0 : 1 : 0]$. If $K(\mathcal{F} \otimes \mathcal{W}(\mathcal{P})) = 0$, then Theorem 7.3 implies that the tangency between $\mathcal{F}$ and $L_1$ is a union of lines through $p_1$. In the affine coordinates $(x, y) = [x : y : 1], L_1 = [dy]$ and the lines through $p_1$ correspond to vertical lines. Therefore, $\mathcal{F} = [a(y)dx + b(x, y)dy]$ for some polynomials $a, b$. 
If \( P \) is also in \( p_2 \)-barycentric general position and \( p_2 = [1 : 0 : 0] \), the same argument shows that \( \mathcal{F} = [a(y)dx + b(x)dy] \) for some polynomials \( a, b \).

Notice that no point \( p \in B(\mathcal{P}) \setminus \{ p_1, p_2 \} \) can be aligned with \( p_1 \) and \( p_2 \). Indeed, suppose the contrary. One can assume that \( p = p_3 = [1 : 1 : 0] \), or equivalently \( L_3 = [dx - dy] \). Then the tangency of \( \mathcal{F} \) and \( L_3 \) is given by vanishing of

\[
(dx - dy) \land (a(y)dx + b(x)dy) = (b(x) + a(y))dx \land dy.
\]

Because \( K(\mathcal{F} \otimes \mathcal{W}(\mathcal{P})) = 0 \), the curve \( \{ b(x) + a(y) = 0 \} \) must be a union of lines passing through \( p_3 \). Explicitly, up to a multiplicative constant, one has

\[
b(x) + a(y) = \prod_{j=1}^{m} (x - y - c_j)
\]

for suitable constants \( c_1, \ldots, c_m \). Such identity is possible if and only if the homogeneous component of higher order of \( a(y)dx + b(x)dy \) is a constant multiple of \( xdy - ydx \). Therefore, \( \mathcal{F} \) has degree zero and consequently is a linear foliation. This contradicts our assumptions on \( \mathcal{F} \).

Suppose now that \( P \) is also in \( p_2 \)-barycentric general position with \( p_3 \notin \overline{PP_2} \). It is harmless to assume that \( p_3 = [0 : 0 : 1] \). Since the tangency of \( \mathcal{F} \) and \( L_3 \) is a union of lines through \( p_3 = (0, 0) \in \mathbb{C}^2 \), then the polynomial \( xa(y) + yb(x) \) must be homogeneous. Thus, for a certain \( d \in \mathbb{N}^* \) and suitable \( c_0, c_1, c_2 \in \mathbb{C} \)

\[
(a(y), b(x)) = (c_1 y^d + c_0 y, c_2 x^d - c_0 x).
\]

It is a simple matter to show that we are in one of the two cases displayed in part (c) of the statement, the first when \( d \geq 2 \) and the second when \( d = 1 \).

Finally, suppose that \( b(\mathcal{P}) \geq 4 \). Since no three points in \( B(\mathcal{P}) \) are aligned, we can assume that \( p_1, p_2, \) and \( p_3 \) are as above and \( p_4 = [1 : 1 : 1] \). Applying again the above argument to \( L_4 \) and discarding the solutions corresponding to degree zero foliations, we prove that

\[
\mathcal{F} = [a(y)dx + b(x)dy] = \left[ y(y - 1) \, dx - x(x - 1) \, dy \right].
\]

Notice that the rational function \( \frac{x(y-1)}{y(x-1)} \) is a first integral of \( \mathcal{F} \), that is, \( \mathcal{F} \) is a pencil of conics through the four points \( p_1, \ldots, p_4 \). Notice also that \( \mathcal{F} \) leaves invariant exactly six lines: the line at infinity and the five affine lines cut out by the polynomial \( xy(x - 1)(y - \ldots \)
1) $(x - y)$. If $\text{tang}(L_p, \mathcal{F})$ is a union of lines through $p$, then $p$ must belong to three of the six $\mathcal{F}$-invariant lines. Since there are only four such points ($p_1$, $p_2$, $p_3$, and $p_4$), $b(P)$ has at most four elements. This concludes the proof.

**Corollary 8.1.** Assume that the cardinality of $P$ is at least four. If it exists, a non-linear foliation $\mathcal{F}$ such that $K(W(P) \boxtimes \mathcal{F}) = 0$ then one of the following two situations occurs:

1. there are three aligned points in $P$;
2. $P$ is the union of four points in general position and $\mathcal{F}$ is the pencil of conics through them.

**Proof.** Assume that we are not in case (1): any line contains at most two points of $P$. Lemma 6.1 item (4) implies that the singularities of $\hat{\mathcal{L}}_p$ coincide with $P \setminus \{p\}$ for any $p \in P$. By assumption, the set of points $P \setminus \{p\}$ is not aligned and, according to Proposition 6.1, $P$ is in $p$-barycentric general position. Thus, $P = B(P)$ and Proposition 8.1 implies the result.

8.3 Aligned points versus invariant lines

Non-generic configurations of points also impose non-trivial conditions on non-linear foliations $\mathcal{F}$ such that the curvature of $\mathcal{F} \boxtimes W(P)$ vanishes identically.

**Proposition 8.2.** Let $P \subset \mathbb{P}^2$ be a set of $k$ points and $\mathcal{F}$ be a non-linear foliation on $\mathbb{P}^2$ such that $K(W(P) \boxtimes \mathcal{F}) = 0$. If $\ell$ is a line that contains at least three points of $P$ then $\ell$ is $\mathcal{F}$ invariant.

**Proof.** Remind that $k_\ell = \text{Card}(P_\ell)$ with $P_\ell = P \cap \ell = \{q_1, \ldots, q_{k_\ell}\}$. By hypothesis, $k_\ell \geq 3$. If $\ell$ is not invariant by $\mathcal{F}$, then

\[ |\text{tang}(\mathcal{F}, \ell)| \subset |\text{tang}(\mathcal{F}, L_i)| \cap \ell \] (35)

for every $i = 1, \ldots, k_\ell$, since $\ell$ is invariant by $L_i = L_{q_i}$.

Notice that for every $i$ ranging from 1 to $k_\ell$, the Riccati foliation $\hat{L}_i$ leaves $\ell$ invariant and its singularities on $\ell$ are $q_i$ and $\hat{q}_i$ according to Lemma 6.1 items (2) and (4).

Theorem 7.3 implies that each irreducible component of $\text{tang}(\mathcal{F}, L_i)$ is invariant by $L_i$ or $\hat{L}_i$. Since the leaves of $L_i$ are lines through $q_i$ and because the algebraic curves...
invariant by $\hat{\mathcal{L}}_i$ must intersect $\ell$ on $\text{sing}(\hat{\mathcal{L}}_i) \cap \ell = \{q_i, \hat{q}_i\}$ (according to Lemma 6.1), it follows from (35) that

$$|\text{tang}(\mathcal{F}, \ell)| \subset \bigcap_{i=1}^{k_\ell} \{q_i, \hat{q}_i\}.$$ 

By Lemma 5.1, this implies that $\text{tang}(\mathcal{F}, \ell) = \emptyset$ or in more geometric terms, that $\mathcal{F}$ is everywhere transverse to $\ell$. Therefore, $\mathcal{F}$ is of degree zero, which is not the case according to our hypothesis. ■

**Proposition 8.3.** Let $\mathcal{F}$ be a non-linear foliation on $\mathbb{P}^2$. Assume that $\ell$ is a line that contains at least three points of a set $\mathcal{P}$ of $k$ points in $\mathbb{P}^2$. If $\mathcal{W}(\mathcal{P}) \times \mathcal{F}$ has curvature zero, then the rational map $F : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ induced by the linear system $\{\text{tang}(\mathcal{F}, \mathcal{L}_p) - \ell\}_{p \in \ell}$ does not contract $\ell$.

**Proof.** First of all, a rephrasing of Proposition 8.2 yields that $\ell$ is a fixed component of the pencil $\{\text{tang}(\mathcal{F}, \mathcal{L}_p)\}_{p \in \ell}$. Thus, $\{\text{tang}(\mathcal{F}, \mathcal{L}_p) - \ell\}_{p \in \ell}$ is indeed a linear system.

Concretely, working with affine coordinates $(x, y)$ such that $\ell$ is the line at infinity and $\mathcal{F}$ is induced by a polynomial 1-form $\omega = a(x, y) \, dx + b(x, y) \, dy$ with isolated zeros then $F(x : y : z) = (B(x, y, z) : -A(x, y, z))$, where $A$ and $B$ are homogenizations of $a$ and $b$ of degree $\max\{\deg(a), \deg(b)\} = \deg(F)$.

Assume that $F$ contracts $\ell$. It means that there exists a point $p \in \ell$ such that $2 \ell \leq \text{tang}(\mathcal{F}, \mathcal{L}_p)$. In other words, the polynomials $A(x, y, 0)$ and $B(x, y, 0)$ are linearly dependent over $\mathbb{C}$. Therefore,

$$|\text{tang}(\mathcal{F}, \mathcal{L}_q) - \ell| \cap \ell = |\text{tang}(\mathcal{F}, \mathcal{L}_{q'}) - \ell| \cap \ell \subset \ell$$

for every $q, q' \in \ell \setminus \{p\}$.

For any $i \in \{1, \ldots, k_\ell\}$, if $C$ denotes an irreducible component of $\text{tang}(\mathcal{F}, \mathcal{L}_i)$ distinct from $\ell$, then Theorem 7.3 implies that $C$ necessarily is $\mathcal{L}_i$ invariant or $\hat{\mathcal{L}}_i$ invariant. Therefore, arguing as in the proof of Proposition 8.2, it follows from (36) that for every $q \in \ell \setminus \{p\}$, in particular for every $q_i \in \mathcal{P}_\ell \setminus \{p\}$, one has

$$|\text{tang}(\mathcal{F}, \mathcal{L}_q) - \ell| \cap \ell \subset \bigcap_{i=1}^{k_\ell} \{q_i, \hat{q}_i\}.$$
But the right-hand side intersection in this formula is empty (according to Lemma 5.1). This implies that \(\text{tang}(\mathcal{F}, \mathcal{L}_q)\) is of degree one hence that \(\mathcal{F}\) has degree zero, which contradicts the assumption that it is a non-linear foliation.

\[\square\]

8.4 A bound for the degree of \(\mathcal{F}\)

Combining Propositions 8.2 and 8.3 with the Riemann–Hurwitz formula, we are able to bound the degree of \(\mathcal{F}\).

**Theorem 8.1.** Let \(\mathcal{P} \subset \mathbb{P}^2\) be a set of \(k \geq 4\) points and \(\mathcal{F}\) be a non-linear foliation on \(\mathbb{P}^2\). If \(K(W(\mathcal{P}) \otimes \mathcal{F}) = 0\), then \(\text{deg}(\mathcal{F}) \leq 4\). Moreover, if \(\text{deg}(\mathcal{F}) \geq 2\), and \(\ell\) is a line containing \(k_\ell\) points of \(\mathcal{P}\) then \(k_\ell \leq 7 - \text{deg}(\mathcal{F})\).

**Proof.** Assume that there is no line that contains at least three points of \(\mathcal{P}\). Then Corollary 8.1 implies that \(\mathcal{P}\) has cardinality four and that \(\mathcal{F}\) is the degree two foliation tangent to the pencil of conics through \(\mathcal{P}\).

From now on, we assume that there exists a line \(\ell\) containing \(k_\ell\) points of \(\mathcal{P}\), with \(k_\ell \geq 3\). Identifying \(\ell\) with \(\mathbb{P}^1\), let us denote by \(f : \mathbb{P}^1 \to \mathbb{P}^1\) the restriction to \(\ell\) of the rational map \(F : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1\) induced by the linear system \(\{\text{tang}(\mathcal{F}, \mathcal{L}_p) - \ell \mid p \in \ell\}\). Proposition 8.3 ensures that \(f\) is a non-constant map.

The map \(f\) is characterized by the following equalities between divisors on \(\ell\)

\[f^{-1}(p) = (\text{tang}(\mathcal{F}, \mathcal{L}_p) - \ell)|_\ell,
\]

with \(p \in \ell\) arbitrary.

Let \(d\) be the degree of \(\mathcal{F}\). Recall from the proof of Proposition 8.3 that \(f\) is defined by degree \(d\) polynomials, that is, \(\text{deg}(f) = d\). Theorem 7.3 implies

\[f^{-1}(q_i) = e_i q_i + (d - e_i) \hat{q}_i\]

for any \(i = 1, \ldots, k_\ell\), where \(e_i\) is an integer satisfying \(0 \leq e_i \leq d\). The contribution of each of these fibers in the Riemann–Hurwitz formula is at least \(d - 2\) therefore

\[\chi(\mathbb{P}^1) = d\chi(\mathbb{P}^1) - (d - 2) k_\ell - r\]
for some non-negative integer \( r \). If \( d > 2 \), then

\[ k_\ell \leq \frac{2d - 2}{d - 2}. \]

If we keep in mind that \( k_\ell \geq 3 \) and \( d \geq 1 \), then we end up with the following possibilities

\[ d = 4 \text{ and } k_\ell = 3, \quad \text{or} \quad d = 3 \text{ and } k_\ell \leq 4, \quad \text{or} \quad 1 \leq d \leq 2 \text{ and } k_\ell \geq 3. \]

If one realizes that for \( d = 2 \) the map \( f \) will have at most three fixed points and two totally ramified points then one sees that in this case \( k_\ell \leq 5 \).

The map \( f : \mathbb{P}^1 \to \mathbb{P}^1 \) used in the proof of Theorem 8.1 codifies a lot of information about the foliation \( \mathcal{F} \). From now on, we will refer to \( f \) as the \( \ell \)-polar map of \( \mathcal{F} \).

### 8.5 The polar map: properties and normal forms

We use here the same notations than in the preceding section and keep the hypotheses of Theorem 8.1.

We first state two properties of the polar map that will be used in the sequel.

**Lemma 8.1.** If the line \( \ell \) is \( \mathcal{F} \) invariant, then the singularities of \( \mathcal{F} \) on \( \ell \) correspond to the fixed points of \( f \).

**Proof.** Let \((x, y) \in \mathbb{C}^2 \subset \mathbb{P}^2\) be affine coordinates and assume that \( \ell \) is the line at infinity. The foliation \( \mathcal{F} \) is induced by a polynomial 1-form \( \omega = a(x, y) \, dx + b(x, y) \, dy \) where \( a(x, y) \) and \( b(x, y) \) are relatively prime polynomials of degree \( d \). If \( a_d(x, y) \) and \( b_d(x, y) \) are the homogeneous components of degree \( d \) of \( a(x, y) \) and \( b(x, y) \) (respectively) then, in the homogeneous coordinates \((x : y : 0) \in \ell\), the polar map \( f \) is

\[ f(x : y) = [b_d(x, y) : -a_d(x, y)]. \]

On the other hand, one has

\[ \text{sing}(\mathcal{F}) \cap \ell = \{ [x : y : 0] \in \mathbb{P}^2 \mid x a_d(x, y) + y b_d(x, y) = 0 \}. \]

Thus, \([x : y : 0] \in \ell\) is a fixed point of \( f \) if and only if it belongs to \( \text{sing}(\mathcal{F}) \).
For $i = 1, \ldots, k_{q_i}$, let $e_{q_i}$ be the non-negative integer appearing in (37).

**Lemma 8.2.** There are exactly $e_{q_i} + 1$ lines invariant by $F$ through $q_i$ counted with the multiplicities that appear in $\text{tang}(F, L_{q_i})$. □

**Proof.** Let $C$ be an irreducible component of $\text{tang}(F, L_{q_i})$ passing through $q_i$. According to Theorem 7.3, $C$ is $L_{q_i}$ invariant or $\widehat{L}_{q_i}$ invariant. Since the only algebraic leaves of $\widehat{L}_{q_i}$ through $q_i$ are lines (see Lemma 6.1 item (6)), $C$ must be a line. This fact together with (37) proves the lemma.

It turns out that relations (37) determine $f$ and $P_\ell$ up to automorphisms of $\mathbb{P}^1$. From Theorem 5.1 stated in Section 5.4 (and proved in the Appendix), one deduces the following corollary (where we still use the notations used above):

**Corollary 8.2.** The pair $(f, P_\ell)$ associated to the flat CDQL web $\mathcal{W}(P) \boxtimes F$ on $\mathbb{P}^2$ can be assumed to be one of the pairs appearing in Table 1 of Section 5.4. □

8.6 Points of $P$ versus singularities of $F$

We start with a simple observation.

**Lemma 8.3.** Let $P$ be a finite collection of points of $\mathbb{P}^2$. If $F$ is a non-linear foliation on $\mathbb{P}^2$ such that $K(\mathcal{W}(P) \boxtimes F) = 0$, then each point of $P$ is contained in an $F$-invariant line. □

**Proof.** The argument used to settle Lemma 8.2 implies that every irreducible component of $\text{tang}(F, L_p)$ containing $p \in P$ must be an $F$-invariant line. ■

Table 1 allows us to restrain the possibilities of $F$ when $K(\mathcal{W}(P) \boxtimes F) = 0$. The next result shows that once $F$ is known there are not many possibilities for $P$.

**Proposition 8.4.** Let $P$ be a finite set of points of $\mathbb{P}^2$. Suppose there exists a line $\ell$ containing at least three points of $P$. If $F$ is a non-linear foliation on $\mathbb{P}^2$ such that the curvature of $\mathcal{W}(P) \boxtimes F$ vanishes identically then $P \setminus \ell \subset \text{sing}(F)$. □
Proof. Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be the $\ell$-polar map of $\mathcal{F}$. Recall that $\mathcal{P}_\ell = \mathcal{P} \cap \ell = \{q_1, \ldots, q_k\}$ where the $q_i$’s are pairwise distinct.

For any distinct $i, j \in \{1, \ldots, k\}$,

$$\text{sing}(\mathcal{F}) \cap (\mathbb{P}^2 \setminus \ell) = |\text{tang}(\mathcal{F}, \mathcal{L}_{q_i})| \cap |\text{tang}(\mathcal{F}, \mathcal{L}_{q_j})| \cap (\mathbb{P}^2 \setminus \ell).$$

(38)

Let $p$ be a point in $\mathcal{P} \setminus \ell$. Assume that $p \notin \text{sing}(\mathcal{F})$. After an eventual reordering, (38) implies that $p$ does not belong to tang$(\mathcal{F}, \mathcal{L}_{q_1})$ nor to tang$(\mathcal{F}, \mathcal{L}_{q_2})$.

Since $p \notin \text{tang}(\mathcal{F}, \mathcal{L}_{q_1})$, the line $\overline{pq_1}$ is not $\mathcal{F}$ invariant. Thus, Proposition 8.2 implies that $\mathcal{P} \cap \overline{pq_1} = \{p, q_1\}$. Consequently, $p \in \text{sing}(\mathcal{L}_{q_1})$ thanks to Lemma 6.1 item (4).

Let $C$ be an irreducible component of tang$(\mathcal{F}, \mathcal{L}_{q_1})$. If $C$ is not $\mathcal{L}_{q_1}$ invariant then it must be $\overline{L_{q_1}}$ invariant by Theorem 7.3 and cannot contain $q_1$ by Lemma 6.1 item (6). Thus, $C$ must intersect the $\overline{L_{q_1}}$-invariant line $\overline{pq_1}$ at $p$. Since $p \notin \text{tang}(\mathcal{F}, \mathcal{L}_{q_1})$, we deduce that every irreducible component of tang$(\mathcal{F}, \mathcal{L}_{q_1})$ is $\mathcal{L}_{q_1}$ invariant. Lemma 8.2 implies that $f^{-1}(q_1) = \text{deg}(\mathcal{F}) q_1$. Similarly, $\mathcal{P} \cap \overline{pq_2} = \{p, q_2\}$ and $f^{-1}(q_2) = \text{deg}(\mathcal{F}) q_2$.

Every rational self-map of $\mathbb{P}^1$ has at most two totally ramified points (or at most two fixed points when the degree is one and the map is not the identity). Consequently, $p$ must belong to tang$(\mathcal{F}, \mathcal{L}_{q_i})$ for every $i \in \{3, \ldots, k\}$. The only possibility is that $k_\ell = 3$ (otherwise $p$ would be in sing$(\mathcal{F})$ according to (38)).

Lemma 8.3 implies that there is a $\mathcal{F}$-invariant line $\ell_p$ through $p$. Since $\overline{pq_1}$ and $\overline{pq_2}$ are not $\mathcal{F}$ invariant, the line $\ell_p$ must be distinct from these. In particular, $\ell_p \cap \ell$ must be contained in (sing$(\mathcal{F}) \cap \ell) \setminus \{q_1, q_2\}$. Therefore, sing$(\mathcal{F}) \cap \ell$ has cardinality at least three and consequently, the degree of $\mathcal{F}$ is at least two. After analyzing Table 1, it follows that $f$ must be as in case (a.2) or as in case (d.3) with $k = 3$.

Let us first consider case (d.3): one has $\mathcal{F} = \{d(xy)\}$. This foliation is of degree one and admits exactly three invariant lines: $\ell$ and the two coordinate axis $\ell_1$ and $\ell_2$ cut out (in affine coordinates) by $y = 0$ and $x = 0$, respectively. So one has $\ell_p = \ell_i$ hence $p \in \ell_i$ for $i = 1$ or $i = 2$. From another hand, we have seen that $p \in \text{tang}(\mathcal{F}, \mathcal{L}_{q_3})$. Since $q_3 = [-1 : 1 : 0]$ with our normalization, it follows that $p$ belongs also to the line $\ell'$ cut out by $x - y = 0$. But $\ell_i \cap \ell'$ is the origin of $\mathbb{C}^2 = \mathbb{P}^2 \setminus \ell$ and this point is a singularity of $\mathcal{F}$, which contradicts the assumption $p \notin \text{sing}(\mathcal{F})$.

We assume now that $f$ is as in case (a.2) of Table 1. Explicitly, the action of $f$ is given by $f^{-1}(q_1) = 2 q_1$, $f^{-1}(q_2) = 2 q_2$, and $f^{-1}(q_3) = q_3 + \check{q}_3$. In particular, $\mathcal{F}$ has degree
two, admits exactly three singularities on $\ell$, namely $q_1$, $q_2$, and $q_3$, and $\ell_p = \overline{pq_3}$ is the unique $F$-invariant line through $p$.

Notice that $\text{tang}(F, L_p)$ is an effective divisor of degree three and its support contains both $\ell_p$ and the singular points of $F$. Since $q_1$, $q_2$, and $p$ are not aligned, there exists an irreducible component $C$ of $\text{tang}(F, L_p)$ distinct from $\ell_p$ and with degree at most two. According to Theorem 7.3, $C$ is invariant by $\widehat{L}_p$ or by $L_p$.

If $C$ contains $p$, then by Lemma 6.1 item (6) it must be a line and therefore is equal to $\ell_p$. This is not possible due to our choice of $C$. Thus, $C$ does not contain $p$ and must be $\widehat{L}_p$ invariant. Recall from above that $\overline{pq_i} \cap \mathcal{P} = \{p, q_i\}$ for $i = 1, 2$. Corollary 6.1 implies that the irreducible curves invariant by $\widehat{L}_p$ that are not lines must have degree at least three. Thus, we can assume that $C$ is a line. Moreover,

$$\text{sing}(\widehat{L}_p) \cap \overline{pq_i} = \{p, q_i\}$$

for $i = 1, 2$ thanks to Lemma 6.1 item (4). Because the intersections of $C$ with $\overline{pq_1}$ and $\overline{pq_2}$ are singularities of $\widehat{L}_p$ that are distinct from $p$, we conclude that $C = \ell$. However $\ell$ is $F$ invariant but not $L_p$ invariant and consequently cannot be in $\text{tang}(F, L_p)$. Thus, the assumption $p \notin \text{sing}(F)$ leads to a contradiction. $\blacksquare$

9 Flat CDQL Webs on $\mathbb{P}^2$

The degree of a web $\mathcal{W}$ on $\mathbb{P}^2$ is, like in the case of foliations, the number of tangencies of $\mathcal{W}$ with a generic line. In particular, the degree of a completely decomposable web is nothing more than the sum of the degrees of its defining foliations and the degree of a CDQL web $\mathcal{W}(\mathcal{P}) \boxtimes F$ is nothing more than the degree of its non-linear defining foliation. If $\ell$ is a line containing (at least) three points of $\mathcal{P}$, the degree of $\mathcal{W}(\mathcal{P}) \boxtimes F$ is equal to the degree of the associated $\ell$-polar map of $F$.

Based on the results of Section 8, we will derive in this section a complete list of (normal forms for) flat CDQL $(k + 1)$-webs on $\mathbb{P}^2$. Up to automorphisms of $\mathbb{P}^2$, there are six countable families of such webs of degree one and exactly sixteen examples of degree strictly bigger: nine of degree two, three of degree three and four of degree four. Once the classification of flat CDQL webs is obtained, it is not difficult to determine which of them is exceptional.

In the next four subsections, we will treat independently the four possibilities for the degree of the non-linear foliation $F$.

We start by considering flat CDQL webs of degree one.
9.1 Flat CDQL webs on $\mathbb{P}^2$ of degree one

9.1.1 Infinitesimal automorphisms

**Proposition 9.1.** Let $W = W(P) \boxtimes F$ be a CDQL $(k+1)$-webs of degree one with $k \geq 4$. If $K(W) = 0$, then it exists a line $\ell$ containing at least $k - 1$ points of $P$. Moreover, there is a system of affine coordinates $(x, y) \in \mathbb{C}^2 \subset \mathbb{P}^2$ where $\ell$ is the line at infinity, $F$ is induced by a homogeneous 1-form $\omega_0$ of degree one, and the radial vector field $R = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$ is an infinitesimal automorphism of $W$.

**Proof.** If $K(W) = 0$, then Corollary 8.1 and Proposition 8.2 imply that there is a $F$-invariant line $\ell$ that contains (at least) three points of $P$. A classical result by Darboux says that a degree $d$ foliation on $\mathbb{P}^2$ has $d^2 + d + 1$ singularities counted with multiplicities. Here $d = 1$ and since at least two of the three singularities of $F$ necessarily lie on $\ell$, it follows that $\text{sing}(F) \setminus \ell$ reduces to a point or is empty. Proposition 8.4 yields that at least $k - 1$ points of $P$ lie on $\ell$. According to Proposition 8.3, the $\ell$-polar map of $F$ does not contract $\ell$ so one of the singularities of $F$ is not contained in $\ell$.

All that said, we can choose affine coordinates where $\ell$ is the line at infinity and $F$ is induced by a homogeneous linear 1-form $\omega_0$ that vanishes only at the origin of $\mathbb{C}^2$. It is then clear that $R$ is a infinitesimal automorphism of $W$. \[\Box\]

From now on in this section, we denote by $W = W(P) \boxtimes F$ a flat CDQL $(k+1)$-web on $\mathbb{P}^2$, of degree one (with $k \geq 4$). We work with the affine coordinates considered at the end of the proof above, noted by $(x, y)$. Then $\ell$ is the line at infinity, $F$ is homogeneous and $R$ is an infinitesimal symmetry of $W$. Finally, let $F_R$ be the radial foliation formed by the closures of the affine lines passing through the origin of $\mathbb{C}^2$.

9.1.2 Flat and exceptional CDQL webs of degree one

Under the assumptions just made, it is a simple exercise to show that one can assume that in the coordinates $x, y$ the 1-form $\omega_0$ that defines $F$ can be written as

$$\omega_0^* = y \, dx - (x - y) \, dy$$

or

$$\omega_0^\kappa = y \, dx - \kappa \, x \, dy$$

with $\kappa \neq 0, 1$. The $\ell$-polar maps of the foliations $[\omega_0^*]$ and $[\omega_0^\kappa]$ are respectively

$$f_0^*(x : y) = [x - y : y] \quad \text{and} \quad f_0^\kappa(x : y) = [\kappa \, x : y].$$
Since \( \mathcal{W}(\mathcal{P}) \boxtimes \mathcal{F} \) is flat, it follows from Corollary 8.2 that there are only three possibilities corresponding to the labels (d.1), (d.2), and (d.3) in Table 1:

(d.1) \( \mathcal{F} = [d(xy)] \) and \( \mathcal{W}(\mathcal{P} \cap \ell) = [dx^k - dy^k]; \)
(d.2) \( \mathcal{F} = [kydx + (k - 2)xdy] \) and \( \mathcal{W}(\mathcal{P} \cap \ell) = [dy][dx^{k-1} - dy^{k-1}]; \)
(d.3) \( \mathcal{F} = [d(xy)] \) and \( \mathcal{W}(\mathcal{P} \cap \ell) = [dxdy] \boxtimes [dx^{k-2} - dy^{k-2}]. \)

One has \( \mathcal{W} = \mathcal{A}^k_1 \) or \( \mathcal{W} = \mathcal{A}^{k-1}_{II} \) in case (d.1) whereas \( \mathcal{W} = \mathcal{A}^{k-2}_{III} \) or \( \mathcal{W} = \mathcal{A}^{k-3}_IV \) in case (d.3). Since these webs are of maximal rank (by Proposition 2.3), they are necessarily flat according to Mihăileanu criterion.

Let us consider the remaining case (d.2): \( \mathcal{W} \) is one of the two webs

\[
\mathcal{A}^{k-4}_V = \mathcal{F} \boxtimes \mathcal{L}_\infty \boxtimes \mathcal{L}_0 \boxtimes \mathcal{L}_1 \boxtimes \cdots \boxtimes \mathcal{L}_{k-2}
\]

or

\[
\mathcal{A}^{k-5}_{VI} = \mathcal{A}^{k-5} \boxtimes \mathcal{F}_R
\]

(39)

where \( \mathcal{L}_\infty = [dy] \) and \( \mathcal{L}_P = [dy - \theta^r dx] \) for \( r \leq k - 2 \) with \( \theta = e^{\frac{2\pi i}{k}}. \)

For \( a \in \mathbb{P}^1 \), let \( L_k(a) \) be the projective line in \( \mathbb{P}^2 \) cut out by \((k - 2)ax + ky = 0\) in the affine coordinates \( x, y \) (with the convention that \( L(\infty) = L_k(\infty) \) is the Zariski closure of the vertical axis \( \{x = 0\} \)). Easy computations give that

\[
\text{tang}(\mathcal{F}, \mathcal{L}_r) = L_k(\theta^r) + \ell \quad \text{for } r = 0, \ldots, k - 2.
\]

\[
\text{tang}(\mathcal{F}, \mathcal{L}_\infty) = L_k(0) + \ell
\]

and

\[
\text{tang}(\mathcal{F}, \mathcal{F}_R) = L_k(0) + L(\infty).
\]

We investigate first the flatness of \( \mathcal{A}^{k-4}_V \). In this case \( \mathcal{P} \subset \ell \). Since the lines \( L_k(0) \) and \( \ell \) are invariant by \( \mathcal{L}_\infty \), it follows from Theorem 7.1 that \( K(\mathcal{A}^{k-4}_V) \) is holomorphic at the generic point of \( |\text{tang}(\mathcal{F}, \mathcal{L}_\infty)| \). For any \( r = 0, \ldots, k - 2 \), the line at infinity \( \ell \) is invariant by \( \mathcal{L}_r \) whereas \( L_k(\theta^r) \) is left invariant by the barycenter

\[
\beta_{\mathcal{L}_r}(\mathcal{W}(\mathcal{P})/\mathcal{L}_r) = [\theta^r(k - 2)dx + kdy].
\]

Again by Theorem 7.1, it follows that \( K(\mathcal{A}^{k-4}_V) \) is holomorphic along any irreducible component of \( \text{tang}(\mathcal{F}, \mathcal{L}_r) \). These arguments show that \( K(\mathcal{A}^{k-4}_V) \) is holomorphic on \( \mathbb{P}^2 \). Hence, \( \mathcal{A}^{k-4}_V \) is flat.

We now consider the web \( \mathcal{A}^{k-4}_{VI} \). Note that it is a \((k + 2)\)-web (this shift is useful: it allows to use the notations just introduced above). It suffices to prove that for every \( r = 0, \ldots, k - 1 \), the line \( L_k(\theta^r) \) is invariant by \( \mathcal{W}(r) = \beta_{\mathcal{L}_r}(\mathcal{A}^{k-4}_{VI} / (\mathcal{F} \boxtimes \mathcal{L}_r)) \). But \( L_k(\theta^r) \) can
also be defined as the line joining the origin \((0, 0)\) of \(\mathbb{C}^2\) (i.e., the singularity of the radial foliation \(\mathcal{F}_r\)) with the \(p_r\)-barycenter \(\hat{p}_r\) of \(\mathcal{P}_r \setminus \{p_r\}\) on \(\ell\) where \(p_r\) stands for the vertex of the pencil \(\mathcal{L}_r\). Since \((0, 0)\) and \(\hat{p}_r\) are the unique singularities of \(\mathcal{W}(r)\) on \(\mathbb{P}^2 \setminus \{p_r\}\) (according to Lemma 6.1), it follows from (the proof of) Proposition 6.1 that \(L(\theta^r)\) is indeed invariant by \(\mathcal{W}(r)\). Since this holds for every \(r = 0, \ldots, k - 1\), this shows that \(A^{k-4}_{VI}\) is flat.

**Proposition 9.2.** Let \(\mathcal{W}\) be a flat CDQL \((k + 1)\)-web on \(\mathbb{P}^2\), with \(k \geq 4\). If \(\mathcal{W}\) has degree one, then it is projectively equivalent to one of the webs in the following list:

\[
A^k_I, \quad A^{k-1}_{II}, \quad A^{k-2}_{III}, \quad A^{k-3}_{IV}, \quad A^{k-4}_{V}, \quad A^{k-5}_{VI}. 
\]

It is certainly possible to classify (up to projective equivalence) the flat CDQL \((k + 1)\)-webs on \(\mathbb{P}^2\) of degree one for \(k \in \{2, 3\}\). It is left to the interested reader.

**Lemma 9.1.** For \(k \geq 4\), the \((k + 1)\)-webs \(A^{k-4}_{V}\) and \(A^{k-5}_{VI}\) are not of maximal rank.

**Proof.** Since \(R\) is an infinitesimal automorphism for \(A^{k-4}_{V}\), it follows from Theorem 2.1 and from (39) that it suffices to prove the lemma for \(A^{k-4}_{V}\) with \(k \geq 4\) fixed.

Let \(z\) be a point of \(\mathbb{C}^2\) lying outside the discriminant of the CDQL \((k + 1)\)-web \(W = [dx] \boxtimes A^{k-4}_{V}\) and consider the \(\mathbb{C}\)-linear map

\[
\varphi : A(W)/A(A^{k-4}_{V}) \longrightarrow \Omega^1(\mathbb{C}, z)
\]

that associates its \([dx]\)-component to any abelian relation of the germification of \(W\) at \(z\). Considering the (polynomial) abelian relations of the parallel \((k + 1)\)-subweb \([dx dy (dx^{k-1} - dy^{k-1})]\) of \(W\), one obtains that \(x^s dx \in \text{Im} \varphi\) for \(s = 0, \ldots, k - 2\).

On the other hand, since \(u = x^k y^{k-2}\) is a first integral for \(\mathcal{F}\), the relation

\[
k \frac{dx}{x} + (k - 2) \frac{dy}{y} - \frac{du}{u} = 0
\]

is an abelian relation for \([dx dy] \boxtimes \mathcal{F}\). Thus, \(x^s dx \in \text{Im} \varphi\) for \(s = -1, 0, \ldots, k - 2\) hence \(\dim_{\mathbb{C}} \text{Im} \varphi \geq k\). This implies that

\[
\operatorname{rk}(W) \geq \operatorname{rk}(A^{k-4}_{V}) + k. \tag{40}
\]
Let \( f \) be the \( \ell \)-polar map of \( \mathcal{F} \) and let \( P \subset \ell \) be the union of the vertices of the \( k + 1 \) linear foliations of \( W \). If \( \mathcal{A}^{k-4}_V \) were of maximal rank, (40) would imply that \( W \) is of maximal rank too. Then this \( (k + 2) \)-web would be flat and consequently, the pair \((f, P)\) would satisfy (27); hence, it would be projectively equivalent to one of the pairs appearing in Table 1. One can verify that it is not the case. Thus, the rank of \( \mathcal{A}^{k-4}_V \) is not maximal. ■

Combined with Proposition 2.3, the preceding lemma gives us the following

**Proposition 9.3.** Let \( W \) be an exceptional CDQL \((k + 1)\)-web on \( \mathbb{P}^2 \), with \( k \geq 4 \). If \( W \) has degree one, then it is projectively equivalent to one of the following webs:

\[
\mathcal{A}^k_1, \quad \mathcal{A}^{k-1}_\II, \quad \mathcal{A}^{k-2}_\III, \quad \mathcal{A}^{k-3}_V. \quad \Box
\]

### 9.2 Flat CDQL webs of degree two

**Proposition 9.4.** Let \( \mathcal{F} \) be a foliation of degree two and \( \mathcal{P} \subset \mathbb{P}^2 \) be a finite set of at least four points. If \( K(\mathcal{F} \boxtimes \mathcal{W}(\mathcal{P})) = 0 \), then \( \mathcal{F} \) is projectively equivalent to one of the following foliations:

\[
\begin{align*}
(a.1.h) \quad & \mathcal{F} = \left[ d \left( xy(x + y) \right) \right]; \\
(a.2.h) \quad & \mathcal{F} = \left[ d \left( \frac{xy}{x+y} \right) \right]; \\
(a.3.h) \quad & \mathcal{F} = \left[ d \left( (4y^2 + xy + 4x^2)^3(x + y) \right) \right]; \\
(a.4.h) \quad & \mathcal{F} = \left[ d \left( x^3 + y^3 \right) \right]; \\
(a.2) \quad & \mathcal{F} = \left[ d \left( \frac{y^2 - 1}{x^2 - 1} \right) \right].
\end{align*}
\]

Moreover, in cases (a.1.h), (a.2.h), and (a.3.h), \( \mathcal{P} \) has cardinality four and is equal to the singular set of \( \mathcal{F} \). In case (a.4.h), there are two possibilities for \( \mathcal{P} \). Either \( \mathcal{P} \) is equal to \( \text{sing}(\mathcal{F}) \cup \{[0 : 1 : 0], [1 : 0 : 0]\} \) or to \( (\text{sing}(\mathcal{F}) \cup \{[0 : 1 : 0], [1 : 0 : 0]\}) \setminus \{[0 : 0 : 1]\} \). Finally, in case (a.2) the set \( \mathcal{P} \) is any of the subsets of \( \text{sing}(\mathcal{F}) \) containing the four base points of the pencil \( < x^2 - z^2, y^2 - z^2 > \). Up to the automorphism group of \( \mathcal{F} \), there are only four possibilities. \( \Box \)

**Proof.** If the points in \( \mathcal{P} \) are in general position then, according to Corollary 8.1, \( \mathcal{F} \) is the pencil generated by two reduced conics intersecting transversally and \( \mathcal{P} \) is the set of base points of this pencil. So \( \mathcal{F} \boxtimes \mathcal{W}(\mathcal{P}) \) is Bol’s web and we are in case (a.2).
From now on, we will assume that there is a line $\ell \subset \mathbb{P}^2$ that contains at least three points of $\mathcal{P}$. Up to the end of the proof, we work with affine coordinates $[x : y : 1]$ on $\mathbb{C}^2 \subset \mathbb{P}^2$, for which $\ell = \{z = 0\}$ is the line at infinity. We will also assume that $\mathcal{P} \cap \ell$ contains $q_1 = [1 : 0 : 0]$, $q_2 = [0 : 1 : 0]$, and $q_3 = [1 : -1 : 0]$.

We will deal separately with each one of the four possibilities given by Table 1 for the $\ell$-polar map $f$ of $\mathcal{F}$.

**Case (a.1).** In this case, $k_\ell = 3$ and $f^{-1}(q_i) = q_i + \hat{q}_i$ for $i = 1, 2, 3$. Notice that $f^{-1}(q_1) = q_1 + \hat{q}_1$ implies that $\text{tang}(\mathcal{F}, \mathcal{L}_{q_1})$ is the union of three lines: the line at infinity $\ell$ together with two other lines, one intersecting $\ell$ at $q_1$ and the other at $\hat{q}_1$. A similar situation occurs for $\text{tang}(\mathcal{F}, \mathcal{L}_{q_2})$ and $\text{tang}(\mathcal{F}, \mathcal{L}_{q_3})$.

Therefore, $f^{-1}(q_1) = q_1 + \hat{q}_1$ and $f^{-1}(q_2) = q_2 + \hat{q}_2$ imply that $\mathcal{F}$ is induced by a 1-form like

$$\omega = (y + c_1)(2x + y + c2) \, dx + (x + c_3)(2y + x + c_4) \, dy,$$

where $c_1$, $c_2$, $c_3$, and $c_4$ are complex constants. After composing with a translation, we can assume that $c_1 = c_3 = 0$.

It remains to consider the conditions imposed by $f^{-1}(q_3) = q_3 + \hat{q}_3$. Notice that $\text{tang}(\mathcal{F}, \mathcal{L}_{q_3})$ is cut out by

$$y(2x + y + c2) - x(2y + x + c_4) = y^2 - x^2 + c_2 y - c_4 x.$$

This latter expression is a product of lines if and only if $c_2 = \pm c_4$. When $c_2 = c_4 = 0$, we arrive at the homogeneous foliation

$$\mathcal{F} = [d(xy(x + y))].$$

We are in case $(a.1.h)$. Because the cardinality of the singular set of $\mathcal{F}$ is four, there is just one possible choice for $\mathcal{P}$: $\mathcal{P} = \text{sing}(\mathcal{F})$.

If $c_2 \neq 0$ then after applying a homothety we can assume that $c_2 = 1$. We arrive at two possibilities for $\omega$, namely

$$\omega_\pm = y(2x + y + 1) \, dx + x(2y + x \pm 1) \, dy.$$

Let $\mathcal{F}_\pm$ be the corresponding foliations. By hypothesis, $k_\ell = 3$ and $\mathcal{P}_\ell = \{q_1, q_2, q_3\}$. If $\mathcal{F}_\pm \otimes \mathcal{U}(\mathcal{P})$ is assumed to be flat, then Proposition 8.4 implies that $\mathcal{P} \setminus \ell$ is included in the
support of \( \text{sing}(\mathcal{F}_\pm) \cap \mathbb{C}^2 \). In particular, there are only a finite number of possibilities for \( \mathcal{P} \). Lengthy, but straightforward, computations show that \( K(\mathcal{F}_\pm \boxtimes \mathcal{W}(Q \cup \{q_1, q_2, q_3\})) \neq 0 \) for any non-empty subset \( Q \subset \text{sing}(\mathcal{F}_\pm) \cap \mathbb{C}^2 \). Therefore, the foliations \( \mathcal{F} = \mathcal{F}_\pm \) are not among the defining foliations of any flat CDQL webs of order at least five.

**Case (a.2).** In this case, \( k_\ell = 3 \), \( f^{-1}(q_i) = 2q_i \) for \( i = 1, 2 \) and \( f^{-1}(q_3) = q_3 + \hat{q}_3 \).

Arguing as in the paragraph above, we conclude that \( \mathcal{F} \) is induced by

\[
\omega = y(y - 1) \, dx + x(x - 1) \, dy \quad \text{or} \quad \omega' = y^2 \, dx + x^2 \, dy.
\]

(41)

Recall that \( \mathcal{P} \setminus \ell \) is included in \( \text{sing}(\mathcal{F}) \cap \mathbb{C}^2 \) (according to Proposition 8.4). If \( \mathcal{F} \) is induced by \( \omega' \), only one possibility can happen, namely \( \mathcal{P} = \{q_1, q_2, q_3, p_4\} \) where \( p_4 = [0 : 0 : 1] \) (since \( \text{sing}(\mathcal{F}) = \{q_1, q_2, q_3, p_4\} \)). By a direct computation, one verifies that the 5-web defined by \( \mathcal{P} \) and \( \omega' \) is indeed flat.

Let us now consider the case when \( \mathcal{F} \) is the foliation induced by \( \omega \). If we set \( p_5 = [1 : 1 : 1] \), \( p_6 = [0 : 1 : 1] \), and \( p_7 = [1 : 0 : 1] \), then

\[
\text{sing}(\mathcal{F}) = \{q_1, q_2, q_3, p_4, p_5, p_6, p_7\}.
\]

Direct computations show that there are exactly four subsets \( \mathcal{P} \) of \( \text{sing}(\mathcal{F}) \) that strictly contain \( \mathcal{P}_\ell = \{q_1, q_2, q_3\} \) and that verify \( K(\mathcal{F} \boxtimes \mathcal{W}(\mathcal{P})) = 0 \), namely

\[
\mathcal{P} = \left\{ \mathcal{P}_\ell \cup \{p_4, p_5\}, \mathcal{P}_\ell \cup \{p_4, p_5, p_6\}, \mathcal{P}_\ell \cup \{p_4, p_5, p_7\}, \mathcal{P}_\ell \cup \{p_4, p_5, p_6, p_7\} \right\}.
\]

Notice that \( \mathcal{F} \boxtimes \mathcal{W}((q_1, q_2, p_4, p_5)) \) is nothing more than Bol’s exceptional 5-web. The second and the third possibilities for \( \mathcal{P} \) are equivalent since they are interchanged by the \( \mathcal{F} \boxtimes \mathcal{W}(\mathcal{P}_\ell) \)-automorphism \((x, y) \mapsto (y, x)\). All the cases above lead to exceptional webs. Indeed, they are the webs \( B_6, B_7 \), and \( B_8 \) of the Introduction.

**Case (a.3).** Here \( k_\ell = 3 \), \( f^{-1}(q_i) = 2\hat{q}_i \) for \( i = 1, 2 \) and \( f^{-1}(q_3) = q_3 + \hat{q}_3 \). Theorem 7.3 tells us that every irreducible component \( C \) of \( \text{tang}(\mathcal{F}, \mathcal{L}_{q_1}) \) is invariant by \( \mathcal{L}_{q_1} \) or \( \mathcal{L}_{q_1}^{-1} \). Because \( f^{-1}(q_1) = 2\hat{q}_1 \), there exists such \( C \) invariant by \( \mathcal{L}_{q_1}^{-1} \) and distinct from \( \ell \). The divisor \( \text{tang}(\mathcal{F}, \mathcal{L}_{q_1}) - \ell \) is effective and of degree two. Consequently, the degree of \( C \) is at
most two. If it is two, then Corollary 6.1 implies that for every point \( p \in \mathcal{P} \setminus \{q_1, q_2, q_3\} \) the line \( \overline{q_1p} \) contains a third point of \( \mathcal{P} \). Proposition 8.2 implies the \( \mathcal{F} \) invariance of \( \overline{q_1p} \) contradicting \( f^{-1}(q_1) = 2\hat{q}_1 \). This proves that for \( i = 1, 2 \) every irreducible component of \( \text{tang}(\mathcal{F}, \mathcal{L}_{q_i}) - \ell \) must be a \( \mathcal{L}_{\hat{q}_i} \)-invariant line through \( \hat{q}_i \). Because \( \text{Card}(\mathcal{P}) \geq 4 \) and \( \mathcal{P} \not\subseteq \ell \), for \( i = 1, 2 \), the foliation \( \hat{\mathcal{L}}_{q_i} \) has only one invariant line through \( \hat{q}_i \) distinct from \( \ell \). Therefore, there exists constants \( c_1 \) and \( c_2 \) for which \( \mathcal{F} = [(2x + y + c_1)^2dx + (x + 2y + c_2)^2dy] \). Modulo a translation, we can assume that \( c_1 = c_2 = 0 \). Thus,

\[
\mathcal{F} = [(2x + y)^2dx + (x + 2y)^2dy] = \left[ d\left((x + y)(4y^2 + xy + 4x^2)^3\right)\right].
\]

We are in case (a.3.h) and necessarily \( \mathcal{P} = \text{sing}(\mathcal{F}) \) since \( \text{Card}(\text{sing}(\mathcal{F})) = 4 \).

**Case (a.4).** We finally arrive at the last case of Table 1 where \( k_\ell = 5 \), \( f^{-1}(q_i) = 2\hat{q}_i \) for \( i = 1, 2 \) and \( f^{-1}(q_j) = q_j + \hat{q}_j \) for \( j = 3, 4, 5 \).

Arguing exactly as in case (a.3) one can show that in this case \( \mathcal{F} \) is also homogeneous. Therefore, \( \mathcal{F} = [x^2dx + y^2dy] = [d(x^3 + y^3)] \) and, as it was shown in Section 2, any of the two possibilities for \( \mathcal{P} \), namely

\[
\mathcal{P} = \{q_1, \ldots, q_5, [0:0:1]\} = \text{sing}(\mathcal{F}) \cup \{[0:1:0], [1:0:0]\}
\]

or

\[
\mathcal{P} = \{q_1, \ldots, q_5\} = \left( \text{sing}(\mathcal{F}) \cup \{[0:1:0], [1:0:0]\} \right) \setminus \{[0:0:1]\}
\]

leads to exceptional, and in particular flat, webs.

9.3 **Flat CDQL webs of degree three**

The classification of flat CDQL webs of degree three is given by the following proposition.

**Proposition 9.5.** Let \( \mathcal{F} \) be a foliation of degree three and \( \mathcal{P} \subset \mathbb{P}^2 \) be a finite set of at least four points. If \( K(\mathcal{F} \boxtimes W(\mathcal{P})) = 0 \), then \( \mathcal{F} \) is projectively equivalent to one of the following foliations:

(a) \( \mathcal{F} = [(x^3 + y^2 + 1 + 6xy^2)dx - (x^3 + y^3 + 1 + 6x^2y)dy] \);

(b) \( \mathcal{F} = [d(x(x^2 + y^3))]. \)

Moreover, \( \mathcal{P} = \text{sing}(\mathcal{F}) \cap \{x - y = 0\} \) in case (a) and \( \mathcal{P} = \text{sing}(\mathcal{F}) \) or \( \mathcal{P} = \text{sing}(\mathcal{F}) \setminus \{[0:0:1]\} \) in case (b).
Proof. Corollary 8.1 implies that there exists a line $\ell$ containing at least three points of $\mathcal{P}$. According to Table 1, $\ell$ must contain indeed four points of $\mathcal{P}$, say $q_1, \ldots, q_4$, and the $\ell$-polar map $f$ of $\mathcal{F}$ is completely determined. It satisfies

$$f^{-1}(q_i) = q_i + 2\widehat{q_i} \quad \text{for } i = 1, \ldots, 4. \quad (42)$$

Recall from [9] that a foliation of degree three has at most four singularities on an invariant line. Therefore, $\text{sing}(\mathcal{F}) \cap \ell = \{q_1, \ldots, q_4\}$. Lemma 8.2 implies that through each $q_i$ there is a $\mathcal{F}$-invariant line $\ell_i$ distinct from $\ell$.

From (42), one deduces that

$$\text{tang}(\mathcal{F}, \ell_{q_i}) = \ell + \ell_i + C_i$$

where $C_i$ is a conic (not necessarily reduced nor irreducible) intersecting $\ell$ at $\widehat{q_i}$ with multiplicity two. Theorem 7.3 implies moreover that $C_i$ is $\widehat{\ell_{q_i}}$ invariant.

Claim 9.1. None of the conics $C_i$ is reduced and irreducible. $\blacksquare$

Proof of the claim. Aiming at a contradiction, suppose that $C_1$ is reduced and irreducible. Then $C_1$ is a $\widehat{\ell_{q_1}}$-invariant curve of degree two. Corollary 6.1 implies that $\mathcal{P}$ is contained in the union of $\ell$ and $\ell_1$ and that $\mathcal{P} \cap \ell_1$ must have the same cardinality of $\mathcal{P} \cap \ell$, that is $\text{Card}(\mathcal{P} \cap \ell_1) = 4$. Recall from above that $\ell_1$ is $\mathcal{F}$ invariant and $\ell \cap \ell_1 = q_1$. Let $p_5$, $p_6$, and $p_7$ be the points of $\mathcal{P}$ in $\ell_1$ distinct from $q_1$, see Figure 3.

![Fig. 3. The singularities of $\widehat{\ell_{q_2}}$ are not aligned.](http://imrn.oxfordjournals.org)
A simple computation shows that $q_2 \neq q_1$ and (42) implies that $q_2$ is contained in at most one $\mathcal{F}$-invariant line different from $\ell$. Therefore, at least two of the three lines $\overline{q_2 p_5}$, $\overline{q_2 p_6}$, and $\overline{q_2 p_7}$ are not $\mathcal{F}$ invariant. Proposition 8.2 combined with item (4) of Lemma 6.1 implies that two of the three points $p_5$, $p_6$, and $p_7$ are singularities of $\widehat{L}_{q_2}$. Therefore, the singularities of $\widehat{L}_{q_2}$ are not aligned. Proposition 6.1 tells us that the only algebraic leaves of $\widehat{L}_{q_2}$ are lines through $q_2$. Theorem 7.3 implies that $\operatorname{tang}(\mathcal{F}, \mathcal{L}_{q_2})$ is constituted of four lines passing through $q_2$. Consequently, $f^{-1}(q_2) = 3q_2$ by Lemma 8.2 contradicting (42).

If each $C_i$ is a union of two distinct lines, then the linear system of cubics

$$\{\operatorname{tang}(\mathcal{F}, \mathcal{L}_p) - \ell\}_{p \in \ell} \tag{43}$$

contains four totally decomposable fibers. These are triangles (three lines in general position) with one of the vertices on $\ell$. This is sufficient (see [44, Section 4.4]) to ensure that (43) is the Hesse pencil and that $\ell$ is one of its nine harmonic lines. Recall from [3] that the Hesse pencil is classically presented as the one generated by the cubic forms $x^3 + y^3 + z^3$ and $xyz$. In these coordinates, the harmonic lines are

$$\{x - y = 0\} \quad \{x - \epsilon y = 0\} \quad \{x - \epsilon^2 y = 0\}$$
$$\{x - z = 0\} \quad \{x - \epsilon z = 0\} \quad \{x - \epsilon^2 z = 0\}$$
$$\{y - z = 0\} \quad \{y - \epsilon z = 0\} \quad \{y - \epsilon^2 z = 0\}.$$

The subgroup of $\operatorname{Aut}(\mathbb{P}^2)$ that preserves the Hesse Pencil is the Hessian group $G_{216}$ isomorphic to $\left(\frac{\mathbb{Z}}{3\mathbb{Z}}\right)^2 \rtimes \operatorname{SL}(2, \mathbb{F}_3)$. The projective transformations

$$a : [x : y : z] \mapsto [y : z : x] \quad \text{and} \quad b : [x : y : z] \mapsto [x : \epsilon y : \epsilon^2 z]$$

generate a subgroup $\Gamma$ of $G_{216}$ isomorphic to $\left(\frac{\mathbb{Z}}{3\mathbb{Z}}\right)^2$ acting transitively on the set of harmonic lines. Thus, we loose no generality by assuming that $\ell = \{x - y = 0\}$.

Notice that the singular set of $\mathcal{F}$ contains the base points of the linear system (43). Thus, the singular set of $\mathcal{F}$ contains the nine base points of the Hesse pencil together with the four fixed points of $f$ on $\ell$. Since $\mathcal{F}$ has degree three, it has at most $3^2 + 3 + 1 = 13$ singular points. Therefore, the singular set of $\mathcal{F}$ has been completely determined.
and each of its points has multiplicity one. In other words, the singular scheme of \( F \) is everywhere reduced.

The main theorem of [12] says that a foliation on \( \mathbb{P}^2 \) of degree greater than one is completely determined by its singular scheme. Therefore, \( F \) is determined and it is equal to the foliation induced by the 1-form

\[
\omega = \left( x^3 + y^3 + 1 + 6xy^2 \right) dx - \left( x^3 + y^3 + 1 + 6x^2y \right) dy.
\]

Therefore, \( F \) is in case \((a)\) of the statement. Concerning the set of points \( P \) it must be equal to \( \{q_1, \ldots, q_4\} \). Otherwise, Corollary 6.1 would imply that there would exist just one \( \hat{L}_{q_i} \)-invariant line through \( \hat{q}_i \) contrary to our assumptions on \( C_i \). A direct computation shows that \( K(F \boxtimes \mathcal{W}(P)) = 0 \).

If at least one of the conics \( C_i \) is non-reduced, then [38, Proposition 3.1] implies that all the \( C_i \)'s are double lines. Therefore, \( F \) is a homogeneous foliation on the affine chart where \( \ell \) is the line at infinity and the singularity of \( F \) corresponding to the unique base point of \( \{\text{tang}(F, L_p) - \ell\}_{p \in \ell} \) is the origin. Thus, \( F \) is defined by a homogenous 1-form with coefficients equal to the coefficients of \( \ell \)-polar map, that is

\[
F = \left[ y(3x + y(1 - \xi_3^2))^2 dx + 3x(x + y(1 - \xi_3^2))^2 dy \right] = \left[ d(xy(x + y)(x - \xi_3y)) \right].
\]

A linear change of coordinates envoys \( F \) to the form presented in case \((b)\) of the statement. Finally, it follows from Proposition 8.4 that there are only two possibilities for \( P \): those mentioned in the statement of the proposition. Both cases are exceptional, and in particular flat, as we have shown in Section 2.

9.4 Flat CDQL webs of degree four

Finally, we turn our attention to the flat CDQL webs \( F \boxtimes \mathcal{W}(P) \) when \( \text{deg}(F) = 4 \) and the cardinality of \( P \) is at least four. Corollary 8.1 implies that \( P \) cannot be in general position and Theorem 8.1 shows that no four points in \( P \) are aligned. Therefore, there exists a line \( \ell \) such that \( P \cap \ell = \{q_1, q_2, q_3\} \).

According to Table 1, there are only two possibilities for the \( \ell \)-polar map \( f \) of \( F \). In both cases, \( f \) has five distinct fixed points that are cut out by the polynomial \( xy(x + y)(x^2 + xy + y^2) \). In particular, \( F \) has exactly five singular points on \( \ell \) according to Lemma 8.1. Notice that \( \text{sing}(F) \cap \ell \) does not intersect \( \{\hat{q}_1, \hat{q}_2, \hat{q}_3\} \).
Lemma 9.2. For $i = 1, 2, 3$, the tangency of $\mathcal{F}$ and $L_{q_i}$ is a union of lines. □

Proof. Let us first consider case (c.2) of Table 1, that is $f^{-1}(q_i) = 3q_i + \hat{q}_i$ for every $i = 1, 2, 3$. By Theorem 7.3, any irreducible component $C$ of the tangency between $\mathcal{F}$ and $L_{q_i}$ is invariant by $L_{q_i}$ or $\hat{L}_{q_i}$. In the former case, $C$ has to be a line as all the irreducible curves left invariant by $L_{q_i}$. In the latter case, $C$ is also a line. This follows from Lemma 6.1 item (2) when $C$ passes through $q_i$ and from $f^{-1}(q_i) = 3q_i + \hat{q}_i$ when $C$ passes through $\hat{q}_i$.

We will now deal with case (c.1) of Table 1, that is $f^{-1}(q_i) = q_i + 3\hat{q}_i$ for every $i = 1, 2, 3$. We can assume that $q_1 = p_1 = [0:1:0]$, $q_2 = p_2 = [1:0:0]$, $q_3 = p_3 = [1:-1:0]$, and $p_4 = [0:0:1] \notin \ell$.

We will deal separately two cases: (a) the cardinality of $\mathcal{P}$ is four, and (b) the cardinality of $\mathcal{P}$ is at least five.

Case (a): $k = \text{Card}(\mathcal{P}) = 4$. In this case, we will work in the affine coordinates $(x, y) = [x:y:1]$. Notice that

$$\hat{L}_1 = \left[d\left(\frac{(x+2y)^3}{x}\right)\right], \quad \hat{L}_2 = \left[d\left(\frac{(2x+y)^3}{y}\right)\right], \quad \text{and} \quad \hat{L}_4 = \left[d(xy(x+y))\right].$$

If we write $\mathcal{F} = [a(x, y)dx + b(x, y)dy]$, where $a$ and $b$ are relatively prime polynomials, then $\text{tang}(\mathcal{F}, L_{q_1})$ is defined by the vanishing of $a(x, y)$. Similarly, $\text{tang}(\mathcal{F}, L_2)$ is defined by the vanishing of $b(x, y)$. Theorem 7.3 implies that

$$\mathcal{F} = \left[(y - \lambda_1)\left((2x + y)^3 - \mu_1 y\right)dx + (x - \lambda_2)\left((x + 2y)^3 - \mu_2 x\right)dy\right]$$

for some $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{C}$.

On the one hand, $\text{tang}(\mathcal{F}, L_4)$ contains the singular points of $\mathcal{F}$ on $\ell$. Theorem 7.3 implies that its irreducible components must be irreducible cubics in the pencil $< z^3, xy(x+y) >$ or lines connecting $p_4$ to one of the five singularities of $\mathcal{F}$ at $\ell$ (corresponding to the five fixed points of the $\ell$-polar map of $\mathcal{F}$). Thus,

$$\text{tang}(\mathcal{F}, L_4) = \{(x^2 + xy + y^2)(xy(x+y) - \lambda_3) = 0\}$$

for a certain $\lambda_3 \in \mathbb{C}$. 

On the other hand, the tangency between $\mathcal{F}$ and $L_4$ is defined by the vanishing of the contraction of the 1-form in (45) with $x\partial_x + y\partial_y$. Explicitly,

$$\text{tang}(\mathcal{F}, L_4) = \left\{ x(y - \lambda_1) \left( (2x + y)^3 - \mu_1 y \right) + y(x - \lambda_2) \left( (x + 2y)^3 - \mu_2 x \right) = 0 \right\}.$$ 

Comparing the homogeneous components of degree two of the two presentations of $\text{tang}(\mathcal{F}, L_4)$, one concludes that $\lambda_3 = \lambda_1 \mu_1 = \lambda_2 \mu_2 = 0$. Plugging $\lambda_3 = 0$ into (46) shows that all the five lines cut out by $xy(x + y)(x^2 + xy + y^2)$ are $\mathcal{F}$ invariant. The $\mathcal{F}$ invariance of $\{x = 0\}$ and $\{y = 0\}$ ensures that $\lambda_1 = \lambda_2 = 0$. Finally, since the homogeneous component of degree three of (46) is zero, $\mu_1 = \mu_2 = 0$. It is then clear that the expression of $\mathcal{F}$ in (45) is homogeneous. Consequently, $\text{tang}(\mathcal{F}, L_q)$ is a union of lines for every $q \in \ell$.

**Case (b):** $k = \text{Card}(\mathcal{P}) \geq 5$. Notice that $\mathcal{P}$ is not in barycentric general position with respect to none of the points $q_1, q_2, q_3$ because $f^{-1}(q_i) \neq 4q_i$ for $i = 1, 2, 3$. Proposition 6.1 implies that all the leaves of $\mathcal{L}_i$ are algebraic. From the proof of Corollary 6.1, one deduces that the leaves of $\mathcal{L}_1$ (for instance) are irreducible components of elements of a pencil of the form $\mathcal{H} = \langle (x + 2y + \lambda z)^{\text{deg}(R)}, R(x, z) \rangle$, where $\lambda \in \mathbb{C}$ and $R$ is a homogeneous polynomial of degree $k - 1$. The irreducible factors of $R$ correspond to the lines $\overline{q_1p}$ where $p$ ranges in $\mathcal{P}_1 = \mathcal{P} \setminus \{q_1\}$ and their multiplicities correspond to number of points of $\mathcal{P}_1$ contained in the respective lines.

If $\text{tang}(\mathcal{F}, L_{q_1})$ has an non-linear irreducible component $C$, then its degree is at most three and is an irreducible component of an element of the pencil $\mathcal{H}$. But $(x + 2y + \lambda z)^{\text{deg}(R)} - \mu R(x, z)$ admits an irreducible factor of degree at most three for some $\mu \in \mathbb{C}^*$ only when $R$ is a square. Indeed, on the one hand, the square of each linear factor of $R$ must divide $R$ otherwise Corollary 6.1 would imply that $C$ has degree $k - 1 \geq 4$. On the other hand, the third power of any linear factor of $R$ cannot divide $R$, otherwise it would exist four points in $\mathcal{P}$ on the same line contradicting Theorem 8.1.

Since $R$ is a square, it must exist a third point $p_5 \in \mathcal{P}$ contained in the line $\overline{q_1p_5}$. From the fact that $\mathcal{P}$ is not in $q_2$-barycentric general position, it follows that $\text{sing}(\mathcal{L}_{q_2}) - \{q_2\}$ is contained in a line. Using $\hat{q}_2 \neq q_1$, one deduces that it must exist a point $p_6 \in \overline{q_2p_5} \cap \mathcal{P}$. Since $R(x, z)$ is a square, the line $\overline{q_1p_6}$ must contain another point of $\mathcal{P}$ (the point $p_7$ in Figure 4).

Proposition 8.2 tells us that any line containing at least three points of $\mathcal{P}$ must be $\mathcal{F}$ invariant. Thus, there are at least three $\mathcal{F}$-invariant lines through $q_1$. This contradicts $f^{-1}(q_1) = q_1 + 3 \hat{q}_i$ and ends the proof of the lemma.
We will also need a classical result of Darboux about the degree of foliations induced by pencil of curves. We state it below as a lemma.

**Lemma 9.3.** If $F, G \in \mathbb{C}[x, y, z]$ are relatively prime homogeneous polynomials of degree $e$, then

$$F \text{d}G - G \text{d}F = \left( \prod_{H} H^{e(H)-1} \right) \cdot \omega$$

where $\omega$ is a homogenous polynomial 1-form with codimension two singular set; $H$ runs over the irreducible components of the polynomials $\{sF + tG = 0\}_{(s,t) \in \mathbb{P}^1}$; and $e(H)$ denotes the maximum power of $H$ that divides the member of the pencil that contains $H$. In particular, if $\mathcal{F} = \left[ \text{d}(F/G) \right]$ then

$$\deg(\mathcal{F}) = 2e - 2 - \sum_{H} \deg(H)(e(H) - 1).$$

**Proof.** See [27, Proposition 3.5.1, pages 110–111].

**Proposition 9.6.** Let $\mathcal{F}$ be a foliation of degree four and $\mathcal{P} \subset \mathbb{P}^2$ be a finite set of at least four points. If $K(\mathcal{F} \boxtimes \mathcal{W}(\mathcal{P})) = 0$ then $\mathcal{F}$ is projectively equivalent to one of the following foliations:

(a) $\mathcal{F} = \left[ \text{d} \left( xy(x + y)(x^2 + xy + y^2)^3 \right) \right]$;

(b) $\mathcal{F} = \left[ \text{d} \left( \frac{xy(x + y)}{x^2 + xy + y^2} \right) \right]$;
(c) \( \mathcal{F} = \left[ d \left( \frac{x^3 + y^3 + 1}{xy} \right) \right] \).

Moreover, \( \mathcal{P} = \{ [1 : -1 : 0], [1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1] \} \) in cases (a) and (b). In case (c), \( \mathcal{P} \) is equal to the nine base points of the pencil \( <xy, x^3 + y^3 + 1> \) or is equal to the three base points of this pencil at the line at infinity union with \([0 : 0 : 1]\). \( \square \)

**Proof.** We keep the notations from the beginning of this section. According to Table 1, there are two possibilities for the \( \ell \)-polar map of \( \mathcal{F} \): (c.1) and (c.2). We will deal with them separately.

**Case (c.1).** We are assuming that the \( \ell \)-polar map of \( \mathcal{F} \) satisfies \( f^{-1}(q_i) = q_i + 3 \hat{q}_i \) for \( i = 1, \ldots, 3 \). According to Lemma 9.2, the tangency between \( \mathcal{F} \) and \( L_{q_i} \) is a union of lines. Since \( \mathcal{P} \) has cardinality at least four, there exists \( p_4 \in \mathcal{P} \setminus \ell \). Moreover, \( \mathcal{P} \) is not in \( q_i \)-barycentric general position. Proposition 6.1 implies that the foliation \( \hat{L}_{q_i} \) admits exactly one invariant line \( \hat{\ell}_i \) through \( \hat{q}_i \). Moreover, \( f(q_i) = q_i + 3 \hat{q}_i \) implies that it exists exactly one \( L_{q_i} \)-invariant line \( \ell_i \) through \( q_i \) distinct from \( \ell \). Thus,

\[
\text{tang}(\mathcal{F}, L_{q_i}) = \ell + \ell_i + 3 \hat{\ell}_i \quad \text{for} \quad i = 1, \ldots, 3.
\]

If \( \mathcal{G} \) is the foliation induced by the pencil \( \{(\text{tang}(\mathcal{F}, L_q) - \ell)\}_{q \in \ell} \) then Lemma 9.3 implies that \( \mathcal{G} \) has degree at most \( 2 \cdot 4 - 2 - 3 \cdot (3 - 1) = 0 \). In an affine coordinate system where \( \ell \) is the line at infinity and the origin belongs to \( \text{sing}(\mathcal{G}) \), the foliation \( \mathcal{F} \) is induced by a polynomial 1-form with homogeneous components. Therefore, it is completely determined by its \( \ell \)-polar map and can be explicitly presented as

\[
\mathcal{F} = \left[ y(2x + y)^3 dx + x(2y + x)^2 dy \right].
\]

A simple computation shows that \( xy(x + y)(x^2 + xy + y^2)^3 \) is a first integral of \( \mathcal{F} \). Since the singular set of \( \mathcal{F} \) has cardinality four, it has to be equal to \( \mathcal{P} \). A direct computation shows that \( K(\mathcal{F} \boxtimes \mathcal{W}(\mathcal{P})) = 0 \). This example corresponds to case (a) of the statement.

**Case (c.2).** Suppose now that the \( \ell \)-polar map of \( \mathcal{F} \) is in case (c.2) of Table 1. Lemma 9.2 implies (for any \( i = 1, \ldots, 3 \)) that \( \text{tang}(\mathcal{F}, L_{q_i}) \) is the union of five lines: \( \ell \), one line through \( \hat{q}_i \) and three lines (counted with multiplicities) through \( q_i \). It follows from [38, Proposition 3.1] that the multiplicities appearing in \( \text{tang}(\mathcal{F}, L_{q_i}) \) do not depend on
the choice of $i \in \{1, 2, 3\}$. Therefore, if $G$ denotes the foliation associated to the pencil \(\{\text{tang}(F, L_p) - \ell\}_{p \in \ell}\) then Lemma 9.3 implies that the degree of $G$ is at most:

\begin{enumerate}
\item[(c.2.1)] zero when there is one line with multiplicity three in tang\((F, L_{q_i})\); \\
\item[(c.2.2)] three when there is one line with multiplicity two in tang\((F, L_{q_i})\); \\
\item[(c.2.3)] six when all the lines in tang\((F, L_{q_i})\) have multiplicity one.
\end{enumerate}

Case (c.2.1). If the degree of $G$ is equal to zero then, as in case (c.1) above, $F$ is completely determined by its $\ell$-polar map. In a suitable affine coordinate system, the foliation $F$ is induced by

$$\omega = y^3(2x + y) \, dx + x^2(x + 2y) \, dy.$$ 

One can verify that $\omega$ admits $\frac{xy(x+y)}{x^2+xy+y^2}$ as a rational first integral and, again as in the case (c.1), $P = \text{sing}(F)$. This example corresponds to case (b) of the statement.

Case (c.2.2). If the degree of $G$ is at most three and distinct from zero then $G$ is tangent to a pencil of quartics with three completely decomposable fibers, each formed by three distinct lines with one of these lines with multiplicity two. Therefore, $G$ has at least nine invariant lines. Since a degree $d$ foliation has at most $3d$ invariant lines (see [36]), it follows that the degree of $G$ is exactly $3$.

It is not hard to show that, up to automorphisms of $\mathbb{P}^2$, there exists a unique foliation $G$ as above. In suitable affine coordinates where $\ell$ is the line at infinity and $q_1 = [1 : 0 : 0]$, $q_2 = [0 : 1 : 0]$, and $q_3 = [1 : -1 : 0]$, the foliation $G$ is defined by the rational function

$$\frac{x^2(x-1)(x+2y-1)}{y^2(y-1)(2x+y-1)}.$$ 

We leave the details to the reader.

It follows that

$$F = \left[ y^2(y-1)(2x+y-1) \, dx + x^2(x-1)(x+2y-1) \, dy \right].$$

By a direct computation, it can be checked that the 4-web $F \boxtimes W((q_1, q_2, q_3))$ has curvature zero. Nevertheless, a lengthy computation shows that there is no set $P$ verifying \{q_1, q_2, q_3\} $\subseteq P \subset \text{sing}(F)$ such that $K(F \boxtimes W(P)) = 0$. 
Case (c.2.3). We are now assuming that for each \( i = 1, \ldots, 3 \), \( \text{tang}(\mathcal{F}, \mathcal{L}_{q_i}) \) consists of five distinct lines, four of them being \( \mathcal{F} \) invariant. This implies that \( \mathcal{F} \) has at least ten invariant lines, \( \ell \) plus nine others.

We will further divide this case in two subcases: (c.2.3.a) when \( k = \text{Card}(\mathcal{P}) = 4 \), and (c.2.3.b) when \( k = \text{Card}(\mathcal{P}) \geq 5 \).

Case (c.2.3.a). Assume that \( \mathcal{P} = \{q_1, q_2, q_3, p\} \) with \( p \notin \ell \). Notice that \( \text{tang}(\mathcal{F}, \mathcal{L}_p) \) intersects \( \ell \) at the five singular points of \( \mathcal{F} \) on \( \ell \): \( q_1, q_2, q_3 \) and two other that we will call \( s_1 \) and \( s_2 \). Recall from Lemma 8.1 that these five points coincide with the fixed points of the \( \ell \)-polar map of \( \mathcal{F} \). With no loss of generality, one can assume that the points of \( \mathcal{P} \) are normalized such that \( q_1 = [1 : -1 : 0] \), \( q_2 = [1 : -\xi_3 : 0] \), \( q_3 = [1 : -\xi_5^2 : 0] \) and \( p = [0 : 0 : 1] \). Then, by Corollary 6.1, the foliation \( \widehat{\mathcal{L}}_p \) admits \( x^3 + y^3 \) as a first integral. Consequently, any irreducible \( \widehat{\mathcal{L}}_p \)-invariant algebraic curve \( C \) is of degree less than three and satisfies \( C \cap \ell \subset \{q_1, q_2, q_3\} \). Observe that \( |\text{tang}(\mathcal{F}, \mathcal{L}_p)| \) must contain all the singularities of \( \mathcal{F} \), in particular \( s_1 \) and \( s_2 \). Because none of the curves \( \{x^3 + y^3 = \text{cst.}\} \) contains \( s_1 \) or \( s_2 \), Theorem 7.3 implies that the lines \( \overline{ps_1} \) and \( \overline{ps_2} \) are \( \mathcal{F} \)-invariant irreducible components of \( \text{tang}(\mathcal{F}, \mathcal{L}_p) \). Therefore, as shown in Figure 5, \( \mathcal{F} \) has at least twelve invariant lines: \( \ell \), \( \overline{ps_1} \), \( \overline{ps_2} \), and the three linear components of \( \text{tang}(\mathcal{F}, \mathcal{L}_i) \) passing through \( q_i \) for each \( i \in \{1, 2, 3\} \). It is well known that a degree \( d \) foliation of \( \mathbb{P}^2 \) has at most \( 3d \) invariant lines (see [36] for instance). Therefore, \( \mathcal{F} \) has exactly twelve invariant lines.

Because \( \mathcal{F} \) has degree four, over each \( \mathcal{F} \)-invariant lines there are at most five singularities of \( \mathcal{F} \). Notice that over the \( \mathcal{F} \)-invariant line \( \overline{ps_1} \) we know already two: \( p \) and \( s_1 \). The three \( \mathcal{F} \)-invariant lines through \( q_1 \) distinct from \( \ell \) must intersect \( \overline{ps_1} \) in three distinct singular points of \( \mathcal{F} \), none of them equals to \( p \) or \( s_1 \) (see Figure 6).

![Fig. 5. The twelve lines invariant by \( \mathcal{F} \).](http://imrn.oxfordjournals.org/)

\[\text{Fig. 5. The twelve lines invariant by } \mathcal{F}.\]
The same being true for the $\mathcal{F}$-invariant lines through $q_2$ and $q_3$ it follows that on $\overline{ps_1}$ there are three singularities of $\mathcal{F}$ distinct from $p$ and $s_1$ such that through each four $\mathcal{F}$-invariant lines pass. Of course, the line $\overline{ps_2}$ has the same property. Thus, we have a set $Q \subset \mathbb{P}^2$ of cardinality 9 such that each of the points of $Q$ is contained in four of the twelve $\mathcal{F}$-invariant lines. It is then a simple combinatorial exercise to show that these twelve lines support a $(4,3)$-net in the sense of Section 3. Therefore (see [44, Section 4.4]), the arrangement of twelve $\mathcal{F}$-invariant lines is projectively equivalent to the Hesse arrangement. Because the foliation determined by the Hesse pencil also has degree four and the tangency of two distinct foliations of degree four has degree nine it follows that $\mathcal{F}$ is the Hesse Pencil.

With the normalizations made above on the points $q_1, q_2, q_3,$ and $p$, we obtain

$$\mathcal{F} = \left[ d \left( \frac{x^3 + y^3 + 1}{xy} \right) \right].$$

This 5-web appears in the Introduction under the label $\mathcal{H}_5$. In Section 3, it is shown that it is an exceptional web and in particular has curvature zero.

**Case (c.2.3.b).** Suppose now that $\mathcal{P}$ has cardinality greater than four. As in case (c.2.3.a), we will denote by $s_1$ and $s_2$ the two other singularities of $\mathcal{F}$ on $\ell$ distinct from $q_1, q_2, q_3$.

**Claim 9.2.** There exists a pair of points $p, s \in \mathcal{P} \setminus \{q_1, q_2, q_3\}$ such that the line $\overline{ps}$ intersects $\ell$ in one of the points $q_1, q_2,$ and $q_3$. ■
Proof. Suppose that the claim is not true and let \( p_4 \) and \( p_5 \) be any two points in \( P \setminus \{q_1, q_2, q_3\} \). Proposition 8.2 combined with Theorem 8.1 implies that the line \( \overline{p_4p_5} \) intersects \( P \) in at most three points. Thus, there are only two possibilities for \( P \): (i) \( \overline{p_4p_5} \cap P = \{p_4, p_5\} \) or (ii) \( \overline{p_4p_5} \cap P = \{p_4, p_5, p_6\} \) for some point \( p_6 \in P \) distinct of \( p_4 \) and \( p_5 \).

If we are in case (i), then \( P \) is in \( p_4 \)- and \( p_5 \)-barycentric general position because, by assumption, the lines \( \overline{p_4q_i} \) and \( \overline{p_5q_i} \) (for \( i = 1, 2, 3 \)) have only two elements of \( P \) each and the points \( q_1, q_2, \) and \( q_3 \) are not aligned with \( p_4 \) nor with \( p_5 \). Theorem 7.3 ensures that \( |\text{tang}(F, L_4)| \) is a union of five \( F \)-invariant lines. Since \( |\text{tang}(F, L_4)| \) contains \( p_4 \) and the singularities of \( F \), these lines have to be \( \overline{p_4s_1} \), \( \overline{p_4s_2} \), \( \overline{p_4q_1} \), \( \overline{p_4q_2} \), and \( \overline{p_4q_3} \). Similarly, the irreducible components of \( \text{tang}(F, L_p) \) are the \( F \)-invariant lines \( \overline{p_5q_i} \) for \( i = 1, \ldots, 3 \) and \( \overline{p_5s_i} \) for \( i = 1, 2 \).

Through at least one of the points \( s_1 \) and \( s_2 \), say \( s_1 \), three \( F \)-invariant lines pass: \( \overline{p_4s_1} \), \( \overline{p_5s_1} \) and \( \ell \). This contradicts the behavior of the \( \ell \)-polar map because \( s_1 \) appears in \( f^{-1}(s_1) \) with multiplicity one as a simple computation shows.

Suppose now that we are in case (ii). Because the barycenter transform of three distinct points in \( \mathbb{P}^1 \) is still three distinct points, \( P \) is in barycentric general position with respect to at least two points in \( \{p_4, p_5, p_6\} \). Exactly as before, we arrive at a contradiction. The claim follows.

By Claim 9.2, we can suppose that \( p_4 \) and \( p_5 \) are two points in \( P \setminus \ell \) such that the line \( \ell' = \overline{p_4p_5} \) intersects \( \ell \) at \( q_1 \), see Figure 7. Notice that \( \ell' \) is \( F \) invariant (by Proposition

\begin{center}
\includegraphics[width=0.5\textwidth]{fig7.png}
\end{center}

**Fig. 7.** The twelve lines invariant by \( F \) in case c.2.3.b.
8.2) and that the \( \ell' \)-polar map of \( \mathcal{F} \) must also be in case (c.2) of Table 1. Therefore, Lemma 8.2 implies that through each of the points \( p_4 \) and \( p_5 \) four \( \mathcal{F} \)-invariant lines pass. Since these intersect \( \ell \) at sing(\( \mathcal{F} \)), there will be one \( \mathcal{F} \)-invariant line through \( s_1 \) (say \( p_4s_1 \)) and one through \( s_2 \), say \( p_5s_2 \). In the total, \( \mathcal{F} \) has the maximal number of invariant lines for a degree four foliation: twelve.

Consider the effective divisor tang(\( \mathcal{F}, \mathcal{L}_4 \)). It has degree five, contains four lines through \( p_4 \) (namely \( p_4q_2, p_4q_3, p_4s_1 \), and \( \ell' = p_4q_1 = p_4p_5 \)) and the point \( s_2 \). Since the four lines through \( p_4 \) do not contain \( s_2 \), there is a line \( \ell'' \subset |\text{tang}(\mathcal{F}, \mathcal{L}_4)| \) through \( s_2 \). By Theorem 7.3, \( \ell'' \) must be \( \mathcal{L}_4 \) invariant and Lemma 6.1 item (4) implies that \( \ell'' \) contains \( \tilde{p}_4 \): the \( p_4 \)-barycenter of \( \{p_5, q_1\} \) in \( \ell' \). In particular, \( \ell'' = \overline{s_2p_4} \). Clearly, \( q_2 \notin \ell'' \). Consequently, Lemma 6.1 item (4) ensures the existence of an extra point in \( \mathcal{P} \), say \( p_6 \), such that \( p_6 \in \overline{p_4q_2} \) and the \( p_4 \)-barycenter of \( \{q_2, p_6\} \) in \( \overline{p_4q_2} \) lies in \( \ell'' \). Similarly, there exists another extra point \( p_7 \in \mathcal{P} \) contained in \( \overline{p_4q_3} \) such that the \( p_4 \)-barycenter of \( \{q_3, p_7\} \) in \( \overline{p_4q_3} \) also lies in \( \ell'' \).

Notice that the line \( \overline{p_4q_2} \) contains three points of \( \mathcal{P} \): \( q_2, 4, p_4 \), and \( p_6 \). Therefore, the \( \overline{p_4q_2} \)-polar map of \( \mathcal{F} \) must be also in case (c.2) of Table 1. Consequently, through \( p_6 \) four \( \mathcal{F} \)-invariant lines pass. Remark that \( p_4, p_5, \) and \( s_1 \) are not aligned and that through \( s_1 \) just two \( \mathcal{F} \)-invariant lines (\( \ell \) and \( \overline{p_4s_1} \)) pass. Thus, one of the four \( \mathcal{F} \)-invariant lines through \( p_6 \) must be the line \( \overline{p_6s_2} \). Similarly, through \( p_7 \) four \( \mathcal{F} \)-invariant lines pass and the line \( \overline{p_7s_2} \) is among these four lines. Since through \( s_2 \) just one \( \mathcal{F} \)-invariant distinct from \( \ell \) passes, it follows that \( \overline{p_6s_2} = \overline{p_7s_2} = \overline{p_5s_2} \).

Changing the role of \( p_4 \) and \( p_5 \) in the preceding argument, it follows that there exist \( p_8 \), \( p_9 \in \mathcal{P} \setminus \{q_1, q_2, q_3, p_4, \ldots, p_7\} \) in the lines \( \overline{p_5q_2} \) and \( \overline{p_5q_3} \), respectively. As before, through each of these points four \( \mathcal{F} \)-invariant lines pass.

Putting all together, we have just proved that \( \mathcal{F} \) leaves invariant an arrangement of twelve lines and \( \mathcal{P} \) contains a subset of at least nine points such that each of these points is contained in four distinct lines of the arrangement. At this point, it is clear that the arrangement is the Hesse arrangement (see [44]), that \( \mathcal{F} \) is projectively equivalent to the Hesse pencil (it is the unique degree four foliation leaving the Hesse arrangement invariant because the tangency of two degree four foliations has degree nine) and that \( \mathcal{P} \) contains the nine base points of it. It remains to show that \( \mathcal{P} \) cannot be larger than the base points of the Hesse pencil. Indeed, if there exists a point \( p_{10} \in \mathcal{P} \) distinct from the nine base points it would exist a line in the arrangement containing four points of \( \mathcal{P} \) contradicting Theorem 8.1. Therefore, there exists only one flat CDQL \((k + 1)\)-web of degree four with \( k \geq 5 \): the 10-web \( \mathcal{H}_{10} \) from the Introduction. ■
9.5 Proof of Theorem 1.2

According to Proposition 9.3, the exceptional CDQL webs of degree one are projectively equivalent to one of the webs $A^k_I, A^k_{II}, A^k_{III},$ and $A^k_{IV}$.

Propositions 9.4, 9.5, and 9.6 put together give a complete classification of flat CDQL $(k + 1)$-webs of degree bigger than two, on the projective plane, when $k \geq 4$. There are only sixteen such webs (up to projective transformations). Thirteen of these have been presented in the Introduction and their exceptionality has been put in evidence in Sections 2 and 3.

It can be verified that the 5-web described in Proposition 9.4 case (a.3.h), the 5-web described in Proposition 9.5 case (a), and the 5-web described in Proposition 9.6 case (a) are not exceptional. For this sake one can use, as we did, the criterion [41, Proposition 4.3] or Hénaut’s curvature as indicated by Ripoll in [42, Theorem 5.1] or even Pantazi’s criterion. Aiming at conciseness, we decided not to reproduce the lengthy computations here.

10 From Global to Local...

10.1 Degenerations

Let $\mathcal{W}_t$ be a holomorphic family of webs in the sense that it is defined by an element

$$W(x, y, t) = \sum_{i+j=k} a_{ij}(x, y, t)dx^idy^j$$

in $\text{Sym}^k \Omega^1(\mathbb{C}^2, 0)$ with coefficients in $\mathcal{O} = \mathbb{C}[x, y, t]$ (convergent power series) and such that $W(\cdot, \cdot, t)$ defines a (possibly singular) $k$-web on $(\mathbb{C}^2, 0)$ for every $t \in (\mathbb{C}, 0)$.

We do not claim originality on the next result. Indeed, the first author, modulo memory betrayals, first heard about it in a talk delivered by Hénaut at CIRM-Marseille in 2003. Anyway it follows almost immediately from the main result of [24]. Since it would take us too far afield to recall the notations and the results of [24], we include a sketchy proof below freely using them. We refer to this work for more precisions.

**Theorem 10.1.** The set $\{t \in (\mathbb{C}, 0) | \mathcal{W}_t$ has maximal rank $\}$ is closed. □
Proof. The differential system $M_t(d)$ can be defined over $\mathcal{O}$ (with $t$ considered as a constant of derivations) and the restriction of $M_t(d)$ to a parameter $t_0$ coincides with the definition of $M_0(d)$.

The prolongations $p_k$ of the associated morphism are morphisms of $\mathcal{O}$-modules and the kernels $R_k$ of the morphisms $p_k$ are $\mathcal{O}$-modules locally free outside the discriminant. Notice that the discriminant is a hypersurface in $(\mathbb{C}^2, 0) \times (\mathbb{C}, 0)$ that does not contain any fiber of the projection $(x, y, t) \mapsto t$ by our definition of family of webs.

If $r_k = \dim R_k$, then Cartan’s Theorem B implies the existence of $r_k$ sections of $R_k$ over a polydisk $D \subset (\mathbb{C}^2, 0) \times (\mathbb{C}, 0)$ that generates $R_k$ on a Zariski open subset of $(\mathbb{C}^2, 0) \times (\mathbb{C}, 0)$. Moreover, this subset can be supposed to contain any given point on the complement of the discriminant. Therefore, we can find a meromorphic inverse of the morphism $\pi_{k-4}$ holomorphic at any given point in the complement of the discriminant.

Following [24], we can construct a holomorphic family of meromorphic connections $\Delta_t$ such that $\mathcal{W}_t$ has maximal rank if and only if $\Delta_t^2 = 0$. The theorem follows. ■

Remark 10.1. The analog for flat webs “the set $\{ t \in (\mathbb{C}, 0) | \mathcal{W}_t \text{ is flat} \}$ is closed” holds true. The proof is left to the reader.

10.2 Singularities of certain exceptional webs

Theorem 10.1 combined with the classification of CDQL exceptional webs in $\mathbb{P}^2$ yields the following result.

Corollary 10.1. (Corollary 1.2 of the Introduction) Let $\mathcal{W}$ be a smooth $k$-web, $k \geq 4$, and $\mathcal{F}$ be a singular holomorphic foliation, both defined on $(\mathbb{C}^2, 0)$, such that $\mathcal{W} \boxtimes \mathcal{F}$ is a $(k+1)$-web with maximal rank. Then one of the following holds:

1. the foliation $\mathcal{F}$ is of the form $[H(x, y)(\alpha dx + \beta dy) + h.o.t.]$ where $H$ is a non-zero homogeneous polynomial and $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{0\}$;
2. the foliation $\mathcal{F}$ is of the form $[H(x, y)(ydx - xdy) + h.o.t.]$ where $H$ is a non-zero homogeneous polynomial;
3. $\mathcal{W} \boxtimes \mathcal{F}$ is exceptional and its first non-zero jet is one of the following webs $A^k_1$, $A^{k-2}_{III}$, $A^d_5$ (only when $k = 4$) and $A^b_6$ (only when $k = 5$).
Proof. Suppose that $\mathcal{W} = [\Omega]$ where $\Omega$ is a germ at the origin of a holomorphic $k$-symmetric 1-form. Consider the expansion of $\Omega$ in its homogeneous components:

$$\Omega = \sum_{i=0}^{\infty} \Omega_i$$

where $\Omega_i$ is a $k$-symmetric 1-form with homogenous coefficients of degree $i$. According to our assumptions $\Omega_0 \neq 0$ and $\Omega_0 = \prod_{i=1}^{k} dL_i$, where the $L_i$'s are linear forms defining the tangent spaces of the leaves of $\mathcal{W}$ at the origin.

Similarly, suppose that $\mathcal{F} = [\omega]$ where $\omega$ is a germ of holomorphic 1-form with codimension two zero set. Let

$$\omega = \sum_{i=i_0}^{\infty} \omega_i \quad \omega_{i_0} \neq 0$$

be the expansion of $\omega$ in its homogeneous components, with $i_0 > 0$ according to the hypothesis made on $\mathcal{F}$. If $\alpha_t(x, y) = (tx, ty)$, then

$$W(x, y, t) = \frac{\alpha_t^*(\Omega \cdot \omega)}{t^{k+i_0+2}} = \left( \sum_{i=0}^{\infty} t^i \Omega_i \right) \left( \sum_{i=i_0}^{\infty} t^{i-i_0} \omega_i \right) = \Omega_0 \cdot \omega_{i_0} + t(\cdots)$$

is an element of $\text{Sym}^k \Omega^1(\mathbb{C}^2)$ with coefficients in $\mathcal{O} = \mathbb{C}[x, y, t]$. For every $t \neq 0$, the web $W_t = [W(\cdot, \cdot, t)]$ is isomorphic to $\mathcal{W} \boxtimes \mathcal{F}$.

If $\omega_{i_0}$ is a multiple of a constant 1-form (equivalently if $\mathcal{F}_0 = [\omega_{i_0}]$ is a smooth foliation), then $\mathcal{F}$ must be like in item (1) of the statement. Notice that when $W(x, y, 0)$ does not define a $(k+1)$-web we are in this situation. Otherwise, the foliation $\mathcal{F}_0 = [\omega_{i_0}]$ has a singularity at the origin and $W(x, y, 0)$ is a $(k+1)$-web. Since for every $t \neq 0$, the web $W_t$ is of maximal rank, $W_0$ also has maximal rank thanks to Theorem 10.1. If $\mathcal{F}_0$ is linear, then we are in case (2) of the statement. Otherwise, $W_0 = [\Omega_0] \boxtimes \mathcal{F}_0$ is the product of a parallel $k$-web with a non-linear foliation. Since $k \geq 4$, Proposition 2.1 implies that $W_0$ is exceptional. Therefore, it must be one of the thirteen sporadic exceptional CDQL webs or it belongs to one of the four infinite families of exceptional CDQL webs. The only ones that are the product of a parallel web with a non-linear foliation are listed in (3). □

From Remark 10.1 and the classification of flat CDQL webs on $\mathbb{P}^2$ obtained in Section 9, it follows an analog for flat webs of Corollary 10.1: one has just to replace “with maximal rank” by “flat” and add the web $A_{k-4}^4$ in the last line of point (3).
11 ...and Back: Quasi-linear Webs on Complex Tori

11.1 First integrals of linear foliations on tori

Let $T$ be a two-dimensional complex torus. The set of linear foliations on $T$ is naturally identified with the one-dimensional projective space $\mathbb{P} H^0(T, \Omega^1_T)$. We are interested in the set $\mathcal{I}(T) \subset \mathbb{P} H^0(T, \Omega^1_T)$ corresponding to linear foliations which admit a holomorphic first integral.

**Proposition 11.1.** The cardinality $i(T)$ of $\mathcal{I}(T)$ is $0$, $1$, $2$ or $\infty$. Moreover,

1. if $i(T) = 0$, then $T$ is a simple complex torus;
2. if $i(T) = 1$, then $T$ is a non-algebraic complex torus;
3. if $i(T) = 2$, then $T$ is isogenous to the product of two non-isogenous elliptic curves;
4. if $i(T) = \infty$, then $T$ is isogenous to the square of an elliptic curve $E$. Moreover, if $\omega_1, \omega_2$ is a pair of linearly independent 1-forms on $T$ admitting rational first integrals then

$$\left\{ \lambda \in \mathbb{C} \mid \omega_1 + \lambda \omega_2 \text{ has a holomorphic first integral} \right\} = \text{End}(E) \otimes \mathbb{Q}. \quad \square$$

**Proof.** Let $\mathcal{F}$ be a linear foliation on $T$. It is induced by a 1-form with constant coefficients $\omega = sdx + tdy$ on $\mathbb{C}^2$ viewed here as the universal covering of $T$.

Notice that $\omega$ is invariant by the action of $T$ on itself. Therefore, since this action is transitive, $\mathcal{F}$ admits a compact leaf if and only if it has a compact leaf through $0$. Notice also that a compact leaf is nothing more than a subtorus of $T$. Reciprocally if $T$ contains a subtorus $T'$, then translations of $T'$ by elements in $T$ form a linear foliation on $T$ admitting a holomorphic first integral given by the quotient map $T \to T/T'$.

Therefore if $i(T)$ is equal to zero, $T$ has no closed subgroups of dimension one that is, $T$ is a simple complex torus. If $i(T)$ is equal to one, then $T$ admits exactly one closed subgroup of dimension one. This implies that $T$ is non-algebraic otherwise $T$ would be isogenous to a product of two elliptic curves (according to Poincaré’s reducibility Theorem) and, consequently, $i(T) > 1$. If $i(T) = 2$, then $T$ admits two closed subgroup $T'$ and $T''$ of dimension one. The natural map

$$(x, y) \in T' \times T'' \mapsto x + y \in T$$
has finite kernel equal to $T' \cap T''$ therefore there is an isogeny between $T' \times T''$ and $T$.

Notice that $T'$ can’t be isogenous to $T''$ otherwise $I(T) = I(T' \times T'') = I(T' \times T')$ and the latter set has infinite cardinality since it is invariant under the induced action of $\text{Aut}(T' \times T') \supseteq \text{PSL}(2, \text{End}(T')) \supseteq \text{PSL}(2, \mathbb{Z})$ on $\mathbb{P}H^0(T, \Omega^1_T) \simeq \mathbb{P}^1$.

If $I(T)$ has cardinality at least three, then there exist three pairwise distinct subtorus $T'$, $T''$, and $T'''$ passing through the origin of $T$. As before, one can get that $T$ is isogenous to $T' \times T''$. The existence of the natural projections $T''' \to T/T'$ and $T''' \to T/T''$ implies that all the three curves are isogenous. Moreover, up to an isogeny, $T$ can be assumed to be $T' \times T'$ with $T'$, $T''$, and $T'''$ identified with the horizontal, vertical, and diagonal subtori, respectively. It follows that $I(T)$ is an orbit of the natural action of $\text{PGL}(2, \text{End}(T'))$, hence $i(T) = \infty$.

\[\blacksquare\]

**Remark 11.1.** Item (4) of Proposition 11.1 can be traced back to Abel, see [2, §X]. According to Markushevich [29, page 158], it is the first appearance of the so-called complex multiplication in the theory of elliptic functions.

\[\blacksquare\]

**Lemma 11.1.** Let $T$ be a complex torus isogenous to the square of an elliptic curve $E$. If $[\omega_1], \ldots, [\omega_4] \in \mathbb{P}H^0(T, \Omega^1_T)$ are linear foliations on $T$ with holomorphic first integral, then the cross-ratio $([\omega_1], [\omega_2] : [\omega_3], [\omega_4])$ belongs to $\text{End}(E) \otimes \mathbb{Q}$.

\[\blacksquare\]

**Proof.** According to the proof of Proposition 11.1, we can assume that $T = E \times E$, $\omega_2 = dx - dy$, $\omega_3 = dy$, and $\omega_4 = dx$. Since the leaves of $\omega_1$ are algebraic, they must be translates of $E_{\alpha, \beta}$ (defined by (10) in Section 4.2) for suitable $\alpha, \beta \in \text{End}(E)$. Thus, $\omega_1 = [\beta dx - \alpha dy]$. Therefore,

$$([\omega_1], [\omega_2] : [\omega_3], [\omega_4]) = \frac{\beta}{\alpha}$$

and the lemma follows.

\[\blacksquare\]

**11.2 Flat CDQL webs on complex tori**

Let $\mathcal{W}$ be a linear $k$-web on $T$. Clearly, it is a completely decomposable web. Thus, we can write $\mathcal{W} = \mathcal{L}_1 \boxtimes \cdots \boxtimes \mathcal{L}_k$ where the $\mathcal{L}_i$’s are linear foliations. For $i = 1, \ldots, k$, set $\hat{\mathcal{L}}_i = \cdots$
\[ \beta_{\mathcal{L}_i}(\mathcal{W} - \mathcal{L}_i) \] and define the polar map of a foliation \( \mathcal{F} \) on \( T \) as the rational map \( P_\mathcal{F} : T \to \mathbb{P}^0(T, \Omega^1_T) \) characterized by the property

\[ P^{-1}_\mathcal{F}(\mathcal{L}) = \text{tang}(\mathcal{F}, \mathcal{L}) \]

for every \( \mathcal{L} \in \mathbb{P}^0(T, \Omega^1_T) \).

Recall from the Introduction that a fiber of a rational map from a two-dimensional complex torus on to a curve is linear if it is set-theoretically equal to a union of subtori.

**Lemma 11.2.** Let \( \mathcal{W} = \mathcal{L}_1 \boxplus \cdots \boxplus \mathcal{L}_k \) be a linear \( k \)-web on \( T \), with \( k \geq 2 \). If \( \mathcal{F} \) is a non-linear foliation on \( T \) such that \( K(\mathcal{W} \boxplus \mathcal{F}) = 0 \), then the rational map \( P_\mathcal{F} \) has at least \( k \) linear fibers, one for each \( \mathcal{L}_i \). Moreover, if \( k \geq 3 \) then each of the fibers \( P^{-1}_\mathcal{F}(\mathcal{L}_i) \) contains at least one elliptic curve invariant by \( \mathcal{L}_i \) and at least one invariant by \( \hat{\mathcal{L}}_i \).

**Proof.** By Theorem 7.2, any irreducible component of \( \text{tang}(\mathcal{F}, \mathcal{L}_i) \) is \( \mathcal{L}_i \) or \( \hat{\mathcal{L}}_i \) invariant. Since \( \mathcal{L}_i \) and \( \hat{\mathcal{L}}_i \) are linear foliations, it follows that the fibers \( P^{-1}_\mathcal{F}(\mathcal{L}_i) \) are linear for \( i = 1, \ldots, k \). This proves the first part of the lemma.

Suppose now that \( k \geq 3 \). Aiming at a contradiction, assume that all the irreducible components of \( \text{tang}(\mathcal{F}, \mathcal{L}_1) \) are \( \hat{\mathcal{L}}_1 \) invariant. Proposition 11.1 implies that \( \hat{\mathcal{L}}_1 \) is tangent to an elliptic fibration.

Since both \( K_T \) and \( NL_i \) are trivial, \( \mathcal{O}_T(\text{tang}(\mathcal{F}, \mathcal{L}_i)) = K_T \otimes N\mathcal{F} \otimes NL_i = N\mathcal{F} \) for every \( i = 1, \ldots, k \). Taking \( i = 1 \), we get that \( N\mathcal{F} \) is linearly equivalent to a divisor supported on some fibers of the fibration \( \hat{\mathcal{L}}_1 \). Taking \( i = 2, \ldots, k \), we see that the divisors \( \text{tang}(\mathcal{F}, \mathcal{L}_i) \) are linearly equivalent to \( N\mathcal{F} \) and consequently to \( \text{tang}(\mathcal{F}, \mathcal{L}_1) \). Therefore, being all of them effective, they also have to be supported on elliptic curves invariant by \( \hat{\mathcal{L}}_1 \).

Since two distinct linear foliations on \( T \) are everywhere transverse, Theorem 7.2 implies that for every \( i = 2, \ldots, k \), \( \mathcal{L}_i \) or \( \hat{\mathcal{L}}_i \) is equal to \( \hat{\mathcal{L}}_1 \). By hypothesis the linear foliations \( \mathcal{L}_1, \ldots, \mathcal{L}_k \) are pairwise distinct. Therefore, at least \( k - 1 \) of the foliations \( \hat{\mathcal{L}}_i \) (\( i = 1, \ldots, k \)) coincide. This contradicts Corollary 5.1.

If one assumes that all the irreducible components of \( \text{tang}(\mathcal{F}, \mathcal{L}_1) \) are invariant by \( \mathcal{L}_1 \), then the same argument with minor modifications also leads to a contradiction. The lemma follows.
Proposition 11.2. Let $\mathcal{W} = \mathcal{L}_1 \boxtimes \cdots \boxtimes \mathcal{L}_k$ be a linear $k$-web on $T$, with $k \geq 3$. If $\mathcal{F}$ is a non-linear foliation on $T$ such that $K(\mathcal{W} \boxtimes \mathcal{F}) = 0$, then

1. $T$ is isogenous to the square of an elliptic curve. In particular, $T$ is an abelian surface;
2. the foliations $\mathcal{L}_1, \ldots, \mathcal{L}_k$ are tangent to elliptic fibrations;
3. the foliations $\hat{\mathcal{L}}_1, \ldots, \hat{\mathcal{L}}_k$ are tangent to elliptic fibrations;
4. $\mathcal{P}_\mathcal{F}$ has $k$ linear fibers.

\[\square\]

Proof. The points (2), (3) and (4) follow from Lemma 11.2 since a linear foliation on $T$ is tangent to an elliptic fibration if and only if it leaves an elliptic curve invariant. Since $k \geq 3$, Proposition 11.1 implies (1).

\[\square\]

11.3 On the number of linear fibers of a pencil on a complex torus

Let $F : T \dashrightarrow \mathbb{P}^1$ be a meromorphic map on a two-dimensional complex torus $T$. We are interested in the number $k$ of linear fibers of $F$.

Theorem 11.1. (Theorem 1.4 of the Introduction) If $k$ is finite then $k \leq 6$. Moreover, if $k = 6$ then every fiber of $F$ is reduced.

\[\square\]

Proof. If $x, y$ are homogeneous coordinates on $\mathbb{P}^1$ then $xdy - ydx \in H^0(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1} \otimes O_{\mathbb{P}^1}(2))$. Therefore, $\omega = F^*(xdy - ydx) \in H^0(T, \Omega^1_T \otimes N^{\otimes 2})$ with $N = F^*O_{\mathbb{P}^1}(1)$.

Let also $X \in H^0(T, T_T \otimes (N^*)^{\otimes 2})$ be dual to $\omega$, that is $\omega = i_X \Omega$ where $\Omega$ is a non-zero global holomorphic 2-form on $T$. The twisted vector field $X$ can be represented by a covering of $\mathcal{U} = \{U_i\}$ of $T$ and holomorphic vector fields $X_i \in T_T(U_i)$ subject to the conditions

$$X_i = g_{ij} X_j$$

on any non-empty $U_i \cap U_j$, where $\{g_{ij}\}$ is a cocycle in $H^1(\mathcal{U}, O^*_T)$ representing $N^{\otimes 2}$.

If $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial w}$ form a basis of $H^0(T, T_T)$, then $X_i = A_i \frac{\partial}{\partial z} + B_i \frac{\partial}{\partial w}$ for suitable holomorphic functions $A_i, B_i \in O(U_i)$. Consider the divisor $\Delta$ locally cut out by

$$\det \begin{pmatrix} A_i & B_i \\ X_i(A_i) & X_i(B_i) \end{pmatrix}.$$ 

Clearly, these local expressions patch together to form an element of $H^0(T, N^{\otimes 6})$. 

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Any divisor corresponding to a fiber of $F$ is defined by the vanishing of a non-zero element of $H^0(T, F^*O_{\mathbb{P}^1}(1)) = H^0(T, N)$. By the very definition of $X$, (the closures of) its one-dimensional orbits are irreducible components of fibers of $F$. Outside the zero locus of $X_i$, the divisor $\Delta_i|_{U_i}$ corresponds to the inflection points of the orbits of $X_i$. Indeed, if $\gamma : (\mathbb{C}, 0) \to U_i$ is an orbit of $X_i$, that is if $X_i(\gamma(t)) = \gamma'(t)$ for $t \in (\mathbb{C}, 0)$, then (with an obvious abuse of notation)

$$\det \left( \begin{array}{cc} A_i & B_i \\ X_i(A_i) & X_i(B_i) \end{array} \right)(\gamma) \equiv \gamma' \wedge \gamma''.$$  

Let $L$ be a linear irreducible component of a fiber of $F$. Its generic point belongs to $\Delta$ since it is an inflection point of $L$ relatively to $X$ (see [36, §6]). It follows that $L \leq \Delta$.

From the preceding discussion, it follows that to prove the theorem it suffices to show that any effective divisor $D_1, \ldots, D_k$ corresponding to a linear fiber of $F$ is smaller than $\Delta$. Indeed, the support of distinct fibers of $F$ does not share irreducible components in common and consequently

$$\sum_{i=1}^{k} D_i \leq \Delta.$$  

Since $\sum_{i=1}^{k} D_i$ is defined by the vanishing of an element in $H^0(T, N \otimes k)$ while $\Delta$ is defined by an element in $H^0(T, N \otimes 6)$, it would follow that $k \leq 6$ as wanted. It remains to show that $D_i \leq \Delta$ for any $i = 1, \ldots, k$.

The divisorial components of the zero locus of $X_i$ correspond to multiple components of the fibers of $F$ just like in Darboux’s Lemma 9.3. If there is a fiber of $F$ containing an irreducible component with multiplicity $a \geq 2$ and locally cut out over $U_i$ by a reduced holomorphic function $f$ then we can write $X_i = f^{a-1} \tilde{X}_i$ with $\tilde{X}_i = \tilde{A}_i \frac{\partial}{\partial z} + \tilde{B}_i \frac{\partial}{\partial w}$ holomorphic. Therefore, $\Delta$ is locally defined by

$$\det \left( \begin{array}{cc} f^{a-1} \tilde{A}_i & f^{a-1} \tilde{B}_i \\ f^{a-1} \tilde{X}_i(f^{a-1} \tilde{A}_i) & f^{a-1} \tilde{X}_i(f^{a-1} \tilde{B}_i) \end{array} \right) = f^{3a-3} \det \left( \begin{array}{cc} \tilde{A}_i & \tilde{B}_i \\ \tilde{X}_i(\tilde{A}_i) & \tilde{X}_i(\tilde{B}_i) \end{array} \right).$$

Since $3a - 3 > a$ when $a \geq 2$, it follows that every linear fiber of $F$ is smaller than $\Delta$ as wanted. Moreover, if $k = 6$ then $F$ cannot have non-reduced fibers. 

Theorem 11.1 combined with Proposition 11.2 yields the following corollary.
Corollary 11.1. Let $\mathcal{W} = L_1 \times \cdots \times L_k$ be a linear $k$-web on $T$. If $\mathcal{F}$ is a non-linear foliation on $T$ such that $K(\mathcal{W} \otimes \mathcal{F}) = 0$, then $T$ is isogenous to the square of an elliptic curve and $k \leq 6$. □

11.4 Constraints on the linear web

Let $\mathcal{W} \otimes \mathcal{F}$ be a flat CDQL $(k + 1)$-web on a complex torus $T$. If $P_{\mathcal{F}}$ denotes the polar map of $\mathcal{F}$ and if $\mathcal{W} = L_1 \times \cdots \times L_k$ then the fibers $P_{\mathcal{F}}^{-1}(L_i)$ are all linear and supported on a union of elliptic curves invariant by $L_i$ or by $\hat{L}_i$ according to Proposition 11.2. From the very definition of $P_{\mathcal{F}}$, it is clear that the singular set of $\mathcal{F}$ coincides with the indeterminacy set of $P_{\mathcal{F}}$.

In order to determine the linear web $\mathcal{W}$ under the assumption that $\mathcal{W} \otimes \mathcal{F}$ has maximal rank, we will take a closer look at the singularities of $\mathcal{F}$. It will be convenient to consider the natural affine coordinates $(x, y)$ on the universal covering $\mathbb{C}^2 \rightarrow T$.

Lemma 11.3. Let $\mathcal{W} = L_1 \times \cdots \times L_k$ be a linear $k$-web, with $k \geq 3$, and $\mathcal{F}$ be a non-linear foliation, both defined on $T$. Suppose that $K(\mathcal{W} \otimes \mathcal{F}) = 0$. If $p \in \text{sing}(\mathcal{F})$ is the origin in the affine coordinate system $(x, y)$ then one of the following two alternatives holds:

1. the foliation $\mathcal{F}$ is locally given by $[xy - ydx + h.o.t.]$. In this case, for each $i = 1, \ldots, k$, the divisor $\text{tang}(\mathcal{F}, L_i)$ has multiplicity one at $p$ and there exists an elliptic curve through $p$ invariant by $L_i$ and by $\mathcal{F}$.
2. the foliation $\mathcal{F}$ is locally given by $[\omega_d + h.o.t.]$ where $\omega_d$ is a non-zero homogeneous 1-form of degree $d \geq 1$ in the coordinates $x, y$ with singular set reduced to $(0, 0)$ and not proportional to $xy - ydx$. In particular, the foliation $[\omega_d]$ is non-linear. □

Proof. According to the proof of Proposition 11.1, one can assume that $L_1 = [dx]$, $L_2 = [dy]$, and $L_3 = [dx - dy]$. If $\mathcal{F}$ is locally given by $[a(x, y)dx + b(x, y)dy]$ where $a$ and $b$ are holomorphic functions without common factors, then $\text{tang}(\mathcal{F}, L_1) = \{ b = 0 \}$, $\text{tang}(\mathcal{F}, L_2) = \{ a = 0 \}$, and $\text{tang}(\mathcal{F}, L_3) = \{ a + b = 0 \}$. Notice that the assumption $p \in \text{sing}(\mathcal{F})$ implies that $a(0, 0) = b(0, 0) = 0$.

Recall from Proposition 11.2 that $\text{tang}(\mathcal{F}, L_1)$ is supported on a union of elliptic curves. Therefore, the first non-zero jet of $b$ will be a constant multiple of $x^k \cdot h(x, y)^l$, $k, l \in \mathbb{N}$, where $h$ is a linear form vanishing defining the tangent space of $sL_1$ at zero. Similarly for $a$ and $a + b$. 
The first non-zero jet of $a, b$ and $a + b$ have the same degree and are pairwise without common factor. Otherwise, the supports of $\text{tang}(F, L_i)$ and $\text{tang}(F, L_j)$ would share an irreducible component in common for some pair $(i, j)$ satisfying $1 \leq i < j \leq 3$. But this is impossible since $\text{tang}(L_i, L_j)$ is empty as soon as $i \neq j$.

Thus, we can write $\omega = \omega_d + h.o.t.$ where $\omega_d$ is homogeneous and with singular set equal to the origin. We are in the first case of the statement when $\omega_d$ is proportional to $x dy - y dx$ and in the second case otherwise.

We are now in position to use Corollary 10.1 to restrict the possibilities of the maximal linear subweb of an exceptional CDQL web on complex tori.

**Proposition 11.3.** Let $\mathcal{W} = L_1 \boxtimes \cdots \boxtimes L_k$ be a linear $k$-web, with $k \geq 4$, and $F$ be a nonlinear foliation on $T$. If $\mathcal{W} \boxtimes F$ has maximal rank then, up to isogenies, one of the following alternatives holds:

1. the torus $T$ is the square of an elliptic curve, $k = 4$ and $\mathcal{W} = [dxdy(dx^2 - dy^2)];$
2. the torus $T$ is $E^2_i$, $k = 6$ and $\mathcal{W} = [dxdy(dx^2 - dy^2)(dx^2 + dy^2)];$
3. the torus $T$ is $E^2_\xi_3$, $k = 5$ and $\mathcal{W} = [dxdy(dx^3 + dy^3)];$
4. the torus $T$ is $E^2_\xi_3$, $k = 4$ and $\mathcal{W} = [dxdy(dx + dy)(dx - \xi_3 dy)].$

**Proof.** Corollary 11.1 tells us that $T$ is isogenous to the square of an elliptic curve $E$ and that $k \leq 6$. Corollary 5.1 implies that we can assume, after an eventual reordering, that $\hat{L}_1 \neq \hat{L}_2$. For $i = 1, 2$, let $\hat{E}_i$ be an elliptic curve contained in $\text{tang}(F, L_i)$ that is $\hat{L}_i$ invariant. Notice that the existence of these curves is ensured by Lemma 11.2.

Since $\hat{L}_1 \neq \hat{L}_2$, there exists $p \in \hat{E}_1 \cap \hat{E}_2$. Notice that $p$ belongs to $\text{sing}(F)$. Moreover, our choice of $p$ implies that it fits in the second alternative of Lemma 11.3. Therefore, we can apply Corollary 10.1 to conclude that the first non-zero jet of $\mathcal{W} \boxtimes L$ at $p$ is equivalent, under a linear change of the affine coordinates $(x, y)$, to one of the following webs:

$$A_4^4, A_1^5, A_1^6, A_3^2, A_3^3, A_3^4, A_5^d, A_6^b.$$  \hspace{1cm} (47)

To prove the proposition, we will analyze the constraints imposed on the torus $T$ by the above local models.
Notice that the 5-web $A^5_4 = [(dx^4 - dy^4)] \otimes [d(xy)]$ is isomorphic (via a linear map) to $[dxdy(dx^2 - dy^2)] \otimes [d(x^2 + y^2)]$. All the defining foliations of $[dxdy(dx^2 - dy^2)]$ are tangent to elliptic fibrations on the square of an arbitrary elliptic curve $E$. Therefore, these local models do not impose restrictions on the curve $E$. Similarly, the 5-web $A^2_3 = [dxdy(dx^2 - dy^2)] \otimes [d(xy)]$ also does not impose restrictions on $E$. Indeed, these two local models coexist in distinct singular points of the exceptional CDQL 5-webs $\mathcal{E}_5$.

The 6-webs $A^3_3 = [dxdy(dx^3 - dy^3)] \otimes [d(xy)]$ and $A^6_6 = [dxdy(dx^3 + dy^3)] \otimes [d(x^3 + y^2)]$ share the same linear 5-web (after the change of coordinates $(x, y) \mapsto (x, -y)$ on $A^6_6$). On the one hand, Proposition 11.2 implies that all the defining foliations of the linear 5-web $[dxdy(dx^3 - dy^3)]$ must be tangent to elliptic fibrations. On the other hand, Lemma 11.1 implies that $\xi_3 \in \text{End}(E) \otimes \mathbb{Q}$. Therefore, $T$ must be isogenous to $E^2_{\xi_3}$. Notice that both local models coexist in distinct singular points of the exceptional CDQL 6-web $\mathcal{E}_6$.

The same argument shows that the 5-web $A^5_6 = [dxdy(dx + dy)(dx - \xi_3 dy)] \otimes [d(xy(x + y)(x - \xi_3 y))]$ can only be a local model for an exceptional CDQL web when $T$ is isogenous to $E^2_{\xi_3}$. Similarly, the 7-web $A^4_3 = [dxdy(dx^4 - dy^4)] \otimes [d(xy)]$ can only be a local model for an exceptional CDQL web when $T$ is isogenous to $E^2_{\xi_4}$.

To conclude the proof of the Proposition, it suffices to show that the two remaining possibilities in the list (47) (namely $A^5_1$ and $A^6_1$) cannot appear as local models for exceptional CDQL webs on a torus.

We will first deal with the 6-web $A^5_1 = [(dx^5 - dy^5)] \otimes [d(xy)]$. If $\xi_5$ is a primitive 5th root of the unity then the cross-ratio $(1, \xi_5 : \xi_5^2, \xi_5^3)$ is a root of the polynomial $p(x) = x^2 - x - 1$. Notice that the roots of $p(x)$ are the golden ratio and its conjugate: $1/2 \pm \sqrt{5}/2$. In particular, they are irrational real numbers and, as such, cannot induce an endomorphism on any elliptic curve $E$. Lemma 11.1 implies that a two-dimensional complex torus $T$ where all the defining foliations of $[(dx^5 - dy^5)]$ are tangent to elliptic fibrations does not exist. Proposition 11.2 implies that $A^5_1$ cannot appear as a local model of an exceptional CDQL web on a torus.

We also claim that the 7-web $A^6_1 = [(dx^6 - dy^6)] \otimes [d(xy)]$ cannot appear as a local model of an exceptional CDQL web on a torus $T$. Using Lemma 11.1, it is a simple matter to show that $T$ is isogenous to $E^2_{\xi_3}$. Assume now that $L_1$ and $L_2$ are such that $L_1 \neq L_2$. Lemma 11.2 ensures that there are: an elliptic curve $E_1$ in $\text{tang}(F, L_1)$ invariant by $L_1$ and an elliptic curve $E_2$ in $\text{tang}(F, L_2)$ invariant by $L_2$. Since $L_1 \neq L_2$, there exits $p \in E_1 \cap E_2$. Since $p \in |\text{tang}(F, L_1)| \cap |\text{tang}(F, L_2)|$, it is a singular point of $F$.

Notice that our choice of $p$ implies that the first non-zero jet of $F$ at $p$ is non-linear, see Lemma 11.3. Since $E_1$ is also $F$ invariant, the linear polynomial defining it
on the affine coordinates \((x, y)\) will be also invariant by the first jet of \(\mathcal{F}\). But for the 7-web \(\mathcal{A}^6_{\mathcal{II}}\) none of the invariant lines through 0 of the non-linear foliation is invariant by any of the linear foliations. Therefore, the local model at \(p\) must be the only other 7-web appearing in the list (47): \(\mathcal{A}^6_{\mathcal{II}} = [\text{d}x \text{d}y (\text{d}x^4 - \text{d}y^4)] \boxtimes [\text{d}(xy)]\). But this implies that \(T\) is isogenous to \(E_i\). Since \(E_i\) is not isogenous to \(E_{i+3}\), the claim follows and so does the proposition.

11.5 The classification of exceptional CDQL webs on tori

To obtain the classification of exceptional CDQL webs on tori, we will analyze in Sections 11.5.1, 11.5.2, 11.5.3, and 11.5.4 the respective alternatives (1), (2), (3), and (4) provided by Proposition 11.3.

11.5.1 The continuous family of exceptional CDQL 5-webs

In case (1) of Proposition 11.3, the torus \(T\) is isogenous to the square of an elliptic curve, \(k = 4\) and the linear web is \(\mathcal{W} = [\text{d}x \text{d}y (\text{d}x^2 - \text{d}y^2)]\). As we have proved in Example 7.1, every flat (in particular exceptional) CDQL 5-web of the form \(\mathcal{W} \boxtimes \mathcal{F}\) must be isogenous to one of the 5-webs \(\mathcal{E}_\tau\) (with \(\tau \in \mathbb{H}\)) presented in the Introduction.

11.5.2 The exceptional CDQL 7-web on \(E_i^2\)

In the second alternative of Proposition 11.3, the torus \(T\) is isogenous to \(E_i^2\), \(k = 6\) and the linear web is \(\mathcal{W} = \mathcal{W}_1 \boxtimes \mathcal{W}_2\) where \(\mathcal{W}_1 = [\text{d}x \text{d}y (\text{d}x^2 - \text{d}y^2)]\) and \(\mathcal{W}_2 = [\text{d}x^2 + \text{d}y^2]\). This decomposition of \(\mathcal{W}\) satisfies the hypothesis of Corollary 7.1. Therefore, a non-linear foliation \(\mathcal{F}\) satisfies \(K(\mathcal{F} \boxtimes \mathcal{W}) = 0\) if and only if \(K(\mathcal{F} \boxtimes \mathcal{W}_1) = K(\mathcal{F} \boxtimes \mathcal{W}_2) = 0\). Thus, the subweb \(\mathcal{F} \boxtimes \mathcal{W}_1\) is isogenous to a web of the continuous family \(\mathcal{E}_\tau\). We lose no generality by assuming that \(\mathcal{F} \boxtimes \mathcal{W}_1 = \mathcal{E}_\tau\) for some \(\tau \in \mathbb{H}\). It remains to determine \(\tau\). Since \(T\) is isogenous to \(E_i^2\), we know that \(\tau = \alpha + \beta i\) for suitable rational numbers \(\alpha, \beta\). Set \(\Gamma = \mathbb{Z} \oplus (\alpha + \beta i)\mathbb{Z}\).

Recall from Section 4.2 that the non-linear foliation \(\mathcal{F}\) is equal to \([\text{d}\mathcal{F}_\tau]\) where

\[
\mathcal{F}_\tau(x, y) = \left(\frac{\vartheta_1(x, \tau)\vartheta_1(y, \tau)}{\vartheta_4(x, \tau)\vartheta_4(y, \tau)}\right)^2.
\]

Recall also that \(\text{Indet}(\mathcal{F}_\tau) = \{(\tau/2, 0), (0, \tau/2)\}\) and that these indeterminacy points correspond to radial singularities of \(\mathcal{F}\).
The tangency of $\mathcal{F}$ with the linear foliation $[dx + idy]$ at $(0, \tau/2)$ has first non-zero jet equal to $(x + iy)$ since $(xdy - ydx) \wedge (dx + idy) = -(x + iy)dx \wedge dy.$ Therefore, Theorem 7.2 implies that there exists an elliptic curve $C$ through $(0, \tau/2)$ invariant by $\mathcal{F}$ and by $[dx + idy].$ Notice that $C$ is the image of the entire map

$$\varphi : C \rightarrow E^2_\tau = (C/\Gamma)^2$$

$$z \mapsto (-iz, z + \tau/2).$$

Thus, $C \cap E_{0,1} = \varphi(i\Gamma).$ The curve $E_{0,1}$ is also $\mathcal{F}$ invariant (but does not coincide with $C$) so the set $C \cap E_{0,1}$ is contained in $\text{sing}(\mathcal{F}).$ But the singularities of $\mathcal{F}$ over $E_{0,1}$ are $(0,0)$ and $(0, \tau/2).$ Moreover, the singularity at $(0,0)$ has only two separatrices, namely $E_{1,0}$ and $E_{0,1}.$ It follows that $C \cap E_{0,1} = \varphi(i\Gamma)$ is equal to the radial singularity $(0, \tau/2)$ of $\mathcal{F}$ on $E_{0,1}.$ Therefore, $i\Gamma + \tau/2 \subset \Gamma + \tau/2.$ Consequently, $i\Gamma \subset \Gamma$ and $-\Gamma \subset i\Gamma.$ Thus, $i\Gamma = \Gamma,$ that is $i \in \text{Aut}(E_\tau).$ This is sufficient to show that the elliptic curve $E_\tau$ is isomorphic to $E_i.$

Recall that

$$\Gamma_0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) \mid b \equiv 0 \mod 2 \right\}.$$

Thus, modulo the action of $\Gamma_0(2)$ we can assume that $\tau \in \{i, 1 + i, (1 + i)/2\}.$ Moreover, the $\mathbb{Z}_2$-extension of $\Gamma(2)$ by the transformation $z \mapsto -2/z$ identifies $1 + i$ with $(1 + i)/2$ because $-2((1 + i)/2)^{-1} = -2 + 2i.$ Therefore, we can assume that $\mathcal{F} \times \mathcal{W}_1$ is equal to $\mathcal{E}_{1+i}$ or to $\mathcal{E}_i.$ If $\tau = i,$ then $(i/2, 0)$ is a radial singularity of $\mathcal{F}$ and, as above, the curve $L_{(i/2,0)}E_{1,i}$ invariant by $[dx + idy]$ is also $\mathcal{F}$ invariant. But this curve intersects the $\mathcal{F}$-invariant curve $E_{0,1}$ at $(0, 1/2)$ which is not a singularity of $\mathcal{F}.$ This contradiction implies that, up to isogenies, $\mathcal{E}_7 = [dx^2 + dy^2] \boxtimes \mathcal{E}_{1+i}$ is the unique exceptional CDQL 7-web on complex tori.

### 11.5.3 The exceptional CDQL 6-web on $E^2_{\xi_3}$

In the third alternative of Proposition 11.3, the torus $T$ is isogenous to $E^2_{\xi_3}, k = 5$ and the linear web is $\mathcal{W} = \mathcal{W}_1 \boxtimes \mathcal{W}_2$ with $\mathcal{W}_1 = [dx dy]$ and $\mathcal{W}_2 = [(dx^2 + dy^2)].$ As in the previous case, this decomposition satisfies the hypothesis of Corollary 7.1. Therefore, $\mathcal{F}$ is a non-linear foliation on $T$ satisfying $K(\mathcal{W} \boxtimes \mathcal{F}) = 0$ if and only if $K(\mathcal{W}_1 \boxtimes \mathcal{F}) = K(\mathcal{W}_2 \boxtimes \mathcal{F}) = 0.$
If \( K(\mathcal{F} \boxtimes [dx\,dy]) = 0 \), then Theorem 7.2 (see also [41]) implies that \( \mathcal{F} = [a(x)dx + b(y)dy] \) for suitable rational functions \( a, b \in \mathbb{C}(E_{\xi_3}) \). Moreover, according to item (3) of Corollary 10.1, we can assume that the singularity of \( \mathcal{W} \boxtimes \mathcal{F} \) at \((0, 0)\) has first non-zero jet equivalent to \( A^b_6 = [dx\,dy(dx^3 + dy^3)] \boxtimes [x^2dx + y^2dy] \). In particular, interpreting \( x, y \) as coordinates on the universal covering of \( T \), we can assume that the meromorphic functions \( a, b \) satisfy \( a(x) = x^2 + O(x^3) \) and \( b(y) = y^2 + O(y^3) \). In particular, \( a(0) = a'(0) = b(0) = b'(0) = 0 \).

A tedious (but trivial) computation shows that \( K(\mathcal{F} \boxtimes [dx^2 + dy^2]) \) is equal to

\[
\frac{ba^3(aa'' - 2a^2) - ab^3(bb'' - 2b^2) + a^4(b^2 + bb'') - b^4(a^2 + aa''}){(a^3 - b^3)^2} \, dx \wedge dy.
\]

After deriving twice the numerator with respect to \( y \), one obtains

\[
a^3b'' \left( 2(a')^2 - a a'' - 3 b''a \right) - 4 a^4b'b''' + bR
\]

where \( R \) is a polynomial in \( a(x), b(y) \) and their derivatives up to order four. Evaluation of this expression at \( y = 0 \) yields the following second-order differential equation identically satisfied by \( a \):

\[
a^3(2(a')^2 - aa'' - 6a) = 0. \tag{48}
\]

Lemma 11.4. If \( a : (C, 0) \to C \) is a germ of solution of (48) satisfying the boundary conditions \( a(0) = a'(0) = 0 \) and \( a''(0) = 2 \), then

\[
a(x) = x^2 \quad \text{or} \quad a(x) = \frac{\lambda^2}{\wp(\lambda^{-1}x, \xi_3)}
\]

for a suitable \( \lambda \in \mathbb{C}^* \). \( \square \)

Proof. Notice that the 6-web \([dx\,dy(dx^3 + dy^3)] \boxtimes [a(x)dx + a(y)dy] \) with \( a(x) = x^2 \) is the 6-web \( A^b_6 \) from the Introduction. Similarly, when \( a(x) = \lambda^2/\wp(\lambda^{-1}x, \xi_3) \) then the 6-web \([dx\,dy(dx^3 + dy^3)] \boxtimes [a(x)dx + a(y)dy] \) can be obtained from \( E_6 \) by the change of coordinates \((x, y) \mapsto (\lambda x, \lambda y) \). Since both \( E_6 \) and \( A^b_6 \) are exceptional, the corresponding \( a \)'s are solutions of (48). Clearly, they all satisfy the boundary conditions. To prove the lemma, it suffices to verify that they are the only solutions.
If \(a(x)\) is a solution of (48) satisfying the boundary conditions, then it is indeed a solution of \(2(a')^2 - a a'' - 6a = 0\). Therefore, \(\gamma(t) = (a(t), a'(t))\) is an orbit of the following vector field

\[
Z(x, y) = y \frac{\partial}{\partial x} + \frac{2y^2 - 6x}{x} \frac{\partial}{\partial y},
\]

that is \(Z(\gamma(t)) = \gamma'(t)\).

Notice that \(Z\) admits as a rational first integral the function \(\frac{y^2 - 4x}{x^4}\). Therefore, every solution \(a(x)\) of (48) satisfying \(a(0) = a'(0) = 0\) and \(a''(0) = 2\) must parameterize (through the map \(t \mapsto (a(t), a'(t))\)) a branch of one of the curves \(y^2 - 4x + \mu x^4\) for some \(\mu \in \mathbb{C}\). When \(\mu = 0\), the corresponding curve is parameterized by \(a(x) = x^2\). For \(\mu \neq 0\), it is parameterized by \(a(x) = \frac{\lambda^2}{\wp} (\lambda^{-1} x, \xi_3)\) with \(\lambda\) satisfying \(\mu \lambda^6 = 1\). Notice that the different choices for \(\lambda\) lead to the same function \(a\). Indeed, the symmetry \(-\xi_3 (Z \oplus \xi_3 Z) = Z \oplus \xi_3 Z\) combined with (13) implies that

\[
\frac{(-\xi_3)^2}{\wp((-\xi_3)^{-1} x, \xi_3)} = \frac{1}{\wp(x, \xi_3)}. \tag{49}
\]

Since each of the curves \(\{y^2 - 4x - \mu x^4 = 0\}\) admits only one parametrization of the form \(t \mapsto (a(t), a'(t))\) with \(a''(0) = 2\), the lemma follows.

Keeping in mind that the coefficients of the defining 1-form of \(\mathcal{F}\) must be doubly periodic functions and the symmetry of our setup, so far we have proved that \(K(\mathcal{F} \boxtimes \mathcal{W}) = 0\) implies that, up to homotheties,

\[
\mathcal{F} = \left[ \frac{dx}{\wp(x, \xi_3)} + \frac{\lambda^2 dy}{\wp(\lambda^{-1} y, \xi_3)} \right]
\]

for a suitable \(\lambda \in \mathbb{C}^\ast\). Computing again, \(K(\mathcal{F} \boxtimes [dx^2 + dy^2])\) yields

\[
\frac{9\lambda^2(\lambda^6 - 1)\wp(x, \xi_3)^2 \wp(\lambda x, \xi_3)^2}{\lambda^{12} \wp(x, \xi_3)^6 - 2\lambda^6 \wp(\lambda x, \xi_3)^3 \wp(x, \xi_3)^3 + \wp(\lambda x, \xi_3)^6} \, dx \wedge dy.
\]

The vanishing of the curvature, taking into account (49), implies that

\[
\mathcal{F} = \left[ \frac{dx}{\wp(x, \xi_3)} + \frac{dy}{\wp(y, \xi_3)} \right].
\]

It follows that the 6-web \(\mathcal{F} \boxtimes \mathcal{W}\) is isogenous to the 6-web \(\mathcal{E}_6\) from the Introduction.
11.5.4 The exceptional CDQL 5-web on $E^2_{\xi^3}$. Combinatorial patchwork

In the last case of Proposition 11.3 (transformed via the change of coordinates $(x, y) \mapsto (y, -x)$), the complex torus $T$ is isogenous to $E^2_{\xi^3}$, $k = 4$ and the linear web $\mathcal{W}$ is $[dx dy(dx - dy)(\xi_3 dx + dy)]$. Unlike in the previous case, the web $\mathcal{W}$ does not admit a decomposition satisfying the hypothesis of Corollary 7.1. We have not succeeded in dealing with this case using analytic methods as in the previous section and in [41]. We were led to adopt a geometrical/combinatorial approach.

If $L_1 = [dx]$, $L_2 = [dy]$, $L_3 = [dx - dy]$, and $L_4 = [\xi_3 dx + dy]$, then straightforward computations using formula (22) show that

$$
\begin{align*}
\widehat{L}_1 &= [dx + (\xi_3^2 - 1) dy] \\
\widehat{L}_2 &= [(\xi_3 - 1) dx + dy] \\
\widehat{L}_3 &= [dx - \xi_3 dy] \\
\widehat{L}_4 &= [dx + \xi_3 dy].
\end{align*}
$$

For $i = 1, 2, 3, 4$, the leaves of $\widehat{L}_i$ are translates of the elliptic curve $\widehat{E}_i$ where

$$
\begin{align*}
\widehat{E}_1 &= E_{1-\xi_3^2,1}, \\
\widehat{E}_2 &= E_{1, 1-\xi_3}, \\
\widehat{E}_3 &= E_{\xi_3,1}, \\
\text{and} \\
\widehat{E}_4 &= E_{\xi_3, -1}.
\end{align*}
$$

Suppose that $\mathcal{F}$ is a non-linear foliation on $T$ such that $\mathcal{W} \boxtimes \mathcal{F}$ has maximal rank. According to Corollary 10.1 and taking into account the change of coordinates $(x, y) \mapsto (y, -x)$, there are only two possibilities for a singularity $p$ of $\mathcal{F}$: either $p$ is a radial singularity or the first non-zero jet of $\mathcal{F} \boxtimes \mathcal{W}$ at $p$ is equivalent to

$$[dx dy(dx - dy)(\xi_3 dx + dy) \boxtimes d(xy(x - y)(\xi_3 x + y))].$$

We will say that the former singularities are of type $A$ whereas the latter are of type $B$. We will write $\text{sing}^A(\mathcal{F})$ (resp. $\text{sing}^B(\mathcal{F})$) for the set of singularities of type $A$ (resp. of type $B$).

By the very definition, the first non-zero jet of $\mathcal{F}$ at a point $p \in \text{sing}^B(\mathcal{F})$ is

$$\mathcal{F}_0 = [d(xy(x - y)(\xi_3 x + y))].$$
Simple computations show that

\[
\text{tang}(\mathcal{F}_0, L_i) = \begin{cases} 
\{x(x + (\xi_3^2 - 1)y)^2 = 0\} & \text{when } i = 1 \\
\{y((\xi_3^3 - 1)x + y)^2 = 0\} & \text{when } i = 2 \\
\{(x - y)(x - \xi_3 y)^2 = 0\} & \text{when } i = 3 \\
\{(\xi_3 x + y)(x + \xi_3 y)^2 = 0\} & \text{when } i = 4.
\end{cases}
\]

Being aware of the first non-zero jets of the singularities of \(\mathcal{F}\), we are able to describe the first non-zero jets of \(\text{tang}(\mathcal{F}, L_i)\). This is the content of the two following lemmata.

**Lemma 11.5.** Let \(p \in \text{sing}^{A}(\mathcal{F})\). For every \(i \in \{1, \ldots, 4\}\), there is a unique irreducible component of the divisor \(\text{tang}(\mathcal{F}, L_i)\) passing through \(p\): it is an irreducible curve \(C\) invariant by \(L_i\). In particular, there is no \(\hat{L}_i\)-invariant curve in \(\text{tang}(\mathcal{F}, L_i)\) passing through \(p\). \(\Box\)

**Proof.** Since the first non-zero jet of \(\text{tang}(\mathcal{F}, L_i)\) at \(p\) coincides with \(\text{tang}([\text{d}x\text{d}y - \text{d}y\text{d}x], L_i)\), the lemma follows from Theorem 7.3. \(\blacksquare\)

**Lemma 11.6.** Let \(p \in \text{sing}^{B}(\mathcal{F})\). For every \(i \in \{1, \ldots, 4\}\), the divisor \(\text{tang}(\mathcal{F}, L_i)\) contains in its support two distinct irreducible curves \(C_i\) and \(\hat{C}_i\) both containing \(p\). Moreover, \(C_i\) (resp. \(\hat{C}_i\)) is invariant by \(L_i\) (resp. by \(\hat{L}_i\)). \(\Box\)

**Proof.** Since the first non-zero jet of \(\text{tang}(\mathcal{F}, L_i)\) at \(p\) coincides with \(\text{tang}(\mathcal{F}_0, L_i)\), the lemma follows from Theorem 7.3 combined with (50) and (51). \(\blacksquare\)

The core of our argument to characterize \(\mathcal{E}_5\) is contained in the next lemma.

**Lemma 11.7.** Let \(\mathcal{F}\) be a non-linear foliation on the torus \(T = E_{\xi_3}^2\). Suppose that the 5-web \(\mathcal{F} \circledast [\text{d}x\text{d}y(\text{d}x - \text{d}y)(\xi_3 \text{d}x + \text{d}y)]\) has maximal rank. If \(0 \in \text{sing}^{B}(\mathcal{F})\), then

(a) \((0, y) \in \text{sing}(\mathcal{F})\) if and only if \((y, 0) \in \text{sing}(\mathcal{F})\);

(b) If \((y, 0) \in \text{sing}(\mathcal{F})\), then \((2y, 0) \in \text{sing}^{B}(\mathcal{F})\);

(c) If \((y, 0) \in \text{sing}(\mathcal{F})\), then \((-\xi_3^2y, 0) \in \text{sing}(\mathcal{F})\);

(d) Both \(\text{sing}(\mathcal{F}) \cap E_{1,0}\) and \(\text{sing}^{B}(\mathcal{F}) \cap E_{1,0}\) are subgroups of \(E_{1,0}\). Similarly, \(\text{sing}(\mathcal{F}) \cap E_{0,1}\) and \(\text{sing}^{B}(\mathcal{F}) \cap E_{0,1}\) are subgroups of \(E_{0,1}\). \(\Box\)
Proof. To prove (a), we will use the curves $E_{0,1}, E_{1,0},$ and $E_{1,1}$ passing through $(0,0)$ that are $\mathcal{F}$ invariant (what is ensured by Lemma 11.7). If $(0,y) \in \text{sing}(\mathcal{F})$, then Lemma 11.5 implies that $L_{(0,y)}E_{1,0}$ is $\mathcal{F}$ invariant. Therefore, $L_{(0,y)}E_{1,0} \cap E_{1,1} = (y,y)$ is the intersection of two distinct leaves of $\mathcal{F}$. It follows that $(y,y) \in \text{sing}(\mathcal{F})$. Consequently, $L_{(y,y)}E_{0,1}$ is also $\mathcal{F}$ invariant. Since $(y,0) = L_{(y,y)}E_{0,1} \cap E_{1,0}$, item (a) follows.

To prove (b), start by noticing that $(0,y) \in \text{sing}(\mathcal{F})$ by (a). Therefore, $L_{(0,y)}E_{1,0}$ is $\mathcal{F}$ invariant according to Lemma 11.5. By hypothesis $(0,0) \in \text{sing}^B(\mathcal{F})$ thus Lemma 11.6 ensures that the elliptic curve $\widehat{E}_1 = E_{1-\xi_3^2,1}$ belongs to $\text{tang}(\mathcal{F}, L_1)$. The curve $L_{(0,y)}E_{1,0}$ being invariant by $L_2$ and $\mathcal{F}$ (since $(0,y) \in \text{sing}(\mathcal{F})$), it is necessarily an irreducible component of $\text{tang}(\mathcal{F}, L_2)$. As a consequence, the intersection $\widehat{E}_1 \cap L_{(0,y)}E_{1,0}$ is included in $\text{sing}^B(\mathcal{F})$. In particular, the point $p = ((1-\xi_3^2)y, y)$ belongs to $\text{sing}^B(\mathcal{F})$. Considering now the $\widehat{L}_3$-invariant curve through $p$, that is $L_pE_{\xi_3,1}$, we see that it intersects $E_{0,1}$ at $(2y,0)$. Thus, $(2y,0) \in \text{sing}^B(\mathcal{F})$ proving item (b).

To prove item (c), recall from the previous paragraph that $L_{(0,y)}E_{1,0}$ is $\mathcal{F}$ invariant. The curve $E_{1-\xi_3}$ intersects $L_{(0,y)}E_{1,0}$ at $p = (-\xi_3^2, y)$. Since $E_{1-\xi_3}$ is $\mathcal{F}$ invariant (by Lemma 11.5), it follows $p \in \text{sing}(\mathcal{F})$. Consequently, $L_pE_{0,1}$ is $\mathcal{F}$ invariant (again by Lemma 11.5) and $(-\xi_3^2, y, 0) = L_pE_{0,1} \cap E_{1,0} \in \text{sing}(\mathcal{F})$. Item (c) follows.

It remains to prove item (d). We will first prove that $S = \text{sing}(\mathcal{F}) \cap E_{1,0}$ is a subgroup of $E_{1,0}$. From item (c) it follows $(y,0) \in S$ if and only if $(-y,0) \in S$. Thus, it suffices to show that, given two elements $(y_1,0)$ and $(y_2,0)$ of $S$, their sum $(y_1 + y_2, 0)$ is also in $S$. Item (a) implies that $(0,y_2) \in \text{sing}(\mathcal{F})$ and consequently the curve $L_{(0,y_2)}E_{1,0}$ is $\mathcal{F}$ invariant (by Lemma 11.5). For the same reason the curve $L_{(y_1,0)}E_{1,1}$ is also $\mathcal{F}$ invariant thus the point $p = (y_2 + y_1, y_2) \in L_{(0,y_2)}E_{1,0} \cap L_{(y_1,0)}E_{1,1}$ belongs to $\text{sing}(\mathcal{F})$. Since $L_pE_{0,1}$ intersects $E_{0,1}$ at $(y_1 + y_2, 0)$ and because these two curves are $\mathcal{F}$ invariant, it follows that $(y_1 + y_2, 0) \in S$. Therefore, $\text{sing}(\mathcal{F}) \cap E_{1,0}$ is a subgroup of $E_{1,0}$.

Consider now the group homomorphism

$$S \rightarrow S$$

$$x \mapsto x + x.$$ 

Item (b) implies that its image is $\text{sing}^B(\mathcal{F}) \cap E_{1,0}$. Therefore, $\text{sing}^B(\mathcal{F}) \cap E_{1,0}$ is also a subgroup of $E_{1,0}$.

Mutatis mutandis we obtain the same statements for $\text{sing}(\mathcal{F}) \cap E_{0,1}$ and $\text{sing}^B(\mathcal{F}) \cap E_{0,1}$; both are subgroups of $E_{0,1}$. ■
Theorem 11.1. Let $\mathcal{F}$ be a non-linear foliation on $T = E^2_{\xi_3}$. If the 5-web $[dxdy(dx-dy)(\xi_3 dx + dy)] \otimes \mathcal{F}$ has maximal rank, then it is isogenous to $E_5$. \hfill \Box

Proof. Let us denote by $\equiv$ the numerical equivalence of divisors on $T$. Since $\mathcal{O}_T(\text{tang}(\mathcal{F}, \mathcal{L}_i)) = N\mathcal{F}$ for $i = 1, \ldots, 4$, all the divisors $\text{tang}(\mathcal{F}, \mathcal{L}_i)$ are pairwise linearly equivalent. Moreover, Theorem 7.2 implies that

$$\text{tang}(\mathcal{F}, \mathcal{L}_i) = a_i E_i + b_i \hat{E}_i$$

for $i = 1, \ldots, 4$, where $E_i$ and $\hat{E}_i$ are elliptic curves in $T$ invariant by $\mathcal{L}_i$ and $\hat{\mathcal{L}}_i$, respectively and $a_i$ and $b_i$ are non-negative integers. Indeed, Lemma 11.2 implies that $a_i$ and $b_i$ are positive integers. In particular, we obtain that

$$a_1 E_{0,1} + b_1 E_{1-\xi_3,1} \equiv a_2 E_{1,0} + b_2 E_{1,1-\xi_3}.$$

Intersecting both members with $E_{0,1}$, $E_{1,0}$, and $E_{1,1}$, we obtain respectively

$$3b_1 = a_2 + b_2, \quad a_1 + b_1 = 3b_2, \quad \text{and} \quad a_1 + b_1 = a_2 + b_2.$$

Thus, $a_1/b_1 = a_2/b_2 = 2$.

Assume, without loss of generality, that $0 \in T$ is point in $\text{sing}^B(\mathcal{F})$. Notice that $E_{1,0}$ is $\mathcal{F}$ invariant and $\text{sing}(\mathcal{F}) \cap E_{1,0}$ is equal to the set of intersection points of $\text{tang}(\mathcal{F}, \mathcal{L}_1)$ with $E_{1,0}$. Moreover, $\text{sing}^B(\mathcal{F})$ corresponds to the intersection with $E_{1,0}$ of the irreducible components of $\text{tang}(\mathcal{F}, \mathcal{L}_1)$ that are invariant by $\hat{\mathcal{L}}_1$. Equation (51) implies that each of the $\hat{\mathcal{L}}_1$-invariant curves in $\text{tang}(\mathcal{F}, \mathcal{L}_1)$ appears with multiplicity two. From $a_1/b_2 = 2$, it follows that the cardinality of $\text{sing}(\mathcal{F}) \cap E_{1,0}$ is four times the cardinality of $\text{sing}^B(\mathcal{F}) \cap E_{1,0}$. Recall from Lemma 11.7 item (d) that $S = \text{sing}(\mathcal{F}) \cap E_{1,0}$ and $S^B = \text{sing}^B(\mathcal{F}) \cap E_{1,0}$ are subgroups of $E_{1,0}$. It is now clear that the kernel of the map $S \to S^B$ given by multiplication by two is the subgroup of two-torsion points of $E_{1,0}$.

Notice that we can reconstruct the divisors $\text{tang}(\mathcal{F}, \mathcal{L}_i)$, for $i = 2, 3, 4$, from the subgroups $S$ and $S^B$. Indeed,

$$\text{tang}(\mathcal{F}, \mathcal{L}_i) = \sum_{p \in S} L_p E_i + 2 \left( \sum_{p \in S^B} L_p \hat{E}_i \right).$$
It follows that the foliation $\mathcal{F}$ is invariant by the natural action of $S^B \subset E_{1,0}$ in $T$, that is,

$$S^B \times T \longrightarrow T$$

$$(g, 0), (x, y) \longmapsto (x + g, y).$$

Indeed, due to the symmetry of our setup, $\mathcal{F}$ is left invariant by the following action of $(S^B)^2$:

$$(S^B)^2 \times T \longrightarrow T$$

$$((g, 0), (h, 0), (x, y)) \longmapsto (x + g, y + h).$$

The quotient of $\mathcal{F} \boxtimes \mathcal{W}$ by this action is a CDQL 5-web on $E_{0,1}^2$ of the form $\mathcal{G} \boxtimes \mathcal{W}$. If $E_{0,1}(2)$ denotes the two-torsion points on $E_{0,1}$ then, by construction,

$$\text{tang}(\mathcal{G}, \mathcal{L}_i) = 2\,\hat{E}_i + \sum_{p \in E_{0,1}(2)} L_pE_i.$$

for $i = 2, 3, 4$. This is sufficient to show that $\mathcal{G} \boxtimes \mathcal{W}$ is the 5-web $\mathcal{E}_5$ of the Introduction.

With Theorem 11.2, we complete the classification of exceptional CDQL webs on complex tori and, consequently, on compact complex surfaces.

**Appendix: Proof of Theorem 5.1**

We classify here (up to projective equivalence) the pairs $(f, Q)$ such that

$$\forall q \in Q, \quad f^{-1}(q) \subset \{q, \hat{q}\}$$

(A.1)

(where $\hat{q}$ stands for the barycenter of $Q \setminus \{q\}$ on $\mathbb{C} \simeq \mathbb{P}^1 \setminus \{q\}$ for every $q \in Q$).

We distinguish several cases according to the values of $k = \text{Card}(Q)$ and of the topological degree $d > 0$ of $f$. Our approach is elementary. We will use repeatedly the two facts stated in the following elementary lemma:

**Lemma A.1.** Taking into account multiplicities, the rational map $f$ admits

(1) $d + 1$ fixed points;

(2) $2d - 2$ critical points.
To prove Theorem 5.1, one translates (A.1) in terms of algebraic equations in the coefficients of \( f \) and in the homogeneous coordinates of the points of \( Q \). Most of these equations are in fact linear and can be easily solved. We do not present here all the details, but a Maple worksheet is available at http://w3.impa.br/~jvp/artigos.html.

We first consider the case when \( d = 1 \).

### A.1 The case of projective automorphisms

Let \( z \) be an affine coordinate on an affine chart \( \mathbb{C} \subset \mathbb{P}^1 \), the complement of which is noted by \( \infty \). For \( Q \subset \mathbb{P}^1 \), a finite subset with \( \geq 3 \) elements, set \( Q_0 = Q \setminus \{\infty\} \) and \( Q^*_0 = Q \setminus \{0, \infty\} \) and consider them as subsets of \( \mathbb{C} \).

Setting \( P(z) = \prod_{\theta \in Q^*_0} (z - \theta) \), one has the following

**Lemma A.2.** For every \( q \in Q_0 \), one has

\[
\hat{t} = t - 2(k - 1) \frac{P'(t)}{P''(t)}
\]

if \( t \) (resp. \( \hat{t} \)) corresponds to \( q \) (resp. to \( \hat{q} \)) in the coordinate \( z \).

For any \( m \geq 1 \), set

\[
Q(m) = \left\{ e^{\frac{2i\pi\ell}{m}} \mid \ell = 0, \ldots, m - 1 \right\}.
\]

In this subsection, we assume that \( f \in \text{Aut}(\mathbb{P}^1) \) is such that (A.1) holds for every \( q \in Q \). The case when \( f \) is the identity is trivial and has to be excluded so we assume that \( f \neq \text{Id} \) in what follows.

Since \( f \) is an automorphism, its fixed points have multiplicity one thus, up to projective equivalence, we are in one of the following two cases (where \( f \) is expressed in the coordinate \( z \)):

\[
i) \quad f(z) = z + c \quad \text{with} \quad c \in \mathbb{C}^*;
\]

\[
ii) \quad f(z) = vz \quad \text{with} \quad v \in \mathbb{C} \setminus \{0, 1\}.
\]

**Remark A.1.** Case \( i) \) corresponds to the automorphisms of \( \mathbb{P}^1 \) admitting exactly one fixed point whereas case \( ii) \) corresponds to those with exactly two fixed points.
We assume first that \( f(z) = z + c \) with \( c \neq 0 \). Since \( \infty \) is the unique point fixed by \( f \), one has \( f^{-1}(q) = \hat{q} \) for every \( q \in Q \setminus \{\infty\} \), which can be written more explicitly as

\[
\forall t \in Q_0, \quad t - c = t - 2(k-1)P'(t)/P''(t).
\]

This implies that \( R(t) = cP''(t) - 2(k-1)P'(t) = 0 \) for every \( t \in Q_0 \). Since \( k \geq 3 \), it follows that \( \text{Card}(Q_0) = \deg(P) > \deg(P') = \deg(R) \). Thus \( R \) vanishes identically and, consequently, \( \deg(P') = \deg(P'') \). But this last equality implies \( \deg(P) \leq 1 \). Therefore \( k = \text{Card}(Q) \leq \deg(P) + 1 \leq 2 \). This contradiction implies that \( f \) cannot be a translation.

Assume now that \( f(z) = vz \) with \( v \neq 0, 1 \). Since 0 and \( \infty \) are the unique fixed points of \( f \), one has \( v^{-1}t = t - 2(k-1)P'(t)/P''(t) \) for every \( t \in Q_0^* \). Explicitly,

\[
\forall t \in Q_0^*, \quad tP''(t) - \rho P'(t) = 0 \quad \text{(A.2)}
\]

where \( \rho \) stands for the non-zero constant \( 2(k-1)/(1-v^{-1}) \).

The following fact will be used repeatedly below:

*Let \( R \) be a polynomial of degree \( \ell \geq 1 \) such that \( zR'(z) - \rho R(z) \equiv 0 \).

Then \( \rho = \ell \) and \( R(z) = rz^\ell \) for a certain non-zero constant \( r \in \mathbb{C} \).* \hspace{1cm} (A.3)

One distinguishes three cases.

- First case: \( Q_0^* = Q \), that is, 0 and \( \infty \) do not belong to \( Q \). Then \( P' \) has degree \( k-1 \) and (A.2) holds for \( k \) distinct values of \( t \). Thus, \( tP''(t) - \rho P'(t) = 0 \). By (A.3), it follows that \( \rho = k-1 \) hence \( \hat{f}(z) = -z \), \( P'(z) = kt^{k-1} \) (since \( P \) is monic) thus \( P(z) = z^k + c \) for a certain complex constant \( c \) distinct from 0 (since \( \deg(P) = 1 \) \( Q \) has cardinality \( k \geq 3 \)). Because \( h^{-1} \circ f \circ h = f \) if \( h \) is the dilatation \( h(z) = -z/c \), one can assume that \( c = -1 \). Then one has \( f(z) = -z \) and \( Q = Q(k) \) in this case.

- Second case: \( \text{Card}(Q \cap [0, \infty)) = 1 \), that is, only one point of \( Q \) is fixed by \( f \). One can assume that it is \( \infty \). Then \( P' \) has degree \( k-2 \). The same arguments as in the previous case give that \( f(z) = (2/k-1)z \) and after conjugation by a dilatation, one can assume that \( Q = Q(k-1) \cup \{\infty\} \).

- Last case: \([0, \infty) \subset Q \). One writes \( P(z) = zR(z) \) with \( R(z) \) of degree \( k-2 \). Thus, \( P'(z) = zR'(z) + R(z) \) and \( P''(z) = zR''(z) + 2R'(z) \). Then (A.2) implies that

\[
t(zR''(z) + 2R'(z)) - \rho(zR'(z) + R(z)) = 0
\]
for every \( t \) such that \( R(t) = 0 \). So \( tR''(t) - (\rho - 2)R'(t) = 0 \) for \( k - 2 \) distinct values of \( t \) hence for every \( t \). Using (A.3), one obtains that \( \rho = k - 1 \) hence \( f(z) = -z \). After conjugation by a dilatation, one can assume that \( Q = Q(k - 2) \cup \{0, \infty\} \).

We have obtained the following

**Proposition A.1.** Let \( Q \subset \mathbb{P}^1 \) be a finite set of cardinality \( k \geq 3 \) and \( f \in \text{Aut}(\mathbb{P}^1) \) be such that (A.1) holds for every \( q \in Q \). If \( f \) is distinct from the identity, then \((f, Q)\) is projectively equivalent to one of the following three pairs:

1. \( f(z) = -z \) and \( Q = Q(k) \);
2. \( f(z) = -z \) and \( Q = Q(k - 2) \cup \{0, \infty\} \);
3. \( f(z) = \left(\frac{2-k}{k}\right)z \) and \( Q = Q(k - 1) \cup \{\infty\} \).

\[ \blacksquare \]

**A.2 The case of rational maps of degree strictly bigger than one**

We assume now that \( f : \mathbb{P}^1 \to \mathbb{P}^1 \) is a rational map of topological degree \( d > 1 \).

**A.3 Some elementary facts**

Let \( Q = \{q_1, \ldots, q_k\} \subset \mathbb{P}^1 \) with \( k \geq 3 \) such that (A.1) holds. For every \( i = 1, \ldots, k \), there exists \( e_i \in \{0, \ldots, d\} \) such that

\[
 f^{-1}(q_i) = e_i q + (d - e_i)\hat{q}_i . \tag{A.4}
\]

Combined with the Riemann–Hurwitz formula, it implies that

\[
2 = 2d - \sum_{i=1}^{k} \left( \max(0, e_i - 1) + \max(0, d - e_i - 1) \right) - n \tag{A.5}
\]

for a certain non-negative integer \( n \). Thus, setting

\[
\tau = \text{Card} \{ i \mid e_i \in \{0, d\} \} \quad \text{and} \quad \tau' = \text{Card} \{ i \mid 0 < e_i < d \},
\]

it follows from (A.5) that

\[
(d - 1)\tau + (d - 2)\tau' \leq 2d - 2 . \tag{A.6}
\]
In particular, when \( d > 2 \), the following inequalities hold true

\[
k \leq 2 + \frac{2}{d - 2} \quad \text{and} \quad d \leq 2 + \frac{2}{k - 2}.
\]  
(A.7)

Since \( k \geq 3 \) by hypothesis, we end up when \( d > 2 \) with the following possibilities

\[ d = 4 \quad \text{and} \quad k = 3 \quad \text{or} \quad d = 3 \quad \text{and} \quad k \leq 4. \]

When \( d = 2 \), (A.6) gives that \( f \) admits at most two totally ramified points hence \( \text{Card} \{ i \mid e_i \neq 1 \} \leq 2 \). From another hand, \( f \) admits at most \( d + 1 = 3 \) fixed points, thus \( \text{Card} \{ i \mid e_i > 0 \} \leq 3 \). Thus, \( k \leq 5 \) in this case.

We are going to determine normal forms for pairs \((f, Q)\) by considering the different cases accordingly to the values taken by the two integers \( k \) and \( d \) where \( k \leq 5 \) and \( d \in \{2, 3, 4\} \) satisfy the conditions obtained above.

Denoting by \( x, y \) some fixed homogeneous coordinates on \( \mathbb{P}^1 \), we will write

\[
f(x : y) = [A(x, y) : B(x, y)]
\]

where \( A \) and \( B \) are degree \( d \) homogeneous polynomials without a common factor.

Because \( Q \) contains at least three distinct points and since we are working up to projective equivalence, we will assume that \((f, Q)\) has been normalized such that

\[
q_1 = [0 : 1], \quad q_2 = [1 : 0] \quad \text{and} \quad q_3 = [1 : -1].
\]  
(A.8)

A.4 Case \( k = 3 \)

Since \( Q = \{q_1, q_2, q_3\} \), easy computations give that

\[
\hat{q}_1 = [2 : -1], \quad \hat{q}_2 = [-1 : 2] \quad \text{and} \quad \hat{q}_3 = [1 : 1].
\]  
(A.9)

Then using (A.4) for \( i = 1, 2 \), it follows that \( A \) and \( B \) have the following form

\[
A(x, y) = \alpha x^{e_1} (x + 2y)^{d-e_1} \quad \text{and} \quad B(x, y) = \beta y^{e_2} (2x + y)^{d-e_2}
\]

where \( \alpha \) and \( \beta \) are non-zero complex constants. Moreover, we can (and will) assume that \( \alpha = 1 \) in what follows. The problem has then been reduced to determine \( \beta \in \mathbb{C}^* \) such that

\[
f^{-1}(q_3) = e_3 q_3 + (d - e_3) \hat{q}_3.
\]
More explicitly, we are looking for integers $d \in \{2, 3, 4\}$, triples $(e_1, e_2, e_3) \in \{0, \ldots, d\}^3$ and complex constants $\beta, \gamma \in \mathbb{C}^*$ such that

$$x^{e_1}(x + 2y)^{d-e_1} + \beta y^{e_2} (2x + y)^{d-e_2} + \gamma (x + y)^{e_3}(x - y)^{d-e_3} \equiv 0. \quad (A.10)$$

Let $d$ and the $e_i$'s be fixed. Expanding (A.10) and considering the coefficients of the components $x^\ell y^{d-\ell}$ for $\ell = 0, \ldots, d$, one obtains a system of linear equations in $\beta$ and $\gamma$. The determination of $\beta$, hence of $f$, amounts to elementary linear algebra.

The results are the following: when $k = 3$, up to projective equivalence, there are three pairs $(f, Q)$ when $d = 2$, none when $d = 3$ and two when $d = 4$. Explicit normal forms for these pairs are labelled (a,1), (a,2), (a,3), (c,1), and (c,2) in Table 5.1.

**A.5 Case $k = 4$**

Assume that $Q = \{q_1, \ldots, q_4\}$ has been normalized such that $q_1, q_2$, and $q_3$ are as in (A.9). Then it exists $\theta \in \mathbb{C} \setminus \{0, -1\}$ such that $q_4 = [\theta : 1]$.

Elementary computations give that

$$\hat{q}_1 = [3\theta : 1 - \theta] \quad \hat{q}_3 = [1 + 2\theta : 2 + \theta]$$

$$\hat{q}_2 = [\theta - 1 : 3] \quad \hat{q}_4 = [-\theta(2 + \theta) : 1 + 2\theta].$$

By (A.7), one has $d = 2$ or $d = 3$. Let us consider separately each of these cases.

**A.5.1 Case $k = 4$ and $d = 3$**

In this situation, (A.6) reads $2\tau + \tau' \leq 4$. Since $\tau + \tau' = 4$, it follows that $\tau = 0$ hence $\tau' = 4$, that is $e_i \in \{1, 2\}$ for every $i = 1, \ldots, 4$. The hypotheses imply also that one can assume that $f$ is of the form

$$f(x : y) = [A(x, y) : B(x, y)] = \left[x^{e_1}(\theta - 1) x + 3 \theta y \right]^{2-e_1} : \beta y^{e_2}(3x + (1 - \theta)y)^{d-e_2}$$

for a certain $\beta \in \mathbb{C}^*$. Since $e_i$ and $3 - e_i$ are positive for every $i$, one has in particular $f(q_j) = f(\hat{q}_j) = q_j$ for $j = 3, 4$. These latter conditions can be written as

$$0 = A(1, -1) + B(1, -1) = A(1 + 2\theta, 2 + \theta) + B(1 + 2\theta, 2 + \theta) \quad (A.11)$$

$$0 = A(\theta, 1) - \theta B(\theta, 1) = A(-\theta(2 + \theta), 1 + 2\theta) - \theta B(-\theta(2 + \theta), 1 + 2\theta).$$
When the $e_i$'s are fixed (in the set \{1, 2\}), these relations can be explicited and are equivalent to a system of polynomial equations in $\beta$ and $\theta$ that are moreover linear in $\beta$. It is then easy to verify that if this system admits a solution $(\beta^*, \theta^*)$ with $\beta^* \neq 0$ and $\theta^* \notin \{0, 1\}$ then necessarily

1. $(e_1, e_2, e_3, e_4)$ belongs to the following list:

   \begin{equation*}
   (1, 1, 1, 1) \quad (1, 1, 1, 2) \quad (1, 1, 2, 1) \quad (1, 1, 2, 2);
   \end{equation*}

2. $z = \beta^* = \theta^*$ verifies $z^2 + z + 1 = 0$ so it belongs to $\left\{ e^{\frac{2i\pi}{3}}, e^{\frac{4i\pi}{3}} \right\}$.

The preceding facts imply that

\begin{equation}
  f(x : y) = \left[ x((\theta - 1)x + 3\theta y)^2 : \theta y(3x + (1 - \theta)y)^2 \right] \tag{A.12}
\end{equation}

with $\theta = e^{\frac{2i\pi}{3}}$ or $\theta = e^{\frac{4i\pi}{3}}$. One verifies that the two possible values for $\theta$ give two projectively equivalent pairs. Taking $\theta = e^{\frac{2i\pi}{3}}$, one obtains easily that

\begin{equation*}
  f^{-1}(q_i) = q_i + 2 \hat{q}_i
\end{equation*}

for $i = 1, \ldots, 4$, which shows that (A.12) when $\theta = e^{\frac{2i\pi}{3}}$ is the unique solution (up to projective equivalence) to the problem considered here when $k = 4$ and $d = 3$.

A.5.2 Case $k = 4$ and $d = 2$

Since $f$ admits at most three fixed points and two totally ramified points, one has $\text{Card}\{i \mid e_i = 1, 2\} \leq 3$ and $\text{Card}\{i \mid e_i = 0, 2\} \leq 2$. A simple analysis gives that $(e_i)_{i=1}^4$ can be assumed to belong to the following list

\begin{equation*}
   (1, 1, 1, 0) \quad (1, 1, 0, 0) \quad (2, 1, 1, 0). \tag{A.13}
\end{equation*}

Then there exists a constant $\beta \in \mathbb{C}^*$ such that $f$ writes

\begin{equation*}
  f(x : y) = \left[ x((\theta - 1)x + 3\theta y) : \beta y(3x + (1 - \theta)y) \right]
\end{equation*}
in the first two cases, and

\[ f(x : y) = \left[ x^2 : \beta y(3x + (1 - \theta)y) \right] \]

in the last one.

The conditions \( f^{-1}(q_i) = e_iq_i + (2 - e_i)\hat{q}_i \) (for \( i = 1, \ldots, 4 \)) lead to a system of polynomial equations in the unknowns \( \beta \) and \( \theta \). One can prove that this polynomial system does not admit any solution in any of the three cases in (A.13).

A.6 Case \( k = 5 \)

According to 11.5.4, one has necessarily \( d = 2 \). There is no loss of generality by assuming that \( 0 \leq e_1 \leq e_2 \leq \ldots \leq e_5 \leq 2 \). Because \( \text{Card}\{i | e_i = 1, 2\} \leq 3 \), one necessarily have \( e_1 = e_2 = 0 \). But \( \text{Card}\{i | e_i = 0, 2\} \leq 2 \) hence \( e_1 = e_2 = 0 \) and \( e_3 = e_4 = e_5 = 1 \). One normalizes \( q_1, q_2, \) and \( q_3 \) as in (A.8). Then \( q_4 = [\theta : 1] \) and \( q_5 = [\epsilon : 1] \) for two distinct constants \( \theta, \epsilon \in \mathbb{C} \setminus \{0, -1\} \) hence

\[ \hat{q}_1 = [4 \theta \epsilon : \theta + \epsilon - \theta \epsilon] \tag{A.14} \]
\[ \hat{q}_2 = [\theta + \epsilon - 1 : 4] \]
\[ \hat{q}_3 = [3 \theta \epsilon + 2(\theta + \epsilon) + 1 : \theta \epsilon + 2(\theta + \epsilon) + 3] \tag{1} \]
\[ \hat{q}_4 = [\theta^3 - 2 \theta^2 \epsilon + 2 \theta^2 - 3 \theta \epsilon : \epsilon - 3 \theta^2 + 2 \theta \epsilon - 2\theta] \]
and \[ \hat{q}_5 = [\epsilon^3 - 2 \epsilon^2 \theta + 2 \epsilon^2 - 3 \theta \epsilon : \theta - 3 \epsilon^2 + 2 \theta \epsilon - 2\epsilon] \]

Since \( f^{-1}(q_i) = 2\hat{q}_i \) for \( i = 1, 2 \), the map \( f \) can be written as

\[ f(x : y) = \left[ ((\theta \epsilon - \theta - \epsilon) x + 4 \theta \epsilon y)^2 : \beta (4 x + (1 - \theta - \epsilon) y)^2 \right] \]

for a non-zero complex constant \( \beta \). Because \( e_3 = e_4 = e_5 = 1 \), one has

\[ f(q_j) = f(\hat{q}_j) = q_j \]

for \( j = 3, 4, 5 \). Using (A.14), it can be shown that these conditions are equivalent to a system of polynomial equations in the variables \( \theta \) and \( \epsilon \), that can be explicited.
For instance, that $f(q_j) = q_j$ for $j = 3, 4, 5$ is equivalent to the fact that the following polynomial equations hold:

\[
0 = \beta (9 + 6\theta + 6\epsilon + \theta^2 + 2\theta\epsilon + \epsilon^2) + 9\theta^2\epsilon^2 + 6\theta^2\epsilon + 6\epsilon^2\theta + \theta^2 + 2\theta\epsilon + \epsilon^2
\]

\[
0 = \beta (9\theta^2 + 6\theta - 6\theta\epsilon + 1 + 2\epsilon + \epsilon^2) - \theta^3\epsilon^2 + 2\theta^2\epsilon - \theta^3 + 6\theta^2\epsilon - 9\epsilon^2\theta
\]

\[
0 = \beta (9\epsilon^2 + 6\epsilon - 6\theta\epsilon + 1 + 2\theta + \theta^2) - \theta^2\epsilon^2 + 2\epsilon^3\theta - 6\epsilon^2\theta^2 - \epsilon^3 + 6\epsilon^2\theta - 9\theta^2\epsilon.
\]

Let $\zeta$ and $\zeta'$ be such that $\{\theta, \epsilon\} = \{\zeta, \zeta'\}$. After having eliminated first $\beta$ then $\zeta'$, one ends up with a polynomial equation on $\zeta$ from what it follows

\[
(\zeta^2 - \zeta + 1) (\zeta^2 + 47 \zeta + 1) (\zeta^2 + 7 \zeta + 1) (4 \zeta^4 + 60 \zeta^3 + 97 \zeta^2 + 60 \zeta + 4) (\zeta^2 + 3 \zeta + 1) = 0
\]

since $\zeta \neq 0, -1$ by hypothesis.

After an elementary (but tedious) analysis, one obtains that necessarily

\[
\{\theta, \epsilon\} = \{- e^{\frac{2\pi i}{3}}, - e^{\frac{4\pi i}{3}}\}
\]

which implies in its turn that

\[
f(x : y) = \left[ y^2 : -x^2 \right].
\]  

(2)

In this case, one has (up to exchanging $q_4$ and $q_5$ that does not matter)

\[
\hat{q}_1 = q_2, \quad \hat{q}_2 = q_1, \quad \hat{q}_3 = [1 : 1], \quad \hat{q}_4 = [e^{\frac{2\pi i}{3}} : 1], \quad \hat{q}_5 = [e^{\frac{4\pi i}{3}} : 1]
\]

and there is no difficulty to verify that (2) provides the unique solution to the problem studied in this Appendix when $k = 5$ and $d = 2$.

Acknowledgments

The first author thanks Jorge Pastore for enlightening discussions. The second author thanks Dominique Cerveau, Frank Loray and the International Cooperation Agreement Brazil–France. Both authors are grateful to Marco Brunella for the elegant proof of Proposition 7.1 and to David Marín for the explicit expression for $\beta_*$ presented in Remark 5.1.

The referee has suggested (sometimes very precisely) several improvements to this paper. Both authors are very grateful to him for his careful reading.
References


