INTRODUCTION:

MODULI PROBLEMS

"DEFINITION"

A moduli problem is a class of geometric objects which one tries to view as another geometric object.

OUR BASIC EXAMPLE will be the

"moduli problem of smooth n-pointed curves of genus g":

For every scheme S, we put

$\mathcal{M}_{g,n}(S)$ = category with

- objects: $(X \xrightarrow{f} S, x_1, \dots, x_n)$ where

  $f : X \to S$ is smooth, proper, with geometric fibres 1-dimensional, connected, of genus g, and

  $x_1, \dots, x_n : S \to X$ are <u>disjoint sections</u>
- morphisms: S-isomorphisms respecting the sections.

$\underline{M}_{g,n}(S)$ = the set of isomorphism classes of objects of $\mathcal{M}_{g,n}(S)$.

The latter is just a set, while by construction $\mathcal{M}_{g,n}(S)$ is a groupoid (= category with all maps invertible).

For every morphism $S' \to S$, we have

base change functors

$$\mathcal{M}_{g,n}(S) \longrightarrow \mathcal{M}_{g,n}(S')$$

whence natural maps

$$\underline{M}_{g,n}(S) \longrightarrow \underline{M}_{g,n}(S')$$

making each $\underline{M}_{g,n}$ into a functor

$$\underline{M}_{g,n} : (\text{Schemes})^{\circ} \longrightarrow (\text{Sets})$$

In general, $\underline{M}_{g,n}$ is NOT (representable by) a scheme.

(In fact, it is a scheme iff $n > 2g+2$).

For instance, one can find two curves

$$\begin{tikzcd} X \arrow[dr] & & X' \arrow[dl] \\ & S & \end{tikzcd}$$

which are not isomorphic, but locally isomorphic over S (for the étale topology, or even the Zariski topology).

Thus, $\underline{M}_{g,0}$ (any g) is not a sheaf for these topologies, hence not representable.

(And, of course, if we take the associated sheaf we lose even more information)

MAIN IDEA:

$\mathcal{M}_{g,n}$ is a BETTER OBJECT to look at than $\underline{M}_{g,n}$:

- it is the natural object one tries to study (no loss of information)
- it has good local-to-global / sheaf-theoretic / descent properties
- it has good approximations by schemes

Of course, in a sense it is a more complicated object ($\mathcal{M}_{g,n}(S)$ is a groupoid, not a set).

FIBERED GROUPOIDS

Let C be a category
(typically: the category of schemes, possibly over a fixed "base scheme")

A **fibered groupoid** $\mathcal{M}$ over C (C-groupoid) consists of the following data:

- for each $U \in \mathrm{ob}\, C$, a groupoid $\mathcal{M}(U)$
- for each map $V \xrightarrow{f} U$ in C, a functor

$$f^*: \mathcal{M}(U) \longrightarrow \mathcal{M}(V)$$

- for each composite map $W \xrightarrow{g} V \xrightarrow{f} U$, an isomorphism

$$g^* f^* \xrightarrow{\sim} (fg)^*$$

of functors $\mathcal{M}(U) \longrightarrow \mathcal{M}(W)$
+ compatibility with the associativity of composition in C.

Examples:

- $\mathcal{C} = (\text{Schemes})$:
  - $U \longmapsto \mathcal{M}_{g;n}(U)$
  - $U \longmapsto$ cat. of all U-schemes + U-isomorphisms
  - $U \longmapsto QCOH(U) :=$ cat. of quasi-coherent $\mathcal{O}_U$-Modules (+isomorphisms)
  - $U \longmapsto BUN_n(U) :=$ cat. of locally free $\mathcal{O}_U$-modules of rank n (+isomorphisms)
    $(n \in \mathbb{N})$

- For any C, any presheaf on C, i.e. any functor

$$F : C^{\circ} \longrightarrow (\text{Sets})$$

defines a C-groupoid (denoted by F):

$F(U) :=$ the discrete category $F(U)$:

set of objects $= F(U)$

maps $=$ identities.

MORPHISMS OF GROUPOIDS

If $\mathcal{M}$, $\mathcal{N}$ are C-groupoids, a **morphism**

$$\Phi : \mathcal{M} \longrightarrow \mathcal{N}$$

consists of the following data:

- For each $U \in \mathrm{ob}\, C$, a functor

$$\Phi(U) : \mathcal{M}(U) \longrightarrow \mathcal{N}(U)$$

- For each $V \xrightarrow{f} U$ in C, consider the diagram

of functors:

$$\begin{tikzcd} \mathcal{M}(U) \arrow[r, "\Phi(U)"] \arrow[d, "f^*"] & \mathcal{N}(U) \arrow[d, "f^*"] \\ \mathcal{M}(V) \arrow[r, "\Phi(V)"] & \mathcal{N}(V) \end{tikzcd}$$

We require (as part of the data) **an isomorphism**

$$f^* \circ \Phi(U) \xrightarrow{\sim} \Phi(V) \circ f^*$$

of functors $\mathcal{M}(U) \longrightarrow \mathcal{N}(V)$.

(+ compatibility with the associativity data).

Examples:

- $\mathcal{M}_{g,n} \longrightarrow \mathcal{M}_{g,n-1}$ $(n>0)$

  "forget the n-th marked point"

- $\mathcal{M}_{1,1} \xrightarrow{j} \mathbb{A}^1$ (viewed as a presheaf)

$$\left(X \rightleftarrows S\right) \longmapsto j(X/S) \in \Gamma(S, \mathcal{O}_S) = \mathbb{A}^1(S)$$

elliptic curve

- Let us define two morphisms

$$\Phi, \Psi : \mathcal{M}_{1,1} \longrightarrow BUN_1$$

by
$$\Phi\left(X \underset{\varepsilon}{\overset{p}{\rightleftarrows}} S\right) := p_* \Omega^1_{X/S}$$
$$\Psi\left(X \underset{\varepsilon}{\overset{p}{\rightleftarrows}} S\right) := \varepsilon^* \Omega^1_{X/S}$$

These are different morphisms, but $\Phi(X)$ and $\Psi(X)$ are known to be canonically isomorphic.

So there should be a notion of (iso)morphism between morphisms!

2. MORPHISMS

The previously defined morphisms $\mathcal{M} \xrightarrow{\Phi} \mathcal{N}$ of fibered groupoids will be called **1-morphisms** (unless no confusion arises).

If $\Phi, \Psi : \mathcal{M} \to \mathcal{N}$ are 1-morphisms, a **2-morphism** $\alpha : \Phi \to \Psi$ is a collection of morphisms of functors

$$U \in \mathrm{ob}\, C \mapsto \begin{tikzcd} \mathcal{M}(U) \arrow[r, bend left, "\Phi(U)"] \arrow[r, bend right, "\Psi(U)"'] \arrow[r, phantom, "\Downarrow \alpha(U)"] & \mathcal{N}(U) \end{tikzcd}$$

(automatically isomorphisms), satisfying natural compatibilities.

In this way, the "set" of C-groupoids becomes a 2-category:

for any two objects $\mathcal{M}, \mathcal{N}$, we have a category of 1-morphisms from $\mathcal{M}$ to $\mathcal{N}$ whose objects are 1-morphisms $\Phi : \mathcal{M} \to \mathcal{N}$ and maps are 2-morphisms between them.

SPECIAL CASE: let U_0 be an object of $\mathbf{C}$, and take $\mathcal{M} = U_0$ (viewed as a presheaf on C, hence as a C-groupoid). Take $\mathcal{N}$ arbitrary.

Any 1-morphism $U_0 \xrightarrow{\Phi} \mathcal{N}$ determines, for each U, a functor

$$\mathrm{Mor}_{\mathbf{C}}(U, U_0) \xrightarrow{\Phi(U)} \mathcal{N}(U)$$

and in particular (taking $U = U_0$) an object

$$\Phi(U_0)(\mathrm{Id}_{U_0})$$

of $\mathcal{N}(U_0)$.

Easy exercise: we obtain in this way an equivalence of categories

$$\{1\text{-morphisms } U \to \mathcal{N}\} \xrightarrow{\simeq} \mathcal{N}(U).$$

For instance: if S is a scheme, a 1-morphism

$$S \longrightarrow \mathcal{M}_{g,n}$$

is "the same thing" as an object of $\mathcal{M}_{g,n}(S)$.

PRODUCTS

Given a diagram of C-groupoids

$$\begin{tikzcd} & \mathcal{N} \arrow[d, "\Psi"] \\ \mathcal{M} \arrow[r, "\Phi"] & \mathcal{P} \end{tikzcd}$$

there is a "fibre product" groupoid, assigning to each $U \in \mathrm{ob}\, C$ the **fibre product category**

$$\mathcal{M}(U) \times_{\mathcal{P}(U)} \mathcal{N}(U)$$

whose objects are triples

$$(X, Y, \alpha)$$

with $\begin{cases} X \in \mathrm{ob}\, \mathcal{M}(U) \\ Y \in \mathrm{ob}\, \mathcal{N}(U) \\ \alpha : \Phi(X) \xrightarrow{\sim} \Psi(Y) \end{cases}$ (isomorphism in $\mathcal{P}(U)$)

Example:

Assume: $\mathcal{M}$ is a C-groupoid,

U_1, U_2 objects of C

$X_i \in \mathrm{ob}\, \mathcal{M}(U_i) \quad (i=1,2)$

Viewing X_i as a 1-morphism $U_i \to \mathcal{M}$, we get a diagram:

$$\begin{array}{ccc} & & U_2 \\ & & \downarrow X_2 \\ U_1 & \xrightarrow{X_1} & \mathcal{M} \end{array}$$

What is $U_1 \times_{X_1, \mathcal{M}, X_2} U_2$?

Answer: it is the presheaf on C given by

$$T \mapsto \left\{ (u_1, u_2, \alpha) : \begin{array}{l} u_1 = \text{a morphism } T \to U_1, \\ u_2 = \text{a morphism } T \to U_2, \\ \alpha : u_1^* X_1 \xrightarrow{\sim} u_2^* X_2 \text{ in } \mathcal{M}(T) \end{array} \right\}$$

or, in standard notations,

$$\underline{\mathrm{Isom}}_{U_1 \times U_2}\left(pr_1^* X_1, pr_2^* X_2\right)$$

Example:

Consider the 1-morphism "forget the marked point"

$$\Phi : \mathcal{M}_{g,1} \longrightarrow \mathcal{M}_{g,0}$$

If S is a scheme and X is an S-curve of genus g, then the fibre product $S \times_{X, \mathcal{M}_{g,0}, \Phi} \mathcal{M}_{g,1}$ is the S-curve X! In other words, we have a Cartesian diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathcal{M}_{g,1} \arrow[d, "\Phi"] \\ S \arrow[r, "X"] & \mathcal{M}_{g,0} \end{tikzcd}$$

which shows that $\mathcal{M}_{g,1}$ can be seen as the universal curve over $\mathcal{M}_{g,0}$.

REPRESENTABLE MORPHISMS

(For safety, assume C has fibre products)

A 1-morphism $\Phi : \mathcal{M} \to \mathcal{N}$

is representable if for each $U \in \mathrm{ob}\, C$ and $X : U \to \mathcal{N}$ (i.e. object of $\mathcal{N}(U)$) the fibre product $U \times_{X, \mathcal{N}, \Phi} \mathcal{M}$ is a presheaf, representable by an object of C.

For instance, the "forgetful" morphism

$$\mathcal{M}_{g,1} \longrightarrow \mathcal{M}_{g,0}$$

is representable.

Another example: for $g \geq 2$, consider

$$3K: \mathcal{M}_{g,0} \longrightarrow BUN_r \qquad (r = 5g-5)$$

$$(X \xrightarrow{f} U) \longmapsto f_* \omega_{X/U}^{\otimes 3}$$

I claim that 3K is representable:

Pick a scheme U and a 1-morphism $U \longrightarrow BUN_r$ (that is, a locally free sheaf $\mathcal{E}$ on U, of rank r).

$$\begin{array}{ccc} \mathcal{N} & \longrightarrow & \mathcal{M}_{g,0} \\ \big\downarrow & & \big\downarrow 3K \\ U & \xrightarrow{\mathcal{E}} & BUN_r \end{array}$$

Then, for a U-scheme T, we have

$\mathcal{N}(T)$ = cat. of curves $X \xrightarrow{f} T$ of genus g, plus isomorphism $f_* \omega_{X/T}^{\otimes 3} \xrightarrow{\sim} \mathcal{E}_T$.

Such a curve is naturally (3-canonically) embedded in $\mathbb{P}(\mathcal{E}_T)$. Putting $P = \mathbb{P}(\mathcal{E})$, we obtain an equivalence:

$\mathcal{N}(T) \xrightarrow{\sim}$ cat. of embedded smooth curves of genus g: $X \hookrightarrow P \times_U T$, plus isomorphism $\mathcal{O}(1)|_X \simeq \omega_{X/T}^{\otimes 3}$

The representability then follows from Hilbert scheme theory.

PROPERTIES OF REPRESENTABLE MORPHISMS

If $\mathcal{P}$ is a property (i.e. a class) of morphisms of $\mathcal{C}$, which is stable by base change, it makes sense to say that a representable 1-morphism $\Phi : \mathcal{M} \to \mathcal{N}$ has property $\mathcal{P}$.

For instance, in the above examples,

$\Phi : \mathcal{M}_{g,1} \longrightarrow \mathcal{M}_{g,0}$ is proper and smooth.

$3K : \mathcal{M}_{g,0} \longrightarrow \mathrm{BUN}_{5g-5}$ is surjective and smooth.

USING THE DIAGONAL

(We assume $\mathcal{C}$ has fibre products)

Proposition For a $\mathcal{C}$-groupoid $\mathcal{M}$, the following conditions are equivalent:

(i) The diagonal 1-morphism

$$\Delta_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{M} \times \mathcal{M}$$
$$X \longmapsto (X, X)$$

is representable.

(ii) For all $U \in \mathrm{ob}\, \mathcal{C}$ and $X, Y \in \mathrm{ob}\, \mathcal{M}(U)$, the presheaf $\underline{\mathrm{Isom}}_{\mathcal{M}}(X, Y)$ is representable (by an object of $\mathcal{C}/_U$).

(iii) For each $U \in \mathrm{ob}\, \mathcal{C}$, every 1-morphism $U \longrightarrow \mathcal{M}$ is representable.

WARNING: $\Delta_{\mathcal{M}}$ is NOT in general a 'monomorphism', i.e. a fully faithful functor.

In fact:

$\Delta_{\mathcal{M}}(U) : \mathcal{M}(U) \longrightarrow (\mathcal{M} \times \mathcal{M})(U)$ is fully faithful for each U

$$\Updownarrow$$

$\mathcal{M}$ is (associated to) a presheaf on $\mathcal{C}$

<u>Note</u>: The properties in the above Proposition are satisfied for $\mathcal{M} = \mathcal{M}_{g,n}$.

In fact,

$$\Delta_{\mathcal{M}_{g,n}} : \mathcal{M}_{g,n} \longrightarrow \mathcal{M}_{g,n} \times \mathcal{M}_{g,n}$$

is representable, separated, of finite type.

If $2g-2+n>0$, it is finite unramified (objects of $\mathcal{M}_{g,n}$ have no infinitesimal automorphisms)

If $n > 2g+2$ then it is a monomorphism, in fact a closed immersion

(objects of $\mathcal{M}_{g,n}$ have no nontrivial automorphisms, and $\mathcal{M}_{g,n}$ is a presheaf in this case)

But for instance, for $g = n = 0$, consider $X : \operatorname{Spec} \mathbb{Z} \to \mathcal{M}_{0,0}$ defined by $\mathbb{P}^1$: then

Then $\operatorname{Spec} \mathbb{Z} \times_{X, \mathcal{M}_{0,0}, X} \operatorname{Spec} \mathbb{Z} \simeq \underline{PGL}_{2,\mathbb{Z}}$

If we take $\mathcal{M}$ to be one of the following groupoids (over $C = (\text{Schemes})$):

$U \longmapsto$ (all U-schemes)

$U \longmapsto$ (quasicoherent $\mathcal{O}_U$-Modules)

then $\Delta_{\mathcal{M}}$ is not representable but has the weaker sheaf property:

for each U and $X, Y \in \text{ob}\, \mathcal{M}(U)$, the presheaf $\underline{\text{Isom}}_{\mathcal{M}}(X, Y)$ is a sheaf for the étale (even for the fpqc) topology:

If $(U_i \longrightarrow U)$ is an étale covering, and $(\varphi_i : X_{U_i} \longrightarrow Y_{U_i})$ is a family of isomorphisms which agree on $U_{ij} := U_i \times_U U_j$ (in the obvious sense), then they come from a unique isomorphism $\varphi : X \to Y$.

SHEAF-THEORETIC PROPERTIES:

PRESTACKS AND STACKS

We now assume given a Grothendieck topology on our category C (e.g. C = (Schemes) + étale topology).

Definition. A C-groupoid $\mathcal{M}$ is a prestack if for all $\begin{cases} U \in \mathrm{ob}\, C \\ X, Y \in \mathrm{ob}\, \mathcal{M}(U) \end{cases}$ the functor $\underline{\mathrm{Isom}}_{\mathcal{M}}(X,Y)$ is a sheaf on C.

Examples:

- C = (Schemes), • $\mathcal{M}(U)$ = (all U-schemes)
  • $\mathcal{M} = \mathcal{M}_{g,n}$
  • (etc.)

- C arbitrary, $\mathcal{M}$ = a presheaf F on C:
  then: C is a prestack

  $\Updownarrow$

  F is a separated presheaf (first axiom of sheaves: two locally equal sections are equal).

Stacks (gluing objects)

Let $\mathcal{M}$ be a C-groupoid and

$$\left(U_i \xrightarrow{\varphi_i} U \right)_{i \in I}$$

a covering family. Any $X \in \mathrm{ob}\, \mathcal{M}(U)$ determines the following data:

- $X_i := \varphi_i^* X = X|_{U_i}$ in $\mathcal{M}(U_i)$
- $\theta_{ij} : \quad X_i|_{U_{ij}} \xrightarrow{\sim} X_j|_{U_{ij}}$ in $\mathcal{M}(U_{ij})$

$$\begin{tikzcd} X_i|_{U_{ij}} \arrow[rr, "\sim"] & & X_j|_{U_{ij}} \\ & X|_{U_{ij}} \arrow[ul, "\sim"] \arrow[ur, "\sim"'] & \end{tikzcd}$$

satisfying the usual cocycle condition (compatibility of θ_{ij}'s on U_{ijk}).

Similarly, any morphism $f : X \to Y$ in $\mathcal{M}(U)$ determines $f_i : X_i \to Y_i$ in $\mathcal{M}(U_i)$, compatible with the θ_{ij}'s.

To say that $\mathcal{M}$ is a prestack just means that

$$\mathrm{Hom}_{\mathcal{M}(U)}(X,Y) \xrightarrow{\sim} \{\text{families } (f_i) \text{ as above}\}$$

If this is the case, we can reconstruct the functor (represented by) X in $\mathcal{M}(U)$ from the "descent datum" (X_i, θ_{ij}).

Definition. A C-groupoid $\mathcal{M}$ is a stack if

(i) $\mathcal{M}$ is a prestack, and

(ii) for each covering $(U_i \xrightarrow{\varphi_i} U)$ in C, every descent datum (X_i, θ_{ij}) as above is effective, i.e. associated to an object of $\mathcal{M}(U)$ (unique up to unique isomorphism, by (i)).

Remarks. One may think of a stack as a "sheaf of groupoids".

- One can (formally) construct the stack associated to a C-groupoid.

Examples :

- If F is a presheaf on C, then :

  F is a stack $\Leftrightarrow$ F is a sheaf

- $C =$ (Schemes) + étale topology :

  - QCOH and BUN_n are stacks

    (étale descent theorem for quasicoherent sheaves)

  - $U \mapsto$ (Schemes$/U$) is a prestack but not a stack (étale descent is not always effective)

  - $\mathcal{M}_{g,n}$ is a stack $\Leftrightarrow (g,n) \neq (1,0)$

    Indeed :

    - if $(g,n) \neq (1,0)$ then for every $(X \to U, x_1, \dots, x_n)$ in $\mathcal{M}_{g,n}(U)$ there is a canonical relatively ample sheaf on X :

      - $T_{X/U}$ if $g = 0$
      - $\Omega_{X/U}$ if $g \geq 2$
      - $\mathcal{O}_X(-x_1)$ if $n \geq 1$.

For $\mathcal{M}_{1,0}$ there are [should be?] examples of noneffective descent data.

If we "want $\mathcal{M}_{1,0}$ to be a stack" we have to relax some conditions in the definition:

$\boxed{\mathcal{M}_{g,n}(U) \text{ (NEW DEFINITION):}}$

objects = $(X \xrightarrow{f} U, x_1, \dots, x_n)$

where $f: X \to U$ is a smooth proper morphism of algebraic spaces, with geometric fibres connected, 1-dimensional, of genus g and $x_1, \dots, x_n : U \to X$ are disjoint sections.

ALGEBRAIC SPACES

Definition. A (quasiseparated) algebraic space is a functor

$$F : (\text{Schemes})^\circ \longrightarrow (\text{Sets})$$

with the following properties: - étale sheaf

(i) The diagonal morphism

$$\Delta_F : F \hookrightarrow F \times F$$

is representable and quasicompact.

(ii) There is a scheme X and a morphism

$$p : X \longrightarrow F$$

(automatically representable) which is étale and surjective.

If $p : X \to F$ is as above, then consider the Cartesian diagram of sheaves

$$\begin{tikzcd} R := X \times_F X \arrow[r, hook] \arrow[d] & X \times X \arrow[d, "p \times p"] \\ F \arrow[r, hook, "\Delta_F"] & F \times F \end{tikzcd}$$

- By (i), R is a scheme and the morphism $R \longrightarrow X \times X$ is quasicompact (in fact, quasiaffine).
- By (ii), both projections $R \rightrightarrows X$ are étale.
- By construction, R is an equivalence relation on X (i.e. $R(U) \subset X(U) \times X(U)$ is an equivalence relation, for each scheme U).
- By (ii), p is an epimorphism of sheaves, hence one can reconstruct F from X and R by

$$\underbrace{X/R}_{\text{(quotient in the category of étale sheaves)}} \xrightarrow{\sim} F.$$

CONVERSELY, any diagram of schemes

$$R \underset{p_2}{\overset{p_1}{\rightrightarrows}} X$$

with the above properties $\begin{cases} \text{equiv. relation} \\ p_1, p_2 \text{ étale} \\ R \xrightarrow{(p_1,p_2)} X \times X \text{ quasicompact} \end{cases}$

defines an algebraic space $F := X/R$.

For instance, if a finite group G acts freely on a scheme X, then X/G is an algebraic space (not a scheme in general).

Other examples:

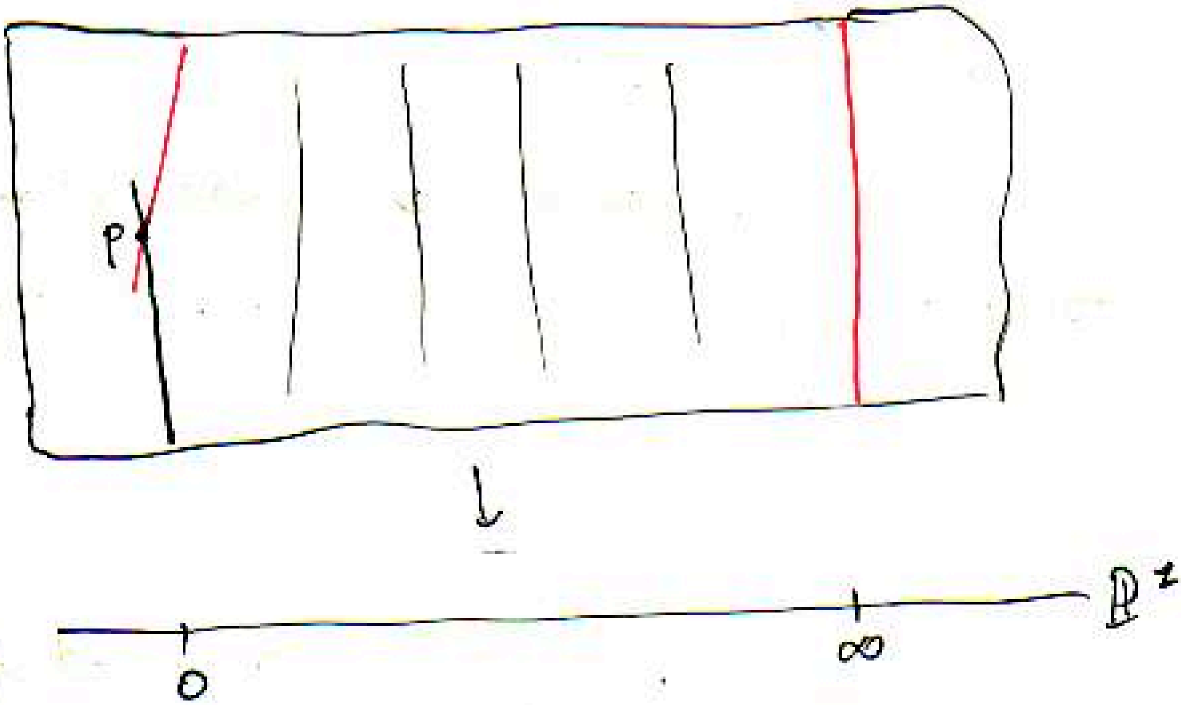
. (M. Artin) If $f: X \to U$ is a morphism of finite presentation, then the Hilbert functor on U-schemes

$$(T \to U) \longmapsto \left\{ \begin{array}{l} \text{closed subschemes of } X_T, \text{ proper and} \\ \text{flat over } T \end{array} \right\}$$

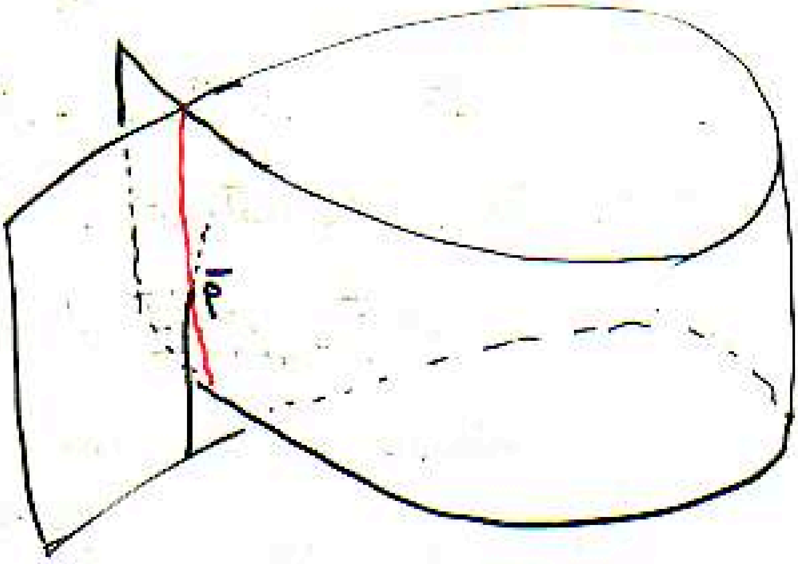
is an algebraic space (but not a scheme in general, unless $X \to U$ is quasiprojective).

Same for symmetric products $(X/U)^{(n)}$.

. Start with $\mathbb{P}^1 \times \mathbb{P}^1$, with one point blown up:



and identify the red lines. We get something like



which (exercise) . is an algebraic space

. is not a scheme ($\bar{p}$ has no affine neighbourhood).

Given a functor $F : (\text{Schemes})^\circ \to \text{Sets}$, it is in general much easier to prove that F is an algebraic space than a scheme.

It is also almost as useful (except if F is a quasiprojective scheme).

Back to our new $\mathcal{M}_{g,n}$:

- if $(g,n) \neq (1,0)$ it's the old one (quasiprojective algebraic spaces are schemes)
- if $X \to U$ is a U-curve of genus 1 (in the new sense) then its Jacobian $E \to U$ is an elliptic curve, and $\underline{\mathrm{Isom}}_U(E,X)$ contains X as a connected component, hence is not necessarily a scheme (but still an algebraic space).

RE-DEFINITION !

If $\Phi : \mathcal{M} \to \mathcal{N}$ is a 1-morphism of stacks (over the cat. of schemes + étale topology), we shall say that Φ is representable [in the sense of algebraic spaces] if:

For every algebraic space U and 1-morphism $X : U \longrightarrow \mathcal{N}$, the fibre product

$$U \times_{X, \mathcal{N}, \Phi} \mathcal{M}$$

is an algebraic space.

Remarks:

- it suffices to test on affine schemes U;
- one could also extend $\mathcal{M}$ and $\mathcal{N}$ to stacks $\overline{\mathcal{M}}, \overline{\mathcal{N}}$ over the category of algebraic spaces: the above definition is then equivalent to the representability of $\overline{\Phi} : \overline{\mathcal{M}} \to \overline{\mathcal{N}}$.

ALGEBRAIC STACKS

Definition. A stack $\mathcal{M}$ (over the cat. of schemes, with the étale topology) is algebraic (in Artin's sense) if:

(i) The diagonal morphism

$$\Delta_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{M} \times \mathcal{M}$$

is representable, quasicompact and separated.

(ii) There exists a scheme Y and a 1-morphism

$$P : Y \longrightarrow \mathcal{M}$$

which is [representable,] smooth and surjective.

A Deligne-Mumford stack is an algebraic stack $\mathcal{M}$ for which there exists $P : Y \to \mathcal{M}$ as in (ii) which is étale.

Example: BUN_r is algebraic:

(i) $E, F \in BUN_r(U)$: $\underline{\mathrm{Isom}}_U(E, F)$ is a U-scheme locally isomorphic to GL_r;

(ii) Take $Y = \mathrm{Spec}\,\mathbb{Z}$ and $P : Y \to BUN_r$ given by the trivial bundle: then for any $U \xrightarrow{E} BUN_r$, the U-scheme $U \times_{E, BUN_r, P} Y = \underline{\mathrm{Isom}}(\mathcal{O}_U^r, E)$ is smooth and surjective over U. Hence $P : Y \to BUN_r$ is smooth and surjective (and affine).

Example: $\mathcal{M}_{g,n}$ is algebraic:

. the representability of the diagonal is known.

If $(g,n) \neq (1,0)$, every $(X \xrightarrow{p} U, x_1, \dots, x_n)$ in $\mathcal{M}_{g,n}(U)$ carries a canonical invertible sheaf $\mathcal{L}(X)$, very ample relative to $p: X \to U$, such that $p_*\mathcal{L}(X)$ is locally free (of rank r, say) and commutes with base change

(e.g. $\omega_{X/U}^{\otimes 3}$ if $g \geq 2$).

Define a stack Y as follows:

$$Y(U) = \text{cat. of } (\underbrace{X \to U, x_1, \dots, x_n}_{\text{in } \mathcal{M}_{g,n}(U)}, \underbrace{\beta}_{\text{basis of } p_*\mathcal{L}(X)})$$

+ isomorphisms respecting the bases.

For such an object of $Y(U)$, we get an embedding:

$$X \hookrightarrow \mathbb{P}(p_*\mathcal{L}(X)) \xrightarrow[\sim]{\beta} \mathbb{P}^{r-1}_U$$

and we can identify $Y(U)$ with the set of all subschemes of $\mathbb{P}^{r-1}_U$ satisfying certain conditions (+ additional data, such as the marked points).

Then we can use the Hilbert scheme theory to show that Y is an algebraic space (in fact, a quasi-projective scheme).

. The natural 1-morphism

$$Y \longrightarrow \mathcal{M}_{g,n} \quad (\text{"forget } \beta\text{"})$$

is obviously smooth and surjective: for a 1-morphism $U \to \mathcal{M}_{g,n}$, corresponding to $(X \to U, \dots) \in \mathcal{M}_{g,n}(U)$, the fibre product

$$Y \times_{\mathcal{M}_{g,n}} U \longrightarrow U$$

is the sheaf of bases of $p_* \mathcal{L}(X)$, which is a GL_r-torsor on U, hence a smooth, affine, surjective U-scheme.

Remark: the above arguments can be put differently: we have a 1-morphism

$$\begin{aligned} \Phi : \mathcal{M}_{g,n} &\longrightarrow BUN_r \\ (X \to U, \dots) &\longmapsto p_* \mathcal{L}(X) \end{aligned}$$

and we check (as in the case of "3K") that Φ is representable. Now use:

Proposition $\Phi : \mathcal{M} \to \mathcal{N}$ representable, $\mathcal{N}$ algebraic (resp. D-M) $\Rightarrow \mathcal{M}$ algebraic (resp. D-M).

If $(g,n)=(1,0)$, consider the 1-morphism

$$\Phi : \mathcal{M}_{1,1} \longrightarrow \mathcal{M}_{1,0} \quad \text{(forget base point)}$$

Since $\mathcal{M}_{1,1}$ is algebraic, there is a $\Psi : Y \longrightarrow \mathcal{M}_{1,1}$, smooth and surjective. But Φ is also smooth and surjective, hence so is $\Phi \circ \Psi : Y \longrightarrow \mathcal{M}_{1,0}$.

Characterising Deligne-Mumford stacks:

Theorem. Let $\mathcal{M}$ be an algebraic stack. Equivalent conditions:

(i) $\mathcal{M}$ is a Deligne-Mumford stack.

(ii) The diagonal $\Delta_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{M} \times \mathcal{M}$ is unramified.

(iii) For every field k and $X \in \mathcal{M}(k)$, the k-group scheme $\underline{\mathrm{Aut}}(X)$ is finite étale ("no infinitesimal automorphisms").

(i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) easy

(ii) $\Rightarrow$ (i) a bit harder.

Corollary (characterisation of algebraic spaces)

Let $F : (\text{Schemes})^{\circ} \to (\text{Sets})$ be a functor.
Then:
F is an algebraic stack $\Leftrightarrow$ F is an algebraic space

Proof: $\Leftarrow$ trivial

$\Rightarrow$ trivial if F is a Deligne-Mumford stack.

In general, if F is an algebraic stack and a presheaf, the diagonal Δ_F is a monomorphism, hence unramified. By the preceding theorem, F is a D-M stack, so we are done ∎

Corollary.

- $\mathcal{M}_{g,n}$ is a D-M. stack $\Leftrightarrow$ $2g-2+n>0$
  $\Leftrightarrow (g,n) \notin \{(0,0),(0,1),(0,2),(1,0)\}$
- $\mathcal{M}_{g,n}$ is an algebraic space $\Leftrightarrow n > 2g+2$. ∎

Remark. To prove that $\mathcal{M}_{g,0}$ is a Deligne-Mumford stack for $g \geq 2$, one can use the moduli scheme $\mathcal{M}_{g,0}^{(r)}$ of curves with level-r structure over $\mathbb{Z}[1/r]$ $(r \geq 3)$:

$$U \longmapsto \left\{ \begin{array}{l} \text{curves } p : X \longrightarrow U \; + \\ \text{isomorphism } (\mathbb{Z}/r\mathbb{Z})_U^{2g} \xrightarrow{\sim} R^1 p_*(\mathbb{Z}/r\mathbb{Z}) \end{array} \right\}$$

This approach is less elementary, but gives a bonus: the natural morphism

$$\mathcal{M}_{g,0}^{(r)} \longrightarrow \mathcal{M}_{g,0} \,|\, \operatorname{Spec} \mathbb{Z}[1/r]$$

is finite étale.

QUOTIENT STACKS

Assume: S is a scheme,

$X \longrightarrow S$ an algebraic space

$G \longrightarrow S$ a sheaf of groups which is a smooth, separated, S-algebraic space of finite type acting on X

We define the quotient stack $\mathcal{M} = [X/G]$, over S, as follows:

$(S\text{-scheme } U) \longmapsto \mathcal{M}(U) =$ category of commutative diagrams

$$\begin{tikzcd} \widetilde{U} \arrow[r, "h"] \arrow[dr] & X_U = X \times_S U \arrow[d] \\ & U \end{tikzcd}$$

where $\left|\begin{array}{l} \widetilde{U} \longrightarrow U \text{ is a } G\text{-torsor over } U \\ h : \widetilde{U} \longrightarrow X_U \text{ is } G\text{-equivariant.} \end{array}\right.$

. There is a natural morphism

$$\pi : X \longrightarrow [X/G]$$

orbit morphism $g \mapsto xg$

$$x \in X(U) \longmapsto \begin{tikzcd} G_U \arrow[r] \arrow[dr] & X_U \arrow[d] \\ & U \end{tikzcd}$$

Now for any object $\tilde{U} \xrightarrow{h} X_U$ defining a point

$$\begin{tikzcd} \tilde{U} \arrow[rr, "h"] \arrow[dr] & & X_U \arrow[dl] \\ & U & \end{tikzcd}$$

$\xi : U \to [X/G]$, the diagram

$$\begin{tikzcd} \tilde{U} \arrow[r, "h"] \arrow[d] & X \arrow[d, "\pi"] \\ U \arrow[r, "\xi"] & {[X/G]} \end{tikzcd}$$

is Cartesian! In particular, π is surjective and smooth, and is a G-torsor in a natural sense (even if the action of G is not free!)

Special case: $X = S$ with trivial action of G.

The resulting stack is just:

$(S\text{-scheme } U) \longmapsto (\text{cat. of } G_U\text{-torsors})$.

This is the classifying stack of G, also denoted by BG.

Note that there is a structural morphism

$$p : BG \longrightarrow S$$

which is not representable unless G is trivial.

On the other hand, this morphism has a section

$$s : S \longrightarrow BG$$

corresponding to the trivial torsor $G \to S$. This section is representable and is in fact the universal G-torsor over BG.

Remark: we have seen some instances of BG before:

$$BUN_r \simeq B(GL_r)$$
$$\mathcal{M}_{0,0} \simeq B(PGL_2)$$
$$\mathcal{M}_{0,1} \simeq B\Gamma \quad (\Gamma = \text{affine transformations of } \mathbb{A}^1)$$
$$\mathcal{M}_{0,2} \simeq B\mathbb{G}_m \simeq BUN_1$$

For $g \geq 2$, we have seen two ways of viewing $\mathcal{M}_{g,0}$ as a quotient:

(1) $\left\{\begin{array}{l}\text{curves } X \xrightarrow{p} U \\ + \text{basis of } p_* \omega^{\otimes 3}\end{array}\right\} \longrightarrow \mathcal{M}_{g,0} \simeq [\widetilde{M}_{g,0} / GL_{5g-5}]$

<u>scheme</u> $\widetilde{M}_{g,0}$ + action of GL_{5g-5}

(2) $\mathcal{M}_{g,0}^{(n)} \longrightarrow \mathcal{M}_{g,0} | \operatorname{Spec} \mathbb{Z}[1/n] \simeq [\mathcal{M}_{g,0}^{(n)} / GL_{2g}(\mathbb{Z}/n\mathbb{Z})]$

(level n structures, $n \geq 3$, over $\mathbb{Z}[1/n]$)

The first morphism is a GL-torsor, hence smooth with geometrically connected fibres. Moreover, for any field (or semilocal ring) k,

$$\widetilde{M}_{g,0}(k) \longrightarrow \mathcal{M}_{g,0}(k)$$ ~~is~~ surjective on objects.

The second morphism is finite étale but

- only over $\mathbb{Z}[1/n]$.
- need a finite extension to lift points of $\mathcal{M}_{g,0}(k)$.
- $\mathcal{M}_{g,0,\mathbb{C}}$ is connected but $\mathcal{M}^{(n)}_{g,0,\mathbb{C}}$ is not.

GROUPOID SPACES
(generalisation of equivalence relations)

Definition A groupoid presheaf on a category C is a set of data:

- two functors $X_1, X_0 : C^\circ \to (\mathrm{Sets})$
  (X_0 = "objects", X_1 = "morphisms")
- morphisms
  $s, t : X_1 \rightrightarrows X_0$ (source + target)
  "identities": $X_0 \to X_1$
  "composition": $X_1 \times_{s, X_0, t} X_1 \to X_1$
  "inverse": $X_1 \to X_1$

which, for every $U \in \mathrm{ob}\, C$, defines a groupoid $X_\bullet(U)$ (whence a fibered groupoid $U \mapsto X_\bullet(U)$)

A groupoid space over $C = (\mathrm{Schemes})$ is a groupoid presheaf $X_\bullet$ where X_1, X_0 are algebraic spaces.

EXAMPLE: if $\mathcal{M}$ is an algebraic stack, X an algebraic space and $P : X \to \mathcal{M}$ a 1-morphism, there is a natural groupoid space

$$X_\bullet = \left(X_1 = X \times_{\mathcal{M}} X \rightrightarrows X_0 = X \right)$$

If P is smooth and surjective, then $\mathcal{M}$ is the étale stack associated to the fibered groupoid $X_\bullet$.

Conversely, given a groupoid space

$$X_\bullet = X_1 \overset{p_1}{\underset{p_2}{\rightrightarrows}} X_0$$

then the stack associated ($\mathcal{M}$) to $X_\bullet$ is algebraic, provided:

- $p_1, p_2 : X_1 \to X_0$ are smooth
- $(p_1, p_2) : X_1 \to X_0 \times X_0$ is quasicompact and separated.

Moreover, the natural morphism $X_0 \to \mathcal{M}$ is smooth and surjective.

Example: $[X/G]$ can be obtained from:

$$X_0 = X, \qquad X_1 = X \times G$$

$p_1(x,g) = x$

$p_2(x,g) = xg$

composition: $((x_1, g_1), (x_1 g_1, g_2)) \mapsto (x_1, g_1 g_2)$

identities: $x \mapsto (x, e)$

inverse: $(x,g) \mapsto (xg, g^{-1})$

Questions about the definition:

- Why the étale topology? (in particular, the fppf topology is very useful)
- Why ask for a smooth $Y \xrightarrow{P} \mathcal{M}$ (and not just a flat one?)

Theorem (M. Artin)

① Every algebraic stack is a stack for the fppf topology (i.e. we have effective fppf descent)

② Let $\mathcal{M}$ be an fppf stack over (Schemes) such that:

- $\Delta_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{M} \times \mathcal{M}$ is representable, quasicompact, separated
- there exists a scheme Y and a 1-morphism

$$P : Y \longrightarrow \mathcal{M}$$

which is faithfully flat, locally of finite presentation.

Then $\mathcal{M}$ is an algebraic stack.

Example: more general quotients:

Let $X_\bullet = X_1 \underset{p_2}{\overset{p_1}{\rightrightarrows}} X_0$ be a groupoid space

with $\begin{cases} p_1, p_2 \text{ flat of finite presentation} \\ (p_1, p_2) : X_1 \longrightarrow X_0 \times X_0 \text{ quasicompact, separated.} \end{cases}$

Then the associated fppf stack is algebraic.

Special case:

$$\begin{tikzcd} G \arrow[dr, "\substack{\text{flat,} \\ \text{finite presentation,} \\ \text{separated}}"'] & \text{acting on} & X \arrow[dl] \\ & S & \end{tikzcd}$$

then $[X/G]$ is an algebraic stack

(in the "torsor" description, one must take fppf torsors!)

Some examples of "geometry on algebraic stacks":

Let $\mathcal{P}$ be a property of schemes which is local in the étale sense, i.e.:

if $X' \longrightarrow X$ is étale surjective,

then X has $\mathcal{P} \Leftrightarrow X'$ has $\mathcal{P}$.

Examples:

- locally Noetherian
- ——— ——— and purely d-dimensional
- reduced
- normal
- regular
- (---)

Then $\mathcal{P}$ carries over to Deligne-Mumford stacks (and algebraic spaces):

if $\mathcal{M}$ is a D-M. stack, choose

$$\Phi : \underset{\text{scheme}}{\underbrace{Y}} \longrightarrow \mathcal{M} \quad \text{étale surjective}$$

and say that $\mathcal{M}$ has $\mathcal{P}$ iff Y does.

This is independent of the choice of Φ:

for another $\Phi' : Y' \longrightarrow \mathcal{M}$ étale surjective,

consider

$$\begin{tikzcd} Y \arrow[r, "\Phi"] & \mathcal{M} \\ Y \times_{\mathcal{M}} Y' =: Z \arrow[u, "g"] \arrow[r, "g'"] \arrow[ur, phantom, "\square"] & Y' \arrow[u, "\Phi'"'] \\ Z_1 \text{ (scheme)} \arrow[u, "\pi"] & \end{tikzcd}$$

π étale surjective (Z is an algebraic space)

Then: Y has $\mathcal{P}$, g and π étale surjective

$$\Downarrow$$

$$Z_1 \text{ has } \mathcal{P}$$

$$\Downarrow$$

$$Y' \text{ has } \mathcal{P}.$$

Remark In this situation, Z is in fact a scheme: for Deligne-Mumford stacks $\mathcal{M}$, the diagonal

$$\Delta_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{M} \times \mathcal{M}$$

is representable in the scheme sense. This boils down to the fact that $\Delta_{\mathcal{M}}$ is separated and quasi-finite (of finite type with finite fibres), hence quasi-affine.

If we try to do the same for Artin stacks, we must restrict to properties which are local in the smooth sense, i.e.:

if $X' \longrightarrow X$ is smooth and surjective, then X has $P \Leftrightarrow X'$ has P.

Examples: all the above, except "purely d-dimensional".

Remark: one can define the dimension of an algebraic stack $\mathcal{M}$ at a point $\mathrm{Spec}(k) \to \mathcal{M}$ (k a field). It may be negative.

For instance if G is an algebraic k-group, the dimension of $BG = [\mathrm{Spec}(k)/G]$ (at the obvious k-point) is $-\dim(G)$.

Definition. If $\mathcal{M}$ is an algebraic stack, a **locally closed** (resp. **open**, **closed**) **substack** of $\mathcal{M}$ is a 1-morphism $\mathcal{Y} \longrightarrow \mathcal{M}$ which is **representable by immersions** (resp. ...)

A closed substack $\mathcal{Y} \hookrightarrow \mathcal{M}$ has an obvious **open complement** $\mathcal{V}$ defined by

$$\mathcal{V}(U) = \{ x : U \longrightarrow \mathcal{M} \text{ in } \mathcal{M}(U) \mid U \times_{x,\mathcal{M}} \mathcal{Y} = \emptyset \}.$$

An opensubstack $\mathcal{V} \hookrightarrow \mathcal{M}$ has a **reduced closed complement** $\mathcal{Z}$ defined as follows: choose

$$P : Y \longrightarrow \mathcal{M} \text{ smooth surjective,}$$

put Z = reduced closed complement of $Y \times_{\mathcal{M}} \mathcal{V}$ in Y and define

$$\mathcal{Z}(U) = \{ x : U \longrightarrow \mathcal{M} \mid U \times_{x,\mathcal{M},P} Y \longrightarrow Y \text{ factors through } Z \}.$$

This does not depend on P, because taking the reduced closed complement commutes with smooth base change.

Definition. An algebraic stack $\mathcal{M}$ is separated if $\Delta_{\mathcal{M}} : \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ is proper.

Proposition (Valuative criterion) $\mathcal{M}$ is separated if and only if:

For every valuation ring Λ, with field of fractions K, and all $x, y \in \mathcal{M}(\Lambda)$,

every isomorphism $x_K \xrightarrow{\sim} y_K$ in $\mathcal{M}(K)$ extends $\underbrace{\text{(uniquely)}}_{\text{automatic since } \Delta_{\mathcal{M}} \text{ is separated.}}$ to $x \xrightarrow{\sim} y$ in $\mathcal{M}(\Lambda)$.

Remarks:
- One has a notion of separated 1-morphism $\mathcal{M} \to \mathcal{N}$ of algebraic stacks.
- If $\mathcal{M}$ is of finite type over a separated Noetherian scheme S, then one can restrict the valuative criterion to discrete valuation rings.
- Many useful stacks (such as BUN_n) are not separated!
- If $\mathcal{M}$ is a Deligne-Mumford stack, then:

$$\mathcal{M} \text{ separated} \Leftrightarrow \Delta_{\mathcal{M}} \text{ finite}$$

PROPER STACKS:

We fix a Noetherian base scheme S.

Definition An algebraic stack $\mathcal{M} \xrightarrow{f} S$ is **proper** (over S) if:

(1) $\mathcal{M}$ is of finite type, separated over S

(2) For each valuation ring V over S, with fraction field K, and every object $x : \operatorname{Spec} K \to \mathcal{M}$ of $\mathcal{M}(K)$, there is a valuation ring $V' \supset V$ dominating V, with fraction field $K' \supset K$, and an object of $\mathcal{M}(V')$ extending $x_{K'}$.

Example: let G be a finite group, and consider $\mathcal{M} = BG$ (over S) $= [S/G]$ (trivial action):

$$\begin{tikzcd} \mathcal{M} = [S/G] \arrow[d, "f"] \\ S \arrow[u, bend left, "s"'] \end{tikzcd} \quad \text{(trivial torsor) } s$$

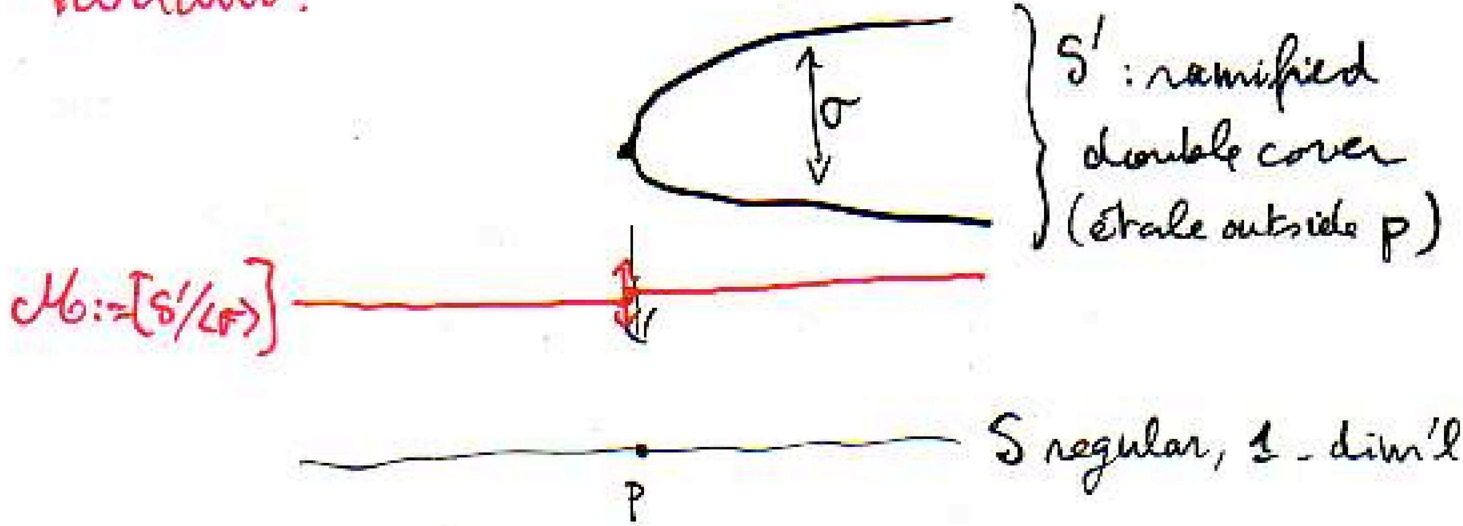
. $\mathcal{M}$ is a separated D.M. stack (the diagonal is finite étale).

. Since $s : S \to \mathcal{M}$ is finite étale surjective, and S is proper over S (!), $\mathcal{M}$ is of finite type, and we expect it to be proper.

Now, if, say, V is a discrete valuation ring, and L a ramified Galois extension of $K = \mathrm{Frac}(V)$, with group G, then $\mathrm{Spec}\, L$ is an object of $\mathcal{M}(K)$ which does not extend to a G-torsor over $\mathrm{Spec}(V)$.

But over $\mathrm{Spec}(L)$ the torsor becomes trivial, hence extends.

Variant:



Then $\mathcal{M} \to S$ is proper, and is an isomorphism over $S - \{p\}$, but is ramified at p.

The section $S - \{p\} \to \mathcal{M}$ does not extend to S (but extends over S').

Remarks.

- Condition (2) of the definition (the valuative criterion) is equivalent to $\mathcal{M} \to S$ being universally closed, in an appropriate sense.
- This notion is hard to use directly. For instance, one would like to restrict to discrete valuation rings, and/or finite extensions K'/K.
- Fortunately, we now have:

Theorem (Gabber-Olsson) Let $S = \mathrm{Spec}(A)$ (A Noetherian). Let $\mathcal{M} \to S$ be a separated algebraic stack, of finite type over S. Then there exists a quasiprojective S-scheme X and an S-morphism

$$p : X \to \mathcal{M}$$

which is proper and surjective.

In particular, $\mathcal{M}$ is proper over S iff X is.

Remarks.

- If $\mathcal{M}$ is a Noetherian Deligne-Mumford stack, there is a scheme X and a morphism $p: X \to \mathcal{M}$ which is finite, surjective, generically étale (no base scheme needed) (Laumon-LMB)
- The theorem implies finiteness of coherent cohomology for proper morphisms of algebraic stacks (other proof by Faltings)

over a field (under some additional assumptions on $\mathcal{M}$) there is a $p : X \longrightarrow \mathcal{M}$ finite and flat.

There are other results in the same vein, asserting the existence of nice morphisms from schemes to a given stack.

Here is such a result, of a local nature (and much easier!):

Theorem Let $\mathcal{M}$ be an Artin (resp. Deligne-Mumford) stack, K a field, $x \in \mathcal{M}(K)$. Then there exists a 2-commutative diagram

$$\begin{tikzcd} & X \arrow[d, "p"] \\ \mathrm{Spec}(K) \arrow[ur] \arrow[r, "x"] & \mathcal{M} \end{tikzcd}$$

where X is a scheme and p is smooth (resp. étale).

What about morphisms from $\mathcal{M}$ to algebraic spaces?

Theorem (Keel-Mori, 1997)

- S: a locally Noetherian scheme
- $\mathcal{M} \to S$: an algebraic stack of finite type over S, such that
$$\Delta_{\mathcal{M}/S} : \mathcal{M} \longrightarrow \mathcal{M} \times_S \mathcal{M} \text{ is finite.}$$

(e.g. $\mathcal{M}$ is a separated Deligne-Mumford S-stack of finite type).

Then $\mathcal{M}$ has a coarse moduli space.

More precisely: there is an S-morphism
$$q : \mathcal{M} \longrightarrow M$$
such that:

1. M is a separated algebraic space of finite type $/S$
2. for each geometric point ξ of S, the natural map
$$\{\text{isom. classes of } \mathcal{M}(\xi)\} \longrightarrow M(\xi)$$
is bijective
3. q is universal for S-morphisms from $\mathcal{M}$ to algebraic spaces
4. for every flat $M' \xrightarrow{\varphi} M$, the pullback
$$M' \times_M \mathcal{M} \longrightarrow M' \text{ is still universal.}$$