Principal Bundles over Valued Fields

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Joint work with

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Summary

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2. Principal bundles and torsors

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4. Admissible valued fields

5. The smooth case and the general strategy
We start with:

- a topological field $k$,
- an algebraic $k$-group $G$,
- a $k$-variety $Y$, and
- a $G$-torsor (principal $G$-bundle) $f : X \to Y$ over $Y$. 
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Taking rational points, we get

- a topological group $G(k)$,
- a continuous free action of $G(k)$ on the space $X(k)$,
- a continuous map $X(k) \rightarrow Y(k)$, invariant for this action.

This map is not surjective in general.

We will consider the following questions, in the case of a henselian valued field:

- What does the image $I$ of this map look like, as a subspace of $Y(k)$?
- Is the induced map $X(k) \rightarrow I$ a principal $G(k)$-bundle?

Remark: the answers are easy and well known in characteristic zero (and more generally if $G$ is smooth).
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Principal bundles in topology

Let $G$ be a topological group. A (left) $G$-bundle consists of the following data:

- a continuous map $f : X \to Y$,
- a (left) action $G \times X \to X$ commuting with $f$ (i.e. $f(g \cdot x) = f(x)$).

A $G$-bundle is trivial if it is isomorphic (in the obvious sense) to $G \times Y \overset{\text{pr}_2}{\to} Y$ with the action of $G$ on itself by left translation.

It is principal if it is locally trivial (on $Y$), in the obvious sense.
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Principal bundles in algebraic geometry: torsors

Let $k$ be a field, $G$ an algebraic group over $k$, and $Y$ a $k$-variety.

A (left) $G$-bundle over $Y$ consists of:
- a $k$-morphism $f : X \to Y$,
- a (left) action of $G$ on $X$, compatible with $f$,

We call it a (left) $G$-torsor if it is locally trivial for the fppf (or flat) topology, i.e. there is a $k$-morphism $h : Y' \to Y$ such that:
- $h$ is flat and surjective,
- $h$ trivializes $f$, i.e. the pullback $G$-bundle $X \times_Y Y' \to Y'$ is trivial.
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A simple example

Let $n$ be a positive integer. Consider the $n$-th power map

$$f : \mathbb{G}_{m,k} \rightarrow \mathbb{G}_{m,k}$$

$$x \mapsto x^n.$$

This is a $\mu_n$-torsor (with the obvious action of $\mu_n = \ker(f)$ on $\mathbb{G}_{m,k}$).

If $n$ is invertible in $k$, then $f$ is even locally trivial for the étale topology, i.e. trivialized by an étale surjective map (e.g. $f$ itself).

More generally, if $G$ is a smooth $k$-group, any $G$-torsor $f : X \rightarrow Y$ is a smooth morphism, hence locally trivial for the étale topology. This holds in particular if $\text{char}(k) = 0$.

But in our example, if $n = \text{char}(k) > 0$, then $f$ is just the Frobenius map on $\mathbb{G}_{m,k}$. 
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Characterization of torsors

A $G$-bundle $f : X \to Y$ in topology (resp. in algebraic geometry) is a $G$-torsor if and only if:

- it is “formally principal” (or a “pseudo-torsor”), i.e. the natural morphism
  \[
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  \]
  
  \[\begin{array}{ccc}
  (g, x) & \mapsto & (g \cdot x, x)
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  is an isomorphism,

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The “pseudo-torsor” property

\[ G \times X \sim \rightarrow X \times_Y X \]

is completely “categorical”, and is preserved by any functor on \( k \)-varieties that commutes with fiber products, such as the functor of rational points \( R : Z \rightarrow Z(k) \).

It follows that if \( f : X \rightarrow Y \) is a \( G \)-torsor over \( k \), then the induced map of sets (or discrete spaces)

\[ R(f) : X(k) \rightarrow Y(k) \]

(which may not be surjective) induces a principal \( G(k) \)-bundle over its image.
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Torsors over topological fields

From now assume that $k$ is a topological field, e.g. a valued field.
For every $k$-variety $Z$, the set $Z(k)$ has a natural topology. The resulting topological space will be denoted by $Z_{\text{top}}$ (or $Z(k)_{\text{top}}$).

In particular, for a $G$-torsor $f : X \rightarrow Y$:

- $G_{\text{top}}$ is a topological group, and
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Torsors over topological fields

Example of the squaring map:

\[ f : \mathbb{G}_m,k \longrightarrow \mathbb{G}_m,k \]

\[ x \longmapsto x^2. \]

If \( k = \mathbb{R} \), the image of \( f_{\text{top}} \) is \( \mathbb{R}_{>0} \) (open and closed in \( \mathbb{R}^\times \)), and \( f_{\text{top}} \) induces a trivial \( \{\pm 1\} \)-bundle over this image.

If \( k = \mathbb{C} \), then \( f_{\text{top}} \) is surjective and induces a nontrivial principal \( \{\pm 1\} \)-bundle over \( \mathbb{C}^\times \).

If \( k = \mathbb{F}_2((t)) \), then \( f_{\text{top}} \) is a homeomorphism onto its image, which is closed in \( k^\times \).
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Back to a general $G$-torsor $f: X \to Y$ over a topological field $k$:

We can factor $f_{\text{top}}: X_{\text{top}} \to Y_{\text{top}}$ as

$$
X_{\text{top}} \quad \to \quad X_{\text{top}}/G_{\text{top}} \quad \to \quad \text{Im}(f_{\text{top}}) \quad \to \quad Y_{\text{top}}
$$

which gives rise to natural questions:
Torsors over topological fields

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X_{\text{top}} \quad \rightarrow \\
\text{quotient map} \\
(\text{open})
\end{array}
\begin{array}{c}
X_{\text{top}}/G_{\text{top}} \quad \rightarrow \\
\text{continuous} \\
\text{bijection}
\end{array}
\begin{array}{c}
\text{Im}(f_{\text{top}}) \quad \rightarrow \\
\text{topological} \\
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quotient map (open) \hspace{2cm} continuous bijection \hspace{2cm} topological embedding

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Torsors over topological fields

\[ X_{\text{top}} \rightarrow X_{\text{top}}/G_{\text{top}} \rightarrow \text{Im}(f_{\text{top}}) \rightarrow Y_{\text{top}} \]

- Is the image of \( f_{\text{top}} \) closed (open, locally closed) in \( Y_{\text{top}} \)?
- Is the middle bijection a homeomorphism? (In other words, is \( f_{\text{top}} \) a strict map?)
- Is \( X_{\text{top}} \rightarrow X_{\text{top}}/G_{\text{top}} \) a principal \( G_{\text{top}} \)-bundle?
Torsors over topological fields

\[ X_{\text{top}} \xrightarrow{\text{quotient}} X_{\text{top}}/G_{\text{top}} \xrightarrow{\text{bijection}} \text{Im}(f_{\text{top}}) \xrightarrow{\text{embedding}} Y_{\text{top}} \]

1. Is the image of $f_{\text{top}}$ closed (open, locally closed) in $Y_{\text{top}}$?

2. Is the middle bijection a homeomorphism? (In other words, is $f_{\text{top}}$ a strict map?)

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Note that a positive answer to both Questions 2 and 3 is equivalent to a positive answer to

4. Is \( X_{\text{top}} \rightarrow \text{Im}(f_{\text{top}}) \) a principal \( G_{\text{top}} \)-bundle?
The main result

**Definition**

A valued field \((K, v)\) is admissible if

- \((K, v)\) is henselian;
- the completion \(\hat{K}\) of \(K\) is a separable extension of \(K\).

**Main Theorem**

Let \((K, v)\) be an admissible valued field, \(G\) an algebraic \(K\)-group, and \(f : X \to Y\) a \(G\)-torsor. Then:

1. \(\text{Im}(f_{\text{top}})\) is locally closed in \(Y_{\text{top}}\).
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- it is open and closed in \(Y_{top}\) if \(G\) is smooth, or if \(K\) is perfect;
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The case of homogeneous spaces

As an example, we can take for $X$ an algebraic group and for $G$ a subgroup of $X$, and consider $f : X \rightarrow Y := X/G$.

Then the image of $f_{top}$ is the orbit $X_{top}.y$ ($y =$ origin of $Y$). The theorem says that

- this orbit is locally closed in $Y_{top}$, and
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When $K$ is a local field, this is due to Bernstein and Zelevinsky (1976).
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An example of a non-closed orbit

Assume char \((K) = p > 0\). Let \(S = \mathbb{G}_a \times \mathbb{G}_m\) be the affine group in dimension 1, acting on \(X = \mathbb{A}^1_K\) transitively “via Frobenius on \(S\)":

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((x, y), u) \longmapsto (x, y).u := x^p + y^p u
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For \(u \in K\), consider the orbit morphism

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f_u : S \rightarrow \mathbb{A}^1, \quad s \mapsto s.u.
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This is a torsor under the stabilizer \(S_u\) of \(u\).

The image of \(f_{u,\text{top}}\) is the orbit \(S(K).u = K^p + (K^\times)^p u \subset K\). In particular:

- if \(u \in K^p\), the orbit is \(K^p\), which is closed in \(K\) if \(K\) is admissible;
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Notation and conventions

- $R$: a valuation ring,
- $K = \text{Frac}(R)$,
- $\nu$: the valuation,
- $\hat{K}$: completion of $K$,
- $K$-variety $= K$-scheme of finite type,
- algebraic $K$-group $= K$-group scheme of finite type,
- $R$ (or $(K, \nu)$) is admissible if $R$ is henselian and the extension $\hat{K}/K$ is separable.
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Properties of admissible valued fields

Assume $(K, \nu)$ is admissible. Then:

- $K$ is algebraically closed in $\hat{K}$.
- If $L$ is a finite extension of $K$, then:
  - $L$ is admissible (for the unique extension of $\nu$),
  - as a topological $K$-vector space, $L$ is free (isomorphic to $K^{[L:K]}$),
  - $\hat{K} \otimes_K L \sim \hat{L}$.
- If $\text{char}(K) > 0$, the Frobenius map $K \to K$ is a closed topological embedding.
- $R$ has the strong approximation property (à la Greenberg).
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Admissible valuations: topological properties of morphisms

**Proposition 1**

Assume \((K, v)\) is admissible, and let \(f : X \rightarrow Y\) be a morphism of \(K\)-varieties. Consider the induced continuous map \(f_{\text{top}} : X_{\text{top}} \rightarrow Y_{\text{top}}\).

1. “Implicit function theorem”: If \(f\) is étale, then \(f_{\text{top}}\) is a local homeomorphism.
2. If \(f\) is smooth, then \(f_{\text{top}}\) has local sections at each point of \(X_{\text{top}}\). (In particular, it is an open map).
3. “Continuity of roots”: If \(f\) is finite, then \(f_{\text{top}}\) is a closed map (hence proper, since it has finite fibers).

**Warning!** If \(f\) is proper, \(f_{\text{top}}\) is not a closed map in general. But its image is closed in \(Y_{\text{top}}\).
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Assume \((K, v)\) is admissible, and let \(f : X \to Y\) be a morphism of \(K\)-varieties. Consider the induced continuous map \(f_{\text{top}} : X_{\text{top}} \to Y_{\text{top}}\).

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Now let us return to the main result:

**Main Theorem**

Let \((K, v)\) be an admissible valued field, \(G\) an algebraic \(K\)-group, and \(f : X \to Y\) a \(G\)-torsor. Then:

1. \(\text{Im}(f_{\text{top}})\) is locally closed in \(Y_{\text{top}}\).

2. The induced map \(X_{\text{top}} \to \text{Im}(f_{\text{top}})\) is a principal \(G_{\text{top}}\)-bundle.
The smooth case

Let us explain the smooth case. If $G$ is smooth, then:

- $f : X \to Y$ is a smooth morphism,
- hence $f_{\text{top}}$ has local sections at each point of $X_{\text{top}}$.
- This proves that
  - $\text{Im}(f_{\text{top}})$ is open, and
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Next, a standard “twisting argument” shows that $Y_{\text{top}} \setminus \text{Im}(f_{\text{top}})$ is a union of subsets similar to $\text{Im}(f_{\text{top}})$. Hence $\text{Im}(f_{\text{top}})$ is also closed.
Strategy for general $G$

Let $K_s$ be a separable closure of $K$. $G$ has a largest smooth subgroup $G^\dagger$, which can be defined as the Zariski closure of $G(K_s)$ in $G$.

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It is easy to check that $(G/G^\dagger)(K_s) = \{e\}$ (in particular $(G/G^\dagger)(K) = \{e\}$).

More generally, if $T$ is a $G$-torsor over $K$, then $T/G^\dagger$ has at most one rational point.

Now let $f : X \rightarrow Y$ be a $G$-torsor. We factor it as

$$X \xrightarrow{\pi} Z := X/G^\dagger \xrightarrow{h} Y.$$ 

The corresponding factorization of $f_{\text{top}}$ looks like

$$X_{\text{top}} \rightarrow \text{Im}(\pi_{\text{top}}) \subset Z_{\text{top}} \xrightarrow{h_{\text{top}}} Y_{\text{top}}.$$
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The hard part of the proof is to show that $h_{\text{top}}$ is in fact a topological embedding, with locally closed image.

This uses:

- strong approximation,
- the construction (due to Gabber) of a remarkable $G$-equivariant compactification of $G/G^\dagger$. 
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Laurent Moret-Bailly (IRMAR) | Principal Bundles over Valued Fields | Oberwolfach, June 2013 | 24 / 24
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