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Topological properties of principal G-bundles over valued fields

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Summary

Orbit maps: the classical case

Let K be a field. Consider the following situation:

- G is an algebraic K-group acting on a K-variety T,
- $t \in T(K)$ is a point, with stabilizer $G_t \subset G$.
- $f : G \to T$ is the orbit map $g \mapsto g.t$.

By a theorem of Chevalley, the image of f is a locally closed subscheme $Y \subset T$, and the induced morphism $G \to Y$ is a principal G_t-bundle, i.e. a G_t-torsor for the fppf topology (even the étale topology, if G_t is smooth).

Orbit maps: the classical case

Now assume that K is $\mathbb{R}$ or $\mathbb{C}$, and look at the K-rational points, with the real/complex topology. The orbit map then factors as

$$G(K) \to G(K).t \hookrightarrow Y(K) \hookrightarrow T(K).$$

From the fact that $G \to Y$ is a principal G_t-bundle, and G_t is smooth, it follows that:

- $G(K).t$ is open in $Y(K)$, and also closed (its complement is a union of orbits);
- in particular $G(K).t$ is locally closed in $T(K)$;
- $G(K) \to G(K).t$ is a principal $G_t(K)$-bundle.

Problem:

do these properties extend

to other topological fields?

Note: the important map here is $G \to Y$, Y being a homogeneous space.

We may also generalize: given a topological field K and a torsor $X \to Y$ under some algebraic K-group G (previously G_t), what can we say about the map $X(K) \to Y(K)$, topologically?

In this talk I will give an answer for certain valued fields.

Notation and conventions

- R: a valuation ring,
- $K = \mathrm{Frac}\,(R)$,
- Γ: the valuation group,
- $v : K \to \Gamma \cup \{+\infty\}$: the valuation,
- $\widehat{R} := \varprojlim_{z \in R \smallsetminus \{0\}} R/zR$ (completion of R),
- $\widehat{K} := Frac\,(\widehat{R})$,
- K-variety = separated K-scheme of finite type,
- algebraic K-group = K-group scheme of finite type.

The valuation topology

Recall that K is a topological field, with a basis of neighborhoods of 0 consisting of the nonzero (principal) ideals of R.

Accordingly, for each K-variety X, $X(K)$ has a natural topology: a basis of open subsets consists of the sets

$$\left\{x \in U(K) \mid v(f_i(x)) \geq 0, i = 1, \ldots, m\right\}$$

for $U \subset X$ affine open, $f_i \in H^0(U, \mathscr{O}_U)$.

The resulting topological space will be denoted by X_{top}.

Admissible valuations

Let us say that (K, v) (or R) is admissible if:

- (K, v) is henselian; i.e. (equivalently):
  - R is a henselian local ring;
  - v extends *uniquely* to every finite extension of K.
- $\widehat{K}$ is a separable extension of K.

(If $\Gamma \cong \mathbb{Z}$, the last condition says that R is excellent).

Properties of admissible valued fields

Assume (K, v) is admissible. Then:

- K is algebraically closed in $\widehat{K}$.
- If L is a finite extension of K, then:
  - L is admissible (for the unique extension of v),
  - as a topological K-vector space, L is free (isomorphic to $K^{[L:K]}$),
  - $\widehat{K} \otimes_K L \xrightarrow{\sim} \widehat{L}$.
- If $\mathrm{char}\,(K) > 0$, the Frobenius map $K \to K$ is a closed topological embedding.
- R has the strong approximation property (à la Greenberg).

Admissible valuations: topological properties of morphisms

Proposition 1

Assume (K, v) is admissible, and let $f : X \to Y$ be a morphism of K-varieties. Consider the induced continuous map $f_{\mathrm{top}} : X_{\mathrm{top}} \longrightarrow Y_{\mathrm{top}}$.

1. "Implicit function theorem": *If f is étale, then f_{top} is a local homeomorphism.*
2. *If f is smooth, then f_{top} has local sections at each point of X_{top}. (In particular, it is an open map).*
3. "Continuity of roots": *If f is finite, then f_{top} is a closed map (hence proper, since it has finite fibers).*

Warning! If f is proper, f_{top} is not a closed map in general. But its image is closed in Y_{top}.

Extension to algebraic spaces

We need to extend the X_{top} construction to algebraic spaces (of finite type over K) because:

- we shall consider G-torsors $f : X \to Y$ where G is an algebraic group. Even if Y is a scheme, a G-torsor over Y is naturally defined as an fppf sheaf. It is always an algebraic space, but not necessarily a scheme.

- We need to construct objects (typically quotients by algebraic group actions) which are always algebraic spaces but not necessarily schemes, even if we start from schemes.

Extension to algebraic spaces

So let K be a topological field, and X an algebraic space of finite type over K.

We equip $X(K)$ with the finest topology making all maps $f_{\mathrm{top}} : X'_{\mathrm{top}} \to X(K)$ continuous, where X' runs through [affine] K-varieties, and $f : X' \to X$ runs through all K-morphisms.

The resulting space is denoted by X_{top} (if X is a scheme, it is the same as before). This construction has the same basic compatibilities as in the case of varieties.

Assume now that K satisfies the implicit function theorem (e.g. K is a henselian valued field).

Then one can define X_{top} using only étale morphisms $f : X' \to X$ in the definition. For such an f, the resulting f_{top} is a local homeomorphism.

Torsors

Let G be an algebraic group over K, and $f : X \to Y$ a (right) G-torsor over a variety (or algebraic space) Y.

The induced map $f_{\mathrm{top}} : X_{\mathrm{top}} \to Y_{\mathrm{top}}$ decomposes as

$$X_{\mathrm{top}} \xrightarrow[\text{(open)}]{\text{quotient map}} X_{\mathrm{top}}/G_{\mathrm{top}} \xrightarrow[\text{bijection}]{\text{continuous}} \mathrm{Im}(f_{\mathrm{top}}) \xrightarrow[\text{embedding}]{\text{topological}} Y_{\mathrm{top}}$$

which gives rise to natural questions:

Torsors

$$X_{\mathrm{top}} \xrightarrow[\text{quotient}]{} X_{\mathrm{top}}/G_{\mathrm{top}} \xrightarrow[\text{bijection}]{} \mathrm{Im}(f_{\mathrm{top}}) \xrightarrow[\text{embedding}]{} Y_{\mathrm{top}}$$

1. Is the image of f_{top} closed (open, locally closed) in Y_{top}?
2. Is the middle bijection a homeomorphism? (In other words, is f_{top} a strict map?)
3. Is $X_{\mathrm{top}} \to X_{\mathrm{top}}/G_{\mathrm{top}}$ a "topological torsor" (i.e. a principal G_{top}-bundle)?
   Equivalently, does this map have continuous local sections everywhere? (Remark: G_{top} acts freely and properly on X_{top}).

Note that a positive answer to both Questions 2 and 3 is equivalent to a positive answer to

4. Is $X_{\mathrm{top}} \to \mathrm{Im}(f_{\mathrm{top}})$ a principal G_{top}-bundle?

The main result

Main Theorem

Let (K, v) be an admissible valued field, G an algebraic K-group, and $f : X \to Y$ a G-torsor. Then:

1. $\mathrm{Im}(f_{\mathrm{top}})$ *is locally closed in* Y_{top}.
2. *The induced map* $X_{\mathrm{top}} \to \mathrm{Im}(f_{\mathrm{top}})$ *is a principal* G_{top}*-bundle.*

Remark. In some cases, we can say more about $\mathrm{Im}(f_{\mathrm{top}})$:

- it is open and closed in Y_{top} if G is smooth, or if K is perfect;
- it is closed in Y_{top} if G°_{red} is smooth, or if G is commutative.

The case of orbit maps

Corollary 1

Let (K, v) be an admissible valued field,
G an algebraic K-group acting on a variety T,
$t \in T(K)$ a rational point, with stabilizer $G_t \subset G$.
Then the orbit map $g \mapsto g.t$ on G_{top} factors as

$$G_{\mathrm{top}} \underset{\substack{\textit{principal} \\ G_t\textit{-bundle}}}{\longrightarrow} G_{\mathrm{top}}.t \underset{\substack{\textit{locally closed} \\ \textit{embedding}}}{\hookrightarrow} T_{\mathrm{top}} .$$

When K is a local field, this is due to Bernstein and Zelevinsky (1976).

An example of a non-closed orbit

Let $S = \mathbb{G}_a \rtimes \mathbb{G}_m$ be the affine group in dimension 1, acting on $X = \mathbb{A}^1_K$ transitively "via Frobenius on S":

$$\begin{array}{rcl} S \times \mathbb{A}^1 & \longrightarrow & \mathbb{A}^1 \\ ((x,y),z) & \longmapsto & (x,y).z := x^p + y^p z \end{array}$$

For $z \in K$, consider the orbit morphism

$$f_z : S \to \mathbb{A}^1, \quad s \mapsto s.z\,.$$

This is a torsor under the stabilizer S_z of z.

The image of $f_{z,\mathrm{top}}$ is the orbit $S(K).z = K^p + (K^\times)^p z \subset K$. In particular:

- if $z \in K^p$, the orbit is K^p, which is closed in K if K is admissible;
- for any choice of z, the orbit has 0 in its closure (consider the action of $\mathbb{G}_m$).

Hence, if $z \notin K^p$, then $\mathrm{Im}(f_{z,\mathrm{top}})$ is not closed in K.

Torsors: the smooth case

Let us explain the smooth case. If G is smooth, then:

- $f : X \to Y$ is a smooth morphism,
- hence f_{top} has local sections at each point of X_{top}.
- This proves that
  - $\mathrm{Im}(f_{\mathrm{top}})$ is open, and
  - $X_{\mathrm{top}} \to \mathrm{Im}(f_{\mathrm{top}})$ is a principal G_{top}-bundle.

Why is $\mathrm{Im}(f_{\mathrm{top}})$ closed?

Consider the "classifying map" $Y(K) \xrightarrow{\partial} \mathrm{H}^1(K, G)$ deduced from f: we know that $\mathrm{Im}(f_{\mathrm{top}}) = \partial^{-1}(1)$ is open in $Y(K)$.
Together with a standard "twisting" argument, this implies that each fiber of ∂ is open.
In other words, ∂ is locally constant and $\mathrm{Im}(f_{\mathrm{top}}) = \partial^{-1}(1)$ is closed. □

Strategy for general G

- Pick a smooth subgroup $G' \subset G$.
- Consider the factorization $X \xrightarrow{\pi} Z := X/G' \xrightarrow{h} Y$.
- The smooth case applies to π (which is a G'-torsor).
- To control h and h_{top}, we need to study G/G', with G' as big as possible.

Strategy for general G

Let K_s be a separable closure of K. G has a largest smooth subgroup $G^\dagger$, which can be defined as the Zariski closure of $G(K_s)$ in G.

This construction is functorial in G and commutes with separable ground field extensions.

Strategy for general G

It is easy to check that $(G/G^\dagger)(K_s) = \{e\}$ (in particular $(G/G^\dagger)(K) = \{e\}$).

More generally, if T is a G-torsor over K, then $T/G^\dagger$ has at most one rational point.

Now let $f : X \to Y$ be a G-torsor, factored as

$$X \xrightarrow{\pi} Z := X/G^\dagger \xrightarrow{h} Y.$$

The corresponding factorization of f_{top} looks like

$$X_{\mathrm{top}} \underset{G^\dagger_{\mathrm{top}}\text{-bundle}}{\longrightarrow} \mathrm{Im}(\pi_{\mathrm{top}}) \underset{\text{open, closed}}{\subset} Z_{\mathrm{top}} \underset{\text{injective}}{\xrightarrow{h_{\mathrm{top}}}} Y_{\mathrm{top}}$$

This leads to a question:

Strategy for general G

Question:

Assume $h : Z \to Y$ is a K-morphism such that h_{top} is injective. Can we say more (topologically) about h_{top}?

The answer is yes if h is proper:

Proposition 2

Let $h : Z \to Y$ be a proper K-morphism. Let $z \in Z(K)$ and $y = h(z)$ be such that $h_{\mathrm{top}}^{-1}(y) = \{z\}$.

Then every neighborhood of z in Z_{top} contains $h_{\mathrm{top}}^{-1}(V)$ for some neighborhood V of y.

(In particular, if h_{top} is injective then it is a closed topological embedding).

(The proof uses strong approximation).

Strategy for general G

In our situation

$$X \xrightarrow{\pi} Z := X/G^{\dagger} \xrightarrow{h} Y$$

h is not proper in general

so we will compactify it.

Gabber's compactification theorem

Let G be an algebraic K-group. Put $Q = G/G^\dagger$, and denote by q the unique point of $Q(K)$.

Theorem (Gabber, 2012)

Q admits a G-equivariant projective compactification $Q \hookrightarrow Q^c$ without separable points at infinity:

if L is a separable extension of K, then $Q^c(L) = Q(L) = \{q\}$.

End of proof

Theorem (Gabber, 2012)

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Back to our diagram $f : X \xrightarrow{\pi} Z \xrightarrow{h} Y$:

Consider the contracted product $Z^c := X \times^G Q^c$. This is a relative compactification (over Y) of $X \times^G Q = X/G^\dagger = Z$. Put $Z^\infty = Z^c \smallsetminus Z$.

End of proof

Now we have a diagram

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[drr, "f"'] & Z \arrow[r, hook, "j"] \arrow[dr, "h"] & Z^c \arrow[d, "h^c"] & Z^\infty \arrow[l, hook', "i"'] \arrow[dl, "h^\infty"] \\ & & Y. & \end{tikzcd}$$

- if $z \in Z(K)$, and $y = h(z)$, then $Z_y(K) = Z_y^c(K) = \{z\}$ (from the properties of Gabber's compactification).
  In particular:
- $Z(K)$ and $Z^\infty(K)$ have disjoint images in $X(K)$. Hence, $\mathrm{Im}(h_{\mathrm{top}}) = \mathrm{Im}(h^c_{\mathrm{top}}) \smallsetminus \mathrm{Im}(h^\infty_{\mathrm{top}})$. But h^c and h^∞ are proper, so these images are closed. Hence $I := \mathrm{Im}(h_{\mathrm{top}})$ is locally closed.
- $h_{\mathrm{top}} : Z(K) \to X(K)$ is injective. Moreover, $Z(K)$ and $Z^c(K)$ "coincide over I", so we can apply Proposition 2 at any point of $Z(K)$. This implies that $Z_{\mathrm{top}} \to \mathrm{Im}(h_{\mathrm{top}})$ is closed and bijective, hence a homeomorphism.

End of proof

Summarizing, we have the following decomposition of f_{top}:

$$X_{\mathrm{top}} \underset{\substack{G^{\dagger}_{\mathrm{top}}\text{-}\\ \text{bundle}}}{\longrightarrow} \mathrm{Im}(\pi_{\mathrm{top}}) \underset{\substack{\text{open}\\ \text{and closed}\\ \text{embedding}}}{\hookrightarrow} Z_{\mathrm{top}} \underset{\text{homeo}}{\longrightarrow} \mathrm{Im}(h_{\mathrm{top}}) \underset{\substack{\text{locally}\\ \text{closed}\\ \text{embedding}}}{\hookrightarrow} Y_{\mathrm{top}}$$

which completes the proof. (Note that $G^{\dagger}_{\mathrm{top}} = G_{\mathrm{top}}!$).