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Topological properties of principal *G*-bundles over valued fields

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Joint work with Ofer Gabber and Philippe Gille (CNRS)

Summary

- 1 Introduction: the classical case
- 2 Topology of varieties over valued fields
- Torsors and the main result
- The smooth case and the general strategy
- 5 Gabber's compactification theorem

Let K be a field. Consider the following situation:

- G is an algebraic K-group acting on a K-variety T,
- $t \in T(K)$ is a point, with stabilizer $G_t \subset G$.
- $f: G \to T$ is the orbit map $g \mapsto g.t$.

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Now assume that K is \mathbb{R} or \mathbb{C} , and look at the K-rational points, with the real/complex topology. The orbit map then factors as

$$G(K) \rightarrow G(K).t \hookrightarrow Y(K) \hookrightarrow T(K).$$

- G(K).t is open in Y(K), and also closed (its complement is a union of orbits);
- in particular G(K).t is locally closed in T(K);
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Problem:

do these properties extend

to other topological fields?

Note: the important map here is $G \rightarrow Y$, Y being a homogeneous space.

We may also generalize: given a topological field K and a torsor $X \to Y$ under some algebraic K-group G (previously G_t), what can we say about the map $X(K) \to Y(K)$, topologically?

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In this talk I will give an answer for certain valued fields.

- R: a valuation ring,
- $K = \operatorname{Frac}(R)$,
- Γ: the valuation group,
- $v: K \to \Gamma \cup \{+\infty\}$: the valuation,
- $\widehat{R} := \varprojlim_{z \in R \setminus \{0\}} R/zR$ (completion of R),
- $\widehat{K} := Frac(\widehat{R}),$
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Recall that K is a topological field, with a basis of neighborhoods of 0 consisting of the nonzero (principal) ideals of R.

Accordingly, for each K-variety X, X(K) has a natural topology: a basis of open subsets consists of the sets

$$\{x \in U(K) \mid v(f_i(x)) \ge 0, i = 1, \dots, m\}$$

for $U \subset X$ affine open, $f_i \in H^0(U, \mathcal{O}_U)$

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Let us say that (K, v) (or R) is admissible if:

- (K, v) is henselian; i.e. (equivalently):
 - R is a henselian local ring;
 - ▶ *v* extends *uniquely* to every finite extension of *K*.
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Properties of admissible valued fields

Assume (K, v) is admissible. Then:

- K is algebraically closed in \widehat{K} .
- If *L* is a finite extension of *K*, then:
 - \triangleright L is admissible (for the unique extension of ν).
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Proposition 1

Assume (K, v) is admissible, and let $f: X \to Y$ be a morphism of K-varieties. Consider the induced continuous map $f_{\text{top}}: X_{\text{top}} \to Y_{\text{top}}$.

- Implicit function theorem": If f is étale, then f_{top} is a local f
 - homeomorphism.
 - If f is smooth, then f_{top} has local sections at each point of X_{top}. (In particular, it is an open map).
- "Continuity of roots": If f is finite, then f_{top} is a closed map (hence proper, since it has finite fibers).

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- we shall consider G-torsors $f: X \to Y$ where G is an algebraic group. Even if Y is a scheme, a G-torsor over Y is naturally defined as an fppf sheaf. It is always an algebraic space, but not necessarily a scheme.
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So let K be a topological field, and X an algebraic space of finite type over K.

We equip X(K) with the finest topology making all maps $f_{\text{top}}: X'_{\text{top}} \to X(K)$ continuous, where X' runs through [affine] K-varieties, and $f: X' \to X$ runs through all K-morphisms.

The resulting space is denoted by X_{top} (if X is a scheme, it is the same as before). This construction has the same basic compatibilities as in the case of varieties.

Assume now that K satisfies the implicit function theorem (e.g. K is a henselian valued field).

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The induced map $f_{ ext{top}}: X_{ ext{top}} o Y_{ ext{top}}$ decomposes as

$$X_{\mathrm{top}} \longrightarrow X_{\mathrm{top}}/G_{\mathrm{top}} \longrightarrow \mathrm{Im}(f_{\mathrm{top}}) \longrightarrow Y_{\mathrm{top}}$$

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 quotient map continuous topological (open) bijection embedding

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- Is the image of f_{top} closed (open, locally closed) in Y_{top} ?
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Note that a positive answer to both Questions 2 and 3 is equivalent to a positive answer to

4 Is $X_{\text{top}} \to \text{Im}(f_{\text{top}})$ a principal G_{top} -bundle?

The main result.

Main Theorem

Let (K, v) be an admissible valued field, G an algebraic K-group, and $f: X \to Y$ a G-torsor. Then:

- **1** Im (f_{top}) is locally closed in Y_{top} .
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Remark. In some cases, we can say more about $Im(f_{top})$:

- ullet it is open and closed in Y_{top} if G is smooth, or if K is perfect;
- it is closed in Y_{top} if G_{red}° is smooth, or if G is commutative.

The case of orbit maps

Corollary 1

Let (K, v) be an admissible valued field, G an algebraic K-group acting on a variety T, $t \in T(K)$ a rational point, with stabilizer $G_t \subset G$. Then the orbit map $g \mapsto g.t$ on G_{top} factors as

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Let $S = \mathbb{G}_a \rtimes \mathbb{G}_m$ be the affine group in dimension 1, acting on $X = \mathbb{A}^1_K$ transitively "via Frobenius on S":

$$\begin{array}{ccc} \mathcal{S} \times \mathbb{A}^1 & \longrightarrow & \mathbb{A}^1 \\ ((x,y),z) & \longmapsto & (x,y).z := x^p + y^p z \end{array}$$

For $z \in K$, consider the orbit morphism

$$f_z: S \to \mathbb{A}^1, \quad s \mapsto s.z.$$

This is a torsor under the stabilizer S_z of z.

The image of $f_{z,\text{top}}$ is the orbit $S(K).z = K^p + (K^{\times})^p z \subset K$. In particular,

- if $z \in K^p$, the orbit is K^p , which is closed in K if K is admissible;
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Why is $Im(f_{top})$ closed?

Consider the "classifying map" $Y(K) \xrightarrow{\partial} H^1(K, G)$ deduced from f: we know that $Im(f_{top}) = \partial^{-1}(1)$ is open in Y(K).

Together with a standard "twisting" argument, this implies that each fiber of ∂ is open.

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- Consider the factorization $X \xrightarrow{\pi} Z := X/G' \xrightarrow{h} Y$.
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It is easy to check that $(G/G^{\dagger})(K_s) = \{e\}$ (in particular $(G/G^{\dagger})(K) = \{e\}$).

More generally, if T is a G-torsor over K, then T/G^{\dagger} has at most one rational point.

Now let $f: X \to Y$ be a G-torsor, factored as

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This leads to a question:

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Assume $h: Z \to Y$ is a K-morphism such that h_{top} is injective. Can we say more (topologically) about h_{top} ?

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Let $h: Z \to Y$ be a proper K-morphism. Let $z \in Z(K)$ and y = h(z) be such that $h_{\text{top}}^{-1}(y) = \{z\}$.

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(The proof uses strong approximation).

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so we will compactify it.

Gabber's compactification theorem

Let G be an algebraic K-group. Put $Q = G/G^{\dagger}$, and denote by q the unique point of Q(K).

Theorem (Gabber, 2012)

Q admits a G-equivariant projective compactification $Q \hookrightarrow Q^c$ without separable points at infinity:

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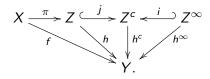
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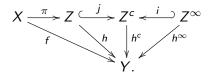
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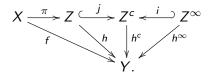
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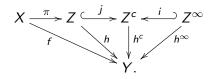
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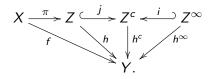
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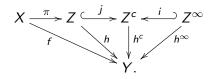
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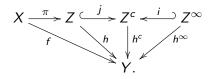
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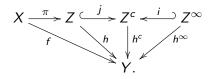
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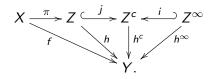
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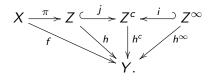
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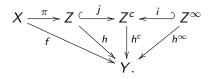
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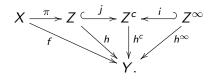
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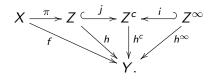
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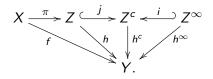
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