FINITE MORPHISMS TO PROJECTIVE SPACE AND CAPACITY THEORY

T. CHINBURG, L. MORET-BAILLY, G. PAPPAS, AND M. J. TAYLOR

Abstract. We study conditions on a commutative ring R which are equivalent to the following requirement: Whenever $\mathscr{X}$ is a projective scheme over $S = \mathrm{Spec}(R)$ of fiber dimension $\leq d$ for some integer $d \geq 0$, there is a finite morphism from $\mathscr{X}$ to $\mathbb{P}^d_S$ over S such that the pullbacks of coordinate hyperplanes give prescribed subschemes of $\mathscr{X}$ provided these subschemes satisfy certain natural conditions. We use our results to define a new kind of capacity for adelic subsets of projective flat schemes X over global fields. This capacity can be used to generalize the converse part of the Fekete-Szegő Theorem.

1. Introduction

Let R be a commutative ring. Suppose $f : \mathscr{X} \to S = \mathrm{Spec}(R)$ is a projective morphism which has fiber dimension $\leq d$ in the sense that every fiber of f has dimension $\leq d$ at each of its points. (If $d < 0$ this just means that $\mathscr{X}$ is empty.) Fix a line bundle $\mathscr{L}$ on $\mathscr{X}$ which is ample relative to f. Suppose i is an integer in the range $0 \leq i \leq d+1$. Let $(h_1, \ldots, h_i)$ be a sequence of sections of $\mathscr{L}$ such that when $\mathscr{X}_j$ is the zero locus of h_j, then $\cap_{j=1}^{i} \mathscr{X}_j$ has fiber dimension $\leq d - i$.

Definition 1.1. *The ring R has the coordinate hyperplane property* (F) *if for all $\mathscr{X}$, $\mathscr{L}$, i, $\{h_j\}_{j=1}^{i}$ and $\{\mathscr{X}_j\}_{j=1}^{i}$ as above, the following is true. There is a finite S-morphism $\pi : \mathscr{X} \to \mathbb{P}^d_S$ and a set of homogenous coordinates $(y_1 : \cdots : y_{d+1})$ for $\mathbb{P}^d_S$ such that for each $1 \leq j \leq i$, the support of $\mathscr{X}_j$ is the same as that of the subscheme of $\mathscr{X}$ defined by $\pi^* y_j = 0$.*

Recall that $\mathrm{Pic}(R)$ is the group of isomorphism classes of invertible sheaves on S, or equivalently, the group of isomorphism classes of rank one projective R-modules under tensor product. Our first Theorem is the following result:

Theorem 1.2. *The following properties of a commutative ring R are equivalent:*

1. (Property F) *The coordinate hyperplane property.*

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2. (Property P) *For every finite R-algebra R',* $\mathrm{Pic}(R')$ *is a torsion group.*
3. (Property S) *If $\mathscr{U}$ is an open subscheme of $\mathbb{P}^1_S$ which surjects onto $S = \mathrm{Spec}(R)$, there is an S-morphism $\mathscr{Y} \to \mathscr{U}$ for which $\mathscr{Y}$ is finite, locally free and surjective over S.*

Property (P) holds, for example, if R is semi-local or the localization of an order inside a ring of integers of a global field. Property (P) is stable under taking finite algebras and quotients. It is stable under filtering direct limits of rings because projective R-modules of rank one are of finite presentation (as direct summands of free finitely generated modules) - see [19, Prop. 1.3]. A zero dimensional ring satisfies (P). A finitely generated algebra R over a field k satisfies (P) if and only if k is algebraic over a finite field and $\dim(R) \leq 1$.

Suppose R has the coordinate hyperplane property. The proof of Theorem 1.2 shows that with the notations of Definition 1.1, one can find a $\pi : \mathscr{X} \to \mathbb{P}^d_S$ such that $\pi^* O_{\mathbb{P}^d_S}(1)$ is a power of $\mathcal{L}$ and the subscheme of $\mathscr{X}$ defined by $\pi^* y_j$ is an effective Cartier divisor which is a positive integral multiple of $\mathscr{X}_j$ for $1 \leq j \leq i$. (See §3.2). If $i = 0$ in Theorem 1.1 one obtains that there is a finite S-morphism $\pi : \mathscr{X} \to \mathbb{P}^d_S$. When R is the ring of integers of a number field and $d = 1$, the existence of such a π was shown by Green in [6] and [7]. Green's result is used in [3] to reduce the proof of certain adelic Riemann-Roch Theorems on surfaces to the case of $\mathbb{P}^1_S$. We have learned that for arbitrary d, the existence of a π as above when R is a Dedekind ring satisfying Property P has been shown by different methods by O. Gabber, Q. Liu and D. Lorenzini in [5].

In §2 we will give some equivalent formulations of properties (P) and (S). The proof of Theorem 1.2 is given in §3. We do not know if the conclusion of property (F) when $i = 0$ is sufficient to imply property (F) holds, i.e. whether this implies the same conclusion for arbitrary i. All of properties (P), (S) and (F) make sense over a scheme S which may not be affine. It would be interesting to consider the relationship of these properties for such a scheme S.

We now discuss an application of Theorem 1.2 to capacity theory.

Suppose X is a projective flat connected normal scheme of dimension $d \geq 1$ over a global field F. Let X_1 be an effective ample Cartier divisor on X. Let $M(F)$ be the set of places of F. Let $\mathbb{C}_v$ be the completion of an algebraic closure $\overline{F}_v$ of the completion F_v of F at $v \in M(F)$. By an adelic set $\mathbb{E} = \prod_{v \in M(F)} E_v$ we will mean a product of subsets of E_v of $(X \setminus X_1)(\mathbb{C}_v)$ which satisfy the following standard hypotheses relative to X_1:

i. Each E_v is nonempty and stable under the group $\mathrm{Gal}^c(\mathbb{C}_v/K_v)$ of continuous automorphisms of $\mathbb{C}_v$ over K_v.
ii. Each E_v is bounded away from $\mathrm{supp}(X_1)(\mathbb{C}_v)$ under the v-adic metric induced by the given projective embedding of X, and for all but

finitely many v, E_v and $\mathrm{supp}(X_1)(\mathbb{C}_v)$ reduce to disjoint sets (mod v).

Let $S_\gamma(\mathbb{E}, X_1)$ be the sectional capacity of $\mathbb{E}$ relative to X_1 (c.f. [2] and [16]). Let $\overline{F}$ be an algebraic closure of F. The main arithmetic interest of $S_\gamma(\mathbb{E}, X_1)$ is that if this number is less than 1, then there is a open adelic neighborhood $\mathbb{U} = \prod_{v \in M(F)} U_v$ of $\mathbb{E}$ such that the set of points $X(\overline{F})$ which have all of their Galois conjugates in $\mathbb{U}$ is not Zariski dense.

In §4 we define a real number $\gamma_{\mathrm{F}}(\mathbb{E}, X_1)$ which will be called the finite morphism capacity of $\mathbb{E}$ relative to X_1. This number gives a higher dimensional converse implication, paralleling the classical Fekete-Szegő Theorem and the work of Cantor and Rumely discussed below when $d = 1$:

Theorem 1.3. *If $\gamma_{\mathrm{F}}(\mathbb{E}, X_1) > 1$ then every open adelic neighborhood $\mathbb{U}$ of $\mathbb{E}$ contains a Zariski dense set of points of $X(\overline{F})$ which have all of their Galois conjugates in $\mathbb{U}$.*

The definition of $\gamma_{\mathrm{F}}(\mathbb{E}, X_1)$ involves considering the set $T(X_1)$ of all finite morphims $\pi : X \to \mathbb{P}^d_F$ such that the pull back via π of a hyperplane H at infinity is a positive integral multiple of X_1. Let $\mathcal{U}(\mathbb{E})$ be the set of open adelic neighborhoods $\mathbb{U}$ of $\mathbb{E}$ which satisfy the standard hypotheses. Roughly speaking, $\gamma_{\mathrm{F}}(\mathbb{E}, X_1)$ is an infimum over $\mathbb{U} \in \mathcal{U}(\mathbb{E})$ of the supremum of the normalized size of the adelic polydiscs B in affine d-space $A^d_F = \mathbb{P}^d_F \setminus H$ for which $\pi^{-1}(B)$ is contained in $\mathbb{U}$ for some $\pi \in T(X_1)$.

Our first result about $\gamma_{\mathrm{F}}(\mathbb{E}, X_1)$ compares it to the outer sectional capacity $S^+_\gamma(\mathbb{E}, X_1) = \inf\{S_\gamma(\mathbb{U}, X_1) : \mathbb{U} \in \mathcal{U}(\mathbb{E})\}$.

Theorem 1.4. *One has*

$$\gamma_{\mathrm{F}}(\mathbb{E}, X_1) \leq S^+_\gamma(\mathbb{E}, X_1)$$

with equality if $\mathbb{E} = \pi^{-1}(B)$ for some $\pi \in T(X_1)$ and some adelic polydisk B in $\mathbb{P}^d$.

It will not in general be the case that $\gamma_{\mathrm{F}}(\mathbb{E}, X_1) = S^+_\gamma(\mathbb{E}, X_1)$. For example, there may be effective Cartier divisors X_1' with the same support as X_1 such that

$$S^+_\gamma(\mathbb{E}, X_1') < 1 < S^+_\gamma(\mathbb{E}, X_1).$$

The first inequality implies the set of points $X(\overline{F})$ which have all their Galois conjugates in some open adelic neighborhood $\mathbb{U}$ of $\mathbb{E}$ is not Zariski dense. So Theorem 1.3 implies $\gamma_{\mathrm{F}}(\mathbb{E}, X_1) \leq 1 < S^+_\gamma(\mathbb{E}, X_1)$.

This suggests that in comparing sectional capacity to finite morphism capacity, one should consider all effective X_1' with the same support as X_1. One would like to minimize sectional capacity and maximize finite morphism capacity. One must also reckon with the fact that if $m > 0$ an integer, replacing X_1' by mX_1' raises each of $S^+_\gamma(\mathbb{E}, X_1')$ and $\gamma_{\mathrm{F}}(\mathbb{E}, X_1')$ to the m^{d+1}-th power. This leads to the following definition:

Definition 1.5. *Suppose X is smooth over F. Let $D(X_1)$ be the countable set of all ample effective Cartier divisors X_1' on X whose support* $\mathrm{supp}(X_1')$ *equals* $\mathrm{supp}(X_1)$. *Let $|X_1'| > 0$ be the d-fold self intersection number of X_1'.*

i. *Let $S_\gamma^+(\mathbb{E}, \mathrm{supp}(X_1))$ be*

$$\inf\{S_\gamma(\mathbb{U}, X_1')^{|X_1'|^{-(d+1)/d}} : \mathbb{U} \in \mathcal{U}(\mathbb{E}), \quad X_1' \in D(X_1)\}$$

ii. *Let $\gamma_{\mathrm{F}}(\mathbb{E}, \mathrm{supp}(X_1))$ be*

$$\inf_{\mathbb{U}\in\mathcal{U}(\mathbb{E})} \left\{ \sup_{X_1' \in D(X_1)} \gamma_{\mathrm{F}}(\mathbb{U}, X_1')^{|X_1'|^{-(d+1)/d}} \right\}.$$

Question 1.6. *If $S_\gamma^+(\mathbb{E}, \mathrm{supp}(X_1)) > 1$ is $\gamma_{\mathrm{F}}(\mathbb{E}, \mathrm{supp}(X_1)) > 1$?*

Note that if Question 1.6 has an affirmative answer for a given X, this and Theorem 1.4 gives a complete Fekete-Szegő Theorem for X and the capacity $S_\gamma^+(\mathbb{E}, \mathrm{supp}(X_1))$. Namely, one would know that if $S_\gamma^+(\mathbb{E}, \mathrm{supp}(X_1)) < 1$, then there is a open adelic neighborhood $\mathbb{U}$ of $\mathbb{E}$ such that the set of points $X(\overline{F})$ which have all of their Galois conjugates in $\mathbb{U}$ is not Zariski dense, while if $S_\gamma^+(\mathbb{E}, \mathrm{supp}(X_1)) > 1$ then every such $\mathbb{U}$ contains a Zariski dense set of such conjugates.

For X of dimension 1 Rumely has compared sectional capacity with his generalization of Cantor's capacity, and he has proved a complete Fekete-Szegő Theorem of the above kind. We will use his work to check that when $d = 1$, Question 1.6 has an affirmative answer in a very strong sense:

Theorem 1.7. *Suppose $d = 1$ and that $\mathbb{E}$ is* $\mathrm{supp}(X)$*-capacitable in the sense of* [15, §6.2]. *If $S_\gamma^+(\mathbb{E}, \mathrm{supp}(X_1)) > 1$ then $\gamma_{\mathrm{F}}(\mathbb{E}, \mathrm{supp}(X_1))$ and $S_\gamma^+(\mathbb{E}, \mathrm{supp}(X_1))$ are both equal to the capacity $\gamma_{\mathrm{CR}}(\mathbb{E}, \mathrm{supp}(X_1))$ of Cantor and Rumely.*

It would be interesting to know if such an equality holds for X of higher dimension when $S_\gamma^+(\mathbb{E}, \mathrm{supp}(X_1)) > 1$. A natural case to consider is when X itself is isomorphic to $\mathbb{P}^d$. Then each map $\phi : X = \mathbb{P}^d \to \mathbb{P}^d$ in the definition of $\gamma_{\mathrm{F}}(\mathbb{E}, X_1)$ together with the line bundle $O_X(X_1)$ gives rise to a polarized dynamical system in the sense of [14, §2]. In a later paper we will return to the problem of how to compute the sectional and finite morphism capacities of filled Julia sets associated to dynamical systems on $\mathbb{P}^d$.

Simple examples show that one cannot expect the conclusion of Theorem 1.7 to hold if $S_\gamma^+(\mathbb{E}, \mathrm{supp}(X_1)) < 1$ (see Example 4.11). These examples suggest that in this case, there may simply not be enough morphisms from X to $\mathbb{P}^d$ to explain the numerical size of $S_\gamma^+(\mathbb{E}, \mathrm{supp}(X_1))$.

We conclude with an application of Theorem 1.2 to finite morphism capacity. If F is a number field, let $M_{\mathrm{f}}(F)$ and $M_\infty(F)$ be the sets of finite and infinite places of F. If F is a global function field, we let $M_\infty(F)$ be a nonempty finite set of places of F, and we let $M_{\mathrm{f}}(F) = M(F) \smallsetminus M_\infty(F)$. For $v \in M_{\mathrm{f}}(F)$ let $O_{\mathbb{C}_v} = \{z \in \mathbb{C}_v : |z|_v \leq 1\}$.

We will say that a product $\mathbb{U}_{\mathrm{f}} = \prod_{v \in M_{\mathrm{f}}(F)} U_v$ of subsets $U_v \subset X(\mathbb{C}_v)$ is an RL-domain for the finite adeles if there is a projective embedding $X \to \mathbb{P}^t_F$ and a hyperplane H_0 in $\mathbb{P}^t_F$ such that X_1 is a multiple of $X \cap H_0$ and

$$U_v = X(\mathbb{C}_v) \cap (\mathbb{P}^t \smallsetminus H_0)(O_{\mathbb{C}_v})$$

for all $v \in M_{\mathrm{f}}(F)$. It is clear that an adelic set $\mathbb{U} = \mathbb{U}_{\mathrm{f}} \times \prod_{v \in M_\infty(F)} U_v$ will satisfy the standard hypotheses provided the U_v associated to $v \in M_\infty(F)$ satisfy conditions (i) and (ii) above.

Note that the dimension d of X may be much smaller than that of the projective space $\mathbb{P}^t_F$ into which we have embedded X. Recall that in $\mathbb{P}^d_F$ we have chosen homogeneous coordinates $y = (y_1 : \ldots : y_{d+1})$ as well as affine coordinates $z_i = y_i/y_1$ for $i = 2, \ldots, d+1$ for the affine space $A^d = \mathbb{P}^d \smallsetminus H$ when $H = \{y : y_1 = 0\}$. We will show:

Theorem 1.8. *Suppose that $\mathbb{U}_{\mathrm{f}}$ is an RL-domain with respect to the finite adeles. There is a finite morphism $\pi_{\mathbb{U}_{\mathrm{f}}} : X \to \mathbb{P}^d_F$ with the following properties:*

a. *The pullback $\pi^{-1}_{\mathbb{U}_{\mathrm{f}}}(H)$ is a positive integral multiple of X_1.*
b. *Let $B_{\mathrm{f}}(1) = \prod_{v \in M_{\mathrm{f}}(F)} B_v(1)$ where $B_v(1)$ is the unit polydisk*

$$\{z = (z_2, \ldots, z_{d+1}) : |z_i|_v \leq 1\} \subset A^d(\mathbb{C}_v).$$

Then

$$\pi^{-1}_{\mathbb{U}_{\mathrm{f}}}(B_{\mathrm{f}}(1)) = \mathbb{U}_{\mathrm{f}}.$$

Corollary 1.9. *Given $\mathbb{U}_{\mathrm{f}}$ as in Theorem 1.8, there is an open adelic set $\mathbb{U} = \mathbb{U}_{\mathrm{f}} \times \prod_{v \in M_\infty(F)} U_v$ which satisfies the standard hypotheses such that $\gamma_{\mathrm{F}}(\mathbb{U}, X_1) > 1$. In consequence the set of global algebraic points of $X(\overline{F})$ which have all of their Galois conjugates in $\mathbb{U}_{\mathrm{f}}$ is Zariski dense.*

We prove Theorem 1.8 by applying Theorem 1.2 to a well chosen integral model of X over the natural choice of Dedekind subring O_F of F.

To obtain more quantitative information about finite morphism capacities and to study Question 1.6, one needs a version of Theorem 1.2 which includes conditions at $v \in M_\infty(F)$. This amounts to a question in function theory, in the sense that one must construct rational functions on X giving sections of a power of $O_X(X_1)$ which define a morphism to $\mathbb{P}^d$ having the required properties.

2. Some properties of rings

Proposition 2.1. *The following conditions on a commutative ring R are equivalent:*

1. *For every R-algebra R' which is integral over R,* $\mathrm{Pic}(R')$ *is a torsion group.*
2. *For every finite R-algebra R',* $\mathrm{Pic}(R')$ *is a torsion group.*
3. *For every finite and finitely presented R-algebra R',* $\mathrm{Pic}(R')$ *is a torsion group.*

We shall say that R satisfies property (P) *if these conditions hold.*

Proof. It is obvious that (1) implies (2) and (2) implies (3). To show that (3) implies (1), note that every integral R'-algebra is a filtering direct limit of finite and finitely presented R-algebras. The result then follows from the fact that the Picard functor commutes with filtering direct limits, as noted in the introduction. □

The second condition on R we will consider is the "Skolem property" (S) (see [13]).

Proposition 2.2. *For a commutative ring R with spectrum S, the following conditions are equivalent:*

1. *For each $n \in \mathbb{N}$ and each open subscheme $\mathscr{U} \subset \mathbb{P}^n_S$ which is surjective over S, there is a subscheme $\mathscr{Y}$ of $\mathscr{U}$ which is finite, free and surjective over S.*
2. *Same as condition* (1)*, with $n = 1$.*
3. *For each $n \in \mathbb{N}$ and each open subscheme $\mathscr{U} \subset \mathbb{P}^n_S$ which is surjective over S, there is an S-morphism $\mathscr{Y} \to \mathscr{U}$ where $\mathscr{Y}$ is finite, locally free and surjective over S.*
4. *Same as condition* (3)*, with $n = 1$.*

We will say that R has property (S) *if these conditions hold.*

To begin the proof of Proposition 2.2, we first note that we can equivalently use $\mathbb{A}^n_S$ instead of $\mathbb{P}^n_S$ in each case. It is trivial that (1) implies both (2) and (3), and that either (2) or (3) implies (4). The fact that (2) implies (1), and that (4) implies (3), is shown by the following result:

Lemma 2.2.1. *Let $\mathscr{U} \subset \mathbb{A}^n_S$ be open and surjective over S. Then there exists an n-tuple of positive integers $m_1 = 1, m_2, \ldots, m_n$ such that if j is the closed immersion $j : \mathbb{A}^1_S \to \mathbb{A}^n_S$ defined by $j(t) = (t, t^{m_2}, \ldots, t^{m_n})$ then the open subset $j^{-1}(\mathscr{U})$ is surjective over S.*

Proof. We may assume that $\mathscr{U}$ is quasi-compact. The complement of $\mathscr{U}$ in $\mathbb{A}^n_S$ is then defined by a finite set of polynomials $f_i \in R[X_1, \ldots, X_n]$. The surjectivity of $\mathscr{U}$ means that the coefficients of the f_i's generate the unit ideal of R. Consider the finite set $\Sigma \subset \mathbb{N}^n$ of all multi-exponents occurring in the f_i's. It is easy to see that one can find positive integers $m_2, \ldots, m_n$ such that the linear form $(x_1, \ldots, x_n) \to x_1 + \sum_{\ell=2}^n m_\ell x_\ell$ maps Σ injectively into $\mathbb{N}$. But this means that for each i the polynomial $f_i(t, t^{m_2}, \ldots, t^{m_n}) \in R[t]$ has the same set of coefficients as f_i. In particular, these coefficients still generate R. □

To complete the proof of Proposition 2.2, it will suffice to show that (4) implies (2). Let $\mathscr{U} \subset \mathbb{A}^1_S$ be open and surjective over S. Choose an S-morphism $\mathscr{Y} \to \mathscr{U}$ as in (4). On passing to a locally free cover of $\mathscr{Y}$, we may assume that $\mathscr{Y} = \mathrm{Spec}(R_1)$ is locally free of constant (positive) rank r over S. The composite map $\mathscr{Y} \to \mathbb{A}^1_R = \mathrm{Spec}(R[t])$ gives rise to a morphism $R[t] \to R_1$ mapping t to an element z. Let $F(t) \in R[t]$ denote the characteristic polynomial of z, and put $\mathscr{Y}' = \mathrm{Spec}(R[t]/(F(t)))$. Then $\mathscr{Y}'$ is finite and free of rank r over R, and it is easy to check that $\mathscr{Y}'$ is set theoretically the image of $\mathscr{Y}$ because $\mathscr{Y} \to \mathscr{U}$ factors through $\mathscr{Y}'$ by the Cayley-Hamilton Theorem. In particular, $\mathscr{Y}' \subset \mathscr{U}$ as desired. □

3. Proof of Theorem 1.2

3.1. **Property (P) implies property (S).** As usual, put $S = \mathrm{Spec}(R)$, and assume R has property (P). Consider an open subscheme $\mathscr{U} \subset \mathbb{P}^1_S$ which is surjective over S. It will suffice to show that there is a subscheme $\mathscr{Y} \subset \mathscr{U}$ which is finite, locally free and surjective over S. We may assume that $\mathscr{U}$ is quasicompact and contained in $\mathbb{A}^1_S$. Let $\mathscr{Z} \subset \mathbb{P}^1_S$ be a closed subscheme of finite presentation with support $\mathbb{P}^1_S \smallsetminus \mathscr{U}$. Then $\mathscr{Z}$ is finite over S. By property (P), there is an integer $m > 0$ such that the invertible $\mathscr{O}_{\mathscr{Z}}$-module $\mathscr{O}_{\mathscr{Z}}(m)$ is trivial, i.e. has a nonvanishing section $s_{\mathscr{Z}}$. Now $\mathscr{Z}$ and $s_{\mathscr{Z}}$ can be defined over a subring $R_0 \subset R$ which is finitely generated over $\mathbb{Z}$. To find $\mathscr{Y}$ we may replace R by R_0, so that we may now assume R is Noetherian. The ideal sheaf $\mathscr{I}_{\mathscr{Z}}$ is coherent, so replacing m by a sufficiently large multiple, we may assume that $\mathrm{H}^1(\mathscr{I}_{\mathscr{Z}}(m)) = 0$. This implies that the restriction map $\mathrm{H}^0(\mathscr{O}_{\mathbb{P}^1}(m)) \to \mathrm{H}^0(\mathscr{O}_{\mathscr{Z}}(m))$ is surjective. In particular, $s_{\mathscr{Z}}$ extends to a section $s \in \mathrm{H}^0(\mathscr{O}_{\mathbb{P}^1}(m))$. We can take $\mathscr{Y}$ to be the scheme of zeros of s. Indeed, the fact that $s_{\mathscr{Z}}$ is a trivialization on $\mathscr{Z}$ means that $\mathscr{Y} \subset \mathscr{U} \subset \mathbb{A}^1_S$. We can therefore view s as a polynomial of degree m in the standard coordinate t of $\mathbb{A}^1_S$. The leading coefficient of this polynomial is invertible because s does not vanish at infinity, so $\mathscr{Y} \cong \mathrm{Spec}(R[t]/(s))$ is free of rank m over S.

3.2. **Property (S) implies property (F).** We will need the following general fact.

Proposition 3.2.1. *Without assumption on the commutative ring R, let $g : \mathscr{Y} \to S = \mathrm{Spec}(R)$ be a projective morphism with fiber dimension $\leq \delta$ for some integer $\delta \geq 0$. Let $\mathscr{L}$ be an invertible sheaf on $\mathscr{Y}$ which is very ample with respect to g. After base change from S to an S-scheme S' which is surjective over S and isomorphic to an open subscheme of an an affine S-space $\mathbb{A}^N_S$, there is a section of $\mathscr{L}$ over $\mathscr{Y}$ whose scheme of zeros has fiber dimension $\leq \delta - 1$. If, moreover, R satisfies condition* (S)*, there is an integer $m > 0$ and a section of $\mathscr{L}^{\otimes m}$ on $\mathscr{Y}$ (without any base change) with the same property.*

Proof. We may assume that $\mathscr{Y} \subset \mathbb{P}^{N-1}_S$ and $\mathscr{L} = \mathscr{O}_{\mathscr{Y}}(1)$ since S is affine. We may identify sections of $\mathscr{O}_{\mathbb{P}^{N-1}_S}(1)$ with sections of the vector bundle $E := \mathbb{A}^N_S$ over S. This identification is compatible with base change. In particular, we have a universal section of $\mathscr{O}_{\mathbb{P}^{N-1}_E}(1)$ whose scheme of zeros in $\mathscr{Y}$ is the universal hyperplane section $\mathscr{H} \subset \mathscr{Y} \times_S E \subset \mathbb{P}^{N-1}_S \times E = \mathbb{P}^{N-1}_E$.

By Chevalley's semi-continuity theorem ([8, 13.1.5]), the locus $S' \subset E$ over which the fibers of $\mathscr{H}$ have dimension $\leq \delta - 1$ is open. Moreover, S' surjects onto S since for each geometric point $\xi : \mathrm{Spec}(k) \to S$ of S there is a hyperplane in $\mathbb{P}_k^{N-1}$ which does not contain any component of the fiber $\mathscr{Y}_\xi$. Thus S' provides the required base change.

Assume now that R satisfies condition (S). Applying this property to the above S', we obtain a finite, free and surjective S-scheme $\pi : T \to S$ contained in S'. By restricting sections to T, we obtain a section h of the pullback of $\mathscr{L}$ to $\mathscr{Y} \times_S T$ whose zero set $\mathcal{Z}$ has fiber dimension $\leq \delta - 1$ over T. Denote by $m > 0$ the degree of π. The natural projection $\pi_{\mathscr{Y}} : \mathscr{Y} \times_S T \to \mathscr{Y}$ is still free of degree m. The norm of h with respect to $\pi_{\mathscr{Y}}$ is a section of $\mathscr{L}^{\otimes m}$ on $\mathscr{Y}$. The zero set of this section is $\pi_{\mathscr{Y}}(\mathcal{Z})$, which has the same fiber dimension as $\mathcal{Z}$ since $\pi_{\mathscr{Y}}$ is finite. This completes the proof. □

We may now show that property (S) implies that R has the coordinate hyperplane property (F). Let d, $f : \mathscr{X} \to S$, $\mathscr{L}$, i and $h_1, \ldots, h_i$ be as in Definition 1.1. By replacing $\mathscr{L}$ by $\mathscr{L}^{\otimes e}$ for some large enough e, and each h_j by $h_j^{\otimes e}$, we may assume that $\mathscr{L}$ is very ample. We apply Proposition 3.2.1 inductively, starting with the S-scheme $\mathscr{Y} = \cap_{j=1}^{i} \mathscr{X}_j$ and the integer $\delta = d - i$. We get (after replacing $\mathscr{L}$ and the h_j's by suitable powers of themselves, which does not change the $\mathscr{X}_j$'s) sections $h_1, \ldots, h_{d+1}$ of $\mathscr{L}$ whose common zero set has fiber dimension ≤ -1, i.e. is empty. Therefore $(h_1 : \cdots : h_{d+1})$ is a well defined S-morphism $q : \mathscr{X} \to \mathbb{P}^d_S$. Moreover, $q^*\mathscr{O}_{\mathbb{P}^d}(1) = \mathscr{L}$ is ample, so q must be finite. By construction, for all $j \leq i$, the pullback of the j^{th} homogeneous coordinate is a power of h_j, hence its zero set is set theoretically equal to $\mathscr{X}_j$ and equal as a Cartier divisor to a multiple of $\mathscr{X}_j$. This completes the proof that property (S) implies property (F).

3.3. **Property (F) implies property (P).** We will need the following general result.

Proposition 3.3.1. *Let S be a scheme, and suppose $\mathscr{X}$ and $\mathscr{Y}$ are two finitely presented S-schemes. Let $\pi : \mathscr{X} \to \mathscr{Y}$ be a finite S-morphism. Assume that $\mathscr{Y} \to S$ is flat, with pure d-dimensional regular fibers. Then π is flat if and only if $\mathscr{X} \to S$ is flat with pure d-dimensional Cohen-Macaulay fibers.*

Proof. Combine [12, Thm. 46, p. 140], applied to the fibers, with "flatness by fibers" [8, IV, 3, 11.3.11]. □

In order to now prove that a ring R has property (P), we will in fact only use property (F) in the special case $d = i = 1$ in the notation of Definition 1.1. Let R' be a finite R-algebra and define $S' = \mathrm{Spec}(R')$. Suppose M is an invertible R'-module. We must show that $M^{\otimes m} \cong R'$ for some $m > 0$. Consider the locally free rank one $\mathscr{O}_{S'}$-module $\mathscr{M}$ associated to M, and the corresponding $\mathbb{P}^1$ bundle $\mathscr{X} = \mathbb{P}(\mathscr{O}_{S'} \oplus \mathscr{M})$. This bundle has two disjoint natural sections over S': The section s_∞ whose complement is isomorphic to the vector bundle $\mathbb{M} = \mathbb{V}(\mathscr{M})$ and the zero section s_0 of $\mathbb{M}$. These sections define divisors D_∞ and D_0 which are ample with respect to S' (and therefore also ample with respect to S). Put $\mathscr{L} = \mathscr{O}_{\mathscr{X}}(D_\infty + D_0)$, and let h be the canonical section of $\mathscr{L}$ having divisor $\mathscr{Y} = D_\infty + D_0$. We can now apply property (F) to this data. We obtain a finite morphism $\mathscr{X} \to \mathbb{P}^1_S$ such that $\mathscr{Y}$ is the set-theoretic inverse image of, say, the section ∞ of $\mathbb{P}^1_S$. Since $\mathscr{X}$ is an S'-scheme, this gives rise to a finite S'-morphism $p : \mathscr{X} \to \mathbb{P}^1_{S'}$ mapping $\mathscr{Y}$ to the section ∞. By Proposition 3.3.1, p must be flat since $\mathscr{X}$ and $\mathbb{P}^1_{S'}$ are smooth and one-dimensional over S'. We conclude that p is in fact locally free since it is finite, flat and of finite presentation. Clearly p is also surjective. The inverse image of the zero section of $\mathbb{P}^1_{S'}$ is therefore a finite locally free S'-scheme T which surjects onto S' and which is contained in the punctured line bundle $\mathscr{X} \smallsetminus \mathscr{Y} = \mathbb{M} \smallsetminus D_0$. This means that $\mathscr{M}$ is trivialized by the base change $T \to S'$, so $\mathscr{M}$ has finite order in $\mathrm{Pic}(S')$.

4. Finite morphism capacities

4.1. **The definition.** Let F be a global field, and let $M(F)$ be the set of places of F. Define $\mathbb{C}_v$ to be the completion of an algebraic closure $\overline{F}_v$ of the completion F_v of F at v. Define $|\ |_v : \mathbb{C}_v \to \mathbb{R}$ to be the unique extension to $\mathbb{C}_v$ of the normalized absolute value on F_v. Define $\mathcal{R}$ to be the set of functions $r : M(F) \to \mathbb{R}$ such that $r(v) \geq 0$ for all v and $r(v) = 1$ for almost all v. We have a norm $|\ | : \mathcal{R} \to \mathbb{R}$ defined by

$$|r| = \prod_{v \in M(F)} r(v).$$

Let $d \geq 1$ be an integer. We let $(y_1 : \cdots : y_{d+1})$ be homogeneous coordinates on $\mathbb{P}^d_F$, and we define H to be the hyperplane $y_1 = 0$. Then we have affine coordinates on $A^d = \mathbb{P}^d \smallsetminus H$ given by $z_i = y_i/y_1$ for $i = 2, \ldots, d+1$. For $r \in \mathcal{R}$ we define the adelic polydisk $B(r) = \prod_{v \in M(F)} B_v(r(v))$ by setting

$$B_v(r(v)) = \{z = (z_2, \ldots, z_{d+1}) \in A^d(\mathbb{C}_v) : |z_i|_v \leq r(v) \quad \text{for all} \quad i\}$$

Let X_1 be an ample effective divisor on the smooth projective variety X over F. Suppose $\mathbb{E} = \prod_{v \in M(F)} E_v$ is an adelic set of points of X satisfying the standard hypotheses described in (i) and (ii) of the introduction. Recall that $\mathcal{U}(\mathbb{E})$ is the set of all open adelic neighborhoods $\mathbb{U} = \prod_v U_v$ of $\mathbb{E}$ which satisfy the standard hypotheses. Let $T(X_1)$ be the set of all finite morphisms $\pi : X \to \mathbb{P}^d$ over F such that $\pi^{-1}(H) = m(\pi)X_1$ as Cartier divisors for some integer $m(\pi) > 0$.

Definition 4.2. *The finite morphism capacity $\gamma_{\mathrm{F}}(\mathbb{U}, X_1)$ of $\mathbb{U} \in \mathcal{U}(\mathbb{E})$ relative to X_1 is the supremum of*

$$|r|^{d \cdot \deg(\pi)/m(\pi)^{d+1}}$$

over all $\pi \in T(X_1)$ and $r \in \mathcal{R}$ such that $\pi^{-1}(B(r)) \subset \mathbb{U}$; this supremum is defined to be 0 if no such r and π exist. Define

$$\gamma_{\mathrm{F}}(\mathbb{E}, X_1) = \inf\{\gamma_{\mathrm{F}}(\mathbb{U}, X_1) : \mathbb{U} \in \mathcal{U}(\mathbb{E})\}. \tag{4.1}$$

4.3. **Generalizing the converse part of the Fekete-Szegő Theorem.** In this paragraph we will prove Theorem 1.3. Suppose $\gamma_{\mathrm{F}}(\mathbb{E}, X_1) > 1$. We must prove that for every $\mathbb{U} \in \mathcal{U}(\mathbb{E})$, the set of points of $X(\overline{F})$ which have all their Galois conjugates in $\mathbb{U}$ is Zariski dense. In view of (4.1), it will suffice to consider the case in which $\mathbb{E} = \mathbb{U}$ is open.

Since $\gamma_{\mathrm{F}}(\mathbb{U}, X_1) > 1$, there are π and r as in Definition 4.2 for which $|r| > 1$. Since $\pi : X \to \mathbb{P}^d_F$ is finite and defined over F, the result now follows from:

Lemma 4.4. *If $r \in \mathcal{R}$ and $|r| > 1$ then there is a Zariski dense set of points of $\mathbb{P}^d(\overline{F})$ which have all their Galois conjugates over F in $B(r)$.*

Proof. By multiplying r by a function from $M(F)$ to $\mathbb{R}$ of the form $v \to |\alpha|_v^{1/n}$ for a suitable $\alpha \in F^*$ and positive integer n, we can reduce to the case in which $r(v) \geq 1$ for all v and $r(v) > 1$ if $v \in M_\infty(F)$. Let C be the set of points $(z_2, \dots, z_{d+1}) \in A^d(\overline{F})$ such that each z_i is a root of unity. Every point of C has all its conjugates in $B(r)$, so it will suffice to show C is Zariski dense in $\mathbb{P}^d_F$, or equivalently in $A^d(\overline{F}) = \overline{F}^d$. This follows from the well-known fact that if I is an infinite subset of a field K, then I^d is Zariski dense in K^d. □

4.5. **Comparing sectional capacity and finite morphism capacity.**

Lemma 4.6. *Suppose $\pi \in T(X_1)$ and $r \in \mathcal{R}$. The sectional capacity $S_\gamma(\pi^{-1}(B(r)), X_1)$ equals $|r|^{d \cdot \deg(\pi)/m(\pi)^{d+1}}$.*

Proof. By the functorial properties of sectional capacity proved in Theorem C of the introduction of [16] we have

$$\begin{aligned} S_\gamma(\pi^{-1}(B(r)), X_1) = & \quad S_\gamma(\pi^{-1}(B(r)), \pi^{-1}(X_1))^{1/m(\pi)^{d+1}} \\ = & \quad S_\gamma(B(r), H)^{\deg(\pi)/m(\pi)^{d+1}}. \end{aligned} \tag{4.2}$$

We are thus reduced to showing that on $\mathbb{P}^d_F$ one has $S_\gamma(B(r), H) = |r|^d$. This follows immediately from Example 4.3 of [9] and an explicit (classical) computation when $d = 1$ (see Examples 4.1 and 4.2 of [9]). □

Proof of Theorem 1.4.

To show $\gamma_{\mathrm{F}}(\mathbb{E}, X_1) \leq S^+_\gamma(\mathbb{E}, X_1)$ it will suffice to consider the case in which $\mathbb{E}$ is an open adelic set $\mathbb{U}$. In view of Lemma 4.6, $\gamma_{\mathrm{F}}(\mathbb{U}, X_1)$ is the supremum of

$$|r|^{d \cdot \deg(\pi)/m(\pi)^{d+1}} = S_\gamma(\pi^{-1}(B(r)), X_1)$$

over $\pi \in T(X_1)$ and $r \in \mathcal{R}$ such that $\pi^{-1}(B(r)) \subset \mathbb{U}$. Since $\pi^{-1}(B(r)) \subset \mathbb{U}$ we have

$$S_\gamma(\pi^{-1}(B(r)), X_1) \leq S_\gamma(\mathbb{U}, X_1) = S^+_\gamma(\mathbb{U}, X_1).$$

Thus $\gamma_{\mathrm{F}}(\mathbb{U}, X_1)$ is the supremum of a set of numbers all of which are bounded by $S_\gamma(\mathbb{U}, X_1)$. Therefore $\gamma_{\mathrm{F}}(\mathbb{U}, X_1) \le S_\gamma(\mathbb{U}, X_1)$.

Suppose now that $\mathbb{E} = \pi^{-1}(B(r))$ for some π and $r \in \mathcal{R}$ above. Then $\pi^{-1}(B(r)) = \mathbb{E}$ is trivially contained in every adelic open neighborhood $\mathbb{U}$ of $\mathbb{E}$. So the above calculations show $\gamma_{\mathrm{F}}(\pi^{-1}(B(r)), X_1) = S_\gamma(\pi^{-1}(B(r)), X_1)$.

4.7. **The case of curves.** In this section we prove Theorem 1.7, whose notations we now assume. Thus X is a regular connected projective curve over F. We may write the effective divisor X_1 on X as a positive integral combination $\sum_i s_i x_i$ of a finite $\mathrm{Gal}(\overline{F}/F)$-stable set of points $\mathscr{T} = \{x_i\}_{i=1}^n$ of $X(\overline{F})$. Let $\mathbb{E}$ be an adelic set which satisfies the standard hypotheses in the introduction and which is $\mathscr{T}$-capacitable in the sense of [15, §6.2]. Rumely has proved in [17] that the sectional capacity $S_\gamma(\mathbb{E}, X_1)$ satisfies

$$-\ln(S_\gamma(\mathbb{E}, X_1)) = s^t \Gamma(\mathbb{E}, \mathscr{T}) s \tag{4.3}$$

when s is the column vector $s = (s_i)_{i=1}^n$ and $\Gamma(\mathbb{E}, \mathscr{T})$ is the real symmetric $n \times n$ Green's matrix arising in the capacity theory of Cantor and Rumely.

Recall that $D(X_1)$ is the set of all ample effective Cartier divisors X_1' on X whose support $\mathrm{supp}(X_1')$ equals $\mathscr{T} = \mathrm{supp}(X_1)$. In the case of curves, $|X_1'|$ is the degree of X_1', and

$$S_\gamma^+(\mathbb{E}, \mathscr{T}) = \inf\{S_\gamma(\mathbb{U}, X_1')^{|X_1'|^{-2}} : \mathbb{U} \in \mathcal{U}(\mathbb{E}), \quad X_1' \in D(X_1)\}.$$

A real vector $r' = (r_i')_{i=1}^n$ will be called F-symmetric if $r_i = r_j$ when x_i and x_j are elements of $\mathscr{T} \subset X(\overline{F})$ in the same orbit under the action of $\mathrm{Aut}(\overline{F}/F)$. Let $\mathcal{P}$ be the set of F-symmetric vectors $r' = (r_i')_{i=1}^n$ such that $0 \le r_i' \in \mathbb{R}$ for all i and $\sum_{i=1}^n r_i' = 1$. Let $\mathcal{P}^0$ be the elements of $\mathcal{P}$ which have only positive entries. In [17], Rumely has shown that sectional capacities are well defined for formal positive real linear integral combinations $X_1' = \sum_i r_i' x_i$ of the points $x_i \in \mathscr{T}$ such that $r = (r_i)_i \in \mathcal{P}^0$.

Lemma 4.8. *Suppose $S_\gamma^+(\mathbb{E}, \mathscr{T}) > 1$. Then $\Gamma(\mathbb{E}, \mathscr{T})$ is a negative definite matrix. There is a unique $\hat{s} = (\hat{s}_i)_i \in \mathcal{P}^0$ such that*

$$\hat{s}^t \Gamma(\mathbb{E}, \mathscr{T}) \hat{s} = -\ln(S_\gamma^+(\mathbb{E}, \mathscr{T})).$$

If $X_1' = \sum_i \hat{s}_i \ x_i$ then

$$S_\gamma^+(\mathbb{E}, \mathscr{T}) = S_\gamma(\mathbb{E}, X_1') = \gamma_{\mathrm{CR}}(\mathbb{E}, \mathscr{T}). \tag{4.4}$$

where $\gamma_{\mathrm{CR}}(\mathbb{E}, \mathscr{T})$ is the Cantor Rumely capacity.

Proof. Suppose $\mathbb{U} \in \mathcal{U} = \mathcal{U}(\mathbb{E})$. It follows from (4.3) that

$$\sup\{\frac{-\ln(S_\gamma(\mathbb{U}, X_1'))}{|X_1'|^2} : X_1' \in D(X_1)\} = \sup_{s \in \mathcal{P}^0} \quad s^t \Gamma(\mathbb{U}, \mathscr{T}) s. \tag{4.5}$$

Thus

$$-\ln(S_\gamma^+(\mathbb{E}, \mathscr{T})) = \sup_{\mathbb{U} \in \mathcal{U}} \sup_{s \in \mathcal{P}^0} \quad s^t \Gamma(\mathbb{U}, \mathscr{T}) s. \tag{4.6}$$

By the approximation theorems of [15, §4.5], for every $\epsilon > 0$, there is a $\mathbb{U} \in \mathcal{U}(\mathbb{E})$ such that every entry of $\Gamma(\mathbb{U}, \mathscr{T}) - \Gamma(\mathbb{E}, \mathscr{T})$ is bounded by ϵ in absolute value.

Let $\mathrm{Val}(\Gamma)$ be the value of a square $n \times n$ matrix Γ as a matrix game (see [15, p. 327]). Then

$$\gamma_{\mathrm{CR}}(\mathbb{E}, \mathscr{T}) = \exp(-\mathrm{Val}(\Gamma(\mathbb{E}, \mathscr{T}))) \tag{4.7}$$

by definition (c.f. [15, Def. 5.1.5]).

Suppose first that $\mathrm{Val}(\Gamma(\mathbb{E}, \mathscr{T})) \geq 0$. Let $\delta > 0$ be given. We can then find a $\mathbb{U} \in \mathcal{U}$ such that $\mathrm{Val}(\Gamma(\mathbb{U}, \mathscr{T})) \geq -\delta$. Let J be the matrix of the same size as $\Gamma(\mathbb{U}, \mathscr{T})$ which has every entry equal to 1. Then $\Gamma(\mathbb{U}, \mathscr{T}) + 2\delta J$ is a symmetric matrix whose only negative entries are on the diagonal, and

$$\mathrm{Val}(\Gamma(\mathbb{U}, \mathscr{T}) + 2\delta J) = \mathrm{Val}(\Gamma(\mathbb{U}, \mathscr{T})) + 2\delta > 0.$$

Hence by [15, Lemma 5.1.7], there is a vector $s_0 \in \mathcal{P}^0$ such that

$$(\Gamma(\mathbb{U}, \mathscr{T}) + 2\delta J)s_0$$

has all positive entries. Then $0 < s_0^t \Gamma(\mathbb{U}, \mathscr{T})s_0 + s_0^t 2\delta J s_0 = s_0^t \Gamma(\mathbb{U}, \mathscr{T})s_0 + 2\delta$, so

$$\sup_{s \in \mathcal{P}^0} s^t \Gamma(\mathbb{U}, \mathscr{T}) s \geq -2\delta.$$

Since $\delta > 0$ was arbitrary, we conclude from (4.6) that $-\ln(S_\gamma^+(\mathbb{E}, \mathscr{T})) \geq 0$. This contradicts the assumption that $S_\gamma^+(\mathbb{E}, \mathscr{T}) > 1$. Hence $\mathrm{Val}(\Gamma(\mathbb{E}, \mathscr{T})) < 0$.

We know from [15, Thm. 5.1.10] that

$$\mathrm{Val}(\Gamma(\mathbb{E}, \mathscr{T})) \geq \mathrm{Val}(\Gamma(\mathbb{U}, \mathscr{T})) \tag{4.8}$$

for $\mathbb{U} \in \mathcal{U}$. Thus if $\mathbb{V} \in \mathcal{U}$ or $\mathbb{V} = \mathbb{E}$ and $\Gamma = \Gamma(\mathbb{V}, \mathscr{T})$, we have $\mathrm{Val}(\Gamma) < 0$. Rumely shows in [15, Lemma 5.1.7] that this implies $s \to s^t \Gamma s$ is a negative definite quadratic form. Hence this quadratic form has a maximum on the space W of all real F-symmetric vectors $s = (s_i)_i$ such that $\sum_i s_i = 1$. The matrix $\Gamma = (\tau_{ij})_{i,j}$ is symmetric and invertible, and $\tau_{i',j'} = \tau_{i,j}$ if $x_{i'} = \sigma(x_i)$ and $x_{j'} = \sigma(x_j)$ for some $\sigma \in \mathrm{Aut}(\overline{F}/F)$. Calculus implies that at each $\hat{s} \in W$ where the maximum of $s \to s^t \Gamma s$ is obtained, the vector $\Gamma \hat{s}$ must have all equal and nonzero components. If s' is another such vector, then $s' - q\hat{s}$ is in the kernel of Γ for some scalar q. Because Γ is invertible, we conclude that $s' = q\hat{s}$ so in fact $s' = \hat{s}$ since $s', \hat{s} \in W$. Hence $\hat{s}$ is unique. Rumely shows in [15, Lemma 5.1.6] that the maximum of $s \to s^t \Gamma s$ is attained at a point $\hat{s} \in \mathcal{P}^0$, where all the components of $\Gamma \hat{s}$ are equal to $\mathrm{Val}(\Gamma)$. Thus the quadratic form $s \to s^t \Gamma s$ achieves its (unique) maximum over the compact set $\mathcal{P}$ at the interior point $\hat{s} \in \mathcal{P}^0$, and this maximum is $\mathrm{Val}(\Gamma)$.

In view of (4.8), and the fact that we can find a sequence of $\mathbb{U} \in \mathcal{U}(\mathbb{E})$ for which the corresponding matrices $\Gamma(\mathbb{U}, \mathscr{T})$ converge to $\Gamma(\mathbb{E}, \mathscr{T})$, we conclude from (4.6) that

$$-\ln(S_\gamma^+(\mathbb{E}, \mathscr{T})) = \mathrm{Val}(\Gamma(\mathbb{E}, \mathscr{T})).$$

The Lemma now follows from (4.7) together with the uniqueness noted above for the $s \in \mathcal{P}$ at which $s \to s^t\Gamma(\mathbb{E}, \mathscr{T})s$ achieves its maximum. $\square$

Lemma 4.9. *Suppose that $\mathbb{U} \in \mathcal{U} = \mathcal{U}(\mathbb{E})$. Let W be the set of all pairs (f, r) of the following kind. The first entry of (f, r) is a finite morphism $f : X \to \mathbb{P}^1$ such that $\hat{X}_1 = f^{-1}(\infty)$ has the same support $\mathscr{T}$ as X_1. The second entry of (f, r) is an element $r \in \mathcal{R}$ such that*

$$f^{-1}(B(r)) \subset \mathbb{U}.$$

Then

$$\sup_{X_1' \in D(X_1)} \gamma_{\mathrm{F}}(\mathbb{U}, X_1')^{|X_1'|^{-2}} = \sup_{(f,r)\in W} \gamma_{CR}(f^{-1}(B(r)), \mathscr{T}) \leq \gamma_{CR}(\mathbb{U}, \mathscr{T}) \tag{4.9}$$

where $D(X_1)$ is the set of all ample effective divisors having support $\mathscr{T}$.

Proof. Since each $\mathbb{U} \in \mathcal{U}$ is open, we have for $X_1' \in D(X_1)$ that

$$\gamma_{\mathrm{F}}(\mathbb{U}, X_1') = \sup\{|r|^{\deg(f)/m(f)^2} : (f, r) \in W \quad \text{and} \quad f^{-1}(\infty) = m(f)X_1'\}$$

On the right hand side we have $m(f)|X_1'| = \deg(f)$. So by the pullback formula for the Cantor-Rumely capacity (c.f. [15, p. 4]) we have

$$(|r|^{\deg(f)/m(f)^2})^{|X_1'|^{-2}} = |r|^{1/\deg(f)} = \gamma_{CR}(B(r), \infty)^{1/\deg(f)} = \gamma_{CR}(f^{-1}(B(r)), \mathscr{T}).$$

Combining the above equalities shows (4.9), where $\gamma_{CR}(f^{-1}(B(r)), \mathscr{T}) \leq \gamma_{CR}(\mathbb{U}, \mathscr{T})$ because $f^{-1}(B(r)) \subset \mathbb{U}$. $\square$

Lemma 4.10. *With the hypotheses and notations of Lemma 4.8, suppose $\mathbb{U} \in \mathcal{U}(\mathbb{E})$ and $\epsilon > 0$. Then there is a finite morphism $f : X \to \mathbb{P}^1$ with the following properties. Let $n(f)$ be the degree of f.*

i. *The pull back divisor $f^{-1}(\infty)$ has support $\mathscr{T}$.*
ii. *There is an adelic polydisk $B(r)$ in $\mathbb{A}^1$ for some $r \in \mathcal{R}$ such that $f^{-1}(B(r)) \subset \mathbb{U}$ and $|r|^{1/n(f)} = \gamma_{CR}(f^{-1}(B(r)), \mathscr{T})$ is within ϵ of $\gamma_{\mathrm{CR}}(\mathbb{E}, \mathscr{T})$.*

In consequence

$$\sup_{X_1' \in D(X_1)} \gamma_{\mathrm{F}}(\mathbb{U}, X_1')^{|X_1'|^{-2}} \geq \gamma_{CR}(\mathbb{E}, \mathscr{T}) \tag{4.10}$$

Proof. By Lemma 4.8, our hypotheses imply that $\gamma_{\mathrm{CR}}(\mathbb{E}, \mathscr{T}) > 1$. The result stated by Rumely in [15, Theorem 6.2.2] is that there is a finite morphism $f : X \to \mathbb{P}^1$ for which (i) holds and such that $f^{-1}(B(r_0)) \subset \mathbb{U}$ when $r_0 \in \mathcal{R}$ has $r_0(v) = 1$ for all $v \in M(F)$. By reading the proof closely, one sees that Rumely shows the sharper quantitative result in (ii). We will make a few comments about how to check this before leaving the details to the reader.

The preliminary reductions started in Step 0 on page 395 of [15] are not completed until Step 7 on page 412. To check that the reduction step which involves a base change to a finite normal extension L of K still applies, one needs to use the behavior of the Cantor-Rumely capacity under such base changes which is stated as (B) on page 4 of [15]. In Step 1 on page 396, constants R_w are defined for each place w in L in the union $S = S_0 \cup \{w_0\}$ of a large finite set of place S_0 with a nonarchimedean place w_0 where the w_0-component E_{w_0} of $\mathbb{E}$ is $\mathscr{T}$-trivial. Define $R_w = 1$ for all $w \in M(L) \smallsetminus S$ and let $\Gamma = \Gamma(\mathbb{E}_L, \mathscr{T})$ be the Green's matrix associated to the pullback of $\mathbb{E}$ to L. Define $R = \prod_{w \in M(L)} R_w$.

In the function field case, Rumely arranges by the end of Case A of Step 1 on page 399 of [15] that $\ln(R)/n = -\mathrm{Val}(\Gamma)$ when n is the common degree of the functions $h_w(z)$ constructed in this step. In the number field case, he shows by the end of Case B of Step 1 on 403 that one can arrange for $\ln(R)/n$ to be as close as one likes to $-\mathrm{Val}(\Gamma)$. Steps 2, 3 and 4 of the proof proceed as stated. In case A of step 5 on page 411 of [15], Rumely shows the lower bound

$$|f_L(z)|_w \geq \frac{1}{2}|h(z)^d|_w \geq \frac{1}{2}R_w^d > 1$$

for $z \in \mathcal{Y}_w$ with the notations introduced there. Here $f_L : L \otimes_K X \to \mathbb{P}^1_L$ is the morphism we need to construct when X is replaced by $L \otimes_K X$ via the first reduction step. Rumely uses only the lower bound $|f_L(z)|_w > 1$ at this step in the proof. One has to carry along the sharper bound $|f_L(z)|_w \geq \frac{1}{2}R_w^d$ and let d go to infinity, where the degree of $f_L(z)$ is d times the degree of the function $h(z)$ appearing in equation (79) on page 411. Since $\ln(R)/n = \sum_{w \in M(L)} \ln(R_w)/n$ can be made as close as we like to $-\mathrm{Val}(\Gamma)$, we find that f_L satisfies the counterparts for X_L of conditions (i) and (ii) because $\gamma_{\mathrm{CR}}(\mathbb{E}_L, \mathscr{T}) = \exp(-\mathrm{Val}(\Gamma))$. We get an f as in (i) and (ii) via the reduction step. Letting $f^{-1}(\infty) = X_1'$, we see that (4.10) follows from the first equality in (4.9) and from that fact that ϵ was an arbitrary positive number in condition (ii) of Lemma 4.10. $\square$

Completion of the proof of Theorem 1.7.

In view of Lemma 4.8, it will suffice to show that

$$\gamma_F^+(\mathbb{E}, \mathscr{T}) = \inf_{\mathbb{U} \in \mathcal{U}} \left\{ \sup_{X_1' \in D(X_1)} \gamma_{\mathrm{F}}(\mathbb{U}, X_1')^{|X_1'|^{-2}} \right\}. \tag{4.11}$$

equals $\gamma_{\mathrm{CR}}(\mathbb{E}, \mathscr{T})$. Lemma 4.9 implies that (4.11) is bounded above by

$$\inf_{\mathbb{U} \in \mathcal{U}} \gamma_{\mathrm{CR}}(\mathbb{U}, \mathscr{T})$$

and this equals $\gamma_{\mathrm{CR}}(\mathbb{E}, \mathscr{T})$ by the approximation Theorems of [15, §4.5]. Thus $\gamma_{\mathrm{F}}^+(\mathbb{E}, \mathscr{T}) \leq \gamma_{\mathrm{CR}}(\mathbb{E}, \mathscr{T})$. Lemma 4.10 and (4.11) give $\gamma_{\mathrm{F}}^+(\mathbb{E}, \mathscr{T}) \geq \gamma_{\mathrm{CR}}(\mathbb{E}, \mathscr{T})$, which completes the proof.

Example 4.11. We give an example showing that if $S_\gamma^+(\mathbb{E}, \mathscr{T}) < 1$ then one may have $\gamma_{\mathrm{F}}^+(\mathbb{E}, \mathscr{T}) = 0$. Suppose that $F = \mathbb{Q}$, $X = \mathbb{P}^1$ and X_1 is the point ∞. Define $\mathbb{E} = \prod_{v \in M(\mathbb{Q})} E_v$ by letting $E_v = \mathbb{A}^1(O_{\mathbb{C}_v})$ if v is finite, and by letting E_v the open disc of radius $1/2$ about $1/2$ in $\mathbb{A}^1(\mathbb{C}_{v_\infty}) = \mathbb{C}$ when v_∞ is the archimedean place. Then $S_\gamma(\mathbb{E}, \infty) = S_\gamma^+(\mathbb{E}, \infty) = 1/2$. Suppose $f : X \to \mathbb{P}^1$ is a finite morphism such that $\mathrm{supp}(f^{-1}(\infty)) = \{\infty\}$

and $f^{-1}(B(r)) \subset \mathbb{E}$ for some $r \in \mathbb{R}$ with $|r| > 0$. Then $f^{-1}(0)$ would contain an element of $\overline{\mathbb{Q}}$ having all its conjugates in $\mathbb{E}$, but no such elements exist. Hence $\gamma_{\mathrm{F}}(\mathbb{E}, \infty) = \gamma_{\mathrm{F}}^{+}(\mathbb{E}, \infty) = 0$.

4.12. **RL-domains for the finite adeles.** We we will assume in this section the notation and hypotheses of Theorem 1.8. We identify $H^0(X, O_X(X_1))$ with the F-subspace of the function field $F(X)$ whose nonzero elements f have the property that $\mathrm{div}(f) + X_1 \geq 0$. We may assume that the projective embedding $X \to \mathbb{P}^t$ is assocated to a generating set $\{f_0, \ldots, f_t\}$ for $H^0(X, O_X(X_1))$ such that $f_0 = 1$ and the f_i for $i > 0$ are nonconstant. Our hypothesis is equivalent to the statement that for $v \in M_{\mathrm{f}}(F)$ one has

$$U_v = \{x \in X(\mathbb{C}_v) : |f_i(x)|_v \leq 1 \quad \text{for all} \quad i\}. \tag{4.12}$$

Let O_F be the ring of integers of F in the number field case and let O_F be the ring of functions regular off of $M_\infty(F)$ in the function field case. Let $A = \bigoplus_{j=0}^{\infty} A_j$ be the graded O_F-algebra for which $A_0 = O_F$, A_1 is the O_F-submodule of $H^0(X, O_X(X_1)) \subset F(X)$ generated by $f_0, \ldots, f_t$ and A_j is the O_F-module generated by all products of j-elements of A_1 if $j \geq 2$. We then have a projective flat model $\mathscr{X}$ of X over O_F given by $\mathscr{X} = \mathrm{Proj}(A)$. This comes with a canonical very ample line bundle $O_{\mathscr{X}}(1)$ whch is associated to the effective horizontal very ample Cartier divisor D whose ideal sheaf is $O_{\mathscr{X}}(-1)$. The zero locus of the degree one element f_0 of A is D. This implies that the general fiber of D is X_1 and that D is the Zariski closure of X_1. In particular, D is horizontal. The description (4.12) implies that for $v \in M_{\mathrm{f}}(F)$, the points $x \in \mathscr{X}(\mathbb{C}_v) = X(\mathbb{C}_v)$ which lie in U_v are exactly those for which the reduction of x at v does not lie on the reduction of D.

We now apply Theorem 1.2 to conclude that there is a finite morphism $\pi : \mathscr{X} \to \mathbb{P}^d_{O_F}$ such that the pullback $\pi^{-1}(H)$ of the hyperplane $H = \{y_0 = 0\}$ in $\mathbb{P}^d_{O_F}$ is a positive multiple of D. The above description of U_v via the reduction of D at $v \in M_{\mathrm{f}}(F)$ now implies $U_v = \pi^{-1}(B_v(1))$. Hence the restriction $\pi_{\mathbb{U}_{\mathrm{f}}} : X \to \mathbb{P}^d_F$ of π to the generic fiber X of $\mathscr{X}$ has the properties stated in Theorem 1.8. Thus if U_v for $v \in M_\infty$ is a large enough open set such that $\mathbb{U} = \mathbb{U}_{\mathrm{f}} \times \prod_{v \in M_\infty(F)} U_v$ is an open adelic set satisfying the standard hypotheses, there will be an adelic polydisk $B(r)$ in A^d such that $|r| > 1$ and $\pi^{-1}(B(r)) \subset \mathbb{U}$. Therefore $\gamma_{\mathrm{F}}(\mathbb{U}, X_1) > 1$, which completes the proofs of Theorem 1.8 and Corollary 1.9.

References

[1] M. Artin, Lipman's proof of resolution of singularities for surfaces, in: Arithmetic Geometry, G. Cornell and J. H. Silverman editors, Springer-Verlag, New York (1986), 267–287.

[2] T. Chinburg, Capacity theory on varieties, Compositio Math. 80 (1991), no. 1, 75–84.

[3] T. Chinburg, G. Pappas and M. J. Taylor, Higher adeles and nonabelian Riemann-Roch, preprint.

[4] W. Fulton. Intersection Theory. 2^{nd} edition, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, 2. Springer-Verlag, Berlin (1998).

[5] O. Gabber, Q. Liu and D. Lorenzini, Hypersurfaces of projective schemes and a moving lemma, preprint (July, 2011).

[6] B. Green, Geometric families of constant reductions and the Skolem property, Transactions of the A. M. S., Vol.350, no. 4 (1998), 1379–1393.

[7] B. Green, On curves over valuation rings and morphisms to P^1. J. Number Theory 59 (1996), no. 2, 262–290.

[8] A. Grothendieck and J. Dieudonné, Éléments de géométrie algébrique, IV.2 and IV.3, Publ. Math. IHES **24** and **28** (1965).

[9] C. F. Lau and R. Rumely, Arithmetic capacities on $\mathbb{P}^N$, Math. Z. 215, (1994) 533-560.

[10] S. Lichtenbaum, Curves over discrete valuation rings. Amer. J. Math. 90, (1968) 380–405.

[11] J. Lipman, Desingularization of two-dimensional schemes, Ann. Math. 107 (1978), 151-207.

[12] H. Matsumura, Commutative Algebra. 2^{nd} edition. Benjamin/Cummings, Reading Mass. (1980).

[13] L. Moret-Bailly, Groupes de Picard et problèmes de Skolem I. Ann. E. N. S., 4^e série, tome 22, no. 2 (1989), 161–179.

[14] C. Petsche, L. Szpiro and M. Tepper, Isotriviality is equivalent to potential good reduction for endomorphisms of $\mathbb{P}^n$ over function fields. J. Algebra 322 (2009), no. 9, 33453365.

[15] R. S. Rumely, Capacity Theory on Algebraic Curves. Springer Lecture Notes in Mathematics 1378. Springer-Verlag, Berlin-Heidelberg-New York (1989).

[16] R. S. Rumely, C. F. Lau and R. Varley, Existence of the sectional capacity. Mem. Amer. Math. Soc. 145 (2000), no. 690, viii+130 pp.

[17] R. S. Rumely, On the relation between Cantor's capacity and the sectional capacity, Duke Math. J. 70 (1993), no. 3, 517–574.

[18] L. Szpiro et al, Séminarie sur les pinceaux de courbes de genre au moins deux, Astérisque, vol. 86.

[19] W. V. Vasconcelos, On projective modules of finite rank, Proc. Amer. Math. Soc. 22 (1969), 430433.

Ted Chinburg, Dept. of Math, Univ. of Penn., Phila. PA. 19104, U.S.A.
E-mail address: ted@math.upenn.edu

Laurent Moret-Bailly, IRMAR, Université de Rennes 1, Campus de Beaulieu, F-35042 Rennes Cedex, France
E-mail address: Laurent.Moret-Bailly@univ-rennes1.fr

Georgios Pappas, Dept. of Math, Michigan State Univ., East Lansing, MI 48824, U.S.A.
E-mail address: pappas@math.msu.edu

Martin J. Taylor, School of Mathematics, Merton College, Univ. of Oxford, Merton Street, Oxford OX1 4JD, U.K.
E-mail address: martin.taylor@merton.ox.ac.uk