Elliptic curves and Hilbert's tenth problem for algebraic function fields over real and p-adic fields

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Abstract

Let k be a field of characteristic zero, V a smooth, positive-dimensional, quasiprojective variety over k, and Q a nonempty divisor on V. Let K be the function field of V, and $A \subset K$ the semilocal ring of Q.

We prove the Diophantine undecidability of: (1) A, in all cases; (2) K, when k is real and V has a real point; (3) K, when k is a subfield of a p-adic field, for some odd prime p.

To achieve this, we use Denef's method: from an elliptic curve E over $\mathbb{Q}$, without complex multiplication, one constructs a quadratic twist $\mathscr{E}$ of E over $\mathbb{Q}(t)$, which has Mordell-Weil rank one. Most of the paper is devoted to proving (using a theorem of R. Noot) that one can choose f in K, vanishing at Q, such that the group $\mathscr{E}(K)$ deduced from the field extension $\mathbb{Q}(t) \xrightarrow{\sim} \mathbb{Q}(f) \hookrightarrow K$ is equal to $\mathscr{E}(\mathbb{Q}(t))$. Then we mimic the arguments of Denef (for the real case) and of Kim and Roush (for the p-adic case).

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Contents

1 Introduction

The aim of this paper is to prove the following result:

1.1 Theorem. *Let k be a field of characteristic zero. Let V be a smooth, positive-dimensional, quasiprojective, irreducible k-scheme, with function field denoted by K.*

(1) (see Theorem 10.3) *Let Q be a nonempty effective divisor on V, and let $A \subset K$ be the semilocal ring of Q (the intersection of the local rings of the maximal points of Q). Then the positive-existential theory of A is undecidable. In other words, Hilbert's tenth problem over A has a negative solution.*

(2) (see Theorem 11.2) *Assume that K is formally real. Then the positive-existential theory of K is undecidable.*

(3) (see Theorem 12.1) *Assume that k is a subfield of a finite extension of $\mathbb{Q}_p$, for some odd prime p. Then the positive-existential theory of K is undecidable.*

1.1.1 Remark. Thus, for instance, the conclusion of (2) and (3) means that there is no algorithm taking as input a polynomial $F \in K[X_1, \ldots, X_n]$ (for some n) and giving a 'yes/no' output according as F has a zero in K^n or not.

In all statements, the positive-existential theory is considered in the language of rings, augmented by a suitable set of constants which can be described.

In each case, a more precise result will be that there is a Diophantine (that is, positive-existentially definable) subset of A^d (resp. K^d) for some d (in fact $d = 2$ in cases (1) and (2)), with a ring structure which is also Diophantine and isomorphic to $\mathbb{Z}$ as a ring. By a standard argument, this together with the negative solution of Hilbert's tenth problem over $\mathbb{Z}$ (Davis-Putnam-Robinson-Matijasevich) implies the result for A (resp. K).

1.1.2 Remark. Note that by enlarging k, we can assume in (1) and (2) that V is a *curve* C: if $r = \dim V$, one can first assume V affine (taking an open subset meeting every component of Q), then choose a k-morphism $V \to \mathbb{A}_k^{r-1}$ whose generic fibre is a smooth curve and such that every component of Q dominates $\mathbb{A}_k^{r-1}$; finally, replace k by the function field of $\mathbb{A}_k^{r-1}$. One can even go further and assume that C is projective and smooth (by completing it) and geometrically connected over k (by replacing k by its algebraic closure in K).

This reduction to the case of curves does not work in case (3): for instance, $\mathbb{Q}_p(x)$ cannot be embedded in a p-adic field. However, in most of this paper, the emphasis will be on curves.

1.1.3 Remark. Several special cases of 1.1 were known before. The case where k is a real field and $K = k(t)$ is due to Denef [D2], as well as the method used here.

Denef's method was also used by Kim and Roush [K-R2] to treat the case where k is as in (3) and $K = k(t)$. We use the proof of Kim and Roush to prove (3), whence the restriction $p \neq 2$.

Some special cases of (2) for non-rational function fields in one variable over real fields were obtained by Zahidi in [Z].

Earlier versions of this paper, without part (3), were circulated before. After completing (3), the author was informed (at the end of August 2004) that K. Eisenträger [E2] had independently proved the p-adic case (3), using one of these versions (specifically, Theorem 1.8 below). She also obtained in [E1] (with the notations of Theorem 1.1) the Diophantine undecidability of K when k is algebraically closed and $\dim V \geq 2$, adapting (again via our Theorem 1.8) the method used in [K-R1] for $K = \mathbb{C}(t_1, t_2)$.

The reader may consult [P-Z] for a review of other related results.

Note that (3) implies in particular:

1.1.4 Corollary. *Let K be a finitely generated, transcendental extension of $\mathbb{Q}$. Then K is positive-existentially undecidable.*

From now on, we shall assume in this introduction that $V = C$ is a smooth, projective, geometrically connected curve over k, and that Q is a finite nonempty set of closed points of C.

To motivate our further setting, we now briefly recall Denef's method.

1.2 Sketch of Denef's method.

Assume $C = \mathbb{P}^1_k$, with standard coordinate t, so that $K = k(t)$. Take an elliptic curve E over k (defined over $\mathbb{Q}$, if we wish), and 'twist' it by the quadratic extension of $K = k(t)$ given by the usual double cover $\pi : E \to \mathbb{P}^1_k$. The result is an elliptic curve $\mathscr{E}$ over K (the 'self-twist' of E), with additive reduction at the branch points of π. An easy computation shows that $\mathscr{E}(K)$ is 'almost' the endomorphism ring of E. More precisely, $2\mathscr{E}(K)$ is canonically isomorphic to $2\operatorname{End}_k(E)$, so if E does not have complex multiplication the group $2\mathscr{E}(K)$ is isomorphic to $\mathbb{Z}$, and the addition is clearly Diophantine since it is induced by the group law of $\mathscr{E}$.

Fixing an isomorphism $2\mathscr{E}(K) \xrightarrow{\sim} \mathbb{Z}$, the less obvious fact that the multiplication is also Diophantine is deduced from the 'additive reduction' properties of $\mathscr{E}$ at branch points of π, in particular at the point ∞. Specifically, we have a 'reduction', or 'specialisation' homomorphism from $2\mathscr{E}(K)$ to the additive group k, which turns out to be nonzero, hence must be (up to a harmless constant) the inclusion of $\mathbb{Z}$ into k. In particular, it must be compatible with multiplication, allowing us to obtain a Diophantine definition of multiplication in $2\mathscr{E}(K)$ — *provided*, however, that the specialisation map has good Diophantine properties, which is where the 'real' (resp. 'p-adic') assumption is used; more precisely, this involves proving the Diophantine definability of certain subsets of K defined by valuation conditions.

1.3 Extending Denef's method to other fields.

We want to extend this argument with $k(t)$ replaced by K. To do this, we simply take a cover $f : C \to \mathbb{P}^1_k$ with reasonable properties, by means of which we identify $k(t)$ with a

subfield of K; we then try to adjust the data in such a way that $\mathscr{E}(K) = \mathscr{E}(k(t))$ (whatever $\mathscr{E}(k(t))$ may be: we shall forget here about the 'no complex multiplication' condition).

As it turns out, we may in fact start with any elliptic curve E over k, and twist it by a quadratic extension of $k(t)$, corresponding to a double cover $\pi : \Gamma \to \mathbb{P}^1_k$ (here π is any double cover of $\mathbb{P}^1_k$ by a smooth curve Γ, not necessarily E itself). The curves C, E, and Γ, and the morphism π, will be fixed throughout (and, therefore, so will the twisted elliptic curve $\mathscr{E}$ over $k(t)$). The only 'variable' piece of data is the morphism f; specifically, we shall allow ourselves to replace the initially given f by λf, for some suitable $\lambda \in k^*$.

But now it is time to fix the notations more precisely. (For the rest of this introduction, we shall concentrate on the algebro-geometric result of the paper; the applications to Hilbert's tenth problem will be considered in Part III).

1.4 Notations.

1.4.1 The ground field. In the rest of this introduction (and in most of the paper), k denotes a field of characteristic $p \geq 0$. Moreover, *unless otherwise specified, we shall always assume that* $p \neq 2$. (For applications to undecidability questions, p will be zero). We fix an algebraic closure of k, denoted by $\overline{k}$.

1.4.2 The fundamental curve C and its function field. We denote by C a smooth projective geometrically connected curve over k, with function field K (thus, K is a finitely generated extension of k, of transcendence degree 1, and k is algebraically closed in K).

If k' is an extension of k, the function field of $C_{k'} = C \times_{\mathrm{Spec}\,(k)} \mathrm{Spec}\,(k')$ will be denoted by $k'(C)$; thus, $\overline{k}(C) = \overline{k} \otimes_k K$, and in general $k'(C)$ is the fraction field of $k' \otimes_k K$.

We are also given a finite nonempty set Q of closed points of C; we assume that their residue fields are *separable* over k (in other words, Q is the spectrum of an étale k-algebra).

1.4.3 The elliptic curve. E denotes an elliptic curve over k. It will be convenient to fix an affine equation of E, of the form

$$y^2 = P(x) \tag{1}$$

for a cubic polynomial $P \in k[T]$ without multiple roots.

1.4.4 The hyperelliptic curve. Γ denotes a smooth projective geometrically connected curve over k, given as a double cover

$$\pi : \Gamma \longrightarrow \mathbb{P}^1_k. \tag{2}$$

We shall always assume that:

- π is étale above $\infty \in \mathbb{P}^1(k)$;
- π is ramified at 0.

(The second assumption is made only to fix ideas and avoid some case discussions, and because it is the relevant case for applications to Hilbert's tenth problem. The first assumption, however, will be essential in our constructions).

Thus, Γ can be described by an affine equation, in coordinates (t, w):

$$w^2 = R(t) \tag{3}$$

where t, the standard coordinate on $\mathbb{P}^1$, is identified with the rational function $t \circ \pi = \pi$ on Γ, and R is a polynomial in $k[T]$, without multiple roots, such that $R(0) = 0$, and $\deg R = 2\,\mathrm{genus}(\Gamma) + 2$.

Clearly, Γ has a unique k-rational point above the point $0 \in \mathbb{P}^1(k)$. We denote this point by 0_Γ.

The natural involution of Γ, sending w to $-w$, will be denoted by σ.

1.4.5 Remarks.

(i) We shall always think of Γ as equipped with the double cover π of (2). In other words, when using the notation Γ we shall often actually mean π. Note that Γ and π are completely determined by the polynomial R of (3); conversely, they determine R up to a square factor in k^*.

(ii) An important special case is when $\Gamma = E$ (thus not 'hyperelliptic', strictly speaking!) and π is the double cover given by x^{-1}, the inverse of the coordinate x in (1) . In this case, the polynomial R of (3) is

$$R(t) = t^4\, P(1/t) \tag{4}$$

and the functions t, w, x, y on E are related by

$$(t, w) = (1/x, y/x^2) \quad (x, y) = (1/t, w/t^2). \tag{5}$$

In fact, this is the important case for applications to Hilbert's tenth problem; however, the author feels that restricting to this special case would only give a less general result without any substantial simplification, while distinguishing between E and Γ actually clarifies the proof.

(iii) We could even have generalised further, by replacing E by any abelian variety over k. But this time, this would make the results we need (essentially those of Section 6 on quadratic twists) slightly less elementary.

From the data 1.4.3 and 1.4.4 we can now perform a well-known construction:

1.4.6 The twisted elliptic curve $\mathscr{E}$ over $k(t)$. We may define it as the $k(t)$-elliptic curve with affine equation

$$y^2 = R(t)\, P(x) \tag{6}$$

(in the affine plane $\mathbb{A}^2_{k(t)}$, with coordinates x, y). We shall give a more intrinsic definition of twists in Section 5, and use slightly different (but of course equivalent) equations.

It is easy to compute the Mordell-Weil group $\mathscr{E}(k(t))$ of $\mathscr{E}$ in terms of morphisms of curves *over* k: as we shall see in 6.3, there is a canonical isomorphism

$$\mathscr{E}(k(t)) \cong \mathrm{Mor}_k^{\mathrm{odd}}(\Gamma, E) \tag{7}$$

where the right-hand side stands for the group of k-morphisms $h : \Gamma \to E$ compatible with involutions, i.e. such that $h \circ \sigma = [-1]_E \circ h$.

1.4.7 Remark. In the case $\Gamma = E$ of 1.4.5 (ii), we get from (7) an isomorphism

$$E[2](k) \times \mathrm{End}_k(E) \cong \mathscr{E}(k(t)) \tag{8}$$

where $E[2]$ is the kernel of multiplication by 2 in E. Concretely, using the coordinates in (1) and (6), this sends $((\xi, 0), 0)$ to $(\xi, 0)$ (where ξ is a zero of P), and $(0_E, \mathrm{Id}_E)$ to $(1/t, t^2P(1/t))$ (recall that $R(t) = t^4P(1/t)$).

1.5 Properties of covers $C \to \mathbb{P}^1_k$: good functions.

Recall that our goal is to extend Denef's argument with $k(t)$ replaced by K. To do this we shall choose a suitable nonconstant rational function g on C; this defines a ramified cover $g : C \to \mathbb{P}^1_k$, and a corresponding field extension $k(t) \to K$, sending t to g.

We denote by K_g the field K viewed as an extension of $k(t)$ via g. Thus, we have an obvious inclusion of abelian groups

$$\mathscr{E}(k(t)) \hookrightarrow \mathscr{E}(K_g) \tag{9}$$

both of which will turn out (see 6.3.3) to be finitely generated with the same torsion subgroup, isomorphic to the kernel $E[2](k)$ of multiplication by 2 in $E(k)$.

We would like (9) to be an equality, for suitable g. If $p > 0$, however, we can only achieve this 'up to p-torsion', which motivates the following definition:

1.5.1 Definition. *Let $u : A \to B$ be a morphism of abelian groups. We shall say that u is* almost bijective *if u is injective and* Coker u *is a finite p-group.*

Of course, if $p = 0$ we take this to mean that u is bijective. In the sequel we shall only apply this notion to morphisms of finitely generated abelian groups.

1.5.2 Definition. *Let k, C, Q, E, Γ be as in 1.4, and let $g : C \to \mathbb{P}^1_k$ be a nonconstant k-morphism.*

(1) *We say that g is* admissible *(for Γ, or for π) if:*

(i) *g has only simple branch points* (i.e. no ramification index ≥ 3);

(ii) *g is étale above ∞ and the branch points of π* (which are the zeros of R);

(iii) *every point of Q is a zero of g* (automatically simple, by (ii)).

(2) *We say that g is* good *for E and Γ if g is admissible and the natural inclusion (9) is almost bijective.*

If k' is an extension of k, we say that g is good over k', *or* k'-good, *if the morphism $g_{k'} : C_{k'} \to \mathbb{P}^1_{k'}$ deduced from g by base change is good for $E_{k'}$ and $\Gamma_{k'}$.*

We say that g is very good *if g is $\overline{k}$-good.*

(3) *Let $f : C \to \mathbb{P}^1_k$ be admissible. For every extension k' of k, define two subsets* $\mathrm{Good}\,(k')$ *and* $\textsc{Good}\,(k')$ *of k'^* by*

$$\begin{array}{rcrcl} \mathrm{Good}\,(k') & = & \mathrm{Good}\,(E,\Gamma,f,k') & = & \{\lambda \in k'^* \mid \lambda f \text{ is good for } E_{k'} \text{ and } \Gamma_{k'}\} \\ \textsc{Good}\,(k') & = & \textsc{Good}\,(E,\Gamma,f,k') & = & \{\lambda \in k'^* \mid \lambda f \text{ is very good for } E_{k'} \text{ and } \Gamma_{k'}\}. \end{array}$$

1.5.3 Remarks.

(i) By definition, g is k'-good if and only if g is admissible and the natural inclusion

$$\mathscr{E}(k'(t)) \hookrightarrow \mathscr{E}(k'(C)_g) \tag{10}$$

is almost bijective; here $k'(C)_g$ is the function field $k'(C)$ of $C_{k'}$, viewed as an extension of $k'(t)$ via $g_{k'}$. In particular, g is very good if and only if g is admissible and $\mathscr{E}(\overline{k}(C)_g) = \mathscr{E}(\overline{k}(t))$, up to p-torsion.

(ii) Of course, the definition of $\textsc{Good}\,(k')$ refers implicitly to some algebraic closure $\overline{k'}$ of k', but is, as usual, independent of it.

(iii) Let f be admissible. Then for all but finitely many $\lambda \in k^*$, the function λf is still admissible. Thus, except for finitely many λ, the 'goodness' property for $g = \lambda f$ just means that (9) is almost bijective.

1.5.4 Proposition. *Assume that $g : C \to \mathbb{P}^1_k$ is admissible, and let k' be an extension of k. Then:*

(i) *If g is k'-good, then it is good. In particular, very good morphisms are good.*

(ii) *Assume that k is separably closed in k'. Then $\mathscr{E}(k'(t)) = \mathscr{E}(k(t))$ and $\mathscr{E}(k'(C)_g) = \mathscr{E}(K_g)$; in particular, g is k-good if and only if it is k'-good.*

(iii) *g is very good if and only if g is F-good for every extension F of k.*

Proof: (i) follows easily from the fact that $k(t) = k(C) \cap k'(t)$.

The proof of (ii) will be postponed until 6.3.4.

The 'if' part of (iii) is trivial. Conversely, assume g is very good, and let F be an extension of k, with an algebraic closure $\overline{F}$ containing $\overline{k}$. Now $g_{\overline{k}}$ is good, hence $g_{\overline{F}}$ is good by (ii). Hence g_F is good by (i). ∎

1.5.5 Corollary. *Let $f : C \to \mathbb{P}^1_k$ be admissible, and let k' be an extension of k. Then:*

(i) $\mathrm{Good}\,(k') \cap k \subset \mathrm{Good}\,(k)$, *with equality if k is separably closed in k'.*

(ii) $\text{GOOD}\,(k') \cap k = \text{GOOD}\,(k)$. ∎

1.5.6 Remark. The existence of admissible morphisms is easy (see 2.3.1, but note that in positive characteristic, this uses our assumption that Q is étale over k). The existence of very good morphisms is the subject of this paper. More precisely, if k is not algebraic over a finite field, we shall prove $\text{GOOD}\,(k) \neq \emptyset$ for any admissible f. This of course implies $\text{Good}\,(k) \neq \emptyset$. The reasons why we need both variants are explained in 1.11 below.

1.6 One last piece of data.

In addition to the data of 1.4, we fix an admissible k-morphism

$$f : C \to \mathbb{P}^1_k. \tag{11}$$

1.7 Main Theorem. *Let k, C, Q, E, Γ, f be as above. Then:*

(i) *Let k' be an extension of k. If $\lambda \in k'$ is transcendental over k, then $\lambda \in \text{GOOD}\,(k')$.*

(ii) *If k is finitely generated over the prime field, then $\text{GOOD}\,(k)$ contains a Hilbert subset of k, in the sense of* [F-J], 11.1*; in other words, its complement in k is a thin set in the sense of* [Se2].

This will be proved in 7.4. Let us now explore some consequences.

1.8 Theorem. *We keep the notations and assumptions of Theorem 1.7.*

(i) *Let k_0 be any subfield of k, finitely generated over the prime field. Then $\text{GOOD}\,(k)$ contains a Hilbert subset of k_0.*

(ii) *If $\operatorname{char} k = 0$, then $\text{GOOD}\,(k) \cap \mathbb{Z}$ is infinite.*

(iii) *If $\operatorname{char} k = p > 0$, and $u \in k$ is transcendental over $\mathbb{F}_p$, then $\text{GOOD}\,(k) \cap \mathbb{F}_p[u]$ is infinite.*

(iv) *The complement of $\text{GOOD}\,(k)$ in k has finite transcendence degree over the prime field. In particular, this complement is countable.*

Proof: (i) If k_0 is finite, the claim is empty. Therefore we may assume that k_0 is either finitely generated over $\mathbb{Q}$, or finitely generated and transcendental over a finite field. In both cases, k_0 is Hilbertian.

There is a subfield k_1 of k, containing k_0, finitely generated over the prime field, and such that C, E, Γ, and f are defined over k_1. Now apply 1.7 with k_1 as ground field: by 1.5.5 (ii), $\text{GOOD}\,(k)$ contains $\text{GOOD}\,(k_1)$ which contains a Hilbert subset of k_1 by 1.7 (ii). Since k_0 is Hilbertian, it follows that $\text{GOOD}\,(k_1) \cap k_0$ contains a Hilbert subset of k_0, by [F-J], 11.7 and 11.8(b).

Assertion (ii) then follows from (i) (with $k_0 = \mathbb{Q}$) and [F-J], Theorem 12.7, and (iii) is similar.

For (iv), take k_1 as in the proof of (i): then 1.7 (i), applied over k_1, shows that the complement of $\mathrm{Good}\,(k)$ is contained in the algebraic closure of k_1 in k, whence the result. ∎

1.9 Remark. Theorem 1.7 may well be true in characteristic 2. Presumably, the arguments of this paper, suitably adapted, might lead to a proof that $\mathrm{Good}\,(k)$ contains a Hilbert set when k is finitely generated over $\mathbb{F}_2$. However, to obtain the same result for $\textsc{Good}\,(k)$ (and hence the general result, for arbitrary k), the present proof makes use of very strong properties of a pencil of curves over $\mathbb{P}^1_k$ considered in Section 7 (namely, that its fibres over $\mathbb{P}^1 \setminus \{0\}$ are semistable and its fibre at 0 is 'tame'). Both these properties fail in general in characteristic 2, which definitely ruins the crucial Lemma 4.5.3.

Of course, to treat the characteristic 2 case one would first have to rewrite the generalities of Section 5 on double covers and twists.

In any case, the applications to Hilbert's tenth problem, which were the prime motivation for this paper, work only in characteristic zero.

1.10 Outline of the proof of the Main Theorem.

For simplicity, we assume $p = 0$ (thus, 'almost bijective' just means 'bijective'). For $\lambda \in k$, we consider the inclusion $\mathscr{E}(k(t)) \hookrightarrow \mathscr{E}(K_{\lambda f})$ (resp. $\mathscr{E}(\overline{k}(t)) \hookrightarrow \mathscr{E}(\overline{k}(C)_{\lambda f})$). The first group is independent of λ, while the second varies with λ, and clearly we have to make the groups $\mathscr{E}(K_{\lambda f})$ and $\mathscr{E}(\overline{k}(C)_{\lambda f}))$ 'as small as possible'.

1.10.1 Our first task is to 'compute' all these groups in terms of 'geometry over k'. For $\mathscr{E}(k(t))$, and similarly for $\mathscr{E}(\overline{k}(t))$, this is achieved by formula (7). To generalise this, we introduce (in 6.3.1) the k-curve

$$\widetilde{C}_{\lambda f} := C \times_{\lambda f, \mathbb{P}^1_k, \pi} \Gamma \tag{12}$$

consisting of pairs (c, γ) in $C \times \Gamma$ such that $\lambda f(c) = \pi(\gamma)$. As a double cover of C, it carries a natural involution, and just as in (7) there is a canonical isomorphism

$$\mathscr{E}(K_{\lambda f}) \cong \mathrm{Mor}^{\mathrm{odd}}_k(\widetilde{C}_{\lambda f}, E). \tag{13}$$

It follows (Proposition 6.4) that $\lambda \in \mathrm{Good}\,(k)$ if and only if every odd k-morphism $\widetilde{C}_{\lambda f} \to E$ is obtained from an odd k-morphism $\Gamma \to E$ by composition with the natural map $\widetilde{C}_{\lambda f} \hookrightarrow C \times \Gamma \to \Gamma$ (and of course there is a similar characterisation of $\textsc{Good}\,(k)$ in terms of $\overline{k}$-morphisms).

The next step consists in translating this condition in terms of Jacobians, and removing the 'odd' restriction. The result is this (Propositions 6.5.3 and 6.5.4): consider the morphism of abelian varieties

$$\mathrm{Jac}\,(C) \times \mathrm{Jac}\,(\Gamma) \longrightarrow \mathrm{Jac}\,(\widetilde{C}_{\lambda f}) \tag{14}$$

deduced from the natural projections from $\widetilde{C}_{\lambda f}$ to C and Γ. Then λ is in $\mathrm{Good}\,(k)$ (resp. in $\textsc{Good}\,(k)$) if every k-morphism (resp. $\overline{k}$-morphism) of abelian varieties $E \to \mathrm{Jac}\,(\widetilde{C}_{\lambda f})$ factors through (14).

1.10.2 So, putting $J_\lambda := \mathrm{Jac}\,(\widetilde{C}_{\lambda f})$, we are led to investigate how the groups $H_\lambda := \mathrm{Hom}_k\,(E, J_\lambda)$ and $\overline{H}_\lambda := \mathrm{Hom}_{\overline{k}}\,(E, J_\lambda)$ vary with λ.

Now J_λ is the fibre at λ of a pencil of abelian varieties parametrised by an open subset $U \subset \mathbb{P}^1_k$. Denote by $\eta = \mathrm{Spec}\,(k(z))$ the generic point of U (here z denotes the natural coordinate on $\mathbb{P}^1_k$): the groups in question have 'generic' values $H_\eta := \mathrm{Hom}_{k(z)}\,(E, J_\eta)$ and $\overline{H}_\eta := \mathrm{Hom}_{\overline{k(z)}}\,(E, J_\lambda)$. There are injective 'specialisation' maps $H_\eta \hookrightarrow H_\lambda$ and $\overline{H}_\eta \hookrightarrow \overline{H}_\lambda$, and a 'specialisation theorem' due to R. Noot asserts that when k is finitely generated over $\mathbb{Q}$, these specialisation maps are isomorphisms for every λ in a Hilbert subset of k.

Hence, to prove the Main Theorem, it suffices to show that our generic groups H_η and $\overline{H}_\eta$ are isomorphic to $\mathrm{Hom}_k\,(E, \mathrm{Jac}\,(C) \times \mathrm{Jac}\,(\Gamma))$ and $\mathrm{Hom}_{\overline{k}}\,(E, \mathrm{Jac}\,(C) \times \mathrm{Jac}\,(\Gamma))$, respectively.

To achieve this, we have to go back to the definition of the curves $\widetilde{C}_{\lambda f}$: as curves on the surface $C \times \Gamma$, they are the fibres of the rational map $C \times \Gamma \cdots\!\to \mathbb{P}^1_k$ sending (c, γ) to $\pi(\gamma)/f(c)$. The result for H_η then follows from more or less standard facts (Theorem 4.4.1) on Jacobians of pencils of curves. The analogous result for $\overline{H}_\eta$ (Theorem 4.5.2) is more delicate and requires a detailed analysis of the degenerations of the pencil, carried out in Section 7.

1.11 Effectivity questions.

Observe that in the proof of 1.8 (i), we have used the inclusion $\textsc{Good}\,(k_1) \subset \textsc{Good}\,(k)$ in an essential way. This explains why we need to consider '$\textsc{Good}$' sets, even to obtain the result for $\mathrm{Good}\,(k)$ for arbitrary k.

Unfortunately, the proof that $\textsc{Good}\,(k) \neq \emptyset$ relies on a highly nonconstructive argument involving infinite Galois groups (this occurs in the proof of the specialisation theorem 3.3).

On the other hand, if we limit ourselves to fields k finitely generated over the prime field (and to proving that $\mathrm{Good}\,(k) \neq \emptyset$), then there is a more effective result, stated below and proved in 7.4 (here $\mathrm{Jac}\,(X)$ denotes the Jacobian of a curve X). We refer to [F-J], Chapter 17 for presented fields and related notions; in particular, recall that k is *presented* over its prime field κ if it is described as $k = \kappa(x_1, \ldots, x_n)$ where, for each $i \geq 1$, the minimal polynomial of x_i over $k_{i-1} := \kappa(x_1, \ldots, x_{i-1})$ is explicitly given (and understood to be zero if x_i is transcendental over k_{i-1}). Many standard algebro-geometric constructions over k can then be carried out 'effectively'; see [F-J] for details.

1.12 Theorem. (Effective version of the Main Theorem) *Assume that k is presented over the prime field, and that the rank of the finitely generated abelian groups* $\mathrm{Hom}_k\,(E, \mathrm{Jac}\,(\Gamma))$ *and* $\mathrm{Hom}_k\,(E, \mathrm{Jac}\,(C))$ *are known.*

Then Good (k) *contains an 'effective' Hilbert subset of* k*, in the following sense:* z *and* y *denoting indeterminates, there is an effectively computable* $\Phi(z,y) \in k(z)[y]$*, with no root in* $k(z)$ *(as a polynomial in* y*), with the property that for all* $\lambda \in k$*, if* $\Phi(\lambda, y) \in k[y]$ *has no root in* k *then* $\lambda \in$ Good (k)*.*

1.13 Organisation of the paper

Apart from this introduction, the paper is divided into three parts.

Part I exposes background material, mostly from algebraic geometry; nothing in this part is really new.

Section 2 contains miscellaneous (and more or less well-known) results and basic definitions.

Section 3 is devoted to Noot's specialisation theorem; we give a proof there because the statement we use is actually a variant of Noot's original result. We also give a proof of the 'effective' variant we need (see 1.11 above).

In section 4, we give some important (and perhaps not so familiar) properties of the relative Jacobian of a surface fibered over the projective line. These properties are at the heart of our proof of the Main Theorem.

Finally, Section 5 presents the basic facts on quadratic twists, especially of elliptic curves.

In Part II, we prove the Main Theorem, as outlined in 1.10 above.

Part III contains the applications to model theory, and the proof of Theorem 1.1. In this part, the Main Theorem is applied in the special case where $E = \Gamma$ (a fixed elliptic curve over k), as explained in 1.4.5 (ii). The resulting twist $\mathscr{E}$ is the 'self-twist' of E; generalities on this construction are exposed in Section 8.

In Section 9, we define the (hopefully Diophantine) model of $\mathbb{Z}$ deduced from $\mathscr{E}$; this ring is denoted by Λ. This is a subset of K^2. To show the Diophantine undecidability of K, we have to prove two things: that Λ is a Diophantine set (this is where we use the Main Theorem), and that the multiplication of Λ is relatively Diophantine (a notion explained in 2.7.6); how we prove the latter depends on the field (or ring) K.

Section 10 contains the proof of part (1) of Theorem 1.1, as well as general notations used in the sequel.

Section 11 contains the proof of part (2) of 1.1, following Denef.

Finally, in Section 12 we prove part (3) of 1.1, adapting the method of Kim and Roush.

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Part I
Geometric background

2 Basic material

Throughout this section, F denotes a field, $\overline{F}$ an algebraic closure of F, and F^{s} the separable closure of F in $\overline{F}$.

2.1 Rings, varieties, morphisms.

All rings are commutative with unit.

If S is a scheme, and X, Y are S-schemes, we shall denote by $\mathrm{Mor}_S(X,Y)$ the set of S-scheme morphisms from X to Y. If $X = \mathrm{Spec}(R)$ is affine, we also use the notation $Y(R)$.

In general we use subscripts to denote base change: thus, if S' is an S-scheme, we write $X_{S'}$ for $X \times_S S'$. By abuse, we sometimes omit some of the subscripts, writing for instance $\mathrm{Mor}_{S'}(X,Y)$ for $\mathrm{Mor}_{S'}(X_{S'},Y_{S'})$.

In these notations, affine base schemes are often denoted by the corresponding ring: thus, if $S = \mathrm{Spec}(F)$, we may write $\mathrm{Mor}_F(X,Y)$ rather than $\mathrm{Mor}_S(X,Y)$.

2.2 Involutions, odd morphisms, algebraic groups.

If X and Y as above are provided with S-involutions σ and τ respectively, we shall denote by $\mathrm{Mor}_S^{\mathrm{odd}}(X,Y)$ the set of S-morphisms $\varphi : X \to Y$ such that $\varphi \circ \sigma = \tau \circ \varphi$.

The involutions considered will in general be clear from the context. In particular, if Y, say, is a commutative S-group scheme, written additively, the involution on Y will be multiplication by -1, unless otherwise specified.

If X and Y are commutative S-group schemes (always assumed to be separated and of finite presentation as S-schemes), we shall denote by $\mathrm{Hom}_S(X,Y)$ the set of morphisms *of S-group schemes* from X to Y (a subgroup of $\mathrm{Mor}_S^{\mathrm{odd}}(X,Y)$), and we write $\mathrm{End}_S(X)$ for $\mathrm{Hom}_S(X,X)$.

If G is a commutative S-group scheme, and $n \in \mathbb{Z}$, we denote by $[n]_G$ or $[n]$ the endomorphism of G given by multiplication by n, and by $G[n]$ its kernel. If n is *invertible on S*, then $[n]$ is an unramified morphism (in fact it is étale along the unit section of G), and $G[n]$ can therefore be written as the disjoint union of the unit section and a subscheme $G[n]^*$: in particular, if n is prime (the only case we shall use is $n = 2$), we may define $G[n]^*$ as the subscheme of G of 'points of exact order n'.

We shall use some 'rigidity' properties of odd morphisms to a commutative group scheme:

2.2.1 Rigidity of odd morphisms: notations. We assume that 2 is invertible on S (equivalently, all residue characteristics of points of S are different from 2).

Let $f : X \to S$ be a morphism of schemes. Assume that f is flat, proper, of finite presentation and that

$$\mathscr{O}_S \xrightarrow{\sim} f_*\mathscr{O}_X \text{ universally}$$

(that is, $\mathscr{O}_{S'} \xrightarrow{\sim} (f_{S'})_*\mathscr{O}_{X_{S'}}$ for every S-scheme S'; these conditions are satisfied in particular when S is Noetherian, f is projective and flat, and all its geometric fibres are irreducible and reduced). Let σ be an S-involution of X.

We also fix a commutative S-group scheme $G \to S$, separated and of finite presentation as an S-scheme. We write G additively.

2.2.2 Proposition. *With the assumptions of 2.2.1, let $u : X \to G$ be an* odd *S-morphism. Then, the following conditions are equivalent:*

(i) $u = 0$;

(ii) $u = 0$ *set-theoretically (that is, u maps the underlying space of X to the unit section of G);*

(iii) *for every point s of S, $u(X_s)$ is contained in an affine open subset of G_s disjoint from $G_s[2]^*$.*

Moreover, the set $\Sigma := \{s \in S \mid u_s : X_s \to G_s \text{ is zero }\}$ is open and closed in S.

Proof: It is trivial that (i)⇒(ii)⇒(iii). Let us prove (iii)⇒(ii). We need to show that, for every $s \in S$ (with residue field $\kappa(s)$), u maps X_s to the unit 0_s of G_s. By (iii), u maps X_s to an affine scheme; but any morphism from X_s to an affine scheme must factor through $\operatorname{Spec}(\Gamma(X_s, \mathscr{O}_{X_s}))$ which is $\operatorname{Spec}\kappa(s)$ by our assumptions on X. In other words, u maps X_s to a rational point γ of G_s which must be fixed by $[-1]$ since u is odd; by the assumption (iii) we must have $\gamma = 0_s$.

To prove (ii)⇒(i) we may assume that S is affine, and (localising further if necessary) that there exists an affine open neighbourhood U of the unit section of G, disjoint from $G[2]^*$. Clearly, (ii) implies that u factors through U. As above, this implies that u must factor as $\gamma \circ f : X \to S \to G$, where $\gamma \in G(S)$ is a section which must be fixed by $[-1]$, hence equal to the unit section by our assumption on U.

Let us now prove the last claim. To see that Σ is open, let us take s in Σ, and show that $u = 0$ over a neighbourhood of s. We may assume that there is an open subset U of G as in the proof of (ii)⇒(i) above. Then $u^{-1}(U)$ is an open subscheme of X containing X_s; since f is proper (hence closed), it must contain $f^{-1}(V)$ for some neighbourhood V of s in S. But then $f_V : X_V \to G_V$ factors through U, hence is zero.

To see that Σ is closed, consider the inverse image in X of the unit section of G: since G is separated, this is a closed subscheme of X, hence its complement $W \subset X$ is open, and so is $f(W) \subset S$ (f is flat of finite presentation, hence open). But the complement of $f(W)$ is just Σ. ∎

2.2.3 Corollary. (Rigidity of odd morphisms) *With the assumptions of* 2.2.1, *let* $T \to S$ *be a faithfully flat quasicompact* S*-scheme, with* geometrically connected fibres. *Then the natural 'base change' map*

$$\mathrm{Mor}_S^{\mathrm{odd}}(X, G) \longrightarrow \mathrm{Mor}_T^{\mathrm{odd}}(X_T, G_T)$$

is an isomorphism.

Proof: Injectivity is clear since $T \to S$ is faithfully flat. Let $u_T : X_T \to G_T$ be an odd T-morphism. By flat descent, it is enough to show that the two morphisms $u_1, u_2 : X_{T\times_S T} \to G_{T\times_S T}$ deduced from u_T by base change via the two projections $T \times_S T \to T$ are equal. But clearly they coincide along the diagonal $T \to T \times_S T$, hence also, by 2.2.3, along an open and closed subscheme of $T \times_S T$ containing the diagonal. By our assumptions, the only such subscheme is $T \times_S T$, and the corollary is proved. ∎

2.2.4 Remark. An interesting special case is when $S = \mathrm{Spec}\,(F)$ and $T = \mathrm{Spec}\,(F')$ where F' is an extension of F. Then T is geometrically connected if and only if F is *separably closed in* F'. Thus, in this case, we have $\mathrm{Mor}_{F'}^{\mathrm{odd}}(X, G) = \mathrm{Mor}_F^{\mathrm{odd}}(X, G)$.

2.3 Existence of admissible morphisms on curves, and of odd projections on varieties.

Let C be a projective, smooth, geometrically connected F-curve, of genus g. If D is a divisor on C, we denote by $|D|$ the linear system associated to D, i.e. the space of effective divisors linearly equivalent to D. In other words, $|D| = \mathbb{P}(\mathrm{H}^0(C, \mathscr{O}_C(D))^*$ is the projective space of lines of the F-vector space $\mathscr{L}(D) = \mathrm{H}^0(C, \mathscr{O}_C(D))$.

Put $d = \deg D$. By Riemann-Roch, $\dim |D| \geq d - g$, with equality if $d \geq 2g - 1$.

2.3.1 Proposition. *With the above assumptions, assume that* F *is infinite, and that* $d = \deg D \geq 2g + 2$. *Then there exists an* F*-morphism* $f : C \to \mathbb{P}^1_F$, *having only simple ramification and such that the divisors* $f^{-1}(\xi)$, *for* $\xi \in \mathbb{P}^1_F$, *belong to* $|D|$ *(equivalently, the invertible sheaf* $f^*\mathscr{O}_{\mathbb{P}^1_F}(1)$ *is isomorphic to* $\mathscr{O}_C(D)$*). In particular,* $\deg f = d$.

Moreover, assume that some F*-rational* $E_0 \in |D|$ *is fixed, without triple points (over* $\overline{F}$*). Then one can choose* f *as above such that* $f^{-1}(0) = E_0$ *(as divisors).*

Proof: The case $g = 0$ is left to the reader; we assume $g > 0$ and in particular $d \geq 4$.

For $n \in \mathbb{N}$, denote by $C^{(n)}$ the n-th symmetric power of C. It is a smooth projective F-scheme, whose $\overline{F}$-points correspond canonically to effective divisors of degree n on $C_{\overline{F}}$. We define a closed subvariety W of $C \times C^{(d-3)}$ by

$$W := \{(P, D_1) \mid 3P + D_1 \sim D\}$$

where $\sim$ denotes linear equivalence. There is an obvious morphism $W \to |D|$ sending (P, D_1) to $3P + D_1$, whose image W' consists of divisors having a triple point.

I claim that $\dim W = d-g-2$. Indeed, consider the natural projection $W \to C$ sending (P, D_1) to P. The fibre of a point $P \in C$ is canonically isomorphic to the projective space

$|D-3P|$, which has dimension $d-3-g$ since by assumption $d-3 \geq 2g-1$. Hence $\dim W = \dim C + d - g - 3 = d - g - 2$, as claimed. In particular, W' has codimension ≥ 2 in $|D|$.

Since F is infinite, we can choose an F-rational point E_0 in $|D| \setminus W'$ (for instance the given one, if necessary). Write $E_0 = P_1 + \cdots + P_d$ (over $\overline{F}$). For each $i \in \{1, \ldots, d\}$, divisors in $|D|$ containing P_i form a linear subspace isomorphic to $|D - P_i|$, hence a hyperplane H_i of $|D|$ (not necessarily defined over F). The divisors meeting E_0 thus form a hypersurface H (the union of the H_i's, which is defined over F). Again, since F is infinite, there is a line $\Delta \subset |D|$ containing E_0, disjoint from W' and not contained in H. Take an F-rational point E_∞ on Δ, distinct from E_0 (hence not in H, because $\Delta \cap H_i = \{E_0\}$ since Δ is a line). There is a rational function f on C with divisor $E_0 - E_\infty$, whence an F-morphism $\varphi : C \to \mathbb{P}^1_F$ whose fibres are precisely the points of Δ. This f satisfies the required conditions. ∎

2.3.2 Remark. The result should also be true if F is finite, possibly with a stronger condition on the degree.

2.3.3 Remark. Proposition 2.3.1 clearly implies the existence of admissible morphisms, in the sense of 1.5.2. The extra information on the degree will be used in Section 12, via the following special case: if C admits a divisor of odd degree, then there is an admissible morphism $C \to \mathbb{P}^1_F$ of odd degree. In the same vein, we shall also need the next proposition.

2.3.4 Proposition. *Assume that* $\operatorname{char} F = 0$, *and let* K *be a finitely generated, regular, transcendental extesion of* F. *Then there is a transcendence basis* $(z_1, \ldots, z_n)$ *of* K *over* F *such that* $K/F(z_1, \ldots, z_{n-1})$ *is a regular exension.*

If, moreover, F *is algebraically closed, then* $(z_1, \ldots, z_n)$ *may be chosen such that, in addition,* $[K : F(z_1, \ldots, z_n)]$ *is odd.*

Proof: Let $V \subset \mathbb{P}^{n+1}_F$ be a projective hypersurface with function field K, and put $d := \deg V$. Since K/F is regular, V is geometrically integral (i.e. $V_{\overline{F}}$ is irreducible and reduced).

By Bertini's theorem, there is a plane $\Pi \subset \mathbb{P}^{n+1}_F$ such that $\Pi \cap V$ is a geometrically integral curve; in fact, this property holds for all Π in a dense open subset of the Grassmannian of planes.

Take an F-rational point $A \in \Pi$, not in V. Consider the projection π from V to the space S_A (isomorphic to $\mathbb{P}^n_F$) of lines through A. This π sends a point $P \in V$ to the line through P and A, and is clearly a finite surjective morphism of degree d. Moreover, $\Pi \cap V = \pi^{-1}(L)$ where the line $L \subset S_A$ is the image of Π. We can choose coordinates $z_1, \ldots, z_n$ on S_A such that L is defined by, say, $z_1 = \ldots = z_{n-1} = 0$. The rational map $\varphi := (z_1, \ldots, z_{n-1}) : V \cdots \to \mathbb{A}^{n-1}_F$ then has the property that for all $\xi = (\xi_1, \ldots, \xi_{n-1}) \in \overline{F}^{n-1}$ except in a proper Zariski closed subset, $\varphi^{-1}(\xi)$ is an integral curve. This implies that the generic fibre of φ is geometrically integral, i.e. that the extension $K/F(z_1, \ldots, z_{n-1})$ is regular.

Note that in the previous construction, $[K : F(z_1, \ldots, z_n)] = d$. Hence to prove the last assertion, we assume d even and F algebraically closed. Again we use a projection, but

this time we take a smooth point $A \in \Pi \cap V$ and project from A. This defines a morphism $\pi : V \setminus \{A\} \to S_A$; for a general point of S_A, corresponding to a line l through A, the fibre $\pi^{-1}(l)$ consists of the $d-1$ points of $l \cap V$ distinct from A, hence π has degree $d-1$, which is odd.

Next, defining $L \subset S_A$ as above, we have $\pi^{-1}(L) = (\Pi \cap V) \setminus \{A\}$ which is geometrically integral, and the same property holds for all lines L' in a dense open subset of the Grassmannian of lines in S_A. Choosing coordinates as above, we again conclude that $K/F(z_1, \ldots, z_{n-1})$ is regular. ∎

2.4 Abelian varieties and schemes.

An *abelian scheme* over a scheme S is a smooth proper S-group scheme $A \to S$, with connected fibres. An abelian scheme over $\mathrm{Spec}\,(F)$ is called an abelian variety over F. Abelian schemes are automatically commutative, and abelian varieties are projective.

2.4.1 Proposition. *Let A be an abelian variety over F, F' an extension of F, and $B \subset A_{F'}$ an abelian subvariety. Then B can be defined over a finite extension of F.*

Proof: We assume $\mathrm{char}\,(F) \neq 2$, which is sufficient for our purposes (but the result holds in general). We may assume F algebraically closed. By standard arguments, there is a finitely generated F-subalgebra $R \subset F'$ and an abelian subscheme $\mathscr{B} \subset A_S$ over $S = \mathrm{Spec}\,(R)$ such that $\mathscr{B}_{F'} = B$. Now S has an F-rational point x; let B_0 be the fibre of $\mathscr{B}$ at x, an abelian subvariety of A. Consider the natural S-morphism $\mathscr{B} \hookrightarrow A_S \twoheadrightarrow (A/A_0)_S$: it is zero at x, hence zero by 2.2.2. Thus, $\mathscr{B} \subset A_{0,S}$. Since the opposite inclusion is proved similarly, we have equality, and in particular $B = A_{0,F'}$. ∎

2.4.2 Torsion points and Tate modules. If A is an abelian scheme over S, of relative dimension g, then $[n]_A$ is a finite locally free morphism of degree n^{2g}, étale above all points of S whose residue characteristic does not divide n.

If $S = \mathrm{Spec}\,(F)$ and $\mathrm{char}\,(F) \nmid n$, then $A[n](\overline{F})$ is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^{2g}$. If $l \neq \mathrm{char}\,(F)$ is a prime number, we define the *l-adic Tate module* $T_l(A)$ of A by

$$T_l(A) := \varprojlim_{n \geq 1} A[l^n](\overline{F}) \tag{15}$$

with transition maps induced by $A[l^{n+1}] \xrightarrow{\times l} A[l^n]$. This is a free $\mathbb{Z}_l$-module of rank $2g$, with a continuous action of $G_F = \mathrm{Gal}\,(F^{\mathrm{s}}/F)$; by the way, note that $A[l^n](\overline{F}) = A[l^n](F^{\mathrm{s}})$ since $A[l^n]$ is étale over F.

It is often convenient to use the corresponding $\mathbb{Q}_l$-vector space:

$$V_l(A) := \mathbb{Q}_l \otimes_{\mathbb{Z}_l} T_l(A) \tag{16}$$

which defines a $2g$-dimensional continuous representation of G_F over $\mathbb{Q}_l$.

2.4.3 Homomorphisms. $T_l(A)$ is obviously functorial in A: thus, if B is another abelian variety over F, there is a canonical homomorphism

$$\mathrm{Hom}_F(A,B) \otimes_{\mathbb{Z}} \mathbb{Z}_l \longrightarrow \mathrm{Hom}_{\mathbb{Z}_l[G_F]}(T_l(A), T_l(B)). \tag{17}$$

We have the following properties (for the first three, see for instance [Mu], §19):

(i) The homomorphism (17) is injective.

(ii) $\mathrm{Hom}_F(A,B)$ is a free finitely generated abelian group.

(iii) If S is a geometrically connected F-scheme, then $\mathrm{Hom}_S(A,B) = \mathrm{Hom}_F(A,B)$. (If $\mathrm{char}(F) \neq 2$, this can also be deduced from 2.2.3). In particular, if Ω is any extension of F^{s}, then $\mathrm{Hom}_\Omega(A,B) = \mathrm{Hom}_{F^{\mathrm{s}}}(A,B)$.

(iv) If F is finitely generated over the prime field, then (17) is an isomorphism.

Property (iv) is Tate's conjecture for homomorphisms of abelian varieties, proved by Faltings: for a proof, see [F-W], VI, §3, Theorem 1 (where it is stated for $A = B$, which implies the general case by considering products).

We shall also use (iv) in the following (seemingly) weaker form: consider the natural homomorphism

$$\mathrm{Hom}_{\overline{F}}(A,B) \otimes_{\mathbb{Z}} \mathbb{Q}_l \longrightarrow H_l(A,B) := \mathrm{Hom}_{\mathbb{Q}_l}(V_l(A), V_l(B)) \tag{18}$$

which is the map (17), taken over $\overline{F}$ and tensored with $\mathbb{Q}_l$, and is therefore injective. Faltings' theorem implies (and, in fact, is equivalent to):

(v) Assume F is finitely generated over the prime field. Then the image of (18) consists of those elements of $H_l(A,B)$ whose stabiliser in G_F is open. In particular, it is determined by the image of G_F in $\mathrm{Aut}_{\mathbb{Q}_l}(H_l(A,B))$.

2.4.4 Elliptic curves. An *elliptic curve* over a scheme S is a pair (E,ω) where E is a smooth proper S-scheme whose fibres are curves of genus 1 and ω is a section of E over S. We shall often drop ω from the notation. Recall that there is a unique (commutative) S-group scheme structure on E with unit section ω: thus, we may equivalently define an elliptic curve to be an abelian scheme of relative dimension 1.

2.5 Picard groups and schemes.

For any scheme X, the Picard group of X, denoted by $\mathrm{Pic}(X)$, is the group of isomorphism classes of invertible $\mathscr{O}_X$-modules. In good cases, this coincides with the group of Cartier (i.e. locally principal) divisors on X, modulo principal divisors: this is the case in particular if X is quasiprojective over F, by ([EGA 4], 21.3.4). If X is regular, $\mathrm{Pic}(X)$ is just the usual group of divisor classes on X.

Now let $f : X \to S$ be a morphism of schemes. We assume that f is proper and flat, and $f_*\mathscr{O}_X \cong \mathscr{O}_S$ universally. In our applications, S will be either an integral regular scheme

of dimension 1 (e.g. a nonsingular curve), or the spectrum of a field F. In the latter case, the condition on $f_*\mathscr{O}_X$ means that $\Gamma(X, \mathscr{O}_X) = F$; this holds whenever X is geometrically integral over F.

One can then define the *Picard functor* $\underline{\mathrm{Pic}}_{X/S}$ of f. As a 'first approximation', we define the *naive Picard functor* from S-schemes to abelian groups, by

$$\mathrm{Pic}^{\mathrm{naive}}_{X/S}(T) := \mathrm{Pic}\,(X \times_S T)/\mathrm{pr}_2^*\,(\mathrm{Pic}\,(T)). \tag{19}$$

Now $\underline{\mathrm{Pic}}_{X/S}$ is another contravariant functor from S-schemes to abelian groups, with the following properties (for which we refer to [B-L-R], Chapter 8):

(i) For any S-scheme T, there is an injective homomorphism, functorial in T:

$$a_{X/S}(T): \quad \mathrm{Pic}^{\mathrm{naive}}_{X/S}(T) \longrightarrow \underline{\mathrm{Pic}}_{X/S}(T). \tag{20}$$

(ii) $\underline{\mathrm{Pic}}_{X/S}$ 'commutes with any base change' $S' \to S$, in the sense that if T is an S'-scheme and $X' = X \times_S S'$ then $\underline{\mathrm{Pic}}_{X/S}\,(T) \cong \underline{\mathrm{Pic}}_{X'/S'}\,(T)$ (where, in the left-hand side, T is viewed as an S-scheme in the natural way).

(iii) If f has a section $\varepsilon : S \to X$, then the morphism $a_{X/S}$ is an isomorphism of functors on S-schemes. Moreover, in this case the functors $\mathrm{Pic}^{\mathrm{naive}}_{X/S}$ and $\underline{\mathrm{Pic}}_{X/S}$ are isomorphic to the functor

$$T \mapsto \mathrm{Pic}^{\varepsilon}_{X/S}(T) := \mathrm{Ker}\left[\mathrm{Pic}\,(X \times_S T) \xrightarrow{(\varepsilon \times \mathrm{Id}_T)^*} \mathrm{Pic}\,(T)\right]$$

of 'invertibles sheaves on X which are trivial along ε'.

(iv) If $S = \mathrm{Spec}\,(F)$, then $\underline{\mathrm{Pic}}_{X/F}$ is representable by an F-group scheme locally of finite type, whose connected component (denoted by $\underline{\mathrm{Pic}}^0_{X/F}$) is an algebraic group over F, which is smooth if $\mathrm{char}\,(F) = 0$ or $\dim X = 1$.

(v) If $S = \mathrm{Spec}\,(F)$ and X is smooth over F, then $\underline{\mathrm{Pic}}^0_{X/F}$ is proper (hence an abelian variety if $\mathrm{char}\,(F) = 0$ or $\dim X = 1$).

(vi) If $S = \mathrm{Spec}\,(F)$ and X is a semistable curve (that is, $X_{\overline{F}}$ has only ordinary double points as singularities), then $\underline{\mathrm{Pic}}^0_{X/F}$ is *semiabelian*, i.e. an extension of an abelian variety by a torus.

Observe that if $S = \mathrm{Spec}\,(F)$, and X has an F-rational point (which is automatic when F is algebraically closed), (iii) implies $\underline{\mathrm{Pic}}_{X/F}\,(F) \cong \mathrm{Pic}\,(X)$. In this case, $\underline{\mathrm{Pic}}^0_{X/F}\,(F)$ is the subgroup consisting of (classes of) invertible sheaves algebraically equivalent to zero; in particular, if X is a (not necessarily irreducible) curve, then $\underline{\mathrm{Pic}}^0_{X/F}\,(F)$ is the group of invertible sheaves having degree 0 on each component of X.

Property (iv), in the general case, is due to Murre; we shall use it only when X is a curve or a nonsingular surface. The situation is more delicate over a more general base S. But of course, we can apply (iv) to the fibres of f, which at least allows us to *define* a subfunctor $\underline{\mathrm{Pic}}^0_{X/S}$ of $\underline{\mathrm{Pic}}_{X/S}$ by

$$\underline{\mathrm{Pic}}^0_{X/S}\,(T) = \{x \in \underline{\mathrm{Pic}}_{X/S}\,(T) \,|\, \forall t \in T, x_t \in \underline{\mathrm{Pic}}^0_{X_t/\kappa(t)}(\kappa(t))\}$$

where, as usual, $\kappa(t)$ is the residue field of t and the subscript t means base change by $\operatorname{Spec}\kappa(t) \to T \to S$. So, loosely speaking, $\underline{\mathrm{Pic}}^0_{X/S}$ parametrises invertible sheaves on X which are algebraically equivalent to zero in the fibres of f.

There are deep representability results for $\underline{\mathrm{Pic}}$ and $\underline{\mathrm{Pic}}^0$; we shall only need the following special case, due to Raynaud:

(vii) In addition to our general hypotheses, assume that S is a regular integral scheme of dimension 1, that X is normal, and that each geometric fibre of f is a curve with at least one reduced irreducible component. Then $\underline{\mathrm{Pic}}^0_{X/S}$ is representable by a smooth separated S-group scheme.

The representability is a special case of Theorem 2 of [B-L-R], 9.4 (which is stated over a discrete valuation ring, but the extension to our case is standard). Smoothness is automatic for the Picard functor of a curve, as explained in [B-L-R], 8.4, Proposition 2 (essentially, the reason is that $\mathrm{H}^2(X_s, \mathscr{O}_{X_s}) = 0$ for all $s \in S$).

Finally we shall need two well-known facts about Picard groups of surfaces. The first one is the birational invariance of $\underline{\mathrm{Pic}}^0$:

(viii) Let Z be a smooth projective geometrically connected surface over F, and let $\rho : Z' \to Z$ be the blowing-up of finitely many (reduced) points. Then ρ^* induces an isomorphism $\underline{\mathrm{Pic}}^0_{Z/F} \xrightarrow{\sim} \underline{\mathrm{Pic}}^0_{Z'/F}$.

The other result we need is the structure of $\underline{\mathrm{Pic}}^0$ of a product surface:

(ix) Let X, Y be two smooth projective geometrically connected varieties over F. Then the natural morphism

$$\mathrm{pr}_1^* \oplus \mathrm{pr}_2^* : \underline{\mathrm{Pic}}^0_{X/F} \times_F \underline{\mathrm{Pic}}^0_{Y/F} \longrightarrow \underline{\mathrm{Pic}}^0_{(X\times_F Y)/F}$$

is an isomorphism.

(Note that the analogues of (viii) and (ix) where $\underline{\mathrm{Pic}}^0$ is replaced by $\underline{\mathrm{Pic}}$ are false).

2.6 Jacobians.

If X is a projective geometrically connected curve over F, we denote by $\mathrm{Jac}\,(X)$ the Jacobian of X, which is by definition $\underline{\mathrm{Pic}}^0_{X/F}$.

If X is smooth, it follows from 2.5 that $\mathrm{Jac}\,(X)$ is an abelian variety over F, and that if L is an extension of F such that $X(L) \neq \emptyset$, then $\mathrm{Jac}\,(X)(L) = \mathrm{Jac}\,(X_L)(L) = \mathrm{Pic}^0(X_L)$, the group of divisor classes of degree zero on X_L.

If X is semistable, then $\mathrm{Jac}\,(X)$ is semiabelian, by 2.5 (vi).

If (E, ω) is an elliptic curve, there is a canonical isomorphism $E \xrightarrow{\sim} \mathrm{Jac}\,(E)$ sending a point x to the divisor class $[x] - [\omega]$. We shall henceforth identify E and $\mathrm{Jac}\,(E)$ in this way.

If $f : X \to Y$ is a k-morphism of smooth projective geometrically connected curves, there is a pullback morphism $f^* : \mathrm{Jac}\,(Y) \to \mathrm{Jac}\,(X)$, corresponding to the usual pullback of divisors. If $Y = E$ is an elliptic curve, the map $f \mapsto f^*$ induces a *group homomorphism*

$$\mathrm{Mor}_F\,(X, E)/E(F) \longrightarrow \mathrm{Hom}_F\,(E, \mathrm{Jac}\,(X)) \tag{21}$$

where of course we identify $E(F)$ with the group of constant morphisms from X to E (or, alternatively, with the group of translations on E acting on $\mathrm{Mor}_F\,(X, E)$). This map is always injective (to see this, use the fact that a morphism $f : X \to E$ also induces $f_* : \mathrm{Jac}\,(X) \to E$ satisfying $f_* \circ f^* = (\deg f)\,\mathrm{Id}_E$). Furthermore, if X has a rational point, then (21) is bijective: this can be seen using self-duality of the Jacobian and the embedding of X into $J(X)$ attached to a rational point of X (if X has genus ≥ 1; otherwise both sides of (21) are zero).

With X and E as above, assume now that X is endowed with an involution σ (and E with the involution $[-1]$). Then (21) induces an injective homomorphism

$$\mathrm{Mor}_F^{\mathrm{odd}}(X, E)/E[2](F) \longrightarrow \mathrm{Hom}_F^{\mathrm{odd}}(E, \mathrm{Jac}\,(X)) \tag{22}$$

where, in the right-hand side, E (resp. $\mathrm{Jac}\,(X)$) is given the involution $[-1]$ (resp. σ^*).

2.6.1 Proposition. *If X has an F-rational point fixed by σ, then (22) is an isomorphism.*

Proof: Let $P \in X(F)$ be such a point, and let $v : E \to \mathrm{Jac}\,(X)$ be an odd morphism. Since (21) is bijective, there is a morphism $u : X \to E$ such that $u^* = v$. Changing u by a translation on E we may assume $u(P) = 0$ (the origin of E). On the other hand, the fact that u^* is odd means that $(u + u \circ \sigma)^* = 0$, hence $u + u \circ \sigma : X \to E$ is constant. By our assumptions it sends P to 0, hence $u + u \circ \sigma = 0$ and u is odd. ∎

2.6.2 Remark. It follows in particular that $\mathrm{Mor}_F^{\mathrm{odd}}(X, E)$ is a finitely generated abelian group, with torsion subgroup $E[2](F)$ and rank $\leq 4\,\mathrm{genus}(X)$.

2.7 Affine Diophantine sets.

Throughout this section, we denote by R a ring and by $\mathscr{O}$ an R-algebra.

We denote by $\mathrm{LR} = \{+, -, ., 0, 1\}$ the language of rings, and by $\mathrm{LR}(R)$ the language LR augmented by the set of constants R.

2.7.1 The case of affine n-space. Let n be a natural integer. A subset X of $\mathscr{O}^n$ will be called *primitive Diophantine* (with respect to R) if there is an integer q and a finite sequence of polynomials $F_1, \ldots, F_r \in R[T_1, \ldots, T_n, Y_1, \ldots, Y_q]$ such that, for $\underline{t} = (t_1, \ldots, t_n) \in \mathscr{O}^n$, we have the equivalence

$$\underline{t} \in X \;\Leftrightarrow\; \exists\, \underline{y} = (y_1, \ldots, y_q) \in \mathscr{O}^q \text{ such that } F_1(\underline{t}, \underline{y}) = \cdots = F_r(\underline{t}, \underline{y}) = 0. \tag{23}$$

With these notations, consider the R-scheme $W := \mathrm{Spec}\,(R[\underline{T}, \underline{Y}]/(F_1, \ldots, F_r)) \hookrightarrow \mathbb{A}_R^{n+q}$, and the R-morphism $\varphi : W \to \mathbb{A}_R^n$ deduced from the projection $(\underline{t}, \underline{y}) \mapsto \underline{t}$: then X is

simply the image of the map $\varphi(\mathscr{O}) : W(\mathscr{O}) \to \mathbb{A}^n_R(\mathscr{O}) = \mathscr{O}^n$ given by φ. Conversely, it is easy to see that if W is any affine R-scheme of finite presentation (that is, of the form $\mathrm{Spec}\,(A)$ where A is a finitely presented R-algebra), and $\varphi : W \to \mathbb{A}^n_R$ is an R-morphism, the image of $\varphi(\mathscr{O}) : W(\mathscr{O}) \to \mathscr{O}^n$ is primitive Diophantine.

A *Diophantine subset* of $\mathscr{O}^n$ is by definition a finite union of primitive Diophantine subsets. It is well known, and easy to see, that these subsets are precisely the *positive-existentially definable* subsets of $\mathscr{O}^n$, in the language $\mathrm{LR}(R)$.

The above remarks justify the following definition:

2.7.2 Definition. *Let V be an affine R-scheme of finite presentation, and let $\mathscr{O}$ be an R-algebra.*

A subset X of $V(\mathscr{O})$ is called primitive Diophantine *(with respect to R) if there is an affine R-scheme W of finite presentation, and an R-morphism $\varphi : W \to V$, such that X is the image of $\varphi(\mathscr{O}) : W(\mathscr{O}) \to V(\mathscr{O})$.*

A Diophantine *subset of $V(\mathscr{O})$ is a finite union of primitive Diophantine subsets.*

The class of Diophantine subsets is closed under finite intersections and unions, images and inverse images by R-morphisms, and various other operations (such as fibre products); these properties are easy to check.

In particular, if $V \subset V'$ is an immersion of affine R-schemes of finite presentation, then $X \subset V(\mathscr{O})$ is Diophantine if and only if it is Diophantine as a subset of $V'(\mathscr{O})$. The most important case is, of course, when $V' = \mathbb{A}^n_R$: thus, in this case, the Diophantine subsets of $V(\mathscr{O})$ are those which are positive-existentially definable in $\mathscr{O}^n$.

2.7.3 Remark. It is not true in general that Diophantine sets are primitive Diophantine; however, this does hold if $\mathrm{Spec}\,(\mathscr{O})$ is connected (or, equivalently, if $\mathscr{O}$ has no idempotent element other than 0 and 1). In particular, this is true if $\mathscr{O}$ is a domain, or a local ring.

2.7.4 Remark. It is of course natural to ask whether Definition 2.7.2 generalises to schemes V which are not necessarily affine.

First, one should probably keep the 'finite presentation' restriction on V: loosely speaking, R-schemes of finite presentation are those which can be defined by a finite set of data from R.

Next, of course the extended notion should specialise to the previous one for affine V; hence, it also seems reasonable to 'use only affine W's' in the definition.

So, our definition of a primitive Diophantine subset of $V(\mathscr{O})$ (for an R-scheme V of finite presentation) would be a subset which is the image of the map $W(\mathscr{O}) \to V(\mathscr{O})$ induced by an R-morphism $W \to V$, where W is some *affine* R-scheme of finite presentation. Of course, a Diophantine subset is a finite union of primitive Diophantine subsets.

The above definition differs from Mazur's ([Ma], Definition 1), who defines a Diophantine subset of $V(\mathscr{O})$ as one which is the image of $W(\mathscr{O})$ for some R-morphism $W \to V$ of R-schemes of finite type.

In any case, we have refrained from including the basic properties of these generalised Diophantine sets in this paper. Of course, in a sense, this would have been the natural framework when working with elliptic curves; but as it turns out, elliptic curves contain very nice affine open subsets which are sufficient for our needs.

2.7.5 Diophantine relations and maps. If V, V' are affine R-schemes of finite presentation, and X, X' are Diophantine subsets of $V(\mathscr{O})$ and $V'(\mathscr{O})$ respectively, a binary relation $Z \subset X \times X'$ is said to be *Diophantine* if it is Diophantine as a subset of $(V \times V')(\mathscr{O})$. (Here products are in the category of of R-schemes, i.e. fibered over Spec (R)).

In particular, a *map* $f : X \to X'$ is Diophantine if its graph is Diophantine in $(V \times V')(\mathscr{O})$. Compositions of Diophantine maps are Diophantine, and images (resp. inverse images) of Diophantine sets by Diophantine maps are again Diophantine sets.

2.7.6 Relative Diophantine sets. If V is an affine R-scheme of finite presentation, and $X \subset Y$ are subsets of $V(\mathscr{O})$, we say that X is *relatively Diophantine* in Y if it is of the form $D \cap Y$, where $D \subset V(\mathscr{O})$ is Diophantine. Of course, if Y is Diophantine, this is equivalent to X being Diophantine in $V(\mathscr{O})$.

This notion will be convenient in the following situation: if Z is a subset of $V(\mathscr{O})$, an n-ary relation on Z will be called relatively Diophantine if it is a relatively Diophantine subset of Z^n (wiewed as a subset of $V^n(\mathscr{O})$). In particular, we can speak of a relatively Diophantine group (or ring) structure on Z, even if Z is not known to be Diophantine.

2.7.7 Diophantine structures. Let V be an R-scheme of finite presentation, and $X \subset V(\mathscr{O})$ a Diophantine set. If $\mathscr{L}$ is a first order language, a *Diophantine $\mathscr{L}$-structure* relative to R and $\mathscr{O}$, or $(R, \mathscr{O})$-Diophantine $\mathscr{L}$-structure, with underlying set X is an $\mathscr{L}$-structure on X such that all subsets of the various product sets X^r, and all maps $X^r \to X$, relevant to the structure are Diophantine.

As an example, take $\mathscr{L} = \mathrm{LR}$, the language of rings. Then an $(R, \mathscr{O})$-Diophantine LR-structure consists of:

(i) a Diophantine set X (w.r.t. R, in some $V(\mathscr{O})$);

(ii) three Diophantine maps $+, -, . : X \times X \to X$;

(iii) two elements 0_X and 1_X of X, which are Diophantine (i.e. $\{0_X\}$ and $\{1_X\}$ are Diophantine in $V(\mathscr{O})$).

Of course, an $(R, \mathscr{O})$-Diophantine *ring* is an $(R, \mathscr{O})$-Diophantine LR-structure which satisfies the axioms of rings (commutative with unit), in the obvious sense. Equivalently, it is a Diophantine set X with a ring structure such that addition and multiplication are Diophantine maps, and the unit is a Diophantine element (the other conditions easily follow from these using the ring axioms).

2.7.8 Proposition. *Let V be an affine R-scheme of finite presentation. Let $\mathscr{L}$ be any first order language, and $X \subset V(\mathscr{O})$ an $(R, \mathscr{O})$-Diophantine $\mathscr{L}$-structure.*

(i) *Let r a nonnegative integer, and $Z \subset X^r$ a subset which is positive-existentially definable in $\mathscr{L}$. Then Z is Diophantine as a subset of $V^r(\mathscr{O})$.*

(ii) *Assume that the positive-existential theory of $\mathscr{O}$ in $\mathrm{LR}(R)$ is decidable. Then the positive-existential theory of X in $\mathscr{L}$ is decidable.*

Proof: (i) By definition, there is an integer m and a quantifier-free and negation-free formula ϕ in $\mathscr{L}$, in $r+m$ variables, such that

$$Z = \{\ (x_1, \ldots, x_r) \in X^r \mid \exists (y_1, \ldots, y_m) \in X^m \text{ such that } \phi(\underline{x}, \underline{y}) \text{ holds } \}.$$

This is the image, by the projection $V^m \to V^r$ to the first r factors, of the set $Z' \subset X^{r+m}$ defined by ϕ. It suffices to prove that Z' is Diophantine, which is done by an easy induction on the length of ϕ, using elementary properties of Diophantine sets and maps.

Now (ii) is an easy consequence of (i). The assumption means that there is a procedure P to decide, for any given affine R-scheme W of finite presentation, whether the set $W(\mathscr{O})$ is empty or not (indeed, $W(\mathscr{O})$ may be described as the set of solutions in some $\mathscr{O}^N$ of a finite system of polynomial equations with coefficients in R). We now need to find another procedure which does the same for subsets of X^r which are $\mathscr{L}$-positive-existentially definable. But by (i) such a set Z is Diophantine in $V^r(\mathscr{O})$, hence there are affine R-schemes $W_1, \ldots, W_s$ of finite presentation and morphisms $\varphi_i : W_i \to V^r$ such that $Z = \bigcup_{i=1}^s \varphi_i(W_i(\mathscr{O}))$. So Z is empty if and only if each $W_i(\mathscr{O})$ is empty, which can be detected by applying P. ∎

In particular, from (ii) and Matijasevich's theorem, we get:

2.7.9 Corollary. *Let R be a ring and $\mathscr{O}$ an R-algebra. Assume that there exists an $(R, \mathscr{O})$-Diophantine ring $\Lambda \subset V(\mathscr{O})$, for some R-scheme V of finite presentation, such that Λ is isomorphic to $\mathbb{Z}$ as a ring.*

Then the positive-existential theory of $\mathscr{O}$ in $\mathrm{LR}(R)$ is undecidable. ∎

3 The specialisation theorem

3.1 The specialisation map.

Let R be a discrete valuation ring with fraction field F, and put $S = \mathrm{Spec}\,(R)$. Denote by k the residue field of R. Choose an algebraic closure $\overline{F}$ of F, and a prime $l \neq \mathrm{char}(k)$. Fix a valuation $\overline{v}$ of $\overline{F}$ extending the valuation v defined by R. The residue field of $\overline{v}$ is an algebraic closure of k, which we denote by $\overline{k}$. The corresponding decomposition group and inertia group are denoted by $D = D_{\overline{v}}$ and $I = I_{\overline{v}}$ respectively.

If A is an abelian scheme over S, then I acts *trivially* on $T_l(A_F)$ (because $A[l^n]$ is finite étale over S, for all n), and we have an isomorphism

$$T_l(A_F) \xrightarrow{\sim} T_l(A_k) \tag{24}$$

(which depends on the choice of $\overline{v}$: observe that $\overline{k}$ is implicit in $T_l(A_k)$).

On the other hand, if B is another abelian S-scheme, every F-homomorphism $A_F \to B_F$ extends (uniquely, of course) to an S-homomorphism $A \to B$ ([B-L-R], 1.2, Proposition 8). So we get a 'specialisation' homomorphism

$$\mathrm{Hom}_F\,(A_F, B_F) \longrightarrow \mathrm{Hom}_k\,(A_k, B_k) \tag{25}$$

which is *injective*: this can be deduced from the isomorphism on Tate modules defined above, or from 2.2.2, or from the 'rigidity lemma' of [Mu-F], Proposition 6.1. Of course, this also applies over finite extensions of F, so that we have an injective homomorphism of free finitely generated $\mathbb{Z}$-modules

$$\mathrm{Hom}_{\overline{F}}\,(A_{\overline{F}}, B_{\overline{F}}) \longrightarrow \mathrm{Hom}_{\overline{k}}\,(A_{\overline{k}}, B_{\overline{k}}) \tag{26}$$

which is compatible with the action of $D_{\overline{v}}$. One recovers (25) from (26) by taking Galois invariants on the left, and $D_{\overline{v}}$-invariants on the right (note that morphisms of abelian varieties are always defined over finite separable extensions of the ground field, hence we are safe from inseparability problems). Changing the choice of $\overline{v}$ does not change the cokernel of (26), up to an isomorphism of abelian groups. Moreover:

3.1.1 Proposition. *The cokernel of the specialisation map* (25) *has no torsion prime to the characteristic of k. Consequently, the same holds for* (26).

Proof: Assume we have homomorphisms $u : A \to B$ and $v_k : A_k \to B_k$ such that $n\,v_k = u_k$, where n is prime to $\mathrm{char}\,(k)$. We need to show that there is a $v : A \to B$ such that $nv = u$ (this v will automatically lift v_k). Now, u induces a morphism $u[n] : A[n] \to B[n]$ of finite étale group schemes. Its kernel must then be open and closed in $A[n]$, but the assumption implies that it contains $A_k[n]$, so it is equal to $A[n]$. This means that u factors through $[n]_A$, as desired. ∎

3.1.2 Remark. If R' is a discrete valuation ring dominating R, with fraction field F' and residue field k', we obtain a specialisation map from $\mathrm{Hom}_{\overline{F'}}\,(A_{\overline{F'}}, B_{\overline{F'}})$ to $\mathrm{Hom}_{\overline{k'}}\,(A_{\overline{k'}}, B_{\overline{k'}})$,

with obvious notations. Due to 2.4.3 (iii), this map is isomorphic to (26), in the obvious sense; in particular, they have isomorphic cokernels.

These constructions also apply when $R = F$ and R' is a discrete valuation ring containing F: of course, (25) and (26) are then just identity maps, and therefore the specialisation map 'over R'' is an isomorphism. Believe it or not, this trivial remark will be used below.

3.2 The specialisation theorem: notations.

Let k be a field of characteristic $p \geq 0$. Let $U \subset \mathbb{P}^1_k$ be a nonempty open set, and let $A \to U$ and $B \to U$ be two abelian schemes over U. Let z denote the standard coordinate on $\mathbb{P}^1_k$, and put $F = k(z)$ (the function field of U). For $x \in U(k)$, we have, as special cases of (25) and (26), specialisation maps

$$\mathrm{sp}_x : \mathrm{Hom}_F(A_F, B_F) \longrightarrow \mathrm{Hom}_x(A_x, B_x) \tag{27}$$

$$\overline{\mathrm{sp}}_x : \mathrm{Hom}_{\overline{F}}(A_{\overline{F}}, B_{\overline{F}}) \longrightarrow \mathrm{Hom}_{\overline{x}}(A_{\overline{x}}, B_{\overline{x}}) \tag{28}$$

where (28) actually depends on some choices (in particular, $\overline{x}$ is a geometric point above x). Of course, if k' is any extension of k, we can do the same for points $x \in U(k')$, using the abelian schemes $A_{U'}$ and $B_{U'}$ over $U' := U \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')$. Now define two 'regular sets' in $U(k')$:

$$\begin{aligned} \mathrm{Reg}(A, B, k') = \mathrm{Reg}(k') &:= \{x \in U(k') \mid \mathrm{Coker}(\mathrm{sp}_x) \text{ is finite}\} \\ &= \{x \in U(k') \mid \mathrm{sp}_x \text{ is almost bijective (1.5.1)}\} \\ &= \{x \in U(k') \mid \mathrm{rk}\,\mathrm{Hom}_F(A_F, B_F) = \mathrm{rk}\,\mathrm{Hom}_x(A_x, B_x)\} \end{aligned} \tag{29}$$

(for the second equality we use 3.1.1). And we have the 'geometric' version of Reg:

$$\textsc{Reg}(A, B, k') = \textsc{Reg}(k') := \{x \in U(k') \mid \mathrm{Coker}(\overline{\mathrm{sp}}_x) \text{ is finite}\} \tag{30}$$

with similar equivalent formulations.

3.2.1 Remark. Observe that the definition of Reg involves the algebraic closure of $k(z)$, which is a much bigger field than $\overline{k}(z)$.

Accordingly, if k is algebraically closed, it is easy to see that $\mathrm{Reg}(k) \subset \textsc{Reg}(k)$ (the point is that, with the above notations, we have $x = \overline{x}$) but in general the inclusion is strict. For instance, let E be a k-elliptic curve without complex multiplication, and take for A the constant abelian scheme $E \times \mathbb{P}^1_k \to \mathbb{P}^1_k$. Then take for B the quadratic twist of A by the double cover of $\mathbb{P}^1_k$ given by $k(\sqrt{z})$ (this extends to an abelian scheme over $U = \mathbb{G}_{m,k}$): it is easy to see that $\textsc{Reg}(k) = U(k)$ while $\mathrm{Reg}(k) = \emptyset$.

3.2.2 Remark. For general k, it is not true that $\mathrm{Reg}(k) \subset \textsc{Reg}(k)$. As an example, take $A = E \times \mathbb{P}^1_k \to \mathbb{P}^1_k$ as before, and assume there is some k-elliptic curve E' which is $\overline{k}$-isomorphic to E but has no nontrivial k-morphism to E. Now take $B \to U$ such that $0 \in U(k)$ and the fibre B_0 is k-isomorphic to E', while the j-invariant of B in $k(z)$ is not constant. Then 0 is in $\mathrm{Reg}(k)$ but not in $\textsc{Reg}(k)$.

3.3 Theorem. (i) *If $k' \subset k''$ are extensions of k, then* $\mathrm{REG}\,(k') = \mathrm{REG}\,(k'') \cap k'$.

(ii) *If $x \in k'$ is transcendental over k, then $x \in \mathrm{Reg}\,(k') \cap \mathrm{REG}\,(k')$.*

(iii) (R. Noot [N]) *If k is finitely generated over the prime field, then* $\mathrm{Reg}\,(k) \cap \mathrm{REG}\,(k)$ *contains a Hilbert set in $\mathbb{P}^1(k)$.*

Proof: parts (i) (which we shall not use) and (ii) follow from Remark 3.1.2: for (ii), observe that if x is transcendental over k, then its local ring in $U_{k'}$ contains F.

Let us sketch the proof of (iii) (the reader can find details in [N], §1, where the context is slightly different: the base is not necessarily an open subset of $\mathbb{P}^1$, but one looks at the specialisation map for closed points, not just rational points).

Choose a prime $l \neq p$, and put $H_l(A_F, B_F) := \mathrm{Hom}_{\mathbb{Q}_l}\,(V_l(A_F), V_l(B_F))$, just as in (18) of 2.4.3. For $x \in U(k)$, we have a similarly defined $\mathbb{Q}_l$-vector space $H_l(A_x, B_x)$, which is isomorphic to $H_l(A_F, B_F)$ via the isomorphism (24); the G_k-action on $H_l(A_x, B_x)$ is compatible with the G_F-action on $H_l(A_F, B_F)$, via the natural group homomorphisms $G_k \twoheadleftarrow D_{\overline{x}} \hookrightarrow G_F$. Let us denote both these $\mathbb{Q}_l$-spaces by H_l.

Now if k (hence also F) is finitely generated over the prime field, we know by Faltings' theorem (2.4.3 (v)) that the rank of $\mathrm{Hom}_{\overline{F}}\,(A_F, B_F)$ (resp. of $\mathrm{Hom}_{\overline{k}}\,(A_x, B_x)$) is determined by the image of G_F (resp. G_k) in $\mathrm{Aut}_{\mathbb{Q}_l}(H_l)$.

In particular, we shall have $x \in \mathrm{REG}\,(k)$ whenever $D_{\overline{x}}$ has the same image in $\mathrm{Aut}_{\mathbb{Q}_l}(H_l)$ as G_F. Moreover, for such an x the groups $D_{\overline{x}}$ and G_F have the same invariants in H_l, hence we can also conclude that $x \in \mathrm{Reg}\,(k)$.

But since the image of G_F is an l-adic Lie group (as a closed subgroup of $\mathrm{Aut}_{\mathbb{Q}_l}(H_l)$), we conclude from a result of Serre ([Se1], or [Se2], 10.6) that for x in a suitable Hilbert set the images of G_F and $D_{\overline{x}}$ are equal. ∎

Let us now prove an effective version of (a weaker form of) (iii):

3.4 Theorem. *With the notations and assumptions of 3.2, assume that k is finitely generated over the prime field, and assume that the rank r of* $\mathrm{Hom}_F\,(A_F, B_F)$ *is known. Then there is an algorithm to construct a polynomial $P \in k[z, u]$, with the following properties:*

(i) *as an element of $k(z)[u]$, P is separable with no roots in $k(z)$;*

(ii) *for any $\lambda \in U(k)$, we have the property: 'if $P(\lambda, u) \in k[u]$ has no root in k, then $\lambda \in \mathrm{Reg}\,(k)$'.*

Proof: Put $d_A = \dim\,(A/U)$, $d_B = \dim\,(B/U)$, and $U = \mathrm{Spec}\ R$, with $R = k[z, D(z)^{-1}]$ for some $D \in k[z]$.

Fix a prime $l \neq \mathrm{char}\,(k)$, and consider, for $n \in \mathbb{N}$, the U-schemes $A[l^n]$ and $B[l^n]$. These are finite étale U-schemes in $\mathbb{Z}/l^n\mathbb{Z}$-modules, locally free of ranks $2\,d_A$ and $2\,d_B$, respectively (for the étale topology on U). It follows that

$$H_n := \underline{\mathrm{Hom}}_{U\text{-group schemes}}\,(A[l^n], B[l^n]) \tag{31}$$

is a similar group scheme, with $\mathbb{Z}/l^n\mathbb{Z}$-rank $4\,d_A\,d_B$.

Note that, for given n, equations for $A[l^n]$ and $B[l^n]$ can be computed from equations of A and B. Thus, one can describe $A[l^n]$ (resp. $B[l^n]$) as the spectrum of a locally free (hence, in fact, free) R-bialgebra $\Lambda_{n,A}$ (resp. $\Lambda_{n,B}$) of rank $l^{2\,n\,d_A}$ (resp. $l^{2\,n\,d_B}$). The scheme H_n is simply the U-scheme of bialgebra morphisms $\Lambda_{n,B} \to \Lambda_{n,A}$, which in turn can be described explicitly from the equations.

Faltings' theorem (2.4.3 (v)) over F can be stated as

$$\mathrm{Hom}_F\,(A_F, B_F) \otimes \mathbb{Z}_l \xrightarrow{\sim} \varprojlim_{n\geq 1} H_n(F).$$

Since $(H_n(F))_{n\geq 1}$ is an inverse system of finite groups, it satisfies the Mittag-Leffler condition. It follows that, for n large enough (say, $n \geq n_0$), the image of $H_n(F)$ in $H_1(F)$ is equal to the image of the projective limit, and hence, by Faltings' theorem, to the image of $\mathrm{Hom}_F\,(A_F, B_F)$. The latter is clearly isomorphic to $\mathrm{Hom}_F\,(A_F, B_F) \otimes_{\mathbb{Z}} \mathbb{Z}/l\mathbb{Z}$, which is a $\mathbb{Z}/l\mathbb{Z}$-vector space of dimension r.

Moreover, since the image of the natural map $\rho_{n,F} : H_n(F) \to H_1(F)$ obviously decreases as n grows, we now see that n_0 is computable: just compute the image of $\rho_{n,F}$ for increasing n, until it has dimension r (the knowledge of r is clearly essential here!).

Consider now H_{n_0}. As a finite étale U-scheme, it decomposes canonically as a disjoint sum of a trivial covering $H_{n_0}^{\mathrm{triv}}$ and a finite étale U-scheme H'_{n_0} with no section over U (or, equivalently, over $\mathrm{Spec}\,(F)$): thus, $H_{n_0}^{\mathrm{triv}}(F) = H_{n_0}(F)$, and in fact $H_{n_0}^{\mathrm{triv}}$ can be identified with $H_{n_0}(F) \times U$. Consider the natural U-morphism

$$\rho_{n_0} : H_{n_0} \to H_1.$$

This is a morphism of étale covers which, by construction, sends $H_{n_0}^{\mathrm{triv}}$ to the trivial subcovering of H_1 whose generic fibre is the image of $H_{n_0}(F)$ in $H_1(F)$. So this subcovering has degree l^r.

Let us now take a look at H'_{n_0}: this is a computable, open and closed subscheme of H_{n_0}. There is a dense open subscheme U_1 of U (of the form $U_1 = \mathrm{Spec}\ R_1$, with $R_1 = k[z, (D\,D_1)^{-1}]$ for some computable $D_1 \in k[z]$) such that $H'_{n_0} \times_U U_1$ is isomorphic to $\mathrm{Spec}\,R_1[u]/(P_1)$ for some $P_1 \in k[z, u]$.

Now put $P = D_1 P_1$. Clearly P satisfies (i), since H'_{n_0} is étale over U and $H'_{n_0}(F) = \emptyset$. Let $\lambda \in U(k)$ be such that $P(\lambda, u)$ has no root in k. Then of course $D_1(\lambda) \neq 0$, hence $\lambda \in U_1(k)$. Hence $H'_{n_0}(\lambda)$ is the set of roots of $P_1(\lambda, u)$ in k, which is empty by assumption. This means that $H_{n_0}(\lambda) = H_{n_0}^{\mathrm{triv}}(\lambda)$. Hence the image of $H_{n_0}(\lambda)$ in $H_1(\lambda)$ has cardinality l^r. But this image contains $\mathrm{Hom}_k(A_\lambda, B_\lambda) \otimes_{\mathbb{Z}} (\mathbb{Z}/l\mathbb{Z})$, hence the $\mathbb{Z}$-rank of $\mathrm{Hom}_k\,(A_\lambda, B_\lambda)$ is at most r, hence equal to r. ∎

3.4.1 Remark. We do not give here a precise definition of 'effective'. Observe, however, that the above procedure only uses the standard constructions of effective algebraic geometry, as explained for instance in [F-J], Chapter 17. In particular they can be carried out effectively if k is 'presented' over the prime field ([F-J], Section 17.2) and A_F, B_F and their group laws are given by explicit equations.

4 The relative Jacobian of a fibered surface

4.1 Notations.

In this section, we consider the following situation: k is a field (with algebraic closure $\overline{k}$ and separable closure k^{s}, as usual), S denotes the k-projective line $\mathbb{P}^1_k$, with standard coordinate z and generic point η (thus, $\eta = \operatorname{Spec} k(z)$). We denote by

$$\theta : X \longrightarrow S = \mathbb{P}^1_k \tag{32}$$

a k-morphism with the following properties:

(i) X is a projective, smooth, geometrically connected surface over k;

(ii) θ is surjective with geometrically connected fibres and smooth generic fibre;

(iii) every geometric fibre of θ has at least one reduced component;

(iv) $\theta_{k^{\mathrm{s}}} : X_{k^{\mathrm{s}}} \to \mathbb{P}^1_{k^{\mathrm{s}}}$ has a section.

(In fact, condition (iii) is easily seen to be a consequence of (i) and (iv)). Observe that all these properties are invariant under ground field extension. Also, (ii) implies that $\mathscr{O}_S \xrightarrow{\sim} \theta_* \mathscr{O}_X$ universally. We put

$$\Pi := \underline{\mathrm{Pic}}^0_{X/k} \quad \text{and} \quad J := \underline{\mathrm{Pic}}^0_{X/S}. \tag{33}$$

Thus, Π is an abelian variety over k, and $J \to S$ is a smooth separated S-group scheme with connected fibres, by 2.5 (vii).

For each extension k' of k and each point $\lambda \in S(k')$ we denote by X_λ the fibre of θ at λ: this is a projective connected curve over k', which is smooth except for finitely many λ. The fibre J_λ of J at λ is the Jacobian $\mathrm{Jac}\,(X_\lambda)$ of X_λ.

We have a canonical morphism of S-group schemes

$$\xi : \Pi_S := S \times_k \Pi \longrightarrow J \tag{34}$$

such that, for k' and λ as above, $\xi_\lambda : \Pi_{k'} \to J_\lambda$ is the 'restriction to X_λ' morphism from $\underline{\mathrm{Pic}}^0_{X_{k'}/k'}$ to $\underline{\mathrm{Pic}}^0_{X_\lambda/k'}$.

4.2 Proposition. ('connected Néron mapping property') *For every smooth group scheme G over S, with* connected fibres, *and every nonempty open subscheme U of S, the obvious restriction homomorphism*

$$\mathrm{Hom}_U\,(G_U, J_U) \longrightarrow \mathrm{Hom}_\eta\,(G_\eta, J_\eta)$$

is an isomorphism.

Proof: This follows from [B-L-R], 9.5, Theorem 4 (b) which says that J is the connected component of the Néron model of J_η. ∎

4.3 Proposition. ('fixed part property') *The morphism ξ of (34) is universal for morphisms from constant abelian S-schemes to J.*

In precise terms, if A is an abelian variety over k, every S-morphism $u : A_S \to J$ of S-group schemes equals $\xi \circ (\mathrm{Id}_S \times v)$, for a uniquely defined k-morphism $v : A \to \Pi$ of abelian varieties.

Proof: By descent it suffices to prove the corresponding universal property for the morphism $\xi_{k^s} : \Pi_{\mathbb{P}^1_{k^s}} \to J_{k^s}$. In other words, we may assume that k is separably closed.

Now let A be an abelian variety over k. We have to prove that the natural map

$$\mathrm{Hom}_S\,(A_S, \Pi_S) \longrightarrow \mathrm{Hom}_S\,(A_S, J) \tag{35}$$

obtained by composition with ξ is an isomorphism.

To prove injectivity, let us first remark that by 2.4.3 (iii), every S-morphism $A_S \to \Pi_S$ is constant (in other words, we have $\mathrm{Hom}_k\,(A, \Pi) \xrightarrow{\sim} \mathrm{Hom}_S\,(A_S, \Pi_S)$). Hence if such a morphism is nonzero, it comes from a k-morphism $u : A \to \Pi$ such that $y := u(x) \neq 0$ for some $x \in A(k)$. It is enough to prove that the image of the constant section y_S by ξ is a nonzero section of J over S. But since k is separably closed, $X(k)$ is nonempty, so we can view y as a (nontrivial) invertible sheaf L_y on X, algebraically equivalent to zero. By inspecting the definition of ξ, one then checks that $\xi(y_S)$ is just the image in $J(S)$ of the class of L_y in $\mathrm{Pic}^{\mathrm{naive}}_{X/S}(S) = \mathrm{Pic}(X)/\theta^*\mathrm{Pic}(S)$, via the morphism (20). Recall that the latter is injective: hence, if $\xi(y_S)$ were zero, then L_y would be the pullback of an invertible sheaf on S, necessarily of degree zero (because L_y is algebraically equivalent to zero), hence trivial (because $S = \mathbb{P}^1$), a contradiction.

(Remark: until this last argument, S could have been any projective smooth curve over k, not just $\mathbb{P}^1$. To generalise the above to such an S, just redefine Π to be the cokernel of $\theta^* : \mathrm{Jac}\,(S) \to \underline{\mathrm{Pic}}^0_{X/k}$).

For surjectivity, we shall use the assumption (iv) of 4.1, which means (since k is separably closed) that θ has a section $\varepsilon : S \to X$. Hence by 2.5 (iii), we can view a morphism $u : A_S \to J$ as an invertible sheaf on $(A \times_k S) \times_S X$ satisfying certain conditions (in particular, the condition of being trivial on $(\{0_A\} \times_k S) \times_S X$ since u is a group morphism). But since $A_S \times_S X = A \times_k X$, we end up with an invertible sheaf on $A \times_k X$, trivial on $\{0_A\} \times_k X$ and hence (since A is connected) algebraically equivalent to 0 in every fibre of the projection $A \times_k X \to A$. In other words, we have found a k-morphism $v : A \to \underline{\mathrm{Pic}}^0_{X/k} = \Pi$, which sends 0 to 0, hence is a morphism of abelian varieties; checking that $\xi \circ v = u$ is then routine. ∎

4.4 The specialisation map.

We now fix a nonempty open subscheme $U \subset S$ over which θ is smooth, and an abelian variety A over k. We denote by A_U the constant U-abelian scheme $A \times_k U$, and by J_U the restriction of J above U (which is also an abelian scheme since X_U is a smooth proper U-curve).

For each $\lambda \in U(k)$, we have an inclusion $j_\lambda : X_\lambda \hookrightarrow X$, whence a k-morphism $j_\lambda^* : \Pi \to J_\lambda$ of abelian varieties, and a group homomorphism

$$\begin{array}{rccc} H(\lambda): & \mathrm{Hom}_k\,(A,\Pi) & \longrightarrow & \mathrm{Hom}_k\,(A,J_\lambda) \\ & u & \longmapsto & j_\lambda^* \circ u. \end{array} \tag{36}$$

For later use, we also have, for any extension k' of k, a homomorphism

$$H(\lambda,k'): \quad \mathrm{Hom}_{k'}\,(A,\Pi) \longrightarrow \mathrm{Hom}_{k'}\,(A,J_\lambda) \tag{37}$$

obtained by base field extension (thus, $H(\lambda) = H(\lambda,k)$).

On the other hand, we have the specialisation map

$$\mathrm{sp}_\lambda : \mathrm{Hom}_\eta(A_\eta, J_\eta) \to \mathrm{Hom}_k(A, J_\lambda) \tag{38}$$

defined in (28) (here, $\eta = \mathrm{Spec}\,(k(z))$ is the common generic point of S and U). We shall now connect the maps (36) and (38):

4.4.1 Theorem. *There is a natural group isomorphism*

$$\nu : \mathrm{Hom}_k\,(A,\Pi) \xrightarrow{\sim} \mathrm{Hom}_\eta(A_\eta, J_\eta)$$

with the property that, for every $\lambda \in U(k)$, we have $\mathrm{sp}_\lambda \circ \nu = H(\lambda)$.

In particular, for each such λ, we have the equivalence:

$$\lambda \in \mathrm{Reg}\,(A_U, J_U, k) \quad \Longleftrightarrow \quad H(\lambda) \text{ is almost bijective.}$$

Proof: Recall from 4.1 that $j_\lambda^* : \Pi \to J_\lambda$ is the fibre at λ of the morphism $\xi_U : \Pi_S := S \times_F \Pi \longrightarrow J$ of (34). This allows us to insert both $H(\lambda)$ and sp_λ in a commutative diagram of groups:

$$\begin{array}{ccc} \mathrm{Hom}_k\,(A,\Pi) & \xrightarrow{H(\lambda)\ (\text{see } (36))} & \mathrm{Hom}_k\,(A,J_\lambda) \\ \downarrow{\scriptstyle \alpha} & & \\ \mathrm{Hom}_S\,(A_S,\Pi_S) & & \| \\ \downarrow{\scriptstyle \beta\ (\text{composition with } \xi)} & & \\ \mathrm{Hom}_S\,(A_S,J) & \xrightarrow{\text{restriction to fibre at } \lambda} & \mathrm{Hom}_k\,(A,J_\lambda) \\ \downarrow{\scriptstyle \gamma\ (\text{restriction to } U)} & & \| \\ \mathrm{Hom}_{\mathbb{P}^1}\,(A_U,J_U) & \xrightarrow{\text{restriction to fibre at } \lambda} & \mathrm{Hom}_k\,(A,J_\lambda) \\ \downarrow{\scriptstyle \delta\ (\text{restriction to } \eta)} & & \| \\ \mathrm{Hom}_\eta\,(A_\eta,J_\eta) & \xrightarrow{\mathrm{sp}_\lambda\ (\text{see } (38))} & \mathrm{Hom}_k\,(A,J_\lambda). \end{array} \tag{39}$$

The commutativity of the diagram is immediate from the definitions of ξ and $H(\lambda)$ (for the top square), and from the definition of the specialisation map 'sp' (for the bottom square). Let us now prove that the four left vertical arrows are isomorphisms.

For α, this is clear: a morphism of constant abelian schemes over a connected k-scheme must be constant (this may be seen as a special case of 2.2.3). For β, this is the fixed part property 4.3; for both γ and δ, this is the Néron property 4.2. This completes the proof. ∎

4.5 The geometric specialisation map.

4.5.1 Notations. Keeping the notations of 4.4, we fix in addition an algebraic closure $\overline{k(z)}$ of $\overline{k}(z)$, and we put $\overline{\eta} := \mathrm{Spec}\,(\overline{k}(z))$ and $\overline{\overline{\eta}} := \mathrm{Spec}\,(\overline{k(z)})$. For $\lambda \in U(k)$ (or even in $U(\overline{k})$), we consider the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\overline{k}}\,(A, \Pi) & \xrightarrow{H(\lambda,\overline{k})\ (37)} & \mathrm{Hom}_{\overline{k}}\,(A, J_\lambda) \\ \downarrow \wr\ \nu & & \| \\ \mathrm{Hom}_{\overline{\eta}}\,(A_{\overline{\eta}}, J_{\overline{\eta}}) & \xrightarrow{\mathrm{sp}_\lambda} & \mathrm{Hom}_{\overline{k}}\,(A, J_\lambda) \\ \downarrow\ \iota\ (\text{field extension } \overline{k}(z) \to \overline{k(z)}) & & \| \\ \mathrm{Hom}_{\overline{\overline{\eta}}}\,(A_{\overline{\overline{\eta}}}, J_{\overline{\overline{\eta}}}) & \xrightarrow{\overline{\mathrm{sp}}_\lambda\ (28)} & \mathrm{Hom}_{\overline{k}}\,(A, J_\lambda) \end{array} \qquad (40)$$

in which the top square is obtained by applying Theorem 4.4.1 after replacing k by $\overline{k}$. The commutativity of the bottom square is clear from the definitions of both specialisation maps.

We cannot expect ι to be an isomorphism in general. However, this is true under additional assumptions on $\theta : X \to S$ which will be satisfied in the case we shall consider:

4.5.2 Theorem. *With the notations of 4.1 and 4.5.1, assume in addition that there is a point $s \in S(k)$ such that, putting $U = S \setminus \{s\}$:*

(i) *for each point y of U, the fibre X_y of θ at y is a* semistable curve (i.e. it has at worst ordinary double points, see 2.5 (vi)*;*

(ii) *writing the geometric fibre $(X_s)_{\overline{k}}$ as $\sum_{i=1}^{s} m_i Y_i$ where the Y_i's are distinct integral divisors, then none of the multiplicities m_i is divisible by p; moreover the reduced divisor $(X_s)_{\mathrm{red}} = \sum_{i=1}^{s} Y_i$ is a semistable curve.*

Then the morphism ι in diagram (40) is an isomorphism. In other words, every $\overline{k(z)}$-morphism from A to J_η is defined over $\overline{k}(z)$.

Consequently, for each $\lambda \in k$, we have the equivalence:

$$\lambda \in \mathrm{Reg}\,(A_U, J_U, k) \quad \Longleftrightarrow \quad H(\lambda, \overline{k}) \text{ is almost bijective.}$$

Proof: We may and will assume $k = \overline{k}$. Let us denote by $k(z)^{\mathrm{s}}$ the separable closure of $k(z)$ in $\overline{k(z)}$, and by η^{s} its spectrum. By 2.4.3 (iii), what we need to show is that every $k(z)^{\mathrm{s}}$-morphism from A to J_η is defined over $k(z)$. So, we consider the abelian group

$$M := \mathrm{Hom}_{\eta^{\mathrm{s}}}\,(A_{\eta^{\mathrm{s}}}, J_{\eta^{\mathrm{s}}}).$$

This is a free finitely generated $\mathbb{Z}$-module (see 2.4.3 (ii)), with a continuous action of $G := \mathrm{Gal}\,(\eta^s/\eta)$; we have to prove that this action is trivial.

For every $y \in S(k)$, let us denote by $I_y \subset G$ one of the inertia groups at y, and by $P_y \subset I_y$ the wild inertia subgroup (i.e. the maximal pro-p-subgroup of I_y, or the trivial subgroup if $p = 0$). It is well known that G is generated (as a normal subgroup) by all the I_y, for $y \in U(k)$, together with P_s (if $p = 0$ this just means that $U \cong \mathbb{A}^1_k$ is simply connected; if $p > 0$, there are nontrivial étale coverings of $\mathbb{A}^1_k$ but they are wildly ramified at infinity). Hence, our claim will follow from:

4.5.3 Lemma. (i) *M is unramified over U (in other words, for all $y \in U(k)$, the subgroup I_y acts trivially on M).*

(ii) *M is tamely ramified at s (in other words, P_s acts trivially on M).*

Proof: Assertion (ii) follows from assumption 4.5.2 (ii), and [Sa], Theorem (3.11) (in fact we only use the 'easier half' of this result). Let us prove (i).

Let $l \neq p$ be a prime, and consider the $\mathbb{Q}_l$-vector spaces $V_l(J_\eta)$ and $V_l(A_\eta)$ (see 2.4.2). Both are finite-dimensional $\mathbb{Q}_l$-vector spaces with continuous actions of G, but in the case of $V_l(A_\eta)$ this action is trivial since A is defined over k and k is algebraically closed.

Consider the natural injective map

$$M \hookrightarrow \mathrm{Hom}_{\mathbb{Q}_l}\,(V_l(A_\eta), V_l(J_\eta)) \tag{41}$$

which is compatible with Galois actions. Since M is a finitely generated $\mathbb{Z}$-module, there is a normal subgroup G' of G of finite index which acts trivially on M, so that the image of M is in fact contained in $\mathrm{Hom}_{\mathbb{Q}_l}\,(V_l(A_\eta), V_l(J_\eta))^{G'}$. But since the action of G on $V_l(A_\eta)$ is trivial, we conclude that

$$M \hookrightarrow \mathrm{Hom}_{\mathbb{Q}_l}\,(V_l(A_\eta), V_l(J_\eta)^{G'}). \tag{42}$$

Now let y be a point of $U(k)$. By assumption 4.5.2 (i), X_y is semistable, hence J_y is semiabelian by 2.5 (vi). Hence, the action of I_y on $V_l(J_\eta)$ is *unipotent*, by [SGA 7], IX, 3.5 (or [Sa], Theorem (3.8)). So, we have a finite group (namely $I_y/I_y \cap G'$) acting unipotently on the $\mathbb{Q}_l$-vector space $V_l(J_\eta)^{G'}$: such an action must be trivial, so the action of I_y on M is trivial too. ∎

5 Double covers, involutions, and twists

In this section, all rings and schemes will be over $\mathbb{Z}[1/2]$ (that is, 2 is invertible in rings, and all residue characteristics in schemes will be $\neq 2$).

5.1 Double covers.

By a *double cover* we mean a morphism $\pi : \widetilde{S} \to S$ of schemes, which is finite locally free of degree 2: in other words, π is affine and the $\mathscr{O}_S$-algebra $\pi_*\mathscr{O}_{\widetilde{S}}$ is locally free of rank 2 as an $\mathscr{O}_S$-module. Recall that since π is affine it is completely determined by this $\mathscr{O}_S$-algebra.

For such a morphism π, there is a canonical $\mathscr{O}_S$-linear projector $\pi_*\mathscr{O}_{\widetilde{S}} \to \mathscr{O}_S$ given by the 'half-trace'. Consequently, $\pi_*\mathscr{O}_{\widetilde{S}} \cong \mathscr{O}_S \oplus L$ (as $\mathscr{O}_S$-module), where $L = \mathrm{Ker}\,(\mathrm{Tr}_{\mathscr{O}_{\widetilde{S}}/\mathscr{O}_S})$ is an invertible $\mathscr{O}_S$-module.

5.1.1 Local description. We can cover S by open affine subsets on which L is trivial. So let us assume $S = \mathrm{Spec}\,(A)$ is affine, and $\widetilde{S} = \mathrm{Spec}\,(B)$ where B has an A-basis of the form $(1, \delta)$ with $\mathrm{Tr}_{B/A}(\delta) = 0$. An immediate computation shows that $D := \delta^2 \in A$, and that $B \cong A[\sqrt{D}] := A[X]/(X^2 - D)$.

Conversely, every morphism which is locally of the form $\mathrm{Spec}\,(A[\sqrt{D}]) \to \mathrm{Spec}\,(A)$ is obviously a double cover.

5.1.2 Globalisation. Returning to the global isomorphism $\pi_*\mathscr{O}_{\widetilde{S}} \cong \mathscr{O}_S \oplus L$, we see from 5.1.1 that multiplication in $\pi_*\mathscr{O}_{\widetilde{S}}$ induces an $\mathscr{O}_S$-linear map $\mu : L^{\otimes 2} \to \mathscr{O}_S$, which of course completely determines the $\mathscr{O}_S$-algebra structure on $\pi_*\mathscr{O}_{\widetilde{S}}$.

In other words, the category of double covers of S (with S-isomorphisms as morphisms) is equivalent to the category of pairs (L, ν) where L is an invertible $\mathscr{O}_S$-module and ν is a global section of $L^{\otimes -2}$ (with the obvious isomorphisms as morphisms).

The local description also shows that if $\pi : \widetilde{S} \to S$ is a double cover, then $\widetilde{S}$ has a *canonical S-involution* $\sigma_\pi = \sigma_{\widetilde{S}/S}$, given locally by $\delta \mapsto -\delta$, and globally on $\pi_*\mathscr{O}_{\widetilde{S}}$ by $x \mapsto x - \mathrm{Tr}(x)$. The subring of invariants is $\mathscr{O}_S$; the $\mathscr{O}_S$-submodule of $\pi_*\mathscr{O}_{\widetilde{S}}$ consisting of anti-invariant elements is L.

In particular, the pair $(\widetilde{S}, \sigma_\pi)$ determines π; geometrically, π is the quotient (in the category of schemes) of $\widetilde{S}$ by the $\mathbb{Z}/2$-action given by σ_π.

5.1.3 Remarks. We use the notations of 5.1.1 and 5.1.2.

(i) The element δ used in 5.1.1 is well defined up to a unit in A, and D is well defined up to the square of a unit; the discriminant of B over A, relative to the basis $(1, \delta)$, is $4D$. The scheme of zeros of D in S is, of course, also well defined, and is the *branch locus* of π.

(ii) The morphism π is étale if and only if (in the local description) D is invertible in A, or (globally) if ν is a trivialisation of $L^{\otimes -2}$. In this case, locally for the étale topology on S, $\widetilde{S}$ is isomorphic to the trivial double cover $S \amalg S$.

(iii) If $\nu = 0$ (or, locally, if $D = 0$) then $\pi_* \mathscr{O}_{\widetilde{S}}$ is the $\mathscr{O}_S$-algebra $\mathscr{O}_S \oplus L$ in which L is an ideal of square zero. In particular, if $L = \mathscr{O}_S$, then $\widetilde{S}$ is the 'scheme of dual numbers' over S.

(iv) In the local description, assume that A is a field (of characteristic $\neq 2$, of course). Then:

- if $D \neq 0$, then B is either $A \times A$ or a quadratic extension of A;
- if $D = 0$, then $B \cong A[X]/(X^2)$.

(v) It is well known that $\widetilde{S}$ is a regular scheme if and only if S is regular and the branch locus of π is a regular divisor in S. For instance, in the local description, if A is a discrete valuation ring, then $A[\sqrt{D}]$ is regular if and only if D has valuation 0 or 1. In fact, the case of interest to us will be when S and $\widetilde{S}$ are Dedekind schemes (mostly, smooth curves over a field or localisations of such curves).

5.2 Weil restriction.

Let us fix a scheme S and a double cover $\pi : \widetilde{S} \to S$. We denote by $\sigma = \sigma_\pi$ the canonical involution of $\widetilde{S}$. If T is an S-scheme, we put $\widetilde{T} := \widetilde{S} \times_S T$: this is a double cover of T, with canonical involution denoted by σ_T, or simply by σ.

For simplicity, we shall restrict our constructions (Weil restrictions and twists) to *quasiprojective* S-schemes $X \to S$: this means, by definition, that S can be covered by open subsets U such that the restriction $X_U \to U$ is a (locally closed) subscheme of a projective U-space $\mathbb{P}^n_U$.

If X is a quasiprojective S-scheme, we shall denote by $\mathrm{WR}(X)$ the functor from S-schemes to sets defined by

$$T \longmapsto \mathrm{WR}(X)(T) := \mathrm{Mor}_S(\widetilde{T}, X). \tag{43}$$

This is known as the *Weil restriction* of the $\widetilde{S}$-scheme $\widetilde{X}$, with respect to π (in general we consider π as fixed once and for all; if necessary we shall use the notation $\mathrm{WR}(X \to S, \pi)$).

By functoriality, we have a canonical involution on $\mathrm{WR}(X)$, deduced from σ, and also denoted by σ if no confusion arises.

We refer to section 7.6 of [B-L-R] for general properties of the Weil restriction. Let us recall some of them:

(i) $\mathrm{WR}(X)$ is representable by a quasiprojective S-scheme (which, as is customary, we shall still denote by $\mathrm{WR}(X)$).

(ii) WR 'commutes with base change': if $S' \to S$ is an S-scheme, and we denote by primes the objects over S' obtained by base change, then $\mathrm{WR}(X' \to S', \pi')$ is canonically isomorphic to $\mathrm{WR}(X \to S, \pi)'$.

(iii) The functor $X \mapsto \mathrm{WR}(X)$ commutes with finite products (of S-schemes, i.e. fibered over S), and in particular takes S-group schemes to S-group schemes.

(iv) If X is the affine n-space $\mathbb{A}^n_S$, *and* if $\pi_*\mathscr{O}_{\widetilde{S}}$ is a free $\mathscr{O}_S$-module (which is always the case locally on S), then $\mathrm{WR}(X) \cong \mathbb{A}^{2n}_S$.
Explicitly, assume for simplicity that $S = \mathrm{Spec}(A)$ is affine, and $\widetilde{S} = \mathrm{Spec}(A[\sqrt{D}])$ for some $D \in A$. Then, for any A-algebra k, we have a natural bijection

$$\begin{array}{rcl} \mathbb{A}^{2n}_S(k) = k^n \times k^n & \longrightarrow & (k[\sqrt{D}])^n = \mathrm{WR}(X)(k) \\ (\underline{x}, \underline{y}) & \longmapsto & \underline{x} + \underline{y}\sqrt{D}. \end{array}$$

The natural involution σ on $\mathrm{WR}(X)$ is given by $(\underline{x}, \underline{y}) \mapsto (\underline{x}, -\underline{y})$.

(v) If Y is a closed (open) subscheme of X, then $\mathrm{WR}(Y)$ is a closed (open) subscheme of $\mathrm{WR}(X)$.

(vi) If X is affine over S, then so is $\mathrm{WR}(X)$.

(vii) If π is the trivial double cover $S \amalg S \to S$, then $\mathrm{WR}(X) \cong X \times_S X$, with the involution exchanging factors.

(viii) If π is the standard 'zero discriminant' cover, i.e. $\pi_*\mathscr{O}_{\widetilde{S}} = \mathscr{O}_S[x]/(x^2)$, then $\mathrm{WR}(X)$ is the relative tangent bundle $T_{X/S}$ of X over S. More generally, if $\pi_*\mathscr{O}_{\widetilde{S}} = \mathscr{O}_S \oplus L$, where L is an ideal of square zero, then $\mathrm{WR}(X)$ is the vector bundle $T_{X/S} \otimes L^{-1}$ over X. The involution σ is the bundle automorphism given by multiplication by -1.

Note that (viii) amounts to nothing but the usual scheme-theoretic definition of the tangent bundle; in particular, the bundle projection $T_{X/S} \to X$ corresponds to the morphism $\mathrm{WR}(X) \to X$ deduced from the obvious section '$x = 0$' of π.

For the unfamiliar reader, let us make this explicit in the affine case: so, assume $S = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(R)$ (where R is an A-algebra). For any A-algebra k, a k-valued point ζ of $\mathrm{WR}(X)$ is an A-algebra morphism $R \to k[\varepsilon] = k \oplus k\,\varepsilon$ (we adopt the traditional notation ε for the class of x modulo x^2). Such a morphism has the form $f \mapsto \varphi(f) + \partial(f)\,\varepsilon$, where $\varphi : R \to k$ is a morphism of A-algebras (thus making k into an R-module), and $\partial : R \to k$ is an A-derivation. Now φ defines a k-valued point $z \in X(k)$ (the projection of ζ on X) and ∂ is a tangent vector at z, in the usual definition by derivations. More precisely, if we view $T_{X/S}$ as $\mathrm{Spec}\,\mathrm{Sym}_R\,\Omega^1_{R/A}$ (where $\Omega^1_{R/A}$ stands as usual for Kähler differentials), then we get a homomorphism $\mathrm{Sym}_R\,\Omega^1_{R/A} \to k$ (that is, a k-valued point of $T_{X/S}$) sending $f\,\mathrm{d}g_1 \otimes \cdots \otimes \mathrm{d}g_n$ to $\varphi(f)\,\partial(g_1)\ldots\partial(g_n)$.

5.3 Twists.

We keep the notations of 5.2. Let (X, τ) be a quasiprojective S-scheme with involution. Consider the functor from S-schemes to sets defined by

$$T \longmapsto X^{(\tau)}(T) := \mathrm{Mor}^{\mathrm{odd}}_S(\widetilde{T}, X), \tag{44}$$

the set of morphisms $\widetilde{T} \to X$ compatible with the involutions. (Here again we omit π from the notation). This is a subfunctor of $\mathrm{WR}(X)$, which we shall call the *(quadratic) twist* of (X, τ) by π.

Let us list some properties of this construction (which are easily deduced from the properties of WR stated in 5.2):

(i) In addition to the involution σ, we now have on $\mathrm{WR}(X)$ an involution deduced from τ by functoriality (and also denoted by τ), which commutes with σ. It is clear from (44) that $X^{(\tau)}$ can be seen as the subfunctor of $\mathrm{WR}(X)$ defined by '$\sigma = \tau$', or, equivalently, as the fixed point subfunctor for $\sigma\tau$. As a consequence, $X^{(\tau)}$ is (representable by) a closed subscheme of $\mathrm{WR}(X)$ (hence is also quasiprojective).

(ii) $X^{(\tau)}$ commutes with any base change $S' \to S$, in a sense analogous to 5.2 (ii).

(iii) The functor $X \mapsto X^{(\tau)}$ commutes with finite products. If X is an S-group scheme and τ is an automorphism of X, then $X^{(\tau)}$ is an S-subgroup scheme of $\mathrm{WR}(X)$.

(iv) Assume that X is the affine $(m+n)$-space $\mathbb{A}_S^{m+n}$, with coordinates $(\underline{x}, \underline{x}')$, and τ acting by $(\underline{x}, \underline{x}') \mapsto (\underline{x}, -\underline{x}')$.
Moreover, assume for simplicity that $S = \mathrm{Spec}(A)$ and $\widetilde{S} = \mathrm{Spec}(A[\sqrt{D}])$, as in 5.1.1.
Then $X^{(\tau)}$ is isomorphic to $\mathbb{A}_S^{m+n}$. Explicitly, with the notations of 5.2 (iv), we have the isomorphism

$$\begin{array}{rcl} \mathbb{A}_S^{m+n}(k) = k^m \times k^n & \xrightarrow{\sim} & X^{(\tau)}(k) \subset k[\sqrt{D}]^{m+n} \\ (\underline{x}, \underline{y}) & \longmapsto & (\underline{x}, \underline{y}\sqrt{D}). \end{array}$$

(v) If Y is a closed (open) subscheme of X, stable by τ, then $Y^{(\tau)}$ is a closed (open) subscheme of $X^{(\tau)}$.

(vi) If X is affine over S, then so is $X^{(\tau)}$.

(vii) If π is the trivial double cover $S \amalg S \to S$, then $X^{(\tau)} \cong X$. Consequently, if π is étale then $X^{(\tau)}$ is an étale twist of X, i.e. locally isomorphic to X for the étale topology on S; explicitly, $X \times_S \widetilde{S}$ is $\widetilde{S}$-isomorphic to $X^{(\tau)} \times_S \widetilde{S}$.

(viii) Assume that π is the standard zero discriminant cover, as in 5.2 (viii), and let $Y \subset X$ be the closed subscheme of fixed points of τ. Then $X^{(\tau)}$ is isomorphic to the normal bundle $N_{Y/X}$ of Y in X. (More generally, if $\pi_* \mathscr{O}_{\widetilde{S}} = \mathscr{O}_S \oplus L$, where L is an ideal of square zero, then $X^{(\tau)}$ is the vector bundle $N_{Y/X} \otimes L^{-1}$ over Y).

Let us explain (viii). In this case we know from 5.2 (viii) that $\mathrm{WR}(X)$ is the tangent bundle $T_{X/S}$. On this bundle, τ acts as the tangent map to $\tau : X \to X$, and σ acts by multiplication by -1 on the fibres. Using, (i), this means that $X^{(\tau)}$ sits above Y, and is in fact the subbundle of $(T_{X/S})_{|Y}$ on which τ (which is now a vector bundle endomorphism) acts by -1. Now since all residue characteristics are $\neq 2$, the action on τ on $(T_{X/S})_{|Y}$ splits it into its $+1$ and -1-subbundles, the former being $T_{Y/S}$ and the latter being canonically isomorphic to the normal bundle of Y in X.

We also have a projective analogue of (iv) in the étale case:

(ix) Assume that X is the projective $(m+n-1)$-space $\mathbb{P}_S^{m+n-1}$, with homogeneous coordinates written as $(Y_1 : \ldots : Y_m : Y'_1 : \ldots : Y'_n) = (\underline{Y} : \underline{Y}')$, and τ acting by $(\underline{Y} : \underline{Y}') \mapsto (\underline{Y} : -\underline{Y}')$.
Moreover, assume (for simplicity) that $S = \mathrm{Spec}\,(A)$ and $\widetilde{S} = \mathrm{Spec}\,(A[\sqrt{D}])$, as in 5.1.1, and that D *is invertible in* A. Then $X^{(\tau)}$ is isomorphic to $\mathbb{P}_S^{m+n-1}$ (with similar homogeneous coordinates denoted by $(\underline{V} : \underline{V}')$), via the map

$$\begin{array}{ccc} \mathbb{P}_S^{m+n-1} & \xrightarrow{\sim} & X^{(\tau)} \\ (\underline{V} : \underline{V}') & \longmapsto & (\underline{V} : \underline{V}'\sqrt{D}). \end{array}$$

5.3.1 Remark. In case (ix), the situation is more complicated if D is not invertible. Assuming for instance that $D = 0$, we can determine $X^{(\tau)}$ by using (viii). Now the fixed locus of τ is the disjoint union of two linear subspaces F and F' of X, given respectively by $\underline{Y} = 0$ and $\underline{Y}' = 0$. Hence $X^{(\tau)}$ is a disjoint union of their normal bundles, isomorphic to $X \setminus F'$ and $X \setminus F$ respectively.

5.4 The case of elliptic curves.

With $\pi : \widetilde{S} \to S$ as in 5.1, we consider the elliptic curve E in the projective S-plane $\mathbb{P}_S^2$, defined in homogeneous coordinates (X, Y, Z) by

$$E : \qquad Y^2\,Z = P(X, Z) \tag{45}$$

where P is homogeneous of degree 3 with coefficients in $\mathscr{O}_S$, monic in X, with nonvanishing discriminant. As usual, we give E the group structure with origin $\omega = (0 : 1 : 0)$; in this way, E becomes a smooth commutative S-group scheme.

The involution $\tau = [-1]_E$ sends $(X : Y : Z)$ to $(X : -Y : Z)$ (or equivalently to $(-X : Y : -Z)$), and the subgroup $E[2]$ is defined by $Y = 0$: this is a finite étale group scheme of degree 4 over S, the disjoint union of ω and the closed subscheme $E[2]^*$ of 'points of exact order 2' (see 2.2). Since $E[2]^*$ is also the intersection of E with the line $Y = 0$, its complement in E is affine, and can be identified (via the affine coordinates $x = X/Y$, $z = Z/Y$) with the closed subscheme

$$E_{\mathrm{aff}} : \qquad z = P(x, z) \tag{46}$$

of the affine plane $\mathbb{A}^2$. Note that the origin ω conveniently has coordinates $(0, 0)$ in E_{aff}, and that E_{aff} is invariant under τ, which sends (x, z) to $(-x, -z)$. This affine model will turn out to be much more useful to us than the 'usual' one (the complement of ω in E).

Our goal is to study the twist $E^{(\tau)}$ of E by π. For simplicity, we shall always assume, as in 5.1.1, that $S = \mathrm{Spec}\,(A)$ and $\widetilde{S} = \mathrm{Spec}\,(B)$ with $B = A[\sqrt{D}]$, for some $D \in A$. We put

$$\begin{array}{rcl} S_{\mathrm{nd}} & := & \mathrm{Spec}\,(A[1/D]) \subset S \\ \Delta & := & \mathrm{Spec}\,(A/DA) \subset S. \end{array} \tag{47}$$

Thus, Δ is the 'branch locus' of π, and S_{nd} is its open complement. The subscript 'nd' stands for 'nondegenerate' and will also be used to denote restriction of S-schemes to S_{nd}.

We can easily give a crude 'qualitative' description of $E^{(\tau)}$: we already know that it is a smooth quasiprojective S-group scheme, whose restriction to S_{nd} is an étale twist of E_{nd} (that is, an S_{nd}-elliptic curve locally isomorphic to E_{nd} for the étale toplogy). On the other hand, if s is a point of Δ, then the fibre $E_s^{(\tau)}$ of $E^{(\tau)}$ at s is isomorphic to the normal bundle of the fixed locus of τ. This fixed locus is $E_s[2]$ which is étale over the residue field $\kappa(s)$, hence $E_s^{(\tau)}$ is the restriction to $E_s[2]$ of the tangent bundle of E, which is trivial. We conclude that there is a canonical isomorphism

$$E_s^{(\tau)} \cong E_s[2] \times \mathbb{G}_{\mathrm{a},\kappa(s)}. \tag{48}$$

so that, geometrically, $E_s^{(\tau)}$ is a disjoint union of four affine lines.

Describing $E^{(\tau)}$ as a scheme is harder, especially if one wants a description by equations and inequations in some projective space (recall that $E^{(\tau)}$ is quasiprojective). What we shall do is describe by equations an open subgroup scheme of $E^{(\tau)}$, sufficiently big for our needs. Specifically, E has two open subschemes whose twists are easy to see, namely, E_{nd} and E_{aff}. So let us twist these first:

5.4.1 The twist of E_{nd} is isomorphic to the elliptic curve $\mathscr{E}_{\mathrm{nd}} \subset \mathbb{P}^2_{S_{\mathrm{nd}}}$ given in homogeneous coordinates (U, V, W) by

$$V^2\, W = D\, P(U, W). \tag{49}$$

The isomorphism is given by

$$\begin{array}{rcl} \mathscr{E}_{\mathrm{nd}} & \xrightarrow{\sim} & E_{\mathrm{nd}}^{(\tau)} \\ (U : V : W) & \longmapsto & (X : Y : Z) = (\sqrt{D}\, U : V : \sqrt{D}\, W) \end{array} \tag{50}$$

which is just a special case of 5.3 (ix), restricted to the appropriate curve.

5.4.2 The twist of E_{aff} is isomorphic to the affine curve $\mathscr{E}_{\mathrm{aff}} \subset \mathbb{A}^2_S$ given in affine coordinates (u, w) by

$$w = D\, P(u, w). \tag{51}$$

The isomorphism is given by

$$\begin{array}{rcl} \mathscr{E}_{\mathrm{aff}} & \xrightarrow{\sim} & (E_{\mathrm{aff}})^{(\tau)} \\ (u, w) & \longmapsto & (\sqrt{D}\, u, \sqrt{D}\, w) \end{array} \tag{52}$$

(recall that τ is induced by multiplication by -1 on the plane).

If s is a point of Δ, then by (51) the fibre $(\mathscr{E}_{\mathrm{aff}})_s$ is the affine line $w = 0$. On the other hand, by definition of E_{aff}, the only point of order two in $(E_{\mathrm{aff}})_s$ is the origin, so the fibre

of $(E_{\mathrm{aff}})^{(\tau)}$ at s is the tangent line to E_s at the origin; from the description (48) we see that this is the connected component of the $\kappa(s)$-group scheme $(E_s)^{(\tau)}$.

Recall that, as a smooth S-group scheme, $E^{(\tau)}$ has a *connected component* $E^{(\tau)\circ}$, which is the largest open subgroup scheme of $E^{(\tau)}$ with connected fibres. We shall now describe $E^{(\tau)\circ}$, first (in 5.4.3) as a subgroup scheme of $E^{(\tau)}$, and then (in 5.4.5) as an S-scheme in its own right (that is, by equations).

5.4.3 Proposition. (i) *$E^{(\tau)\circ}$ is the open subset of $E^{(\tau)}$ given by*

$$E^{(\tau)\circ} = (E_{\mathrm{aff}})^{(\tau)} \cup (E_{\mathrm{nd}})^{(\tau)}.$$

(ii) *The multiplication by two in $E^{(\tau)}$ factors through $E^{(\tau)\circ}$. Equivalently, for any A-algebra k, we have*

$$2\,E^{(\tau)}(k) \subset E^{(\tau)\circ}(k).$$

(iii) *Let $x \in E^{(\tau)}(A)$ correspond to the odd morphism $\widetilde{x} : \widetilde{S} \to E$. Then the following are equivalent:*

(a) *$x \in E^{(\tau)\circ}(A)$;*

(b) *for each point $s \in \Delta$, $\widetilde{x}$ sends the only point $\widetilde{s}$ of $\widetilde{S}$ above s to the origin of E_s;*

(c) *for s and $\widetilde{s}$ as in (b), $\widetilde{x}(\widetilde{s})$ is not a nontrivial 2-division point of E_s (equivalently, $\widetilde{x}(\widetilde{s}) \in E_{\mathrm{aff}}$).*

Proof: (i) Since both sides are open subschemes of $E^{(\tau)}$ we need only show that they coincide set-theoretically, or that they have the same fibre over S. If s is a point of S_{nd}, then $E_s^{(\tau)} = (E_{\mathrm{nd}})_s^{(\tau)}$ is an elliptic curve, hence connected and equal to $E_s^{(\tau)\circ}$. If $s \in \Delta$, we have seen in 5.4.2 that $E^{(\tau)\circ}$ and $(E_{\mathrm{aff}})^{(\tau)}$ have the same fibres at s.

(ii) Again, since $\mathscr{E}^\circ$ is an open subscheme it suffices to see that the set-theoretic image of $[2]_{\mathscr{E}}$ is contained in $\mathscr{E}^\circ$. This is clear above S_{nd}, and above Δ it follows from the description (48).

(iii) Clearly, the restriction of x to S_{nd} is in $E^{(\tau)\circ}(S_{\mathrm{nd}})$ by (i). Thus, $x \in E^{(\tau)\circ}(S)$ if and only if, for each $s \in \Delta$, we have $x(s) \in E_s^{(\tau)\circ}$. So we may, by base change, assume that A is a field (with spectrum $S = \{s\}$) and $D = 0$. But by (i) (or by 5.4.2), $E^{(\tau)\circ}$ is then equal to $(E_{\mathrm{aff}})^{(\tau)}$, so this is equivalent to the condition that $\widetilde{x}$ factors through E_{aff}, which in turn is equivalent to (c) because E_{aff} is an open subscheme of E. Hence, we have (a)⇔(c).

Obviously, (b) implies (c); conversely, since $\widetilde{x}$ is odd the point $\widetilde{x}(\widetilde{s})$ must be a 2-division point of E, hence (c) implies (b) because the only 2-division point of E_{aff} is the origin. ∎

5.4.4 Gluing. It follows from 5.4.3 (i) and the descriptions 5.4.1 and 5.4.2 that $E^{(\tau)\circ}$ can be constructed by gluing $\mathscr{E}_{\mathrm{nd}}$ and $\mathscr{E}_{\mathrm{aff}}$ in some way. The gluing is in fact the obvious one: the affine equation (51) defining $\mathscr{E}_{\mathrm{aff}}$ is the affine form (obtained by putting $u = U/V$, $w = W/V$) of the homogeneous equation (49) defining $\mathscr{E}_{\mathrm{nd}}$ (observe, however, that $\mathscr{E}_{\mathrm{nd}}$

is restricted to $D \neq 0$). So we are led to define closed subschemes $\overline{\mathscr{E}^\circ}$ and $\mathscr{F}$ of $\mathbb{P}^2_S$ (in homogeneous coordinates U, V, W) by

$$\begin{array}{ll} \overline{\mathscr{E}^\circ}: & V^2\,W = D\,P(U,W) \\ \mathscr{F}: & V = D = 0. \end{array} \tag{53}$$

Clearly, $\mathscr{F} \subset \overline{\mathscr{E}^\circ}$. Now define $\mathscr{E}^\circ \subset \mathbb{P}^2_S$ by

$$\mathscr{E}^\circ := \overline{\mathscr{E}^\circ} \setminus \mathscr{F}. \tag{54}$$

It is clear from the equations (53) that:

- over S_{nd}, we have $\mathscr{E}^\circ \times_S S_{\mathrm{nd}} = \overline{\mathscr{E}^\circ} \times_S S_{\mathrm{nd}} = \mathscr{E}_{\mathrm{nd}}$, and
- $\mathscr{E}^\circ \cap (V \neq 0) = \mathscr{E}_{\mathrm{aff}}$.

Moreover the isomorphisms (50) and (52), viewed as open immersions from $\mathscr{E}_{\mathrm{nd}}$ and $\mathscr{E}_{\mathrm{aff}}$ into $E^{(\tau)}$, obviously glue together to form a morphism

$$\begin{array}{rccc} j: & \mathscr{E}^\circ & \longrightarrow & E^{(\tau)} \\ & (U:V:W) & \longmapsto & (X:Y:Z) = (\sqrt{D}\,U : V : \sqrt{D}\,W). \end{array} \tag{55}$$

5.4.5 Proposition. *The morphism j of (55) is an isomorphism of $\mathscr{E}^\circ$ with the connected component $E^{(\tau)\circ}$ of $E^{(\tau)}$.*

Proof: it is clear from 5.4.3 (i) and the construction of j that the image of j is $E^{(\tau)\circ}$. We know that j induces an isomorphism of $\mathscr{E}_{\mathrm{nd}}$ with $E^{(\tau)}_{\mathrm{nd}}$ (namely, (50)) and an isomorphism of $\mathscr{E}_{\mathrm{aff}}$ with $E^{(\tau)}_{\mathrm{aff}}$ (namely, (52)). To conclude, we only have to check that (50) maps $\mathscr{E}_{\mathrm{nd}} \cap \mathscr{E}_{\mathrm{aff}}$ *onto* $E^{(\tau)}_{\mathrm{nd}} \cap E^{(\tau)}_{\mathrm{aff}}$, which is immediate if one observes that $E^{(\tau)}_{\mathrm{nd}} \cap E^{(\tau)}_{\mathrm{aff}} = (E_{\mathrm{nd}} \cap E_{\mathrm{aff}})^{(\tau)}$. ∎

We now prove a useful property of $E^{(\tau)}$, which follows directly from its definition:

5.4.6 Proposition. ('Néron mapping property') *Assume that S is integral with generic point η, and that $\widetilde{S}$ is regular. Then the natural map*

$$E^{(\tau)}(S) \longrightarrow E^{(\tau)}(\eta)$$

is an isomorphism.

Proof: We have to show that the natural map

$$\mathrm{Mor}^{\mathrm{odd}}(\widetilde{S}, E) \longrightarrow \mathrm{Mor}^{\mathrm{odd}}(\widetilde{\eta}, E)$$

is bijective. Injectivity is clear because $\widetilde{\eta}$ is dense in $\widetilde{S}$ (indeed, $\widetilde{S}$ is flat over S and η is dense in S). Next, take any odd S-morphism $x : \widetilde{\eta} \to E$. It suffices to extend x to an S-morphism $\widetilde{S} \to E$ (automatically odd by density). Now, x certainly extends to an S-morphism $x_1 : U \to E$ where U is a dense open subset of $\widetilde{S}$. We can view x_1 as a section over U of the $\widetilde{S}$-elliptic curve $E \times_S \widetilde{S}$. But since $\widetilde{S}$ is regular, the fact that x_1 extends to a section over $\widetilde{S}$ is a standard property of elliptic curves (and, more generally, abelian schemes): see ([B-L-R], 1.2, Proposition 8).

(Remark: we shall only use 5.4.6 when $\dim S = 1$. In this case, it is easy to replace the reference to [B-L-R] by the valuative criterion of properness). ∎

Part II
Proof of the main theorem

6 Twisted elliptic curves over function fields

6.1 Notations.

In this section we apply the constructions of Section 5 to the following situation:

- k is a field of characteristic different from 2;
- the base scheme S is $\mathbb{P}^1_k$ (with standard coordinate t);
- the double cover $\pi : \widetilde{S} \to S$ is
$$\pi : \Gamma \longrightarrow \mathbb{P}^1_k \tag{56}$$
as in 1.4.4; thus Γ is a smooth curve over k, and π is étale above ∞ and ramified at 0. The unique point above 0 is denoted by 0_Γ, and the natural involution on Γ by σ.

  (Note that S is not affine, while results of Section 5 are often presented in the affine case; however the extension to this case is usually obvious).

  The function field of Γ is isomorphic to $k(t, \sqrt{R(t)})$. It will be convenient (and compatible with the notations of 5.1) to denote by δ one of the square roots of $R(t)$ in this field; the affine curve $\pi^{-1}(\mathbb{A}^1_k)$ is then $\operatorname{Spec} k[t,\delta]$, where $k[t,\delta] \cong k[t][s]/(s^2 - R(t))$, with δ corresponding to the class of s.
- E denotes an elliptic curve over k, with equation
$$E : \qquad Y^2\, Z = P(X, Z) \tag{57}$$
in projective coordinates X, Y, Z, and $E_S \to \mathbb{P}^1_k$ denotes the constant $\mathbb{P}^1_k$-elliptic curve $E \times_k \mathbb{P}^1_k \to \mathbb{P}^1_k$, with its natural involution $\tau = [-1]_{E_S}$.

From these data, we deduce a $\mathbb{P}^1_k$-group scheme
$$\mathscr{E} = E_S^{(\tau)} \longrightarrow \mathbb{P}^1_k, \tag{58}$$
the twist of E_S by π, defined in 5.3 and studied in 5.4. This is a smooth commutative $\mathbb{P}^1_k$-group scheme of relative dimension 1; over the generic point of $\mathbb{P}^1_k$ it can be given by the homogeneous equation (in $\mathbb{P}^2_{k(t)}$, with projective coordinates U, V, W)
$$V^2\, W = R(t)\, P(U, W) \tag{59}$$
as in (49); this description is in fact valid above $\mathbb{A}^1_k \cap S_{\mathrm{nd}}$ where S_{nd} (notation of (47)) is the complement of the zeros of R in S.

6.2 Points of $\mathscr{E}$.

Let $T \to S = \mathbb{P}^1_k$ be any S-scheme. By definition of a twist (see (44)), the group $\mathscr{E}(T) = \mathrm{Mor}_S(T, \mathscr{E})$ can be described as $\mathrm{Mor}_S^{\mathrm{odd}}(\widetilde{T}, E_S)$, where $\widetilde{T} = \widetilde{S} \times_S T$ is the double cover of T deduced from π.

But here, by definition of E_S, an S-morphism $\widetilde{T} \to E_S = E \times_k S$ is the same thing as a k-morphism $\widetilde{T} \to E$, so we get a canonical isomorphism

$$\mathscr{E}(T) \xrightarrow{\sim} E(\widetilde{T})^{\mathrm{odd}} := \mathrm{Mor}_k^{\mathrm{odd}}(\widetilde{T}, E). \tag{60}$$

Explicitly, assume that, say, $T = \mathrm{Spec}\,(L)$ where L is an extension of $k(t)$: we can describe an element of $\mathscr{E}(T)$ as a nontrivial solution $(U, V, W) \in L^3$ of (59). The isomorphism (60) maps this element to $(X : Y : Z) = (\delta\, U : V : \delta\, W)$: this is indeed an $L[\delta]$-valued point of E, which is odd since changing δ to $-\delta$ amounts to applying $[-1]_E$.

6.3 The twisted elliptic curve: points over function fields.

6.3.1 The curve $\widetilde{C}_g$. Consider now our k-curve C from 1.4.2, and assume given a nonconstant rational function g on C, which we view as a k-morphism $C \to S$. We denote by C_g the S-scheme thus obtained, and accordingly by K_g (as in 1.4.2) the function field K of C viewed as a finite extension of $k(t)$, via g.

Denote by $\widetilde{C}_g$ the curve $C \times_{g,\mathbb{P}^1_k,\pi} \Gamma$ (recall that our plan is to let g vary). Thus, we have a Cartesian diagram of projective k-curves

$$\begin{array}{ccc} \widetilde{C}_g & \xrightarrow{\varphi} & \Gamma \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ C = C_g & \xrightarrow{g} & S = \mathbb{P}^1_k \end{array} \tag{61}$$

with all maps finite and flat. The ring of rational functions on $\widetilde{C}_g$ (its function field, if it is irreducible) is $\widetilde{K}_g = K[\delta]$.

In any case, π' is a double cover and $\widetilde{C}_g$ has a canonical involution, which we denote by $\widetilde{\sigma}$ (of course, this $\widetilde{\sigma}$ induces the involution on $\widetilde{K}_g$ sending δ to $-\delta$).

As explained in Section 1, we are interested in the group $\mathscr{E}(K_g)$ of K_g-rational points of $\mathscr{E}$.

6.3.2 Proposition. *We keep the notations and assumptions of 6.3.1.*

(i) *$\widetilde{C}_g$ is geometrically connected over k.*

(ii) *If the ramification loci of g and π in $\mathbb{P}^1_k$ are disjoint, then $\widetilde{C}_g$ is smooth over k.*

(iii) *If g has only simple ramification, then $\widetilde{C}_g$ is semistable (see 2.5 (vi)).*

(iv) *If g is admissible in the sense of 1.5.2, then $\widetilde{C}_g$ is smooth, and for all (resp. all but finitely many) $\lambda \in k^*$, the curve $\widetilde{C}_{\lambda g}$ is semistable (resp. smooth).*

(v) *There is a canonical isomorphism of groups*

$$\mathrm{Mor}_S\,(C_g, \mathscr{E}) \xrightarrow{\sim} \mathrm{Mor}_k^{\mathrm{odd}}(\widetilde{C}_g, E). \tag{62}$$

(vi) *If $\widetilde{C}_g$ is smooth, the natural map $\mathrm{Mor}_S\,(C_g, \mathscr{E}) \to \mathscr{E}(K_g)$ is an isomorphism. In particular, by (62), we have a canonical isomorphism*

$$\mathscr{E}(K_g) \cong \mathrm{Mor}_k^{\mathrm{odd}}(\widetilde{C}_g, E). \tag{63}$$

Proof: (i) is clear because C is geometrically connected and π' is a *ramified* double cover since π is.

(ii) Let q be a point of $\widetilde{C}_g$. The assumption implies that either g is étale at $\pi'(q)$, or π is étale at $\varphi(q)$. In the first (resp. second) case, φ (resp. π') is étale at q by base change, hence $\widetilde{C}_g$ is smooth at q because Γ (resp. C) is smooth.

(iii) We may assume k algebraically closed. The proof of (ii) shows that if q is a singular point of $\widetilde{C}_g$, then $c = \pi'(q)$ and $e = \varphi(q)$ must be branch points of g and π respectively, mapping to the same point x of $\mathbb{P}^1_k$. If z denotes a local coordinate at x, the completed local ring of x in $\mathbb{P}^1_k$ is isomorphic to $k[[z]]$, and the completed local rings of c and e in C and Γ are respectively isomorphic, as $k[[z]]$-algebras, to $k[[z]][u]/(u^2 - z)$ and $k[[z]][v]/(v^2 - z)$. So, the completed local ring of q is isomorphic to $k[[z]][u, v]/(u^2 - z, v^2 - z)$, hence to $k[[u]]/((u^2 - v^2)$, which proves that q is an ordinary double point.

(iv) is an immediate consequence of (ii) and (iii), and (v) is a special case of (60) (applied with $T = C_g$).

(vi) Since $\widetilde{C}_g$ is a smooth curve and E is projective, every k-morphism $u : \mathrm{Spec}\,\widetilde{K}_g \to E$ extends uniquely to a k-morphism $\widetilde{C}_g \to E$, and the condition on involutions is of course preserved. (Alternatively, we can invoke the Néron property 5.4.6). ∎

6.3.3 Remark. It follows from (63) and Remark 2.6.2 that if $\widetilde{C}_g$ is smooth, then $\mathscr{E}(K_g)$ is a finitely generated abelian group, with torsion subgroup $E[2](k)$.

We can now prove assertion (ii) of 1.5.4:

6.3.4 Corollary. *With the same notations and assumptions, let k' be an extension of k. If k is separably closed in k', then $\mathscr{E}(k'(t)) = \mathscr{E}(k(t))$ and $\mathscr{E}(k'(C)_g) = \mathscr{E}(K_g)$.*

Proof: It suffices to prove the second equality. By 6.3.2 (vi) this is equivalent to

$$\mathrm{Mor}_{k'}^{\mathrm{odd}}(\widetilde{C}_g, E) = \mathrm{Mor}_k^{\mathrm{odd}}(\widetilde{C}_g, E)$$

which follows from the rigidity property 2.2.3 (see Remark 2.2.4). ∎

6.3.5 Remark. From 6.3.2 (vi) we obtain a description of the group $\mathscr{E}(K_g)$, entirely in terms of (morphisms of) projective curves *over* k. This will be our viewpoint in the subsequent sections, where no 'geometry over K' will be involved; in particular we shall then forget about $\mathscr{E}$ completely.

Let us now compare $\mathscr{E}(k(t))$ and $\mathscr{E}(K_g)$:

6.4 Proposition. *Assume that $g : C \to \mathbb{P}^1_k$ is admissible. Then the following conditions are equivalent:*

(i) *g is good for E and Γ, i.e. $\mathscr{E}(K_g) = \mathscr{E}(k(t))$ (see 1.5.2);*

(ii) *the homomorphism*

$$\begin{array}{ccc} \mathrm{Mor}_k^{\mathrm{odd}}(\Gamma, E) & \longrightarrow & \mathrm{Mor}_k^{\mathrm{odd}}(\widetilde{C}_g, E) \\ j & \longmapsto & j \circ \varphi \end{array} \tag{64}$$

is almost bijective (see 1.5.1);

(iii) *the homomorphism*

$$\mathrm{Mor}_k^{\mathrm{odd}}(\Gamma, E)/E[2](k) \hookrightarrow \mathrm{Mor}_k^{\mathrm{odd}}(\widetilde{C}_g, E)/E[2](k) \tag{65}$$

(deduced from composition with φ) is almost bijective.

Proof: Of course, the subgroups $E[2](k)$ appearing in (65) are simply the groups of *constant* odd morphisms from Γ (resp. $\widetilde{C}_g$) to E. Since they are also the torsion subgroups of $\mathrm{Mor}_k^{\mathrm{odd}}(\Gamma, E)$ and $\mathrm{Mor}_k^{\mathrm{odd}}(\widetilde{C}_g, E)$, the equivalence of (ii) and (iii) follows.

The equivalence (i)⇔(ii) is clear because the maps (9) and (64) correspond to each other via the isomorphism of 6.3.2 (vi). ∎

6.5 Reduction to a problem about Jacobians.

We shall now reformulate the 'goodness' property of 1.5.2 (or rather, its equivalent form 6.4 (iii)) in terms of Jacobians. So let $g : C \to \mathbb{P}^1_k$ be an admissible morphism. By functoriality of Jacobians, $\varphi : \widetilde{C}_g \to \Gamma$ gives rise to $\varphi^* : \mathrm{Jac}\,(\Gamma) \to \mathrm{Jac}\,(\widetilde{C}_g)$, hence to a morphism of abelian groups

$$\begin{array}{ccc} \mathrm{Hom}_k^{\mathrm{odd}}(E, \mathrm{Jac}\,(\Gamma)) & \longrightarrow & \mathrm{Hom}_k^{\mathrm{odd}}(E, \mathrm{Jac}\,(\widetilde{C}_g)) \\ j & \longmapsto & j \circ \varphi^* \end{array} \tag{66}$$

where $\varphi^* : \mathrm{Jac}\,(\Gamma) \to \mathrm{Jac}\,(\widetilde{C}_g)$ is deduced from φ, and the 'odd' superscripts refer to the involution $[-1]$ on E, and the involutions induced by the double cover structures on Γ and $\widetilde{C}_g$.

Clearly, the map (66) is connected to (65) via the morphism (22) of 2.6: we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Mor}_k^{\mathrm{odd}}(\Gamma, E)/E[2](k) & \xrightarrow{\sim} & \mathrm{Hom}_k^{\mathrm{odd}}(E, \mathrm{Jac}\,(\Gamma)) \\ \downarrow{\scriptstyle (65)} & & \downarrow{\scriptstyle (66)} \\ \mathrm{Mor}_k^{\mathrm{odd}}(\widetilde{C}_g, E)/E[2](k) & \xrightarrow{\sim} & \mathrm{Hom}_k^{\mathrm{odd}}(E, \mathrm{Jac}\,(\widetilde{C}_g)) \end{array} \tag{67}$$

where all maps are injective, and the horizontal maps are isomorphisms by 2.6.1 (indeed, Γ has a rational point fixed by σ, namely 0_Γ). So, 6.4 immediately implies:

6.5.1 Proposition. *With the notations and assumptions of 6.4, g is good if and only if (66) is almost bijective.* ∎

Let us now get rid of the annoying $^{\mathrm{odd}}$ superscript in (66). By its very definition as a fibre product, $\widetilde{C}_g$ is contained in the product surface $C \times \Gamma$; let us denote by $i_g = (\pi', \varphi) : \widetilde{C}_g \hookrightarrow C \times \Gamma$ the inclusion. This induces a morphism of k-abelian varieties

$$i_g^* : \underline{\mathrm{Pic}}^0_{C\times\Gamma/k} \longrightarrow \mathrm{Jac}\,(\widetilde{C}_g) \tag{68}$$

and, in turn, a group homomorphism

$$\begin{array}{rcl} \mathrm{Hom}_k\,(E, i_g^*) : \mathrm{Hom}_k\,(E, \underline{\mathrm{Pic}}^0_{C\times\Gamma/k}) & \longrightarrow & \mathrm{Hom}_k\,(E, \mathrm{Jac}\,(\widetilde{C}_g)) \\ u & \longmapsto & i_g^* \circ u. \end{array} \tag{69}$$

6.5.2 Lemma. *The involution σ of Γ induces $[-1]$ on $\mathrm{Jac}\,(\Gamma)$.*

Proof: We may assume k algebraically closed. Then a point of $\mathrm{Jac}\,(\Gamma)$ corresponds to a divisor ξ of degree 0. Now $\xi + \sigma(\xi)$ is the pullback of a divisor of degree 0 on $\mathbb{P}^1_k$ (namely $\pi_*(\xi)$). Such a divisor is principal, hence so is $\xi + \sigma(\xi)$. ∎

6.5.3 Proposition. *Let $g : C \to \mathbb{P}^1_k$ be an admissible morphism.*

Assume that the homomorphism $\mathrm{Hom}_k\,(E, i_g^*)$ of (69) is almost bijective. Then g is good.

Proof: Clearly, the canonical involution $\widetilde{\sigma}$ on $\widetilde{C}_g$ is induced by $\tau := \mathrm{Id}_C \times \sigma$ on $C \times \Gamma$, hence $\widetilde{\sigma}^*$ on $\mathrm{Jac}\,(\widetilde{C}_g)$ is compatible (via i_g^*) with the involution τ^* on $\underline{\mathrm{Pic}}^0_{C\times\Gamma/k}$. Now, the assumption of the proposition clearly implies (by restriction to the 'odd' parts) that

$$\begin{array}{rcl} \mathrm{Hom}_k^{\mathrm{odd}}(E, \underline{\mathrm{Pic}}^0_{C\times\Gamma/k}) & \longrightarrow & \mathrm{Hom}_k^{\mathrm{odd}}(E, \mathrm{Jac}\,(\widetilde{C}_g)) \\ u & \longmapsto & i_g^* \circ u \end{array} \tag{70}$$

is almost bijective. But we know from 2.5 (ix) that $\underline{\mathrm{Pic}}^0_{C\times\Gamma/k}$ is canonically isomorphic to $\mathrm{Jac}\,(C) \times \mathrm{Jac}\,(\Gamma)$; under this isomorphism, τ^* corresponds (by Lemma 6.5.2) to $\mathrm{Id}_{\mathrm{Jac}\,(C)} \times [-1]_{\mathrm{Jac}\,(\Gamma)}$. It follows that the *odd* homomorphisms from E to $\underline{\mathrm{Pic}}^0_{C\times\Gamma/k}$ are those which factor through the inclusion $\mathrm{Jac}\,(\Gamma) \xrightarrow{\mathrm{pr}_2^*} \underline{\mathrm{Pic}}^0_{C\times\Gamma/k}$. Hence we have a chain of almost bijective homomorphisms

$$\begin{array}{rcl} \mathrm{Hom}_k\,(E, \mathrm{Jac}\,(\Gamma)) & \xrightarrow{\sim} & \mathrm{Hom}_k^{\mathrm{odd}}(E, \underline{\mathrm{Pic}}^0_{C\times\Gamma/k}) \xrightarrow{(70)} \mathrm{Hom}_k^{\mathrm{odd}}(E, \mathrm{Jac}\,(\widetilde{C}_g)) \\ u & \longmapsto & \mathrm{pr}_2^* \circ u; \qquad\qquad v \longmapsto i_g^* \circ v. \end{array} \tag{71}$$

Since the composite of these maps is just (66), the proposition follows. ∎

Of course, we can apply this 'over $\overline{k}$', to obtain:

6.5.4 Proposition. *Let $g : C \to \mathbb{P}^1_k$ be an admissible morphism. Assume that the natural homomorphism*

$$\begin{array}{rcl} \mathrm{Hom}_{\overline{k}}(E, i_g^*) : \mathrm{Hom}_{\overline{k}}(E, \underline{\mathrm{Pic}}^0_{C\times\Gamma/k}) & \longrightarrow & \mathrm{Hom}_{\overline{k}}(E, \mathrm{Jac}(\widetilde{C}_g)) \\ u & \longmapsto & i_g^* \circ u. \end{array} \tag{72}$$

is almost bijective. Then g is very good. ∎

7 Geometry of a pencil of curves

7.1 Notations.

In this section we keep k, C, Q, E, Γ, π as in 1.4, and an admissible k-morphism $f : C \to \mathbb{P}^1_k$ as in 1.6. Recall that $0_\Gamma \in \Gamma(k)$ denotes the unique zero of π. We put $d = \deg(f)$ and

$$c_0 := f^{-1}(0) \quad \text{and} \quad c_\infty := f^{-1}(\infty).$$

These are viewed interchangeably as closed subschemes or as effective divisors on C; note that both are reduced of degree d, and that c_0 contains the finite set Q.

Similarly, we define divisors on Γ by

$$\gamma_0 := \pi^{-1}(0) \quad \text{and} \quad \gamma_\infty := \pi^{-1}(\infty).$$

Here, of course, $\gamma_0 = 2[0_\Gamma]$ as a divisor, while γ_∞ is a reduced divisor of degree 2 because of the assumptions on π in 1.4.4.

For $\lambda \neq 0$ in k (or, more generally, in an extension of k) we put

$$X_\lambda := \widetilde{C}_{\lambda f}$$

where $\widetilde{C}_{\lambda f}$ is defined in 6.3.1. Thus, X_λ is the curve in $C \times \Gamma$ defined by '$\lambda f(a) = \pi(b)$' ($a \in C$, $b \in \Gamma$). By abuse, we shall still denote by f (resp. π) the composed map $f \circ \mathrm{pr}_1$ (resp. $\pi \circ \mathrm{pr}_2$) on $C \times \Gamma$.

We want to view X_λ as a 'family of curves with parameter λ'. This leads to consider the rational map

$$\theta_0 := \pi/f : \ C \times \Gamma \ \cdots \longrightarrow \mathbb{P}^1_k.$$

Roughly speaking, X_λ is '$\theta_0^{-1}(\lambda)$'. We have to make this precise, including when λ is 0 or ∞; this amounts to 'make the rational map θ_0 into a morphism'. Now the divisors of zeros and poles of f and π on $C \times \Gamma$ are

$$\begin{aligned} (f)_0 &= c_0 \times \Gamma, & (f)_\infty &= c_\infty \times \Gamma, \\ (\pi)_0 &= C \times \gamma_0 = 2(C \times 0_\Gamma), & (\pi)_\infty &= C \times \gamma_\infty. \end{aligned}$$

Geometrically (i.e. over $\overline{k}$) $(f)_0$ and $(f)_\infty$ are (disjoint) unions of d copies of Γ; $(\pi)_\infty$ is a union of two copies of C, and $(\pi)_0$ is a 'double C'. The function θ_0 is undefined at the finite sets $S_0 := (f)_0 \cap (\pi)_0 = c_0 \times \gamma_0$ and $S_\infty := (f)_\infty \cap (\pi)_\infty = c_\infty \times \gamma_\infty$. To make θ_0 defined everywhere we have to perform some blowups. The situation is quite simple at S_∞, because $\theta_0 = \pi/f$ and $(1/\pi, 1/f)$ is a local coordinate system at points of S_∞. It is more complicated at S_0, where π has a double zero: at each point of S_0 there are local coordinates of the form (u, f) and a unit ε such that $\pi = \varepsilon u^2$, so that $\theta_0 = \varepsilon u^2/f$. So we need a two-step modification of $C \times \Gamma$, detailed below.

7.2 The blown-up surface X.

Let $\rho_1 : X' \to C \times \Gamma$ be the surface obtained by blowing up S_0 and S_∞, viewed as reduced subschemes. This gives rise to exceptional divisors $D_{1,0} = \rho_1^{-1}(S_0)$ and $D_{1,\infty} = \rho_1^{-1}(S_\infty)$.

If Y is a curve on $C \times \Gamma$, let us denote by $\rho_1^\bullet(Y)$ its proper transform on X'. Then the divisor of $\theta_1 := \theta_0 \circ \rho_1$ on X' is given by

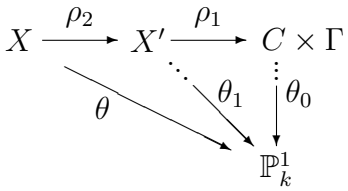
$$\begin{array}{lrcl} \text{poles:} & (\theta_1)_\infty & = & \rho_1^\bullet(C \times \gamma_\infty) + \rho_1^\bullet(c_0 \times \Gamma), \\ \text{zeros:} & (\theta_1)_0 & = & \rho_1^\bullet(c_\infty \times \Gamma) + 2\rho_1^\bullet(C \times 0_\Gamma) + D_{1,0}. \end{array}$$

So θ_1 is defined except at $S_0' := D_{1,0} \cap \rho_1^\bullet(c_0 \times \Gamma)$ where the zeros and poles meet: this finite set maps isomorphically onto S_0 in $C \times \Gamma$.

Now let $\rho_2 : X \to X'$ be the blowup of S_0', with exceptional divisor $D_{2,0}$, and let $\rho = \rho_1 \circ \rho_2 : X \to C \times \Gamma$. The divisor of $\theta := \theta_0 \circ \rho$ is given by

$$\begin{array}{lrcl} \text{poles:} & (\theta)_\infty & = & \rho^\bullet(C \times \gamma_\infty) + \rho^\bullet(c_0 \times \Gamma), \\ \text{zeros:} & (\theta)_0 & = & \rho^\bullet(c_\infty \times \Gamma) + 2\rho^\bullet(C \times 0_\Gamma) + \rho^\bullet(D_{1,0}). \end{array}$$

Since these divisors have disjoint supports, θ is a morphism to $\mathbb{P}^1$. Thus, we have a commutative diagram



and the properties of X and the family of curves θ are summarised in the following proposition, where we assume k separably closed to alleviate notations; note that all our constructions commute with ground field extension. (The picture is of course simplified: for instance, $C \times \gamma_\infty$ in $C \times \Gamma$ consists in two copies of C, not one).

7.3 Proposition. *Assume that k is separably closed.*

(i) *X is a smooth projective surface, and ρ is birational.*

(ii) *ρ induces an isomorphism*

$$\rho^* : \underline{\mathrm{Pic}}^0_{C\times\Gamma/k} \xrightarrow{\sim} \underline{\mathrm{Pic}}^0_{X/k} \tag{73}$$

of abelian varieties over k.

(iii) *θ is projective and flat with geometrically connected fibres. For $\lambda \in k^*$, the fibre $\theta^{-1}(\lambda)$ maps isomorphically via ρ to the curve X_λ in $C \times \Gamma$.*

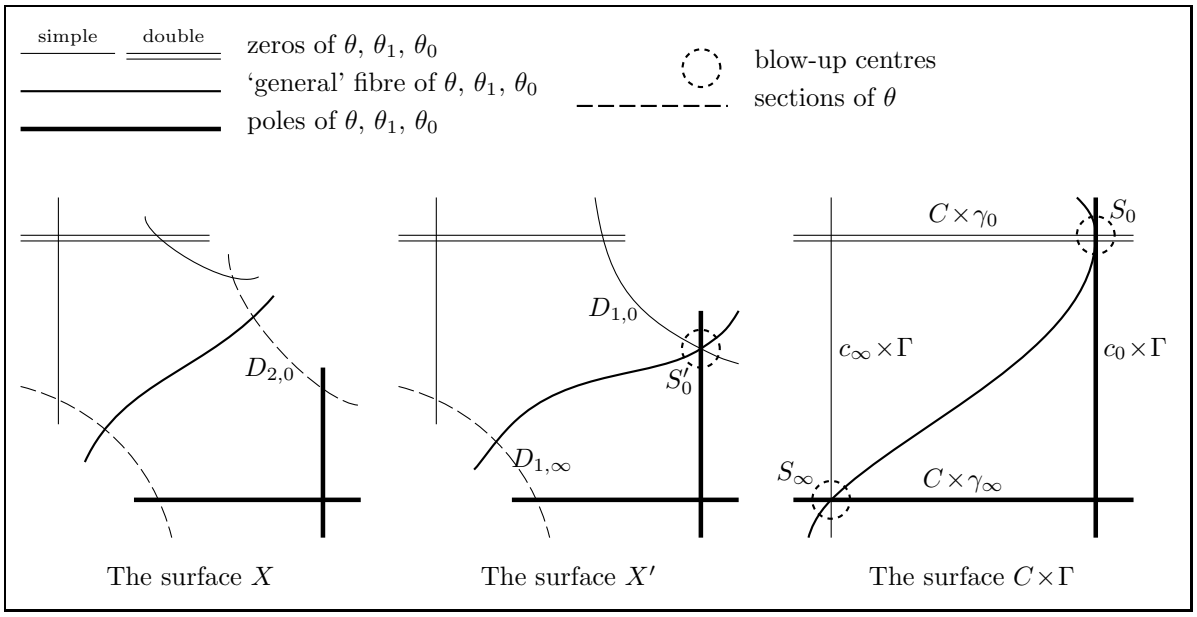
(iv) *The fibre $\theta^{-1}(\infty)$ maps isomorphically to $(C \times \gamma_\infty) \cup (c_0 \times \Gamma) \subset C \times \Gamma$. It is a union of d disjoint copies of Γ and 2 disjoint copies of C, each Γ of the first set meeting each C of the second transversally at one point.*

(v) *The fibre $X_0 = \theta^{-1}(0)$ is a union of:*

- *a copy of C, with multiplicity two in the fibre, mapping isomorphically to the divisor $2(C \times 0_\Gamma)$ in $C \times \Gamma$;*
- *d disjoint copies of Γ, each attached to the above copy of C by identifying $0_\Gamma \in \Gamma$ to one of the poles of f on C;*
- *d disjoint copies of $\mathbb{P}^1$, each attached to the above copy of C by identifying one point with a zero of f.*

(vi) *The fibre X_λ is semistable for each $\lambda \in \mathbb{P}^1_k \setminus \{0\}$.*

(vii) *Every component of $D_{2,0}$ (the exceptional divisor of ρ_2) maps isomorphically to $\mathbb{P}^1_k$ via θ, hence defines a section of θ. The same holds for every component of $\rho_2^{-1}(D_{1,\infty}) = \rho_2^\bullet(D_{1,\infty})$.*



Proof: Most assertions follow from a careful look at the construction of X, so we leave the details to the reader. For (ii), use 2.5 (viii) twice. For (vi), use (iv) for the fibre at ∞, and 6.3.2 (iii) for the other fibres. (Also, note that if $p > 0$ we use the assumption of 1.4.2 that the points of Q are separable over k, hence, under our assumptions, rational). ∎

Proposition 7.3 allows us to apply the results of Section 4 to θ:

7.3.1 Proposition. *Let $U \subset \mathbb{P}^1_k$ be the smooth locus of θ (or any nonempty open subset of it). We have the inclusions*

(i) $\mathrm{Reg}\,(E_U, J_U, k) \subset \mathrm{Good}\,(E, \Gamma, f, k)$,

(ii) $\textsc{Reg}\,(E_U, J_U, k) \subset \textsc{Good}\,(E, \Gamma, f, k)$

where the sets Good *and* Good *(resp.* Reg *and* Reg *) are defined in* 1.5.2 *(resp.* (29) *and* (30) *of Section* 3*).*

Proof: (i) Let λ belong to $\mathrm{Reg}\,(E_U, J_U, k)$. We have a commutative diagram

$$\begin{array}{ccc} \theta^{-1}(\lambda) & \xrightarrow{j_\lambda} & X \\ \downarrow\cong & & \downarrow\rho \\ X_\lambda = \widetilde{C}_{\lambda f} & \xrightarrow{i_{\lambda f}} & C \times \Gamma \end{array} \tag{74}$$

where j_λ and $i_{\lambda f}$ are the natural inclusions, and the left vertical map is an isomorphism by 7.3 (iii). Applying the '$\underline{\mathrm{Pic}}^0_{?/k}$' functor, we get a commutative diagram

$$\begin{array}{ccc} J_\lambda & \xleftarrow{j_\lambda^*} & \underline{\mathrm{Pic}}^0_{X/k} \\ \uparrow\cong & & \uparrow{\scriptstyle \rho^*}\cong \\ \mathrm{Jac}\,(\widetilde{C}_{\lambda f}) & \xleftarrow{i_{\lambda f}^*} & \underline{\mathrm{Pic}}^0_{C\times\Gamma/k} \end{array} \tag{75}$$

where ρ^* is an isomorphism by 7.3 (ii).

By Proposition 7.3, θ satisfies all the assumptions of 4.1. So we can apply Theorem 4.4.1 (with $A = E$) to conclude that the group homomorphism

$$H(\lambda) : \mathrm{Hom}_k\,(E, \underline{\mathrm{Pic}}^0_{X/k}) \to \mathrm{Hom}_k\,(E, J_\lambda),$$

deduced from j_λ^* by functoriality, is almost bijective. But from diagram (75) we see that the same holds for the homomorphism

$$\begin{array}{ccc} \mathrm{Hom}_k\,(E, \underline{\mathrm{Pic}}^0_{C\times\Gamma/k}) & \longrightarrow & \mathrm{Hom}_k\,(E, \mathrm{Jac}\,(\widetilde{C}_{\lambda f})) \\ u & \longmapsto & i_{\lambda f}^* \circ u \end{array}$$

which is the morphism (69) with $g = \lambda f$. Hence we conclude by Proposition 6.5.3 that λf is good, i.e. $\lambda \in \mathrm{Good}\,(E, \Gamma, f, k)$.

The proof of (ii) is completely similar: just apply 6.5.4 instead of 6.5.3, and Theorem 4.5.2 instead of 4.4.1, observing that the extra assumptions of 4.5.2 are satisfied here (in particular the fibre at infinity does have a multiple component, but its multiplicity is 2; this of course would ruin our argument in characteristic 2). ∎

7.4 Proofs of Main Theorem 1.7 and Theorem 1.12.

Our Main Theorem now readily follows from Proposition 7.3.1 and the specialisation theorems 3.3 and 3.4. Let us first prove 1.7 (i): let k' be an extension of k, and let $\lambda \in k'$ be transcendental over k. Then $\lambda \in \mathrm{Reg}\,(E_U, J_U, k')$ by 3.3 (ii), hence $\lambda \in \mathrm{Good}\,(k')$ by 7.3.1 (ii).

Assume now that k is finitely generated over the prime field. By 3.3 (iii), there is a Hilbert subset of k contained in $\mathrm{Reg}\,(k)$, hence in $\mathrm{Good}\,(k)$ by 7.3.1 (ii). This proves 1.7 (ii).

The proof of Theorem 1.12 is similar. Namely, we assume here that k is presented over the prime field ([F-J], 17.2), and that E and C are explicitly given. By 7.3.1 (i) it suffices to find an effective Hilbert set in $\mathrm{Reg}\,(E_U, J_U, k)$. This is possible by the effective version 3.4 of the specialisation theorem, provided that the rank of $\mathrm{Hom}_\eta\,(E_\eta, J_\eta)$ is known (here η is the generic point of $\mathbb{P}^1_k$) and that we have explicit equations for E_η and J_η. For E_η, just take the equations of E. For J_η, a procedure for finding equations for the Jacobian of a curve is given in [A].

For the rank of $\mathrm{Hom}_\eta\,(E_\eta, J_\eta)$, we have a chain of isomorphisms

$$\mathrm{Hom}_\eta\,(E_\eta, J_\eta) \xrightarrow[(4.4.1)]{\cong} \mathrm{Hom}_k\,(E, \underline{\mathrm{Pic}}^\circ_{X/k}) \xrightarrow[(7.3\,(\mathrm{ii}))]{\cong} \mathrm{Hom}_k\,(E, \underline{\mathrm{Pic}}^\circ_{C\times\Gamma/k})$$
$$\xrightarrow[(2.5\,(\mathrm{ix}))]{\cong} \mathrm{Hom}_k\,(E, \mathrm{Jac}\,(C)\times\mathrm{Jac}\,(\Gamma)) \xrightarrow{\cong} \mathrm{Hom}_k\,(E, \mathrm{Jac}\,(C))\times\mathrm{Hom}_k\,(E, \mathrm{Jac}\,(\Gamma))$$

which completes the proof since the rank of the right-hand side is known by assumption. ∎

Part III
Applications to undecidability

8 Self-twisted elliptic curves.

8.1 Notations.

We denote by κ a field of characteristic different from 2; the important cases in applications will be $\kappa = \mathbb{Q}$ and $\kappa =$ our ground field k of characteristic zero.

We fix an elliptic curve E over κ (in our applications, E will be defined over $\mathbb{Q}$). We have a canonical double cover

$$\pi = E \to L \tag{76}$$

which is the quotient of E by the involution $[-1]$. The curve L is smooth projective of genus zero, with a rational point (the image of the origin of E), hence is isomorphic to $\mathbb{P}^1_\kappa$. For the moment we refrain from fixing a coordinate on L (or equations of E), to emphasise the intrinsic character of our constructions. However, we can safely denote by $0 \in L(\kappa)$ the image of the origin.

Thus, the branch locus of π consists of the point 0 plus three other (geometric) points of L, the images of the points of order two of E.

We denote by $\kappa(L)$ the function field of L, by $\mathscr{O} \subset \kappa(L)$ the local ring of 0 in L, and by $\mathfrak{m}$ its maximal ideal. Finally we put $S = \mathrm{Spec}\,(\mathscr{O})$.

8.2 The self-twist $\mathscr{E}$.

We denote by $E_L = E \times_\kappa L \to L$ the constant L-elliptic curve deduced from E by base change, and we consider the quadratic twist

$$\mathscr{E} \longrightarrow L \tag{77}$$

of E_L by π, as defined in 5.3: this is the *self-twist* of E. It is a smooth quasiprojective group scheme over L, which induces an elliptic curve over the complement L_{nd} of the branch locus of π (notation of 5.4, (47)).

Note that if, say, $\xi \in L(\kappa)$ is a rational point, then $\pi^{-1}(\xi)$ is a double cover of $\mathrm{Spec}\,(\kappa)$ (the spectrum of a two-dimensional κ-algebra), and the fibre $\mathscr{E}_\xi$ is the twist of E by $\pi^{-1}(\xi)$.

Recall from 5.4 that we have important open subschemes

$$\mathscr{E}_{\mathrm{aff}} \subset \mathscr{E}^\circ \subset \mathscr{E}$$

where $\mathscr{E}_{\mathrm{aff}}$ is affine over L and $\mathscr{E}^\circ$ is a subgroup scheme of $\mathscr{E}$ with connected fibres.

8.3 Sections of $\mathscr{E}$; the canonical section.

From the definition (44) of a twist, we have in particular a canonical isomorphism of groups

$$\mathscr{E}(L) \xrightarrow{\sim} \mathrm{Mor}_L^{\mathrm{odd}}(E, E_L) \tag{78}$$

where E is viewed as an L-scheme via π. But by definition of E_L, this boils down to

$$\begin{array}{rcl} \mathscr{E}(L) & \xrightarrow{\sim} & \mathrm{Mor}_\kappa^{\mathrm{odd}}(E, E) \\ & \xrightarrow{\sim} & \mathrm{End}_\kappa(E) \times E[2](\kappa) \end{array} \tag{79}$$

(clearly, odd morphisms $E \to E$ have the form $\tau \circ u$ where u is a group scheme endomorphism and τ is translation by a 2-division point). In particular, we have a canonical section

$$\gamma : L \to \mathscr{E} \tag{80}$$

corresponding to Id_E under the first isomorphism of (79) (and to $(\mathrm{Id}_E, 0)$ under the second one). We call γ the *canonical* element of $\mathscr{E}(L)$. It has the following 'tautological' description: if, say, $\xi \in L(\kappa)$ is a rational point, then we have an inclusion $\pi^{-1}(\xi) \hookrightarrow E$ which obviously respects involutions (by definition of π). But this is precisely the definition of a κ-point of the fibre of $\mathscr{E}$ at ξ, and this point is just $\gamma(\xi)$.

We shall denote by γ_S (or $\gamma_{\mathscr{O}}$) the section of $\mathscr{E}_S$ induced by γ; similarly we have $\gamma_{\kappa(L)} \in \mathscr{E}(\kappa(L))$.

Note that by construction γ has infinite order in $\mathscr{E}(S)$; we take this opportunity to prove the following result, which will be used in Section 12.

8.3.1 Lemma. *Assume $k = \mathbb{Q}$. Then for all but finitely many $\xi \in L(\mathbb{Q})$, the fibre $\mathscr{E}_\xi$ of $\mathscr{E}$ at ξ is an elliptic curve, and $\gamma(\xi) \in \mathscr{E}_\xi(\mathbb{Q})$ has infinite order.*

In particular, every elliptic curve over $\mathbb{Q}$ has a quadratic twist with positive rank.

Proof: Clearly, $\mathscr{E}_\xi$ is an elliptic curve for almost every ξ. For such a ξ, let $F_\xi \subset \overline{\mathbb{Q}}$ be the field of rationality of the two points $\pm\zeta$ of $\pi^{-1}(\xi)$: we have $[F_\xi : \mathbb{Q}] \leq 2$. By definition of γ, $\gamma(\xi)$ has finite order in $\mathscr{E}_\xi(\mathbb{Q})$ if and only if ζ has finite order in $E(F_\xi)$. So, all we have to show is that the set T of torsion points of $E(\overline{\mathbb{Q}})$ which are rational over some quadratic extension of $\mathbb{Q}$ is finite. But this is an easy consequence of the theory of heights, for which we refer to [Lan], Chapter 5 or to [Se2], Chapters 2 and 3. Namely, the canonical height of any point of T is zero, while bounding both the canonical height and the degree defines a finite subset of $E(\overline{\mathbb{Q}})$. ∎

8.3.2 Proposition. (i) *The canonical homomorphisms*

$$\mathscr{E}(L) \longrightarrow \mathscr{E}(\mathscr{O}) \longrightarrow \mathscr{E}(\kappa(L))$$

are isomorphisms. In particular, by (79), we have an isomorphism

$$\mathrm{Mor}_\kappa^{\mathrm{odd}}(E, E) \xrightarrow{\sim} \mathscr{E}(\mathscr{O}). \tag{81}$$

(ii) *The isomorphism* (81) *above maps* $\mathrm{End}_\kappa(E)$ *onto* $\mathscr{E}^\circ(\mathscr{O})$. *In particular,* $\gamma_\mathscr{O} \in \mathscr{E}^\circ(\mathscr{O})$.

(iii) *The element* $\gamma(0) \in \mathscr{E}_0^\circ(\kappa)$ *is nonzero.*

(iv) $\mathscr{E}_{\mathrm{aff}}(\mathscr{O}) = \mathscr{E}^\circ(\mathscr{O})$. *(Hence, by* (ii)*,* $\mathbb{Z}\,\gamma_\mathscr{O} \subset \mathscr{E}_{\mathrm{aff}}(\mathscr{O})$*).*

(v) *If* E *does not have complex multiplication over* κ *(that is,* $\mathrm{End}_\kappa(E) \cong \mathbb{Z}$*), then* $\mathbb{Z}\,\gamma_\mathscr{O} = \mathscr{E}_{\mathrm{aff}}(\mathscr{O}) = \mathscr{E}^\circ(\mathscr{O})$.

Proof: (i) follows from the Néron property 5.4.6, since E is a regular scheme, and (v) is a trivial consequence of (i), (ii) and (iv).

Let us prove (ii). Let $u : E \to E$ be an odd κ-morphism, and let $\mu \in \mathscr{E}(\mathscr{O})$ be the corresponding section of $\mathscr{E}$. First, since $\mathscr{E}_{\kappa(L)} = \mathscr{E}^\circ_{\kappa(L)}$, we have $\mu \in \mathscr{E}^\circ(\mathscr{O})$ if and only if $\mu(0) \in \mathscr{E}_0^\circ$, the fibre of $\mathscr{E}^\circ$ at zero. Now, $\mu(0)$ is obtained as follows. Consider $j : \pi^{-1}(0) \hookrightarrow E$. This is simply the first infinitesimal neighbourhood of the origin ω of E, isomorphic as a scheme to $\mathrm{Spec}\,(\kappa[\varepsilon])$ (with $\varepsilon^2 = 0$). The composition $u \circ j : \pi^{-1}(0) \to E$ is an odd morphism, hence by definition a point of $\mathscr{E}_0(\kappa)$, which is precisely $\mu(0)$. But $u \circ j$ sends the closed point of $\pi^{-1}(0)$ to $u(\omega)$; hence, by the criterion 5.4.3 (iii) (b), $\mu(0)$ is in the connected component if and only if $u(\omega) = \omega$, that is, if and only if u is a group endomorphism. This proves (ii).

The previous computation, applied with $u = \mathrm{Id}_E$ (or, equivalently, the tautological desription of γ) shows that $\gamma(0)$ is the point of $\mathscr{E}_0^\circ(\kappa)$ corresponding to the inclusion $j : \pi^{-1}(0) \hookrightarrow E$; this is clearly nonzero, which proves (iii).

It remains to prove (iv). We know that $\mathscr{E}_{\mathrm{aff}}$ is an open subscheme of $\mathscr{E}^\circ$, and that they have the same fibre at the closed point 0 of $\mathrm{Spec}\,(\mathscr{O})$ (cf. 5.4.2). It follows that if $z : \mathrm{Spec}\,(\mathscr{O}) \to \mathscr{E}^\circ$ is a section, then $z^{-1}(\mathscr{E}_{\mathrm{aff}})$ is an open subscheme of $\mathrm{Spec}\,(\mathscr{O})$ which contains the closed point, hence is equal to $\mathrm{Spec}\,(\mathscr{O})$. ∎

8.3.3 Remark. Assertion (i) generalises (with the same proof) in the following way: if $\mathscr{O} \subset \mathscr{O}'$, where $\mathscr{O}'$ is a regular semilocal domain of dimension 1, whose Jacobson radical is generated by the maximal ideal of $\mathscr{O}$, and if K denotes the fraction field of $\mathscr{O}'$, then $\mathscr{E}(\mathscr{O}') \longrightarrow \mathscr{E}(K)$ is an isomorphism.

This applies in particular if C is a smooth curve over (some extension of) κ, given with a morphism $g : C \to L$, and $\mathscr{O}'$ is the semilocal ring of C at some set of simple poles of g.

8.3.4 Remark. Assertion (v) has the following consequence. Assume in addition that E does not have complex multiplication over the algebraic closure of κ, and hence over any extension of κ. For an extension κ' of κ, let $\mathscr{O}_{\kappa'}$ be the local ring at infinity on $L_{\kappa'}$. Then it follows from (v) that $\mathscr{E}^\circ(\mathscr{O}) \xrightarrow{\sim} \mathscr{E}^\circ(\mathscr{O}_{\kappa'})$ since both are generated by the same element $\gamma_\mathscr{O}$. In other words, the group $\mathscr{E}^\circ(\mathscr{O})$, which is isomorphic to $\mathbb{Z}$, is essentially independent of the ground field κ.

8.3.5 Remark. It follows from (iv) that $\mathscr{E}_{\mathrm{aff}}(\mathscr{O})$ is a subgroup of $\mathscr{E}(\mathscr{O})$, even though $\mathscr{E}_{\mathrm{aff}}$ is not a subgroup scheme of $\mathscr{E}$.

9 The ring Λ and its multiplication.

9.1 Notations, definition of Λ.

In this section we keep the notations and assumptions of Section 8 (including the self-twist $\mathscr{E}$ and its canonical section γ), but now we assume $\operatorname{char}\kappa = 0$.

We fix a ring $\mathscr{O}'$ containing $\mathscr{O}$, with the aim of proving that the Diophantine theory of $\mathscr{O}'$ (with constants $\mathscr{O}$) is undecidable.

We identifiy $\mathscr{E}(\mathscr{O})$ with a subset of $\mathscr{E}(\mathscr{O}')$, and similarly for $\mathscr{E}_{\mathrm{aff}}$, $\mathscr{E}^{\circ}$, etc. In particular we have a subgroup

$$\Lambda := \mathbb{Z}\,\gamma_{\mathscr{O}} \subset \mathscr{E}^{\circ}(\mathscr{O}) \subset \mathscr{E}^{\circ}(\mathscr{O}') \tag{82}$$

which is, in fact, contained in $\mathscr{E}_{\mathrm{aff}}(\mathscr{O})$ by 8.3.2 (iv), and therefore also in $\mathscr{E}_{\mathrm{aff}}(\mathscr{O}')$.

The group isomorphism $\mathbb{Z} \xrightarrow{\sim} \Lambda$ sending n to $n\gamma_{\mathscr{O}}$ defines a ring structure on Λ. To prove the Diophantine undecidability of $\mathscr{O}'$, it suffices to prove that, for suitable E, this ring $\Lambda \subset \mathscr{E}_{\mathrm{aff}}(\mathscr{O}')$ is Diophantine (this makes sense since $\mathscr{E}_{\mathrm{aff},\mathscr{O}}$ is an affine $\mathscr{O}$-scheme of finite presentation; we shall be more explicit in 9.3.2 below).

Proving that Λ is a Diophantine ring involves two tasks:

- show that the ring structure on Λ is relatively Diophantine, in the sense of 2.7.6,
- show that $\Lambda \subset \mathscr{E}_{\mathrm{aff}}(\mathscr{O}')$ is Diophantine.

Concerning the second property, note that by 8.3.2 (v), it is true for $\mathscr{O}' = \mathscr{O}$ if E has no complex multiplication, which we shall always assume in applications. In fact this extends to the situation mentioned in 8.3.3. For other rings (specifically for function fields of curves) our standard weapon will be Theorem 1.8.

But this will come later; this section is devoted to the first property, which involves checking several points. Here are the easy ones:

9.1.1 Proposition. (i) *The graph of the addition law on Λ is relatively Diophantine in Λ^3, with respect to $\mathscr{O}$ (here Λ is seen as a subset of $\mathscr{E}_{\mathrm{aff}}(\mathscr{O}')$).*

(ii) *The unit $\{\gamma_{\mathscr{O}'}\}$ of Λ is a Diophantine subset of $\mathscr{E}_{\mathrm{aff}}(\mathscr{O}')$.*

Proof: (ii) is obvious since $\gamma \in \mathscr{E}_{\mathrm{aff}}(\mathscr{O})$.

For (i), we have to be careful because $\mathscr{E}_{\mathrm{aff}}$ is not a group scheme; however, the graph G of 'addition' in $\mathscr{E}^3_{\mathrm{aff},\mathscr{O}}$ makes sense, as the intersection of $\mathscr{E}^3_{\mathrm{aff},\mathscr{O}}$ with the graph of addition on the $\mathscr{O}$-group scheme $\mathscr{E}^3_{\mathscr{O}}$. Moreover, G is clearly a closed $\mathscr{O}$-subscheme of $\mathscr{E}^3_{\mathscr{O}}$, hence defines a ternary relation on $\mathscr{E}_{\mathrm{aff}}(\mathscr{O}')$ which is Diophantine with respect to $\mathscr{O}$. The conclusion follows by restriction to Λ. ∎

Thus (as always with this method) the hard part is the Diophantine character of multiplication in Λ, which will occupy the rest of this section.

9.2 Evaluating at zero.

Recall that by (48) the fibre $\mathscr{E}^\circ_0$ of $\mathscr{E}^\circ$ at zero is isomorphic to the additive group $\mathbb{G}_{\mathrm{a},\kappa}$. Hence (once such an isomorphism is fixed, which we assume from now on), evaluating sections at 0 defines a group homomorphism

$$\mathrm{ev}_0 : \quad \mathscr{E}^\circ(\mathscr{O}) \longrightarrow \mathscr{E}^\circ(\mathscr{O}/\mathfrak{m}) \xrightarrow{\sim} \kappa. \tag{83}$$

If we restrict this map to $\mathscr{E}_{\mathrm{aff}}(\mathscr{O})$, embedded, say, in the affine plane $\mathbb{A}^2_{\mathscr{O}}$, this simply consists in reducing coordinates modulo $\mathfrak{m}$, and then applying the isomorphism with $\mathbb{G}_{\mathrm{a},\kappa}$ (which is algebraic, hence must be given by a polynomial in the coordinates, with coefficients in κ).

The restriction of ev_0 to Λ necessarily has the form

$$n\gamma_{\mathscr{O}} \longmapsto n\,\mathrm{ev}_0(\gamma_{\mathscr{O}}) = n\gamma(0). \tag{84}$$

Since $\mathrm{char}\,\kappa = 0$ and $\gamma(0) \neq 0$ by 8.3.2 (iii), this map is injective. Therefore we can 'encode' the multiplication on Λ as follows: if $z_i = n_i\gamma_{\mathscr{O}}$ $(i = 1, 2, 3)$ are three elements of Λ, then:

$$\begin{aligned} z_3 = z_1 z_2 \ \ (\text{in } \Lambda) \ \ &\Leftrightarrow \ \ n_3 = n_1 n_2 \ \ (\text{in } \mathbb{Z}) \\ &\Leftrightarrow \ \ \mathrm{ev}_0(z_3)\,\mathrm{ev}_0(\gamma) = \mathrm{ev}_0(z_1)\,\mathrm{ev}_0(z_2) \ \ (\text{in } \kappa). \end{aligned} \tag{85}$$

The last condition involves the relation $t_3\,\mathrm{ev}_0(\gamma) = t_1\,t_2$ in κ. This is a polynomial relation (in which $\mathrm{ev}_0(\gamma)$ is a constant), which is good news. But it also involves ev_0, i.e. essentially a reduction modulo $\mathfrak{m}$, which is rather bad news. In fact, from now on, all the hard work will consist in showing, in various contexts, that in some sense reduction modulo $\mathfrak{m}$ has good Diophantine properties.

9.3 Explicit equations.

Assume now that $E \subset \mathbb{P}^2_\kappa$ is given by an equation

$$Y^2Z = P(X,Z) = X^3 + a\,X^2\,Z + b\,X\,Z^2 + c\,Z^3 \tag{86}$$

in homogeneous coordinates (X, Y, Z) (in our applications, a, b, c will be in $\mathbb{Q}$). We may, and will, identify $\mathbb{P}^1_\kappa$ with L via the double cover $Z/X : E \to \mathbb{P}^1_\kappa$ (also called π); this is the inverse of the 'usual' coordinate $x := X/Z$, for which we shall have little use. We denote by t the standard coordinate on $\mathbb{P}^1_\kappa$. Thus, the branch locus of π consists of the three (geometric) zeros of $P(1, t)$ and the point 0. The ring $\mathscr{O}$ is then $\kappa[t]_{(t)}$, with maximal ideal $\mathfrak{m} = t\mathscr{O}$.

9.3.1 Remark. Unlike 0, the 'point at infinity' of L (the pole of t in $L = \mathbb{P}^1_\kappa$) has no intrinsic meaning; in fact, by a change of coordinates it can be chosen arbitrarily in $L\setminus\{0\}$. In particular, assume that E is defined over $\mathbb{Q}$; then, by 8.3.1, we can choose the equation (86) (with $P \in \mathbb{Q}[X, Z]$) in such a way that the fibre $\mathscr{E}_\infty$ of $\mathscr{E}$ at ∞ is an elliptic curve (this simply means $c \neq 0$) and, moreover, that the point $\gamma(\infty) \in \mathscr{E}_\infty(\kappa)$ has *infinite order*.

Let us now give equations for (some pieces of) $\mathscr{E}$.

9.3.2 Equations for $\mathscr{E}^\circ_S$ and $\mathscr{E}_{\mathrm{aff},S}$. We only give the results derived from Section 5, leaving details to the reader.

The traditional way of describing π as a double cover of $\mathbb{P}^1_\kappa$ is by 'extracting the square root of $P(x,1)$'; however, $P(x,1) = t^{-3}P(1,t)$ does not belong to $\mathscr{O}$ (it has a triple pole at zero), so instead we put

$$\rho := t/P(1,t) = t/(1 + a\,t + b\,t^2 + c\,t^3); \tag{87}$$

this is a uniformising parameter of $\mathscr{O}$, such that the double cover π is given above S by $\mathrm{Spec}\,(\mathscr{O}[\sqrt{\rho}\,])$. Accordingly, by 5.4.5, $\mathscr{E}^\circ_S \to S$ can be described in $\mathbb{P}^2_S$ by

$$\mathscr{E}^\circ_S = \overline{\mathscr{E}^\circ} \setminus \mathscr{F}, \text{ with } \begin{cases} \overline{\mathscr{E}^\circ}: & V^2\,W = \rho\,P(U,W) \\ \mathscr{F}: & V = \rho = 0 \end{cases} \tag{88}$$

in projective coordinates U, V, W. (In fact, this is not just a description of $\mathscr{E}^\circ_S$, but also of the restriction of $\mathscr{E}^\circ$ above the complement of ∞ in $\mathbb{P}^1_\kappa$.) The unit section of the group scheme $\mathscr{E}^\circ_S$ is $(0:1:0)$.

The open subscheme $\mathscr{E}_{\mathrm{aff}}$ corresponds to $V \neq 0$; in affine coordinates $u = U/V$, $w = W/V$, it is given by

$$\mathscr{E}_{\mathrm{aff}}: \quad w = \rho\,P(u,w). \tag{89}$$

The fibre $\mathscr{E}^\circ_0$ at 0 is the affine line $W = 0$, $V \neq 0$ in $\mathbb{P}^2_\kappa$; it is isomorphic to $\mathbb{G}_{\mathrm{a},\kappa}$ via the map

$$\begin{array}{ccc} \mathscr{E}^\circ_0 & \xrightarrow{\sim} & \mathbb{G}_{\mathrm{a},\kappa} \\ (U:V:0) & \longmapsto & U/V \end{array} \tag{90}$$

or, using the affine coordinates of $\mathscr{E}_{\mathrm{aff}}$, via the coordinate u. With the above identification, the evaluation at infinity is given on $\mathscr{E}_{\mathrm{aff}}$ (in these affine coordinates u, v) by the very simple formula, where u is viewed as a function on $\mathscr{E}_{\mathrm{aff}}$:

$$\begin{array}{rcl} \mathrm{ev}_0 : \mathscr{E}_{\mathrm{aff}}(\mathscr{O}) & \longrightarrow & \kappa \\ z & \longmapsto & u(z) \bmod \mathfrak{m}. \end{array} \tag{91}$$

The canonical section γ is given by $(U:V:W) = (1:1:t)$; in particular, its value at 0 is $(1:1:0)$, which is indeed a nonzero element of $\mathscr{E}^\circ_0$, as predicted by 8.3.2 (iii). In fact, using (90) to identify $\mathscr{E}^\circ_0(\kappa)$ with κ, we have $\mathrm{ev}_0(\gamma) = 1$, hence the restriction of is ev_0 to Λ (identified with $\mathbb{Z}$) is just the natural inclusion of $\mathbb{Z}$ into κ.

In particular, for the multiplication on Λ, property (85) boils down to the following: if z_i $(i = 1,2,3)$ are three elements of Λ, then

$$z_3 = z_1 z_2 \text{ (in } \Lambda) \;\Leftrightarrow\; u(z_3) \equiv u(z_1)\,u(z_2) \pmod{\mathfrak{m}}. \tag{92}$$

This has the following consequence:

9.4 Proposition. *Assume there exists an additive subgroup X of $\mathscr{O}$ with the following properties:*

(i) *X contains all elements of the form $u(z_1)\,u(z_2)$ with $z_1, z_2 \in \Lambda$;*

(ii) *the inclusion of $X_+ := X \cap \mathfrak{m}$ into X is relatively Diophantine (as subsets of $\mathscr{O}'$, with respect to $\mathscr{O}$).*

Then the multiplication (hence the whole ring structure) on Λ is relatively Diophantine.

Proof: By assumption, there is a Diophantine subset $\mathscr{D} \subset \mathscr{O}'$ such that $\mathscr{D} \cap X = X_+$. It follows that if $z_i \in \Lambda$ $(i = 1, 2, 3)$ we have:

$$z_3 = z_1 z_2 \;\Leftrightarrow\; u(z_3) - u(z_1)\,u(z_2) \in \mathfrak{m} \;\Leftrightarrow\; u(z_3) - u(z_1)\,u(z_2) \in \mathscr{D}$$

since, by our assumptions, $u(z_3) - u(z_1)\,u(z_2) \in X$ (note that $u(z_3) = u(z_3)\,u(\gamma)$). ∎

The simplest choice for X is, of course, $X = \mathscr{O}$, which gives:

9.4.1 Corollary. *Assume that t is not invertible in $\mathscr{O}'$ (in other words, $\mathfrak{m}\mathscr{O}' \neq \mathscr{O}'$). Then the ring structure on Λ is relatively Diophantine.*

Proof: We have $\mathfrak{m}\mathscr{O}' \cap \mathscr{O} = \mathfrak{m}$ since it is a proper ideal of $\mathscr{O}$ containing $\mathfrak{m}$. But of course $\mathfrak{m}\mathscr{O}' = t\mathscr{O}'$ is Diophantine in $\mathscr{O}'$, hence we can apply 9.4 with $X = \mathscr{O}$ (and $X_+ = \mathfrak{m}$). ∎

9.4.2 Remark. This of course applies to $\mathscr{O}' = \mathscr{O}$. In fact, at this point we can already conclude that $\mathscr{O}$ *is positive-existentially undecidable*; in other words, for any field k of characteristic zero, the local ring $k[t]_{(t)}$ is positive-existentially undecidable with respect to $\mathbb{Q}[t]_{(t)}$. Indeed, choose any E over $\mathbb{Q}$ without complex multiplication: then, from assertions (v) and (iv) of 8.3.2 we have $\Lambda = \mathscr{E}_{\mathrm{aff}}(\mathscr{O})$, so Λ is Diophantine, hence is a Diophantine ring by 9.4.1. Of course this will be generalised later.

9.5 Description of $\mathscr{E}$ at infinity.

The results below will be needed in Section 12 to treat the p-adic case, because the Kim-Roush method involves controlling the order of certain functions at ∞.

Denote by $\mathscr{R} = \kappa[t^{-1}]_{(t^{-1})}$ the local ring of $\mathbb{P}^1_\kappa$ at ∞, and put $T = \mathrm{Spec}\,(\mathscr{R})$. Assume that $c \neq 0$: then ∞ is not a branch point of π, and $\mathscr{E}_T$ is an elliptic curve. Accordingly, $P(t^{-1}, 1)$ is a unit of $\mathscr{R}$, so we can view π (above T) as $\mathrm{Spec}\,(\mathscr{R}[\sqrt{\rho'}])$ where $\rho' = P(t^{-1}, 1)^{-1}$ (this will give nicer coordinate changes than using $\sqrt{P(t^{-1}, 1)}$).

We can then describe $\mathscr{E}_T$ by the homogeneous equation

$$\mathscr{E}_T : \quad V'^2\,W' = \rho'\,P(U', W') \tag{93}$$

in homogeneous coordinates U', V', W'. The canonical section is given by $(t^{-1} : 1 : 1)$; the corresponding affine model is

$$(\mathscr{E}_{\mathrm{aff}})_T : \quad w' = \rho'\,P(u', w') \tag{94}$$

in affine coordinates $u' = U'/V'$, $w' = W'/V'$.

Of course, over $\operatorname{Spec}\kappa(t)$ (the intersection of S and T in $\mathbb{P}^1_\kappa$), equation (93) defines the same curve as (88); the isomorphism between the two models is readily checked to be given by

$$U' = t^{-1}U, \quad V' = V, \quad W' = t^{-1}W. \tag{95}$$

In particular, the rational functions $u = U/V$ and $u' = U'/V'$ on $\mathscr{E}$ are related by

$$u' = t^{-1}u. \tag{96}$$

This implies:

9.5.1 Proposition. (i) *Let $z \in \mathscr{E}(\mathbb{P}^1_\kappa)$ be a section. Assume that $z(\infty) \in \mathscr{E}_{\mathrm{aff}}$. Then the value $u(z) \in \kappa(t)$ of the rational function u at z has order ≥ -1 at ∞.*

(ii) *Assume that the condition of 9.3.1 is satisfied, i.e. $\gamma(\infty)$ has infinite order in $\mathscr{E}_\infty$. Then for every $z \in \Lambda$, the value $u(z)$ of the function u at z has order ≥ -1 at ∞.*

Proof: (i) The condition implies that z maps $T = \operatorname{Spec}(\mathscr{R})$ into $\mathscr{E}_{\mathrm{aff}}$. In particular, $u'(z)$ belongs to $\mathscr{R}$, i.e. has nonnegative order at ∞: the assertion then follows from (96).

(ii) The condition in (i) just means that $z(\infty)$ is not a point of order 2 on $\mathscr{E}_\infty$. With the assumption of (ii), this will be satisfied for $z = n\gamma$ (any $n \in \mathbb{Z}$), so (ii) follows from (i). ∎

9.5.2 Remark. Without explicitly computing the coordinate change (95), it was a priori clear that u must be a polynomial in u', v' with coefficients in $\kappa(t)$; it follows that 9.5.1 had to hold with -1 possibly replaced by some unspecified integer independent of z in (i) (resp. of n in (ii)). With some care, this 'computation-free' approach would be sufficient for our purposes.

9.5.3 Corollary. *Let X (resp. X_+) be the set of rational functins in $\kappa(t)$ having order ≥ -2 at ∞ and nonnegative (resp. positive) order at 0. Assume that X_+ is a relatively Diophantine subset of X (in $\mathscr{O}'$, with respect to $\mathscr{O}$).*

Then, if $\gamma(\infty)$ has infinite order in $\mathscr{E}_\infty$, the ring structure on Λ is Diophantine.

Proof: Clearly, X is a subgroup of $\mathscr{O}$ and $X_+ = X \cap \mathfrak{m}$. Also, it follows from 9.5.1 that X contains all products $u(z_1)u(z_2)$ for z_1, z_2 in Λ. So this is a special case of 9.4. ∎

10 Diophantine undecidability of semilocal rings of curves.

10.1 Notations.

10.1.1 The function field side. We denote by k a field of characteristic zero, by C a smooth projective geometrically connected curve over k, and by K the function field of C.

Let Q be a finite nonempty set of closed points of C. We choose f in K having simple ramification, simple zeros and simple poles on C, and vanishing at Q.

We denote by A the semilocal ring of C at Q:

$$A := \bigcap_{q \in Q} \mathscr{O}_{C,q}. \tag{97}$$

Thus, A is a regular semilocal domain of dimension 1 with fraction field K. It contains f, which generates its Jacobson radical; since $Q \neq \emptyset$, the intersection $A \cap \mathbb{Q}(f)$ is the ring

$$A_0 := A \cap \mathbb{Q}(f) = \mathbb{Q}[f]_{(f)}. \tag{98}$$

All Diophantine sets (in some affine space over A) will be relative to A_0.

10.1.2 The elliptic curve side. Let us fix an elliptic curve E over $\mathbb{Q}$. We choose an isomorphism $E/\{\pm \mathrm{Id}_E\} \xrightarrow{\sim} \mathbb{P}^1_{\mathbb{Q}}$ sending the origin to 0; we denote by t the standard coordinate on $\mathbb{P}^1$.

Applying the constructions of 8.1 and 8.2 with $\kappa = \mathbb{Q}$, we obtain a group scheme

$$\mathscr{E} \longrightarrow \mathrm{Spec}\,(\mathscr{O})$$

where $\mathscr{O} = \mathbb{Q}[t]_{(t)}$ is the local ring of $\mathbb{P}^1_{\mathbb{Q}}$ at 0 (this $\mathscr{E}$ was denoted by $\mathscr{E}_S$ or $\mathscr{E}_{\mathscr{O}}$ in 8.3 but we shall not need the original $\mathscr{E}$, which was over $\mathbb{P}^1_{\mathbb{Q}}$). Inside $\mathscr{E}$ we have open subschemes

$$\mathscr{E}_{\mathrm{aff}} \subset \mathscr{E}^\circ \subset \mathscr{E}$$

where $\mathscr{E}_{\mathrm{aff}} \subset \mathbb{A}^2_{\mathscr{O}}$ is affine. Recall also from 8.3 that we have a canonical section $\gamma \in \mathscr{E}_{\mathrm{aff}}(\mathscr{O})$, generating a subgroup $\Lambda = \mathbb{Z}\gamma$ of $\mathscr{E}(\mathscr{O})$, which is contained in $\mathscr{E}_{\mathrm{aff}}(\mathscr{O})$. We give Λ the ring structure deduced from the obvious isomorphism $\mathbb{Z} \xrightarrow{\sim} \Lambda$.

10.1.3 Where both sides meet. For any $\lambda \in \mathbb{Q}^*$, we can send $k[t]$ to A by mapping t to (λf). This gives a diagram of injective ring homomorphisms

$$\begin{array}{ccccc}
\mathscr{O} & & \mathscr{O}_k & & \\
\| & & \| & & \\
\mathbb{Q}[t]_{(t)} & \hookrightarrow & k[t]_{(t)} & \hookrightarrow & A \\
\bigcap & & \bigcap & & \bigcap \\
\mathbb{Q}(t) & \hookrightarrow & k(t) & \hookrightarrow & K \\
 & & t & \longmapsto & \lambda f.
\end{array}$$

I claim that

$$\mathscr{E}_{\mathrm{aff}}(\mathscr{O}_k) = \mathscr{E}(k(t)) \cap \mathscr{E}_{\mathrm{aff}}(A). \tag{99}$$

This is in fact obvious: if we embed $\mathscr{E}_{\mathrm{aff}}$ into, say, $\mathbb{A}^2$ in the usual way, then a point of $\mathscr{E}_{\mathrm{aff}}(A)$ is in $\mathscr{E}(k(t))$ if and only if its coordinates are in $k(t)$, hence in $k(t) \cap A = \mathscr{O}_k$.

Now, Theorem 1.8 (ii) (applied with $\Gamma = E$) implies that for suitable λ, we have

$$\mathscr{E}(k(t)) = \mathscr{E}(K) \tag{100}$$

(just take λ in $\mathrm{Good}\,(k) \cap \mathbb{Q}$).

We choose λ once and for all with this property, and identify t with λf, thus viewing all the maps in the above diagram as inclusions. Note that, independently of λ, the image of $\mathscr{O}$ in A is A_0.

10.2 Proposition. *With the notations and assumptions of 10.1, assume that E has no complex multiplication over $\mathbb{C}$.*

Then Λ is a Diophantine subset of A^2, and of K^2 (with respect to A_0).

Proof: By 8.3.2 (v) we have $\Lambda = \mathscr{E}_{\mathrm{aff}}(\mathscr{O}) = \mathscr{E}_{\mathrm{aff}}(\mathscr{O}_k)$. By (99) and (100) we have $\Lambda = \mathscr{E}(K) \cap \mathscr{E}_{\mathrm{aff}}(A) = \mathscr{E}_{\mathrm{aff}}(A)$ hence Λ is Diophantine in A^2.

Let us show that Λ is Diophantine in K^2. Recall that $\mathscr{E}(k(t)) = \Lambda \times E[2](k)$ by (79), hence $2\Lambda = 2\mathscr{E}(k(t)) = 2\mathscr{E}(K)$. This is also equal to $2\mathscr{E}_{\mathrm{aff}}(K)$ because the complement of $\mathscr{E}_{\mathrm{aff}}$ in $\mathscr{E}$ consists of the nontrivial 2-torsion points. But the graph of addition is Diophantine in $\mathscr{E}_{\mathrm{aff}}(K)^3$ (as in the proof of 9.1.1 (i)). Hence $2\Lambda = 2\mathscr{E}_{\mathrm{aff}}(K)$ is Diophantine in K^2, and so is $\Lambda = (2\Lambda) \cup (\gamma + 2\Lambda)$. ∎

We can now prove part (1) of Theorem 1.1 (more precisely the 1-dimensional case, which implies the general case as explained in the introduction):

10.3 Theorem. *With the notations and assumptions of 10.1.1, there is a Diophantine ring $\Lambda \subset A^2$, isomorphic to $\mathbb{Z}$. In particular, the positive-existential theory of A in $\mathrm{LR}(A_0)$ is undecidable.*

Proof: Choose any elliptic curve E over $\mathbb{Q}$, without complex multiplication over $\mathbb{C}$. Applying the constructions of 10.1.2 and 10.1.3, we conclude from 10.2 that Λ is Diophantine in $\mathscr{E}_{\mathrm{aff}}(A)$, hence is a Diophantine ring by 9.4.1, applied with $\mathscr{O}' = A$ (the fact that $Q \neq \emptyset$ is used here!). ∎

10.3.1 Remark. The Diophantine ring in 10.3 has a very simple explicit definition, following from the computations in 9.3. Let E_{aff} be given by the equation

$$z = P(x,z) = x^3 + a\,x^2\,z + b\,x\,z^2 + c\,z^3$$

(which is the affine form of (86)), with $a, b, c \in \mathbb{Q}$; of course we assume that the discriminant of $P(x,1)$ is nonzero, and also that E has no complex multiplication over $\mathbb{C}$ (this is easy to ensure; for instance it is true if the j-invariant of E is not an integer).

We choose $\lambda \in \mathbb{Q}^*$ as in 10.1.3, i.e. such that (100) holds, and we define $\mathscr{E}_{\mathrm{aff}} \subset \mathbb{A}^2_{A_0}$ by the equation (in affine coordinates (u, w)):

$$w = \frac{\lambda f}{P(1, \lambda f)}\, P(u, w).$$

Now, the Diophantine ring Λ is defined as follows:

(i) the underlying set is the subset $\mathscr{E}_{\mathrm{aff}}(A)$ of A^2 (that is, the set of solutions $(u, w) \in A^2$ of the above equation),

(ii) the zero element is $0_\Lambda = (0, 0)$,

(iii) the ring unit is $1_\Lambda = \gamma = (1, \lambda f)$,

(iv) the addition is given by the elliptic curve law, or equivalently by:

$$(u'', w'') = (u, w) + (u', w') \;\Leftrightarrow\; \exists\, \alpha,\ u'' = u + u' + \alpha f,$$

(v) the multiplication is given by:

$$(u'', w'') = (u, w)(u', w') \;\Leftrightarrow\; \exists\, \alpha,\ u'' = uu' + \alpha f.$$

10.3.2 Remark. Assume that there is a Diophantine subset $\mathscr{D}$ of A such that $\mathbb{Z} \subset \mathscr{D} \subset k$ (or, more generally, that $\mathbb{Z} \subset \mathscr{D}$ and the composite $\mathscr{D} \hookrightarrow A \to A/fA$ is injective). Then we have the stronger property that $\mathbb{Z}$ is Diophantine in A: indeed, for $\alpha \in A$, we have $\alpha \in \mathbb{Z}$ if and only if $\alpha \in \mathscr{D}$ and there exists $(u, w) \in \mathscr{E}_{\mathrm{aff}}(A)$ such that $\alpha - u \in fA$.

10.3.3 Corollary. *Assume that A (or equivalently, its Jacobson radical fA) is Diophantine in K. Then there is a Diophantine ring in K^2, isomorphic to $\mathbb{Z}$.*

In particular, the positive-existential theory of K in $\mathrm{LR}(A_0)$ is undecidable. ∎

10.3.4 Corollary. *Let k be a* real closed *field, and let C be a smooth k-curve having a rational point. Then the function field K of C is positive-existentially undecidable.*

Proof: Let $q \in C(k)$ be a rational point. One can find $\varphi \in K$ having only simple zeros, and vanishing at q. If Q is the set of k-rational zeros of φ, then Theorem (1.8) of [Z], Chapter V shows that the semilocal ring of Q is Diophantine in K. Hence we can apply 10.3.3. ∎

Apart from this case (which will be superseded by 11.2), the Diophantine definability of A in K seems, in general, rather difficult to prove.

10.4 Remarks on effectivity.

As explained in the introduction, the choice of λ satisfying (100) is not effective in general. Let us describe a procedure for finding such a λ if k is finitely generated over the prime field.

More precisely, we assume here that:

- k is presented over $\mathbb{Q}$ (in the sense of [F-J], Section 17.2),
- K is presented over k (which essentially means that the curve C is explicitly described),
- f is explicitly given.

Assume we have effectively constructed an elliptic E over $\mathbb{Q}$, without complex multiplication over $\mathbb{C}$, and such that $\mathrm{Hom}_k(E, \mathrm{Jac}(C)) = 0$. Then by Theorem 1.12, we can effectively find $\lambda \in \mathrm{Good}(k) \cap \mathbb{Z}$: simply list all integers until such an element is found, which can be checked effectively since it reduces to deciding whether a given polynomial in $k[y]$ has no root in k. The rest of the proof of 10.3 involves only effective constructions.

Let us now construct E with the required properties. For an indeterminate z, fix an elliptic curve $E_{\mathbb{Q}(z)}$ over $\mathbb{Q}(z)$ with j-invariant equal to z. By ground field extension to $k(z)$ we obtain an elliptic curve $E_{k(z)}$ with the property that $\mathrm{Hom}_{k(z)}(E_{k(z)}, \mathrm{Jac}(C)_{k(z)})$ is zero: indeed, by 2.4.1, any abelian subvariety of $\mathrm{Jac}(C)_{k(z)}$ is defined over $\overline{k}$, while $E_{k(z)}$ (or any nontrivial quotient of it) is not. Hence we can apply Theorem 1.12 to find $\zeta \in \mathbb{Q}$ such that $E_{\mathbb{Q}(z)}$ specialises to an elliptic curve E_ζ over $\mathbb{Q}$ with the property that $\mathrm{Hom}_k(E_\zeta, \mathrm{Jac}(C)) = 0$. Moreover we can certainly find such a ζ which is not an integer, which implies that the resulting E_ζ (whose j-invariant is ζ) has no complex multiplication.

11 Diophantine undecidability of real function fields.

The following lemma combines results of [D2] (for the real case) and [K-R2] (for the p-adic case, which will be considered later):

11.1 Lemma. *Let k be a field of characteristic zero, and let K be a finitely generated regular extension of k.*

(i) *There is an elliptic curve E_1 over $\mathbb{Q}$ with the following properties:*

  (a) *$E_1(\mathbb{Q})$ is infinite (i.e. E has positive rank over $\mathbb{Q}$);*

  (b) *$E_1(K) = E_1(k)$.*

(ii) *Let Σ be a finite set of independent absolute values on $\mathbb{Q}$. Denote by $\mathbb{Q}_\Sigma = \prod_{v\in\Sigma}\mathbb{Q}_v$ the Σ-completion of $\mathbb{Q}$. Then there is a $\mathbb{Q}$-Diophantine subset $\mathscr{C}$ of K such that $\mathscr{C} \subset k$ and $\mathscr{C} \cap \mathbb{Q}$ is dense in $\mathbb{Q}_\Sigma$.*

Proof: (i) We may assume K transcendental aver k (otherwise, $K = k$). By 2.3.4 there is a transcendence basis $(z_1, \dots, z_n)$ of K/k such that K is a regular extension of $k(z_1, \dots, z_{n-1})$. For any elliptic curve E_1 over k, we have $E_1(k(z_1, \dots, z_{n-1})) = E_1(k)$ (immediate by induction on n since there is no nonconstant rational map from an elliptic curve to $\mathbb{P}^1$). Thus, to prove (i), we may replace k by $k(z_1, \dots, z_{n-1})$ and assume $n = 1$.

Let C be the (projective, smooth, geometrically connected) k-curve with function field K. The elliptic curves with a nonconstant morphism from C are those appearing (up to isogeny) as factors of the Jacobian of C, which form a finite set of isogeny classes. So we can choose an elliptic curve E_0 over $\mathbb{Q}$ which is not $\overline{\mathbb{Q}}$-isogenous to any of these, ad then apply 8.3.1 to find a twist E_1 of E_0 with positive rank over $\mathbb{Q}$, thus satisfying both conditions.

(ii) Choose E_1 as in (i), and write it in affine coordinates as

$$E_{1,\mathrm{aff}}: \qquad w = P(u, w)$$

with $P \in \mathbb{Q}[u, w]$, homogeneous of degree 3 and monic in u. Let $D \subset K$ be the set of u-coordinates of points of $E_{1,\mathrm{aff}}(K)$, and let $\mathscr{C}$ be the set of quotients u_1/u_2 with $u_1 \in D$ and $u_2 \in D \setminus \{0\}$. Let us show that $\mathscr{C}$ has the required properties.

Clearly, D and $\mathscr{C}$ are $\mathbb{Q}$-Diophantine in K, and property (b) implies that $D \subset k$, hence $\mathscr{C} \subset k$ as well. To prove the density property, it suffices to show that the closure of $D \cap \mathbb{Q}$ in $\mathbb{Q}_\Sigma$ contains a neighbourhood of 0. Since u is a local coordinate at the origin of $E_{1,\mathrm{aff}}$, this will follow if we prove that the closure of $E_{1,\mathrm{aff}}(\mathbb{Q})$ in $E_{1,\mathrm{aff}}(\mathbb{Q}_\Sigma)$ contains a neighbourhood of the origin. This is equivalent to the analogous statement with E_1 instead of $E_{1,\mathrm{aff}}$ since the latter is E_1 minus finitely many nonzero points.

Now, for each $v \in \Sigma$, $E_1(\mathbb{Q}_v)$ is a compact one-dimensional Lie group over $\mathbb{Q}_v$, hence it has an open subgroup of finite index U_v isomorphic to $\mathbb{Z}_v$ (if v is p-adic) or to the circle group S^1 (if v is real). Let $U \subset E_1(\Sigma)$ be the product of the U_v's. By property (a), $E(\mathbb{Q})$ has an element γ of infinite order; replacing it by some multiple we may assume that $\gamma \in U$.

For each v, the projection of γ in $U_v \subset E_1(\mathbb{Q}_v)$ still has infinite order, hence generates a subgroup whose closure is open. By weak approximation, it easily follows that the closed subgroup of $E_1(\Sigma)$ generated by γ is also open. This completes the proof. ∎

We can now prove part (2) of Theorem 1.1.

11.2 Theorem. *Let k be a formally real field., and let K be a finitely generated transcendental extension of k, which is also formally real. Then there is a Diophantine ring in K^2, isomorphic to $\mathbb{Z}$. In particular, the Diophantine theory of K is undecidable.*

Proof: By replacing k by a bigger subfield of K, we may assume that K has transcendence degree one over k, and that k is algebraically closed in K. Then K is the function field of a projective, smooth, geometrically connected k-curve C. Moreover, the assumption that K is formally real means that C has a closed point q with formally real residue field. Putting $Q = \{q\}$, we are in the situation of 10.1.1. Choosing any elliptic curve E over $\mathbb{Q}$, without complex multiplication, we can perform the constructions of 10.1.2 and 10.1.3, and we keep the same notations. In particular we have a ring $\Lambda \cong \mathbb{Z}$ in A^2, where A is the local ring of q. Moreover, we know from 10.2 that Λ is Diophantine in K^2, and it remains only to prove that the multiplication in Λ is relatively Diophantine.

To do this, we apply Proposition 9.4 with $\kappa = \mathbb{Q}$ (hence $\mathscr{O} = A_0 = \mathbb{Q}[f]_{(f)}$), $X = A_0$, and $\mathscr{O}' = K$. Thus, all we have to prove is that the maximal ideal $\mathfrak{m}_0$ of A_0 is a relatively Diophantine subset of A_0 in K.

First take $\mathscr{C} \subset K$ as provided by 11.1, applied with our k and K, and with Σ consisting of the ordinary absolute value (thus, $\mathbb{Q}_\Sigma = \mathbb{R}$). Consider the following formula in one variable x:

$$\varphi(x): \quad \exists\, \alpha, \beta, x_1, \ldots, x_5 : \ \alpha \in \mathscr{C} \wedge \beta \in \mathscr{C} \wedge (\alpha - f^{-1})\, x^2 + \beta = x_1^2 + \ldots + x_5^2.$$

I claim that the set $\mathscr{D} \subset K$ defined by φ satisfies $\mathscr{D} \cap A_0 = \mathfrak{m}_0 = f\, A_0$.

First, let us show that $\mathscr{D} \subset f\, A$ (which implies that $\mathscr{D} \cap \mathbb{Q}(f) \subset f\, A_0$). Indeed, for some $x \in K$, assume that $\varphi(x)$ holds and x does not vanish at q. The elements α and β in $\varphi(x)$ must be in k, hence $(\alpha - f^{-1})\, x^2 + \beta$ has negative odd order at q. Since the residue field of q is real, this cannot be a sum of squares in K, which contradicts $\varphi(x)$.

Let us now prove that $f\, A_0 \subset \mathscr{D}$. We view A_0 as the local ring of 0 in $\mathbb{P}^1_{\mathbb{Q}}$, with standard coordinate f. If $x \in f\, A_0$, then x is a rational function on $\mathbb{P}^1_{\mathbb{Q}}$, vanishing at 0; hence, so does $f^{-1}x^2$. In particular, $|f^{-1}x^2| \leq 1$ on $I := [-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$. We can choose α and β in $\mathscr{C}$, and such that $\beta > 1$ and $\alpha > 1/\varepsilon$. Then, by our choices (recall that f is the standard coordinate on $\mathbb{P}^1_{\mathbb{Q}}$, hence $|f| > \varepsilon$ on $\mathbb{R} \setminus I$):

- on $\mathbb{R} \setminus I$, we have $|f^{-1}| < 1/\varepsilon < \alpha$, hence $(\alpha - f^{-1})\, x^2 + \beta \geq \beta > 0$;
- on I, we have $(\alpha - f^{-1})x^2 + \beta = \alpha\, x^2 - f^{-1}x^2 + \beta \geq \alpha\, x^2 - 1 + \beta > 0$.

This implies that $(\alpha - f^{-1})\, x^2 + \beta$ is a sum of squares in $\mathbb{Q}(f)$, and in fact a sum of five squares by [Po]. Hence $\varphi(x)$ is satisfied, and the proof is complete. ∎

12 Diophantine undecidability of p-adic function fields.

In this section we prove part (3) of Theorem 1.1:

12.1 Theorem. *Let p be an odd prime. Let κ be a subfield of a finite extension of $\mathbb{Q}_p$, and let K be a finitely generated transcendental extension of k.*

Then there is a Diophantine ring $\Lambda \subset K^d$, for some d, isomorphic to $\mathbb{Z}$ as a ring. In particular, K is positive-existentially undecidable.

12.2 Extending the ground field.

First, we may replace κ by its algebraic closure in K and assume that K/k is a regular extension.

Next, assume there is a finite extension L of K, of degree n, and a Diophantine ring isomorphic to $\mathbb{Z}$ in L^2. Then by using the Weil restriction (i.e. by fixing a K-basis of L and identifying L^d with K^{2n}) we obtain a Diophantine ring isomorphic to $\mathbb{Z}$ in K^{2n}. In particular, to prove 12.1 we may replace κ by a finite extension (which we shall always view as embedded in some finite extension of $\mathbb{Q}_p$, and in particular equipped with a p-adic valuation, normalised in such a way that its value group is $\mathbb{Z}$).

Thus, replacing if necessary κ by a finite extension κ' and K by $K \otimes_\kappa \kappa'$, we shall assume from now on that:

(i) there is a transcendence basis $(z_1, \dots, z_n)$ of K over κ such that $[K : \kappa(z_1, \dots, z_n)]$ is odd and the extension $K/\kappa(z_1, \dots, z_{n-1})$ is regular,

(ii) κ contains elements i, a, ϖ such that:

- (a) $i^2 = -1$,
- (b) a is a root of unity,
- (c) ϖ is algebraic over $\mathbb{Q}$, and has odd p-adic valuation,
- (d) the 4-dimensional quadratic form

$$\langle 1, a\rangle\langle 1, \varpi\rangle = x^2 + \varpi\, y^2 + a\, z^2 + a\varpi\, w^2 \tag{101}$$

  is anisotropic over κ,
- (e) the quadratic form (101) is isotropic at all 2-adic primes of the field $\mathbb{Q}(i, a, \varpi)$.

Indeed, (i) holds over some finite extension of κ by 2.3.4. The fact that κ can be further enlarged to satisfy (ii) is proved in [K-R2], Proposition 8; here we denote by ϖ what was (somewhat confusingly) called p in [K-R2].

(In the left-hand side of (101) we use the standard notation $\langle d_1, \dots, d_n\rangle$ for the diagonal quadratic form $\sum_{j=1}^n d_j\, x_j^2$, and the product is the 'Kronecker product', or tensor product.)

From now on we fix $z_1, \dots, z_{n-1}$ as in (i), i, a, ϖ as in (ii), and we put

$$k := \kappa(z_1, \dots, z_{n-1}).$$

Thus, K is a one-variable function field over k; we denote by C the smooth, projective, geometrically connected k-curve with function field K. By condition (i), C is a cover of $\mathbb{P}^1_k$ of odd degree, hence admits a divisor of odd degree. By 2.3.1, this implies:

(iii) there is an element f of K which, viewed as a k-morphism $C \to \mathbb{P}^1_k$, has simple ramification, simple zeros, simple poles, and odd degree.

From now on we fix f as in (iii); in fact we may replace z_n by f and consider the tower of extensions

$$\kappa \subset k \subset k(f) = \kappa(z_1, \ldots, z_{n-1}, f) \subset K \tag{102}$$

in which the first two inclusions are purely transcendental and the last is finite and odd.

We have done all this to use our results on curves while ensuring the following property:

12.2.1 Lemma. *Every anisotropic quadratic form over $\kappa(f)$ remains anisotropic over K.*

Proof: Let φ be such a quadratic form. Clearly, φ is still anisotropic over $k(f)$ which is purely transcendental over $\kappa(f)$ ([Lam], Chapter 9, Lemma 1.1). Since K is finite of odd degree over $k(f)$, we conclude from Springer's theorem ([Lam], Chapter 7, Theorem 2.3) that φ is also anisotropic over K. ∎

12.3 Defining the ring Λ.

With $f : C \to \mathbb{P}^1_k$ as in 12.2 (iii), we choose a zero q of f on C (not necessarily k-rational), put $Q = \{q\}$, and we adopt the notations of 10.1.1.

We choose an elliptic curve E over $\mathbb{Q}$, without complex multiplication, and we identifiy $L := E/\{\pm \mathrm{Id}_E\}$ with $\mathbb{P}^1_{\mathbb{Q}}$ in such a way that the origin of E goes to 0, and the condition of 9.3.1 is satisfied; in other words, we choose the equation (86) in such a way that $c \neq 0$ and the points $(0 : \pm\sqrt{c} : 1)$ of $E(\overline{\mathbb{Q}})$ have infinite order.

We then proceed with the constructions of 10.1.2 and 10.1.3. We obtain a subset $\Lambda \subset A^2$ with a ring structure isomorphic to $\mathbb{Z}$, which is a Diophantine subset of K^2 by 10.2. To prove that the multiplication of Λ is Diophantine, we need the following refinement of 9.5.3 (cf. [K-R2], Theorem 6, where t corresponds to our f):

12.4 Lemma. *Denote by v_∞ (resp. v_0) the valuation on $\mathbb{Q}(f) \subset K$ such that $v_\infty(f) = -1$ (resp. $v_0(f) = +1$). Define subsets Y_0, Y_1, Y of $\mathbb{Q}(f)$ by*

$$\begin{array}{rcl} Y_i & := & \{r \in \mathbb{Q}(f) \mid v_\infty(r) = -2 \text{ and } v_0(r) = i\} \quad (i = 0,\ 1) \\ Y & := & Y_0 \cup Y_1. \end{array} \tag{103}$$

Assume that Y_1 is a relatively Diophantine subset of Y (in K). Then the ring structure of Λ is Diophantine. Hence K is positive-existentially undecidable.

Proof: By assumption, there is a Diophantine set $\mathscr{D} \subset K$ such that $\mathscr{D} \cap Y = Y_1$. Put

$$\begin{array}{rcl} X & := & \{r \in \mathbb{Q}(f) \mid v_\infty(r) \geq -2 \text{ and } v_0(r) \geq 0\} \\ X_+ & := & \{r \in \mathbb{Q}(f) \mid v_\infty(r) \geq -2 \text{ and } v_0(r) > 0\}. \end{array}$$

We shall prove that X_+ is relatively Diophantine in X, which by 9.5.3 implies the result.

If $r \in X$, then $\frac{1}{1+f^2}\, r$ has nonnegative v_∞ and the same v_0 as r. It follows that if we put

$$s := f + f^2 + \Big(\frac{1}{1+f^2}\, r\Big)^2,$$

then we have

$$\begin{aligned} v_\infty(s) &= -2, \\ v_0(s) &= \begin{cases} 0 & \text{if } v_0(r) = 0 \\ 1 & \text{if } v_0(r) > 0. \end{cases} \end{aligned}$$

Hence, for any $r \in X$, we have $s \in Y$, and $s \in Y_1$ if and only if $r \in X_+$. Consequently, $X_+ = X \cap \mathscr{D}_1$, where

$$\mathscr{D}_1 = \Big\{\, r \in K \;\mid\; f + f^2 + \Big(\frac{1}{1+f^2}\, r\Big)^2 \in \mathscr{D} \,\Big\},$$

which proves the lemma. ∎

12.5 Isotropy of quadratic forms.

It remains to prove that the assumption of Lemma 12.4 is satisfied, i.e. Y_1 is relatively Diophantine in Y; we follow [K-R2], indicating only the changes to be made.

To stick to the notations of [K-R2], we put $t = f$ from now on. (Thus we forget the t of 10.1.2, which corresponds to λf for some $\lambda \in \mathbb{Q}$).

Applying Lemma 11.1, we fix a $\mathbb{Q}$-Diophantine subset $\mathscr{C}$ of K, contained in κ and such that $\mathbb{Q} \cap \mathscr{C}$ is dense in $\mathbb{Q}_p$.

To every $r \in K$ we associate two elements u_0, u_1 of K and two quadratic forms φ_0, φ_1 over K depending on parameters c_3, c_5, by the formulas

$$\begin{aligned} u_e &:= a^e\,((1+t)^3\, r + c_3\, t^3 + c_5\, t^5) \quad (e = 0, 1) \\ \varphi_e &:= \langle t, at, -1, -u_e\rangle\langle 1, \varpi\rangle. \end{aligned} \tag{104}$$

We define a Diophantine set $\mathscr{D} \subset K$ by

$$r \in \mathscr{D} \;\Leftrightarrow\; \exists\, c_3, c_5 \in \mathscr{C} \text{ such that } \varphi_0 \text{ and } \varphi_1 \text{ are isotropic over } K.$$

and claim that $\mathscr{D} \cap Y = Y_1$. This amounts to proving that $Y_1 \subset \mathscr{D}$ and $Y_0 \cap \mathscr{D} = \emptyset$.

12.5.1 The relation $Y_0 \cap \mathscr{D} = \emptyset$. Assume that $r \in Y_0$. Then r is in $\mathbb{Q}(t)$ and has order 0 at 0, so the same holds for u_0, for any choice of c_3 and c_5 in k (and in particular in $\mathscr{C}$). By the first assertion of [K-R2], Proposition 7 (applied with $b = \varpi$, $g =$ our u_0, and $a =$ our $-a$), this implies that one of the forms φ_0, φ_1 is anisotropic over $\kappa(t)$, hence also over K by Lemma 12.2.1.

12.5.2 The inclusion $Y_1 \subset \mathscr{D}$. Assume that $r \in Y_1$. We have to show that for some choice of c_3 and c_5 in $\mathscr{C}$ (in fact, we can take them in $\mathscr{C} \cap \mathbb{Q}$) both forms φ_0, φ_1 are isotropic over K (and in fact, over $\kappa(t)$). We refer to [K-R2] for the details: first, it is shown in the proof of [K-R2], Theorem 9 that for suitable c_3 and c_5 in $\mathscr{C} \cap \mathbb{Q}$, some condition on the Newton polygons of u_0 and u_1 is satisfied (the only thing that matters about $\mathscr{C} \cap \mathbb{Q}$ is p-adic density). Then, the results of [K-R2], Section 3 (in particular Theorem 21) imply that this Newton polygon condition in turn implies isotropy. This completes the proof. ■

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