

An extension of Greenberg's theorem to general valuation rings

Laurent Moret-Bailly *

IRMAR (Institut de Recherche Mathématique de Rennes, UMR 6625, CNRS)

Université de Rennes 1, campus de Beaulieu, F-35042 Rennes Cedex

laurent.moret-bailly@univ-rennes1.fr

<http://perso.univ-rennes1.fr/laurent.moret-bailly/>

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Abstract

We extend Greenberg's strong approximation theorem to schemes of finite presentation over valuation rings with arbitrary value group. As an application, we prove a closed image theorem (in the strong topology on rational points) for proper morphisms of varieties over valued fields.

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To Jan Denef, on the occasion of his 60th birthday

1 Introduction

1.1 Notations

Throughout this paper, we denote by R a valuation ring, by K its fraction field, and by Γ the valuation group (written additively). The valuation is denoted by $\text{ord} : K \rightarrow \Gamma \cup \{\infty\}$. We put $\Gamma^+ := \{\alpha \in \Gamma \mid \alpha \geq 0\}$.

The completion of R is denoted by \widehat{R} , with fraction field \widehat{K} ; recall that \widehat{R} is a valuation ring with group Γ .

For each $\alpha \in \Gamma^+$, we put $I_\alpha := \{x \in K \mid \text{ord}(x) \geq \alpha\}$. This is a principal ideal of R , with quotient $R_\alpha := R/I_\alpha$.

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If X is an R -scheme, the sets $X(R_\alpha)$ ($\alpha \in \Gamma^+$) form an inverse system, whose limit $\varprojlim_{\alpha \in \Gamma^+} X(R_\alpha)$ is easily seen (although we shall not really need it) to be $X(\widehat{R})$. Indeed, this is immediate if X is affine; in general, since each R_α is local, every element of the projective limit belongs to $\varprojlim_{\alpha \in \Gamma^+} V(R_\alpha)$ where $V \subset X$ is an affine open subscheme.

Clearly, we have a natural map $X(R) \rightarrow \varprojlim_{\alpha \in \Gamma^+} X(R_\alpha)$.

Our main result is the following:

1.2 Theorem (strong approximation) *With the above notations, assume further that R is Henselian and that \widehat{K} is a separable extension of K . Let X be an R -scheme of finite presentation.*

Then there exist a positive integer N and an element $\delta \in \Gamma^+$ with the following property: for each $\alpha \in \Gamma^+$ and each $x \in X(R_{N\alpha+\delta})$, there is an $x' \in X(R)$ such that x and x' have the same image in $X(R_\alpha)$.

Equivalently:

$$(1.2.0.1) \quad \forall \alpha \in \Gamma^+, \quad \text{Im}(X(R_{N\alpha+\delta}) \rightarrow X(R_\alpha)) = \text{Im}(X(R) \rightarrow X(R_\alpha)).$$

1.2.1 Corollary (weak approximation) *We keep the notations and assumptions of Theorem 1.2. Then:*

- (i) $X(R)$ is dense in $X(\widehat{R})$ for the valuation topology.
- (ii) If V is a K -scheme locally of finite type, then $V(K)$ is dense in $V(\widehat{K})$ for the valuation topology.

Proof: Theorem 1.2 immediately implies that $X(R)$ and $X(\widehat{R})$ have the same image in $X(R_\alpha)$, for each $\alpha \in \Gamma^+$. This proves (i). To prove (ii), observe that $V(K)$ has an open covering by subsets of the form $j(\mathcal{U}(R))$ where each \mathcal{U} is an affine R -scheme of finite presentation with an open immersion $j : \mathcal{U}_K \hookrightarrow V$ (see 4.1). Thus, (ii) follows from (i) since $V(\widehat{K})$ is then covered by the corresponding sets $\mathcal{U}(\widehat{R})$. ■

1.2.2 Corollary (“infinitesimal Hasse principle”) *With the notations and assumptions of Theorem 1.2, we have the equivalence:*

$$X(R) \neq \emptyset \quad \Leftrightarrow \quad \forall \gamma \in \Gamma^+, X(R_\gamma) \neq \emptyset.$$

Proof: Taking $\alpha = 0$ in (1.2.0.1), we see that if $X(R_\delta) \neq \emptyset$ then $X(R) \neq \emptyset$. ■

The author’s original motivation for proving these results is that they have deep consequences for the topology of varieties over valued fields. For instance, as an easy consequence of Corollary 1.2.2, we obtain:

1.3 Theorem *Assume R is Henselian and \widehat{K} is separable over K . Let $f : X \rightarrow Y$ be a proper morphism of K -schemes of finite type. Then the induced map $f_K : X(K) \rightarrow Y(K)$ has closed image (for the topology defined by the valuation).*

1.3.1 Remark Theorem 1.3 is of course trivial if K is a *local* field (i.e. locally compact), since f_K is then a proper map. But apart from this case, and even if R is a discrete valuation ring, f_K is not a closed map in general.

1.4 Related results

Theorem 1.2 of course generalizes Greenberg’s strong approximation theorem [9], which is the special case where R is a discrete valuation ring (the separability of \widehat{K} meaning in this case that R is excellent). In fact, Greenberg’s original proof extends rather easily to valuation rings of height one provided the fraction field has characteristic zero.

The method used here is due to Becker, Denef, Lipshitz and van den Dries [2]: in fact, most of our proof is shamelessly copied from there, with the exception of the separability property 2.4 (ii) which is proved in [2] by a ramification index argument which breaks down for nondiscrete valuations.

Similar methods are used in [2], and also by Denef and Lipshitz in [7] to obtain strong approximation theorems more general than Greenberg’s (of the kind considered by Artin, Popescu and others); typically, these are derived from the corresponding “weak” approximation theorems. The ground rings in these results are subrings of power series rings over discrete valuation rings.

Schoutens [15, Theorem 2.4.1] has proved the weak approximation theorem 1.2.1 for certain subrings of $A[[T_1, \dots, T_n]]$ where A is a complete valuation ring of height one.

The approach of Elkik [8] leads to strong approximation results without excellence assumption on the base ring, but with (generic) smoothness assumptions on the scheme X . The extension to certain non-Noetherian bases (including Henselian valuation rings of height one), outlined in [8, Remarque 2, p. 587], is carried out in [1, 1.16].

1.5 Organization of the paper

In Section 2 we review some basic facts about ultraproducts, in particular about ultrapowers of R . We then explain how Theorem 1.2 reduces to two technical results involving such ultrapowers (namely, the separability theorem 2.4 (ii) and the lifting theorem 2.5). This reduction is the “formal” part of the proof. Theorems 2.4 and 2.5 are proved in section 3, and Theorem 1.3 in section 4.

1.6 Acknowledgments

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2 Basic constructions

We keep the notations of the introduction.

2.1 Ultraproducts: basic definitions

(For details on these constructions, see [2, §2] or [16]).

We fix an infinite set W and a nonprincipal ultrafilter \mathcal{U} of subsets of W . We shall say that a property $P(w)$ holds “for almost all $w \in W$ ” if the corresponding subset of W belongs to \mathcal{U} .

If $A = (A_w)_{w \in W}$ is a family of algebraic structures (sets, rings, groups, ordered groups. . .) indexed by W , we denote by $A_{\mathfrak{h}}$ (notation taken from [16]) the corresponding ultraproduct. It is *usually* defined as the quotient of $\prod_{w \in W} A_w$ by the equivalence relation “equality almost everywhere”. It will often be convenient to use the more explicit notation (inspired from the same source) $\text{ulim}_{\mathcal{U}, w} A_w$ for $A_{\mathfrak{h}}$.

However, the above definition works as expected only if either the sets A_w are nonempty, or almost all are empty. In general, the correct definition (which we shall use here) is

$$(2.1.0.1) \quad \text{ulim}_{\mathcal{U}, w} A_w := \varinjlim_{U \in \mathcal{U}} \prod_{w \in U} A_w$$

where \mathcal{U} is ordered by reverse inclusion and the transition maps are the obvious projections.

Given $U \in \mathcal{U}$ and an element $x = (x_w)_{w \in U} \in \prod_{w \in U} A_w$, we denote its class in $A_{\mathfrak{h}}$ by $[x_w]_{w \in U}$, or $\text{ulim}_{\mathcal{U}, w} x_w$.

If the family is constant ($A_w = A$, independent of w) we obtain the \mathcal{U} -ultrapower of A , denoted by

$$(2.1.0.2) \quad \text{upw}_{\mathcal{U}} A = \text{ulim}_{\mathcal{U}, w} A.$$

2.2 Ultraproducts and the functor of points

If A is a ring and Y is an A -scheme, we wish to know whether the functor of points $B \mapsto Y(B)$ (from A -algebras to sets) “commutes with ultraproducts”. Given the definition of ultraproducts, this must involve compatibility properties of this functor with products and with direct limits. Now, recall the following facts:

2.2.1 Proposition (special case of [10, (8.8.2)]) *Let A be a ring, $(B_\lambda)_{\lambda \in \Lambda}$ a filtering inductive system of A -algebras, and Y an A -scheme. Consider the natural map*

$$(2.2.1.1) \quad \alpha : \varinjlim_{\lambda \in \Lambda} Y(B_\lambda) \longrightarrow Y\left(\varinjlim_{\lambda \in \Lambda} B_\lambda\right).$$

If Y is locally of finite type (resp. locally of finite presentation) over A , then α is injective (resp. bijective). \blacksquare

2.2.2 Proposition *Let A be a ring, $(B_i)_{i \in I}$ a family of A -algebras, and Y an A -scheme. Consider the natural map*

$$(2.2.2.1) \quad \beta : Y\left(\prod_{i \in I} B_i\right) \longrightarrow \prod_{i \in I} Y(B_i).$$

- (i) If Y is affine, β is bijective.
- (ii) If Y is quasiseparated, β is injective.
- (iii) If Y is quasicompact and quasiseparated, and each B_i is a local ring, then β is bijective.

Proof: (i) is immediate since the Spec functor takes arbitrary products of rings to sums in the category of affine schemes.

A proof of (ii) and (iii) can be found embedded in the proof of [6, Theorem 3.6]. Note that (iii) appears in [12, Lemme 3.2], although the quasiseparated assumption is missing there. ■

Now, let A be a ring, and let $(A_w)_{w \in W}$ be a family of A -algebras indexed by W , with ultraproduct $A_{\mathfrak{h}}$. If Y is an A -scheme, we want to compare the sets $Y(A_{\mathfrak{h}})$ and $\text{ulim}_{\mathcal{U}, w} Y(A_w)$.

For each $U \in \mathcal{U}$, put $A_U := \prod_{w \in U} A_w$. Since $A_{\mathfrak{h}} = \varinjlim_U A_U$, we have a natural map $\alpha : \varinjlim_{U \in \mathcal{U}} Y(A_U) \rightarrow Y(A_{\mathfrak{h}})$, to which Proposition 2.2.1 applies.

On the other hand, for each U we have a map $\beta_U : Y(A_U) \rightarrow \prod_{w \in U} Y(A_w)$ (of the type considered in Proposition 2.2.2) and, passing to the limit, a map $\beta : \varinjlim_{U \in \mathcal{U}} Y(A_U) \rightarrow \text{ulim}_{\mathcal{U}, w} Y(A_w)$. Finally, we have constructed a diagram of sets

$$(2.2.2.2) \quad \begin{array}{ccc} \varinjlim_{U \in \mathcal{U}} Y(A_U) & \xrightarrow{\alpha} & Y(\varinjlim_{U \in \mathcal{U}} A_U) = Y(A_{\mathfrak{h}}) \\ & & \downarrow \beta \\ \text{ulim}_{\mathcal{U}, w} Y(A_w) & \xlongequal{\quad} & \varinjlim_{U \in \mathcal{U}} \prod_{w \in U} Y(A_w). \end{array}$$

Combining Propositions 2.2.1 and 2.2.2, we obtain:

2.2.3 Proposition *With the above notations and assumptions, assume that:*

- Y is finitely presented over A , and
- Y is affine, or each A_w is a local ring.

Then the maps α and β in (2.2.2.2) are bijective. In particular, we have a natural bijection

$$\text{ulim}_{\mathcal{U}, w} Y(A_w) \xrightarrow{\sim} Y(A_{\mathfrak{h}}). \quad \blacksquare$$

2.2.4 Remarks

(1) In the affine case, $Y(B)$ (for an A -algebra B) is the set of B -valued solutions of a given finite system of polynomial equations with coefficients in A . Thus, Proposition 2.2.3 in this case will be seen by model theorists as an instance of Łoś' theorem.

(2) There is no “direct” map between the two sets in 2.2.3, valid for *all* A -schemes Y . To see this, take for W the set of prime numbers (and for \mathcal{U} any nonprincipal ultrafilter), and take $A = \mathbb{Z}$, $A_p = \mathbb{F}_p$ for all $p \in W$. In particular, $A_{\mathfrak{h}}$ is a field of characteristic zero.

First, try $Y = \text{Spec } \mathbb{Q}$. Then $Y(A_{\mathfrak{h}})$ has one element, while $Y(A_p) = \emptyset$ for each $p \in W$. Hence in this case there is no map from $Y(A_{\mathfrak{h}})$ to $\text{ulim}_{\mathcal{U},p} Y(A_p)$.

Now, take $Y = \coprod_{p \in W} \text{Spec } \mathbb{F}_p$. Then $Y(A_{\mathfrak{h}}) = \emptyset$, and $Y(A_p)$ has one element for each p , so $\text{ulim}_{\mathcal{U},p} Y(A_p)$ also has one element and there is no map from $\text{ulim}_{\mathcal{U},p} Y(A_p)$ to $Y(A_{\mathfrak{h}})$.

(3) Proposition 2.2.3 would fail with the traditional definition of ultraproducts: it may happen that $Y(A_{\mathfrak{h}}) \neq \emptyset$ but $Y(A_w) = \emptyset$ for some $w \in W$.

(4) With the assumptions of 2.2.3, we have in particular:

$$(2.2.4.1) \quad Y\left(\text{ulim}_{\mathcal{U},w} A_w\right) \neq \emptyset \quad \Leftrightarrow \quad Y(A_w) \neq \emptyset \text{ for almost all } w.$$

An interesting special case is when $A_w = B$, a fixed local A -algebra: we then have the equivalence

$$(2.2.4.2) \quad Y\left(\text{upw}_{\mathcal{U}}(B)\right) \neq \emptyset \quad \Leftrightarrow \quad Y(B) \neq \emptyset.$$

2.3 Ultrapowers of valuation rings

Take our valuation ring R , and consider $R_{\mathfrak{h}} = \text{upw}_{\mathcal{U}} R$. Using Łoś’ theorem, one checks that this is a valuation ring with fraction field $K_{\mathfrak{h}} = \text{upw}_{\mathcal{U}} K$ and valuation group $\Gamma_{\mathfrak{h}} = \text{upw}_{\mathcal{U}} \Gamma$; moreover, $R_{\mathfrak{h}}$ is Henselian if R is.

We have canonical embeddings $R \hookrightarrow R_{\mathfrak{h}}$, $\Gamma \hookrightarrow \Gamma_{\mathfrak{h}}$. We shall denote the valuation on $K_{\mathfrak{h}}$ by $\text{ord}_{\mathfrak{h}} : K_{\mathfrak{h}} \rightarrow \Gamma_{\mathfrak{h}} \cup \{\infty\}$. Thus, if $z = (z_w)_{w \in W} \in K^W$, we have $\text{ord}_{\mathfrak{h}}[z_w]_{w \in W} = [\text{ord}(z_w)]_{w \in W}$.

2.3.1 Some quotients of $R_{\mathfrak{h}}$: principal ideals. Each element $\alpha_{\mathfrak{h}}$ of $\Gamma_{\mathfrak{h}}^+$ defines a principal ideal $I_{\alpha_{\mathfrak{h}}} \subset R_{\mathfrak{h}}$ and a quotient ring $(R_{\mathfrak{h}})_{\alpha_{\mathfrak{h}}}$, which we denote by $R_{\mathfrak{h},\alpha_{\mathfrak{h}}}$. If we write $\alpha_{\mathfrak{h}} = [\alpha_w]_{w \in W}$ for some family $(\alpha_w) \in (\Gamma^+)^W$, we immediately check that the canonical surjection $R^W \twoheadrightarrow \prod_{w \in W} R_{\alpha_w}$ induces an isomorphism

$$R_{\mathfrak{h},\alpha_{\mathfrak{h}}} \xrightarrow{\sim} \text{ulim}_{\mathcal{U},w} R_{\alpha_w}.$$

2.3.2 Some quotients of $R_{\mathfrak{h}}$: prime ideals. Recall that a subset C of an ordered set (S, \leq) is *convex* if whenever $a \in C$, $b \in C$, $x \in S$ and $a \leq x \leq b$, then $x \in C$. (Convex subgroups of a totally ordered group are called *isolated* in [4]).

Let C be a convex subgroup of $\Gamma_{\mathfrak{h}}$. We denote by $P_C \subset R_{\mathfrak{h}}$ the ideal

$$P_C = \{x \in R_{\mathfrak{h}} \mid \text{ord}_{\mathfrak{h}}(x) \notin C\} = \{x \in R_{\mathfrak{h}} \mid \text{ord}_{\mathfrak{h}}(x) > C\},$$

the latter condition meaning of course that $\text{ord}_{\mathfrak{h}}(x) > \alpha$ for all $\alpha \in C$. This is a prime ideal of $R_{\mathfrak{h}}$ (they are all of this form), and the quotient $R^{(C)} := R_{\mathfrak{h}}/P_C$ is a valuation ring with group C . If C contains Γ , the canonical map $R \rightarrow R_{\mathfrak{h}} \rightarrow R^{(C)}$ is injective.

The ideal P_C is not principal in general, but it is the (totally ordered) union of the principal ideals contained in it. So we can write (as $R_{\mathfrak{h}}$ -algebras)

$$R^{(C)} = \varinjlim_{\alpha_{\mathfrak{h}} > C} R_{\mathfrak{h}, \alpha_{\mathfrak{h}}}.$$

We shall be interested only in convex subgroups C satisfying $\Gamma \subset C \subsetneq \Gamma_{\mathfrak{h}}$. (If $C = \Gamma_{\mathfrak{h}}$, then $P_C = \{0\}$, and the above direct limit runs over the empty set; otherwise we have an honest filtering colimit). Any such subgroup contains the convex hull Γ_c of Γ in $\Gamma_{\mathfrak{h}}$. We can think of P_{Γ_c} as the ideal of elements of $R_{\mathfrak{h}}$ with “infinitely large” valuation.

Thus we have a diagram of valuation rings

$$\begin{array}{ccc} R_{\mathfrak{h}} & \twoheadrightarrow & R^{(\Gamma_c)} \\ \uparrow & \nearrow & \\ R & & \end{array}$$

and our general strategy for solving equations over R will be “find solutions in $R^{(\Gamma_c)}$, lift them to $R_{\mathfrak{h}}$, and then extract solutions in R ”. As we shall see now, the first and third steps are essentially trivial.

2.3.3 Proposition *Let C be a proper convex subgroup of $\Gamma_{\mathfrak{h}}$ containing Γ , and let X be an R -scheme of finite presentation. Then we have the implications*

$$X(R) \neq \emptyset \quad \Leftrightarrow \quad X(R_{\mathfrak{h}}) \neq \emptyset \quad \Rightarrow \quad X(R^{(C)}) \neq \emptyset \quad \Leftrightarrow \quad \forall \alpha \in \Gamma^+, X(R_{\alpha}) \neq \emptyset.$$

Proof: The first two “ \Rightarrow ” are obvious, and the first equivalence follows from (2.2.4.2).

Assume $X(R^{(C)}) \neq \emptyset$, and take $\alpha \in \Gamma^+$. The ideal P_C is contained in P_{Γ_c} , hence in $I_{\alpha}R_{\mathfrak{h}}$, and therefore $R_{\mathfrak{h}}/I_{\alpha}R_{\mathfrak{h}}$ is a quotient of $R^{(C)}$, which implies that $X(R_{\mathfrak{h}}/I_{\alpha}R_{\mathfrak{h}}) \neq \emptyset$. But $R_{\mathfrak{h}}/I_{\alpha}R_{\mathfrak{h}}$ is immediately seen to be the ultrapower $\text{upw}_{\mathcal{U}} R_{\alpha}$, whence $X(R_{\alpha}) \neq \emptyset$ by (2.2.4.2) again. This proves the last “ \Rightarrow ”.

Finally, assume the last condition in the chain. Since $C \subsetneq \Gamma_{\mathfrak{h}}$ by assumption, we can pick some $\alpha_{\mathfrak{h}} > C$ in $\Gamma_{\mathfrak{h}}$, and it suffices to show that $X(R_{\mathfrak{h}, \alpha_{\mathfrak{h}}}) \neq \emptyset$ since $R^{(C)}$ is a quotient of $R_{\mathfrak{h}, \alpha_{\mathfrak{h}}}$. Now represent $\alpha_{\mathfrak{h}}$ as $\text{ulim}_{\mathcal{U}, w} \alpha_w$ for some $(\alpha_w) \in \Gamma^W$: then from 2.3.1 we have $R_{\mathfrak{h}, \alpha_{\mathfrak{h}}} = \text{ulim}_{\mathcal{U}, w} R_{\alpha_w}$, whence, by 2.2.3, $X(R_{\mathfrak{h}, \alpha_{\mathfrak{h}}}) = \text{ulim}_{\mathcal{U}, w} X(R_{\alpha_w}) \neq \emptyset$ since each $X(R_{\alpha_w})$ is nonempty. \blacksquare

2.3.4 Remark It may happen that $\Gamma_c = \Gamma_{\mathfrak{h}}$, in which case there is no C as in the proposition. This is the case in particular if W is “too small” in the sense that Γ has no cofinal subset of cardinality $\leq \text{Card } W$.

On the other hand, if we restrict ourselves to those (W, \mathcal{U}) such that $\Gamma_c \neq \Gamma_{\mathfrak{h}}$, then 2.3.3 shows that the condition $X(R^{(C)}) \neq \emptyset$ is equivalent to $X(R^{(\Gamma_c)}) \neq \emptyset$, hence independent of C (and even independent of (W, \mathcal{U}) , subject to the above restriction).

Let us now state the technical results from which 1.2 will be derived. First, a structure theorem for the fraction fields of the rings $R^{(C)}$:

2.4 Theorem *Let C be a convex subgroup of $\Gamma_{\mathfrak{h}}$ containing Γ . Consider the extension $K^{(C)} := \text{Frac}(R^{(C)})$ of K .*

- (i) *If R is complete, then $K^{(C)}$ is a regular extension of K , i.e. $K^{(C)}$ is linearly disjoint from every finite extension of K . (In other words, $K^{(C)}$ is a geometrically integral K -algebra).*
- (ii) *If \widehat{K} is separable over K , then so is $K^{(C)}$. (In other words, $K^{(C)}$ is a geometrically reduced K -algebra).*
- (iii) *If K is separably closed in \widehat{K} (e.g. if R is Henselian [14, F, Th. 4, Cor. 2 p. 190]), then it is separably closed in $K^{(C)}$. (In other words, $K^{(C)}$ is a primary extension of K , or equivalently a geometrically connected K -algebra).*

A word of warning may be appropriate here: for a valuation ring, “complete” does *not* imply “Henselian”, except if the value group Γ has height one, i.e. is isomorphic to a subgroup of \mathbb{R} with the induced ordering.

Theorem 2.4 will be proved in section 3. As we shall see, assertions (ii) and (iii) are easy consequences of (i). For us, the useful one is the separability property (ii), which will be used, also in section 3, to prove the following result:

2.5 Theorem (Lifting theorem) *Assume that R is Henselian and that \widehat{K} is separable over K . For each convex subgroup $C \subset \Gamma_{\mathfrak{h}}$ containing Γ , the canonical map $X(R_{\mathfrak{h}}) \rightarrow X(R^{(C)})$ is onto.*

We shall end this section by deducing Theorem 1.2 from the lifting theorem.

2.6 Proof of Theorem 1.2 (from Theorem 2.5)

We argue by contradiction. Thus, assume that for all $N \in \mathbb{Z}_{>0}$ and $\delta \in \Gamma^+$ there exist $\alpha_{N,\delta} \in \Gamma^+$ and $\xi_{N,\delta} \in X(R_{N\alpha_{N,\delta}+\delta})$ such that the image of $\xi_{N,\delta}$ in $X(R_{\alpha})$ does not lift to $X(R)$. Using the axiom of choice we fix such families $(\alpha_{N,\delta})$ and $(\xi_{N,\delta})$. For simplicity, put $\beta_{N,\delta} := N\alpha_{N,\delta} + \delta$.

Now, let us choose our ultrafilter: we take $W := \mathbb{Z}_{>0} \times \Gamma^+$, and pick an ultrafilter \mathcal{U} on W , containing all the sets $w + W$ ($w \in W$). The ultrapowers $\mathbb{Z}_{\mathfrak{h}}$ and $\Gamma_{\mathfrak{h}}$ contain in particular the “diagonal” elements

$$H := \text{ulim}_{\mathcal{U}, (N,\delta)} N, \quad \Delta := \text{ulim}_{\mathcal{U}, (N,\delta)} \delta$$

and our choice of \mathcal{U} implies that

$$(2.6.0.1) \quad H > \mathbb{Z} \text{ (in } \mathbb{Z}_{\mathfrak{h}}) \quad \text{and} \quad \Delta > \Gamma \text{ (in } \Gamma_{\mathfrak{h}}).$$

For $w = (N, \delta) \in W$, we of course write α_w for $\alpha_{N, \delta}$. Now we consider the elements

$$\begin{aligned}\alpha_{\mathfrak{h}} &:= \operatorname{ulim}_{\mathcal{U}, w} \alpha_w \in \Gamma_{\mathfrak{h}}^+, \\ \beta_{\mathfrak{h}} &:= \operatorname{ulim}_{\mathcal{U}, w} \beta_w \in \Gamma_{\mathfrak{h}}^+, \\ \xi_{\mathfrak{h}} &:= \operatorname{ulim}_{\mathcal{U}, w} \xi_w \in \operatorname{ulim}_{\mathcal{U}, w} X(R_{\beta_w}) \cong X\left(\operatorname{ulim}_{\mathcal{U}, w} R_{\beta_w}\right) = X(R_{\mathfrak{h}, \beta_{\mathfrak{h}}})\end{aligned}$$

where in the last line we have used 2.2.3 and 2.3.1. An equivalent definition of $\beta_{\mathfrak{h}}$ is of course $\beta_{\mathfrak{h}} = H\alpha_{\mathfrak{h}} + \Delta$ (note that $\Gamma_{\mathfrak{h}}$ is an ordered $\mathbb{Z}_{\mathfrak{h}}$ -module in a natural way); in particular, from (2.6.0.1) (and since $\alpha_{\mathfrak{h}} \geq 0$) we see that $\beta_{\mathfrak{h}} \geq \mathbb{Z}\alpha_{\mathfrak{h}} + \Gamma$ in $\Gamma_{\mathfrak{h}}$, and consequently

$$\beta_{\mathfrak{h}} \geq C := \text{convex hull of } \mathbb{Z}\alpha_{\mathfrak{h}} + \Gamma \text{ in } \Gamma_{\mathfrak{h}}.$$

This means that $R^{(C)}$ is a quotient of $R_{\mathfrak{h}, \beta_{\mathfrak{h}}}$. In turn, $R_{\mathfrak{h}, \alpha_{\mathfrak{h}}}$ is a quotient of $R^{(C)}$, by definition of C . Thus we have a diagram of sets

$$\begin{array}{ccccc} & & X(R_{\mathfrak{h}}) & & \\ & & \downarrow & & \\ \xi_{\mathfrak{h}} \in X(R_{\mathfrak{h}, \beta_{\mathfrak{h}}}) & \longrightarrow & X(R^{(C)}) & \longrightarrow & X(R_{\mathfrak{h}, \alpha_{\mathfrak{h}}}) \end{array}$$

By Theorem 2.5, the image of $\xi_{\mathfrak{h}}$ in $R^{(C)}$ lifts to an element $\eta_{\mathfrak{h}} \in X(R_{\mathfrak{h}})$. By construction, $\xi_{\mathfrak{h}}$ and $\eta_{\mathfrak{h}}$ have the same image in $X(R_{\mathfrak{h}, \alpha_{\mathfrak{h}}})$. This means, using 2.3.1, that, for almost all $w \in W$, the image of ξ_w in $X(R_{\alpha_w})$ lifts to $X(R)$. This contradicts our initial choices. ■

3 Proof of Theorems 2.4 and 2.5

3.1 Finite extensions

In this section we assume R complete. Let K_1 be a finite extension of K , of degree d . Then the valuation ord has an extension ord_1 to K_1 , with group $\Gamma_1 \supset \Gamma$ (this is true for any extension). Moreover, we know that the index $(\Gamma_1 : \Gamma)$ is finite (and in fact $\leq d$) [4, VI, §8, n° 3, th. 1]; in particular, Γ is cofinal in Γ_1 .

(Note that, unless it has height 1, ord may have several extensions to K_1 ; however, they are all dependent, i.e. they define the same topology on K_1 [4, §8, n° 2, cor. 1]).

We denote by $R_1 \subset K_1$ the ring of ord_1 . The situation is complicated by the fact that R_1 is not necessarily a finitely generated R -module. To address this, we shall define substitutes for R_1 and ord_1 as follows: choose a K -basis \mathcal{B} of K_1 whose elements are integral over K (hence $\mathcal{B} \subset R_1$), and put $R_0 := R[\mathcal{B}] \subset R_1$. Then R_0 is a finite R -algebra with fraction field K_1 . Since R is a valuation ring, R_0 is a free R -module of rank d , so we can fix a basis $\mathcal{B}_0 = (e_1 = 1, e_2, \dots, e_d)$ of R_0 over R . Now, for each $z = \sum_{i=1}^d x_i e_i \in K_1$ (with the x_i 's in K) we can put

$$\operatorname{ord}_0(z) := \min_i \operatorname{ord}(x_i).$$

3.1.1 Lemma *With the above assumptions and notations, the function*

$$f := \text{ord}_1 - \text{ord}_0 : K_1^* \longrightarrow \Gamma_1$$

is bounded.

Proof: Let us introduce the “balls”

$$\begin{aligned} B_0(\alpha) &:= \{z \in K_1 \mid \text{ord}_0(z) \geq \alpha\} \quad (\alpha \in \Gamma) \\ B_1(\alpha) &:= \{z \in K_1 \mid \text{ord}_1(z) \geq \alpha\} \quad (\alpha \in \Gamma_1). \end{aligned}$$

Thus, we have $B_0(0) = R_0$ and $B_1(0) = R_1$. The family $(B_1(\alpha))_{\alpha \in \Gamma_1}$ (resp. $(B_0(\alpha))_{\alpha \in \Gamma}$) is a basis of neighbourhoods of 0 in K_1 for the topology defined by ord_1 (resp. for the product topology on K_1 , identified with K^d via \mathcal{B}_0); since Γ is cofinal in Γ_1 we can even restrict the first family to Γ . Note that we have $tB_0(\alpha) = B_0(\alpha + \text{ord}(t))$ for $\alpha \in \Gamma$ and $t \in K$, and similarly for B_1 .

Since K is complete, our two topologies are in fact the same [4, chap. 6, §5, n° 2, prop. 4]. In fact, we trivially have $B_0(\alpha) \subset B_1(\alpha)$ for all α (look at the definition of ord_0); but by [4] we also have, say, $B_1(\lambda) \subset B_0(0)$ for some $\lambda \in \Gamma$, whence $B_1(0) \subset B_0(-\lambda)$ and, by scaling,

$$B_0(\alpha) \subset B_1(\alpha) \subset B_0(\alpha - \lambda)$$

for all $\alpha \in \Gamma$.

Returning to the function f , observe that $f(tz) = f(z)$ for $t \in K^*$. Next, since Γ has finite index in Γ_1 , there is a finite subset $\Sigma \subset \Gamma_1$ such that each $z \in K_1^*$ can be written $z = tz_1$ with $t \in K$ and $\text{ord}_1(z_1) \in \Sigma$. It follows that it is enough to bound $f(z)$ whenever z is in the “annulus” $U := B_1(r) \setminus B_1(r')$, for any fixed $r \leq \min(\Sigma)$ and $r' > \max(\Sigma)$, which we may (and do) take in Γ . Now from the above inclusions we have $B_0(r') \subset B_1(r') \subset B_1(r) \subset B_0(r - \lambda)$, whence $U \subset B_0(r - \lambda) \setminus B_0(r')$. In other words, ord_0 (hence also f) is bounded on U , which completes the proof. ■

3.2 Proof of Theorem 2.4

We adopt the notations and assumptions of Theorem 2.4. Let us first show that assertion (i) easily implies (ii) and (iii). Consider the commutative diagram of fraction fields:

$$\begin{array}{ccccc} K & \hookrightarrow & K_{\mathfrak{h}} & \hookrightarrow & K^{(C)} \\ \cap & & \cap & & \cap \\ \widehat{K} & \hookrightarrow & \widehat{K}_{\mathfrak{h}} & \hookrightarrow & \widehat{K}^{(C)}. \end{array}$$

Assuming (i) (applied to \widehat{R}), the extension of fraction fields $\widehat{K} \hookrightarrow \widehat{K}^{(C)} := \text{Frac}(\widehat{R}^{(C)})$ (bottom line) is regular. This proves that $\widehat{K}^{(C)}/\widehat{K}$ is separable (resp. primary) if \widehat{K}/K is, and the same holds for the subextension $K^{(C)}/K$. This implies (ii) and (iii), as promised.

From now on, we assume R complete. Let K_1 be a finite extension of K : we need to prove that $K^{(C)}$ and K_1 are linearly disjoint over K . Put $d := [K_1 : K]$.

We now apply the constructions (and keep the notations) of 3.1. We also have ultra-powers $R_{0,\mathfrak{h}} \subset R_{1,\mathfrak{h}} \subset K_{1,\mathfrak{h}}$, and a valuation $\text{ord}_{1,\mathfrak{h}}$ on $K_{1,\mathfrak{h}}$ with ring $R_{1,\mathfrak{h}}$ and group $\Gamma_{1,\mathfrak{h}}$. Since $\Gamma \subset \Gamma_1$ has finite index, so does $\Gamma_{\mathfrak{h}} \subset \Gamma_{1,\mathfrak{h}}$. We denote by C_1 the convex hull of C in $\Gamma_{1,\mathfrak{h}}$ (a convex subgroup containing Γ_1): this defines a prime ideal P_{C_1} of $R_{1,\mathfrak{h}}$, with quotient $R_{1,\mathfrak{h}}^{(C_1)}$. We immediately see that $C = \Gamma_{\mathfrak{h}} \cap C_1$ (since C is already convex in $\Gamma_{\mathfrak{h}}$) and $P_C = R_{\mathfrak{h}} \cap P_{C_1}$.

The following lemma clearly implies that the fields $K_1 = \text{Frac}(R_0)$ and $K^{(C)} = \text{Frac}(R^{(C)})$ are linearly disjoint over K , thus completing the proof of 2.4 (i):

3.2.1 Lemma $R_0 \otimes_R R^{(C)}$ is an integral domain.

Proof: Since R_0 is finite free over R , the natural map $R_0 \otimes_R R^W \rightarrow R_0^W$ is an isomorphism, and one readily checks that the same holds for $R_0 \otimes_R R_{\mathfrak{h}} \rightarrow R_{0,\mathfrak{h}}$. Hence $R_0 \otimes_R R^{(C)}$ is isomorphic to $R_{0,\mathfrak{h}}/P_C R_{0,\mathfrak{h}}$. So, our task is to show that $P_C R_{0,\mathfrak{h}}$ is a prime ideal of $R_{0,\mathfrak{h}}$. In fact we shall prove that $P_C R_{0,\mathfrak{h}} = R_{0,\mathfrak{h}} \cap P_{C_1}$, which implies the claim since $P_{C_1} \subset R_{1,\mathfrak{h}}$ is prime.

Since (e_1, \dots, e_d) is a basis of $R_{0,\mathfrak{h}}$ over $R_{\mathfrak{h}}$, every element of $R_{0,\mathfrak{h}}$ can be written as

$$x = \sum_{i=1}^d x_{\mathfrak{h}}^{(i)} e_i \quad (x_{\mathfrak{h}}^{(i)} \in R_{\mathfrak{h}})$$

and this x is in $P_C R_{0,\mathfrak{h}}$ if and only if each coordinate $x_{\mathfrak{h}}^{(i)}$ is in P_C , or equivalently in P_{C_1} . In other words:

$$\begin{aligned} x_{\mathfrak{h}} \in P_C R_{0,\mathfrak{h}} &\Leftrightarrow \forall i \in \{1, \dots, d\}, \text{ord}_{\mathfrak{h}}(x_{\mathfrak{h}}^{(i)}) > C_1 \\ &\Leftrightarrow \min_{1 \leq i \leq d} \text{ord}_{\mathfrak{h}}(x_{\mathfrak{h}}^{(i)}) > C_1 \\ &\Leftrightarrow \text{ord}_{0,\mathfrak{h}}(x_{\mathfrak{h}}) > C_1. \end{aligned}$$

But it follows from Lemma 3.1.1 that the difference $\text{ord}_1 - \text{ord}_0$ is uniformly bounded by elements of Γ : this property extends to the function $\text{ord}_{1,\mathfrak{h}} - \text{ord}_{0,\mathfrak{h}}$ on $R_{0,\mathfrak{h}}$. Since $\Gamma \subset C_1$, the last condition is therefore equivalent to $\text{ord}_{1,\mathfrak{h}}(x_{\mathfrak{h}}) > C_1$, hence to $x_{\mathfrak{h}} \in P_{C_1}$, which completes the proofs of 3.2.1 and 2.4. \blacksquare

3.3 Proof of the lifting theorem 2.5

The following proposition and its proof are essentially taken from [2, Lemma 2.2].

3.3.1 Proposition Consider a commutative diagram of integral domains

$$\begin{array}{ccc} A & \subset & V \\ \downarrow i & & \downarrow \pi \\ A' & \subset & V/P \end{array}$$

where:

- V is a Henselian valuation ring, P is a prime ideal of V and π is the canonical map;
- i is injective and the extension $\text{Frac}(A')/\text{Frac}(A)$ admits a separating transcendence basis.

Then A' lifts to V , i.e. there is a subring of V containing A and mapping isomorphically to A' by π .

Proof: put $F = \text{Frac}(A)$ and $F' = \text{Frac}(A')$. First, we may replace A' by $F' \cap (V/P)$, which is a valuation ring because V/P is. If B is a separating transcendence basis for F'/F , then for each $b \in B$ we have $b \in A'$ or $b^{-1} \in A'$. So, by modifying B we may assume that $B \subset A'$. Now the ring $A[B]$ lifts trivially to V (just lift B arbitrarily), so we assume from now on that F' is separably algebraic over F . By Zorn's lemma, we are reduced to the case $A' = A[x]$ where x is a root of $g \in A[X]$, irreducible and separable over F . So we have $g(x) = 0$ and $g'(x) \neq 0$. Let $\tilde{x} \in V$ be a lift of x : we have $g(\tilde{x}) \in P$ and $g'(\tilde{x}) \notin P$, whence $g'(\tilde{x})^2 \notin P$ since P is prime. So $e := g(\tilde{x})/g'(\tilde{x})^2$ belongs to the maximal ideal of V . By the ‘‘Hensel-Rychlik lemma’’ (following from the Hensel property applied to the polynomial $G(h) = \frac{1}{g'(\tilde{x})} g(\tilde{x} + e g'(\tilde{x}) h)$) there exists $\bar{x} \in V$ with $g(\bar{x}) = 0$ and $\bar{x} \equiv \tilde{x} \pmod{e g'(\tilde{x})}$. In particular we have $\pi(\bar{x}) = \pi(\tilde{x}) = x$. Put $\bar{A} := A[\bar{x}] \subset V$: then \bar{A} lifts A' , because $\pi(\bar{A}) = A[x] = A'$ and A' and \bar{A} can both be seen as subrings of $F[X]/(g(X))$. ■

3.3.2 Corollary *With $A \subset V \twoheadrightarrow V/P$ as in 3.3.1, assume that the composite map $A \rightarrow V/P$ is injective and that the extension $\text{Frac}(V/P)/\text{Frac}(A)$ is separable. Let Y be an A -scheme locally of finite type. Then the natural map $Y(V) \rightarrow Y(V/P)$ is onto.*

Proof: Since V/P is a local ring, every morphism $y : \text{Spec}(V/P) \rightarrow Y$ factors through an affine open subset of Y . So we may assume that $Y = \text{Spec}(B)$ with B finitely generated over A . Then y corresponds to $\varphi : B \rightarrow V/P$. If $A' \subset V/P$ is the image of φ , then $\text{Frac}(A')/\text{Frac}(A)$ is a finitely generated separable extension, hence admits a separating transcendence basis [3, V, §9, n° 3, th. 2]. The conclusion then follows from 3.3.1. ■

3.3.3 Remark Another noteworthy special case of 3.3.1 (already mentioned in [2]) is when $\text{Frac}(V/P)$ (hence also $\text{Frac}(V)$) has characteristic zero: we may then take $A = \mathbb{Z}$ and $A' = V/P$ and conclude that π has a section.

3.3.4 End of the proof of Theorem 2.5. With R , X and C as in the theorem, we deduce from 2.4 (ii) that $\text{Frac}(R^{(C)})$ is separable over K . Therefore we may apply Corollary 3.3.2 with $A = R$, $V = R_{\mathfrak{q}}$, $P = P_C$ and $Y = X$. This completes the proof.

4 Application: a closed image theorem

4.1 Basic topological facts

Recall that if F is any Hausdorff topological field, we can uniquely define a topology on $X(F)$ for every F -scheme X locally of finite type, in such a way that (X and Y denoting

arbitrary F -schemes locally of finite type):

- if $X = \mathbb{A}_F^1$ we obtain the given topology on $X(F) = F$;
- every F -morphism $f : X \rightarrow Y$ gives rise to a continuous map $X(F) \rightarrow Y(F)$ which, moreover, is an open (resp. closed) topological embedding if f is an open (resp. closed) immersion;
- the natural bijection $(X \times Y)(F) \rightarrow X(F) \times Y(F)$ is a homeomorphism.

In the sequel we keep the notations $(R, K, \Gamma, \text{ord})$ of 1.1 and we take $F = K$ with the topology defined by the valuation. Thus, if X is a K -scheme locally of finite type, we can characterize the topology on $X(K)$ as follows: for $x \in X(K)$, fix an affine open neighborhood $U = \text{Spec}(A)$ of x in X and a finite sequence (f_1, \dots, f_n) generating A as a K -algebra. We obtain a basis of neighborhoods of x in $X(K)$ by taking the “balls” $B(x, \gamma) = \{y \in U(K) \mid \text{ord}(f_i(x) - f_i(y)) \geq \gamma, i = 1, \dots, n\}$ for all $\gamma \in \Gamma$.

Note that in the above description, $B(x, \gamma)$ is the image of $\mathcal{U}(R)$ in $U(K)$, where \mathcal{U} is the spectrum of the R -algebra $\mathcal{A} = R[\frac{f_1 - f_1(y)}{t}, \dots, \frac{f_n - f_n(y)}{t}] \subset A$ and we denote by t any element of K with valuation γ . (More generally, it can be checked that if X is of finite type over K , we obtain a basis of open sets for $X(K)$ by taking the sets $\text{Im}(\mathcal{X}(R) \rightarrow X(K))$ where \mathcal{X} runs through all R -schemes of finite type with generic fiber X).

If \mathcal{X} is a *separated* R -scheme of finite type, then we can identify $\mathcal{X}(R)$ with a subset of $\mathcal{X}(K) = \mathcal{X}_K(K)$, which is easily seen to be open; we can then endow $\mathcal{X}(R)$ with the induced topology. It is in fact possible to define the topology on $\mathcal{X}(R)$ directly, even if \mathcal{X} is not separated; however, this takes some more care (see [6, Proposition 3.1]) and the present definition will be sufficient for our purposes.

With \mathcal{X} as above, denote by \mathcal{X}_0 the Zariski closure of \mathcal{X}_K in \mathcal{X} , with its reduced subscheme structure. Then \mathcal{X}_0 and \mathcal{X} have the same R -points (resp. K -points), and it is easy to see that \mathcal{X}_0 is flat over R (recall that every torsion-free R -module is flat). It is also of finite type, as a closed subscheme of \mathcal{X} , and hence of *finite presentation* by [13, (3.4.7)]. To summarize, when using “ R -models” to study the topology of a given K -scheme of finite type X , we only need models of X_{red} which are flat of finite presentation over R .

The following result is essentially equivalent to Corollary 1.2.2 (in a more general setting, see also [11, Proposition 4.1.1]):

4.2 Proposition *Assume that R is Henselian and \widehat{K} is separable over K , and let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of R -schemes of finite presentation, with \mathcal{Y} separated.*

Then the induced map $f_R : \mathcal{X}(R) \rightarrow \mathcal{Y}(R)$ has closed image.

Proof: The question is local on \mathcal{Y} , so we may assume that \mathcal{Y} is affine, and even that $\mathcal{Y} = \mathbb{A}_R^n = \text{Spec } R[T_1, \dots, T_n]$ for some n , by choosing a closed immersion $\mathcal{Y} \hookrightarrow \mathbb{A}^n$. Using a finite affine open covering of \mathcal{X} , we may assume that

$$\mathcal{X} = \text{Spec} \left(R[T_1, \dots, T_n, Z_1, \dots, Z_m] / (F_1, \dots, F_r) \right)$$

for suitable polynomials $F_j \in R[T, Z]$. We may further assume that the origin $0 \in \mathcal{Y}(R) = R^n$ is in the closure of the image of f_R .

This means the following: for each $\gamma \in \Gamma$, there exist $t_1, \dots, t_n, z_1, \dots, z_m$ in R such that $F_j(t, z) = 0$ ($1 \leq j \leq r$) and $\text{ord}(t_i) \geq \gamma$ ($1 \leq i \leq n$). Since F_j has coefficients in R , this implies $\text{ord}(F_j(0, z)) \geq \gamma$. In other words, the fibre \mathcal{Y}_0 of f at 0 (which is an R -scheme of finite presentation) has R_γ -valued points for all $\gamma \in \Gamma^+$. By 1.2.2, $\mathcal{Y}_0(R) \neq \emptyset$. In other words, 0 is in the image of f_R . ■

4.2.1 Remark We have assumed \mathcal{Y} separated only to avoid using the general definition of the topology on $\mathcal{Y}(R)$, alluded to in 4.1 above. Surprisingly (at least to the author), this assumption is not necessary, and in fact $\mathcal{Y}(R)$ is always a Hausdorff space, even if \mathcal{Y} is not separated.

4.3 Proof of Theorem 1.3

Consider $f : X \rightarrow Y$ as in Theorem 1.3. To prove that the image of f_K is closed, we may assume Y affine. Fix a flat, affine, finitely presented R -scheme \mathcal{Y} with generic fiber Y . It suffices to show that $f_K(X(K)) \cap \mathcal{Y}(R)$ is closed in $\mathcal{Y}(R)$, because the sets $\mathcal{Y}(R) \subset Y(K)$, for varying \mathcal{Y} , form a basis of open subsets.

Since K is the increasing union of its subrings $R[t^{-1}]$, where t runs through nonzero elements of R , we can apply the results of [10, §8] and find, for suitable such t , a scheme \mathcal{X}_1 , separated of finite presentation over $R[t^{-1}]$ (hence also over R), such that $(\mathcal{X}_1)_K = X$, and an R -morphism $f_1 : \mathcal{X}_1 \rightarrow \mathcal{Y}$ extending f .

By Nagata's compactification theorem [5, Theorem 4.1] the morphism f_1 factors as $\mathcal{X}_1 \xrightarrow{j} \mathcal{X} \xrightarrow{\bar{f}} \mathcal{Y}$ where j is a dense open immersion and \bar{f} is proper; in particular, we have $\mathcal{X}_K = (\mathcal{X}_1)_K = X$ since $(\mathcal{X}_1)_K$ is assumed proper over \mathcal{Y}_K . Thus, by the valuative criterion of properness, we have $f_K(X(K)) \cap \mathcal{Y}(R) = \bar{f}(\mathcal{X}(R))$. Hence the result follows from 4.2, applied to \bar{f} . ■

4.3.1 Remark Of course, if f is assumed projective, Nagata's theorem is not needed: in the proof, we can choose \mathcal{X}_1 quasiprojective over R .

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