The relative Fujita-Zariski theorem

Laurent Moret-Bailly*†

April 3, 2025

Abstract

We prove, with no claim to originality, a relative version of the Fujita-Zariski theorem. When the base is a field, this result is due to Fujita [1] and states that if an invertible sheaf on a proper variety is ample on its base locus, its sufficiently high powers are globally generated. The special case where the base locus is finite was proved by Zariski [5], whence the name.

AMS 2020 classification: 14A15, 14C20.

Contents

1 Definitions, notation, and statement

Throughout the paper we consider a noetherian ring R and a proper morphism of schemes $f : X \to S := \mathrm{Spec}(R)$.

Definition 1.1. *A* linear system $\Lambda = (\mathscr{L}, V, \rho)$ *on* X *consists of:*

(i) *an invertible* $\mathscr{O}_X$*-module* $\mathscr{L}$*,*

(ii) *an* R*-module* V *of finite type, and*

(iii) *an* R*-linear map* $\rho : V \to \mathrm{H}^0(X, \mathscr{L})$*, or (equivalently) an* $\mathscr{O}_X$*-linear map* $\widetilde{\rho} : f^*V \to \mathscr{L}$*.*

From such a Λ we derive an exact sequence $f^*V \otimes \mathscr{L}^{-1} \to \mathscr{O}_X \to \mathscr{O}_B \to 0$ where B is a closed subscheme of X called the *base locus* of the system.

*Univ Rennes, CNRS, IRMAR - UMR 6625, F-35000 Rennes, France

†`laurent.moret-bailly[AT]univ-rennes.fr`

Theorem 1.2. *Let Λ be a linear system on X as above. Assume that the restriction of $\mathscr{L}$ to B is ample. Then there exists an integer t_0 such that $\mathscr{L}^{\otimes t}$ is globally generated for all $t \geq t_0$.*

Remarks 1.3. (1) Note that the conclusion of the theorem is about $\mathscr{L}$ only. The assumption is preserved if we replace V by its image in $\mathrm{H}^0(X, \mathscr{L})$ or a bigger submodule (since this can only shrink B). So we could as well assume that $V = \mathrm{H}^0(X, \mathscr{L})$. However, this does not make the proof any simpler. Moreover, given an element $v \in V$, it is advisable to distinguish v (living in an R-module) from $\rho(v)$ which is a section of $\mathscr{L}$ and thus lives on X.

(2) When R is a field, this result is due to Fujita [1, Theorem 1.10]. The special case when B is finite was proved earlier by Zariski [5, Theorem 6.2].

(3) Our proof follows Fujita's strategy rather faithfully. The only difficulties were to find the right substitutes for induction on dimension of a subscheme, on dimension of V, and the notion of a "general element" of V.

(4) The interested reader will have no trouble adapting the proof to the case where X is a proper algebraic space over S.

(5) To the author's knowledge, and to this date, Theorem 1.2 is not in the literature. It is used without proof in the paper [4], with a reference to Fujita's article.

1.4 Graded rings and modules

By convention, all our graded rings and modules are $\mathbb{N}$-graded. For modules, this may seem somewhat unnatural, but it is harmless as the properties we shall be concerned with can be checked "in large enough degree".

Let $X \to S$ and Λ be as above. To a coherent $\mathscr{O}_X$-module $\mathscr{F}$ and an integer $q \geq 0$ we associate the graded R-module

$$\mathrm{H}^q_*(\mathscr{F}) = \mathrm{H}^q(X, \mathscr{F} \otimes \underline{\mathrm{Sym}}_{\mathscr{O}_X}(\mathscr{L})) = \bigoplus_{t \geq 0} \mathrm{H}^q(X, \mathscr{F} \otimes \mathscr{L}^{\otimes t})$$

whose graded components are finite R-modules. Note that $\mathrm{H}^0_*(\mathscr{O}_X) = \mathrm{H}^0(X, \underline{\mathrm{Sym}}_{\mathscr{O}_X}(\mathscr{L}))$ is a graded R-algebra, and each $\mathrm{H}^q_*(\mathscr{F})$ is a graded $\mathrm{H}^0_*(\mathscr{O}_X)$-module. Moreover, $\rho : V \to \mathrm{H}^0(X, \mathscr{L})$ gives rise to a morphism

$$\mathrm{Sym}_R(V) \to \mathrm{H}^0_*(\mathscr{O}_X)$$

of graded R-algebras, thus turning $\mathrm{H}^q_*(\mathscr{F})$ into a graded $\mathrm{Sym}_R(V)$-module. Technically, these modules will be our main object of study; observe that $\mathrm{Sym}_R(V)$ is a finitely generated R-algebra and thus a noetherian ring, unlike $\mathrm{H}^0_*(\mathscr{O}_X)$ in general.

We shall make use of the following easy fact: a graded $\mathrm{Sym}_R(V)$-module $M = \bigoplus_{t \geq 0} M_t$ is finitely generated if and only if each M_t is finitely generated over R, and the natural map $V \otimes_R M_t \to M_{t+1}$ (where V is viewed as the degree 1 component of $\mathrm{Sym}_R(V)$) is surjective for t large enough.

Definition 1.4.1. *A* Λ-module $(\mathscr{F}, q, M_*)$ *consists of*

(i) *a coherent $\mathscr{O}_X$-module $\mathscr{F}$,*

(ii) *an integer* $q \geq 0$,

(iii) *a graded* $\mathrm{Sym}_R(V)$*-submodule* $M_* = \bigoplus_{t\geq 0} M_t \subset \mathrm{H}^q_*(\mathscr{F})$.

This differs slightly from the corresponding notion of a "Λ-module system" in [1], where negative degrees are allowed. Such a triple $(\mathscr{F}, q, M_*)$ will usually be called M_* for simplicity. In particular, we say that $(\mathscr{F}, q, M_*)$ is *finitely generated* if M_* is a finitely generated $\mathrm{Sym}_R(V)$-module.

Remark 1.4.2. To prove Theorem 1.2, an essential tool is to prove that (under its assumptions) certain Λ-modules are finitely generated. We shall use induction arguments typically involving the choice of a "general" element of V. The next section explains what we mean by this.

1.5 Fat subsets in modules, general position results

Definition 1.5.1. *Let A be a ring, M an A-module. A subset $E \subset M$ is (A-)*fat *if $M \smallsetminus E$ is contained in a finite union of proper submodules of M.*

Lemma 1.5.2. *Let A be a ring.*

(1) *The following are equivalent:*

(i) *For every A-module M, every A-fat subset of M is nonempty.*

(ii) *For every finitely generated A-module M, every A-fat subset of M is nonempty.*

(iii) *For each maximal ideal $\mathfrak{m}$ of A, the residue field $A/\mathfrak{m}$ is infinite.*

(iv) *For each prime ideal $\mathfrak{p}$ of A, the residue field $\kappa(\mathfrak{p})$ is infinite.*

(2) *There is a faithfully flat A-algebra A' satisfying the conditions of* (1)*. Moreover we can take A' noetherian if A is.*

Proof. (1) The implication (i)⇒(ii) is trivial. For the converse, let M be an A-module and $(N_i)_{1\leq i\leq r}$ a finite family of submodules. Pick an element in $M \smallsetminus N_i$ for each i, and apply (ii) to the submodule generated by these.

(ii)⇒(iii): Assume A has a finite residue field κ. Then the finitely generated A-module κ^2 is the union of its finitely many proper submodules.

(iii)⇔(iv) is immediate. Now we assume (iii) and prove (ii). Let M and $(N_i)_{1\leq i\leq r}$ be as above, with M finitely generated, and let us prove that $\bigcup_{1\leq i\leq r} N_i \subsetneq M$. For each i, M/N_i is finitely generated, so there is a maximal ideal $\mathfrak{m}_i$ and an epimorphism $\pi_i : M/N_i \twoheadrightarrow \kappa_i := A/\mathfrak{m}_i$. Clearly we may replace N_i by $\operatorname{Ker} \pi_i$. In other words, N_i is now the preimage of a hyperplane H_i in the κ_i-vector space $M \otimes_A \kappa_i$.

First assume that all the $\mathfrak{m}_i$'s are equal, with residue field κ. Since κ is infinite by assumption, we have $\bigcup_{1\leq i\leq r} H_i \subsetneq M \otimes_A \kappa$, whence the result.

Otherwise we renumber the κ_i's as $\kappa_1, \ldots, \kappa_s$ (assumed pairwise distinct), and the N_i's as N_{jk} in such a way that $M/N_{jk} \cong \kappa_j$. For each j we pick an element m_j of $M \otimes \kappa_j$ which is not in the image of any N_{jk}, as in the previous step. Then we observe that the $\mathfrak{m}_j$'s are pairwise coprime, and therefore $M \to \prod_j M \otimes \kappa_j$ is surjective, so there is $m \in M$ which reduces to m_j for all j and thus cannot belong to any N_{jk}.

(2) We can take $A' = U^{-1}A[T]$ where U is the multiplicative set of all monic polynomials. (If A is noetherian, another choice is $A' = A[[T]][T^{-1}]$). □

Proposition 1.5.3. *Let R, $f : X \to S$ and $\Lambda = (\mathscr{L}, V, \rho)$ be as in* 1.1, *with base locus $B \subset X$. Let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module, and let $\Sigma \subset X \smallsetminus B$ be a finite set of points. Then, for all δ in a suitable fat subset of V, the following conditions are satisfied:*

(i) *The section $\rho(\delta)$ of $\mathscr{L}$ does not vanish at any $x \in \Sigma$.*

(ii) *The morphism $\varphi : \mathscr{F} \otimes \mathscr{L}^{-1} \to \mathscr{F}$ induced by tensoring with $\rho(\delta)$ is injective on $X \smallsetminus B$; in other words,* $\mathrm{Supp}\,(\mathrm{Ker}\,\varphi) \subset B$.

Proof. We can enlarge Σ and assume it contains $\mathrm{Ass}_{\mathscr{O}_X}(\mathscr{F}) \smallsetminus B$. Then condition (ii) is a consequence of (i). Now for $x \in \Sigma$, $V_x := \{\delta \in V \mid \rho(\delta)(x) = 0\}$ is an R-submodule of V, and $V_x \subsetneq V$ since $x \notin B$. The result follows. □

Here is a first consequence:

Corollary 1.5.4. *Let R be a noetherian ring, V a finitely generated R-module, M_* a finitely generated graded* $\mathrm{Sym}(V)$*-module. There exists $t_0 \geq 0$ and a fat subset E of V such that for all $\delta \in E$, the map $M_t \xrightarrow{\times\delta} M_{t+1}$ is injective for all $t \geq t_0$.*

Proof. We apply Proposition 1.5.3 to the following data: let $P := \mathbb{P}(V) = \mathrm{Proj}(\mathrm{Sym}(V))$, $\mathscr{M} = \widetilde{M_*}$ the coherent $\mathscr{O}_P$-module associated to M_*. We take for $\mathscr{L}$ the canonical sheaf $\mathscr{O}_P(1)$; note that the base locus B is empty. By the proposition, for δ in a fat subset of V the corresponding $\mathscr{M}(-1) \to \mathscr{M}$ is injective, so of course the same holds for all maps $\mathscr{M}(t) \to \mathscr{M}(t+1)$ and $\mathrm{H}^0(P, \mathscr{M}(t)) \to \mathrm{H}^0(P, \mathscr{M}(t+1))$.

On the other hand, by [2, (2.3.1)] there exists t_0 such that for all $t \geq t_0$ the canonical map $M_t \to \mathrm{H}^0(P, \mathscr{M}(t))$ is an isomorphism, which completes the proof. □

2 Proof of the theorem

2.1 A finiteness result

A key step in the proof is the following proposition (which will be used for $q = 1$ only!):

Proposition 2.1.1. *With the assumptions of Theorem* 1.2, *let $(\mathscr{F}, q, M_*)$ be a Λ-module with $q > 0$. Then M_* is finitely generated.*

The proof relies on a dévissage lemma:

Lemma 2.1.2. *Let $\Lambda = (\mathscr{L}, V, \rho)$ be a linear system on X. Let $\delta \in V$ be fixed, and let $\varphi : \mathscr{F} \otimes \mathscr{L}^{-1} \to \mathscr{F}$ denote the morphism given by multiplication by $\rho(\delta)$. Consider the self-defining exact sequence*

$$0 \longrightarrow \mathscr{K} \longrightarrow \mathscr{F} \otimes \mathscr{L}^{-1} \xrightarrow{\varphi} \mathscr{F} \xrightarrow{\pi} \mathscr{C} \longrightarrow 0$$

of sheaves on X. We fix an integer $q \geq 0$, and we introduce the following Λ-modules:

$$\begin{array}{rcl} M_* & := & \mathrm{H}^q_*(\mathscr{F}) \\ N_* & := & \mathrm{Im}\,(M_* \to \mathrm{H}^q_*(\mathscr{C})) \\ K^+_* & := & \mathrm{H}^{q+1}_*(\mathscr{K}) \end{array}$$

and we assume the following conditions:

(i) $K_t^+ = 0$ *for* $t \gg 0$;

(ii) N_* *is a finitely generated* Λ*-module.*

Then M_* *is finitely generated.*

Proof of 2.1.2. Condition (ii) means that $VN_{t-1} = N_t$ for large t. We fix t such that this holds in addition to $K_t^+ = 0$, and we proceed to show that $VM_{t-1} = M_t$, which will prove the result.

To achieve this we already observe that we have a natural surjective morphism $M_* \twoheadrightarrow N_*$ of graded $\mathrm{Sym}(V)$-modules. This and the condition $VN_{t-1} = N_t$ imply that $VM_{t-1} \hookrightarrow M_t \to N_t$ is surjective. Thus it suffices to prove that $\mathrm{Ker}(M_t \to N_t) \subset VM_{t-1}$. We shall see that, more precisely, $\mathrm{Ker}(M_t \to N_t) = \delta M_{t-1}$.

Le $\mathscr{I} \subset \mathscr{F}$ be the image of φ. Put $I_* = \mathrm{H}^q_*(\mathscr{I})$. The map $M_{t-1} \to M_t$ induced by φ is just multiplication by δ for the $\mathrm{Sym}(V)$-module structure on M_* (in particular its image is δM_{t-1}), and it factors as $M_{t-1} \xrightarrow{\alpha} I_t \xrightarrow{\beta} M_t$. From the short exact sequence $0 \to \mathscr{K} \to \mathscr{F} \otimes \mathscr{L}^{-1} \to \mathscr{F} \to \mathscr{I} \longrightarrow 0$ (twisted by $\mathscr{L}^{\otimes t}$) and the condition on K_t^+ we infer that $\alpha : M_{t-1} \to I_t$ is surjective. Thus $\mathrm{Im}\,\beta = \mathrm{Im}(\beta \circ \alpha) = \delta\, M_{t-1}$. On the other hand, the short exact sequence $0 \to \mathscr{I} \to \mathscr{F} \to \mathscr{C} \to 0$ yields an exact sequence $I_t \xrightarrow{\beta} M_t \to N_t \to 0$. Combining these we conclude that $\mathrm{Ker}(M_t \to N_t) = \delta\, M_{t-1}$, as promised. □

Proof of 2.1.1. We first note that the question is local on S for the fpqc topology. In particular, applying Lemma 1.5.2 (2), we may assume that all the residue fields of R are infinite, so that fat subsets of R-modules are always nonempty.

Next, we may and will assume that M_* is the full $\mathrm{H}^q_*(\mathscr{F})$ since $\mathrm{Sym}(V)$ is a noetherian ring.

We now choose a point $s \in S$ and work locally, i.e. we shall find an affine neighborhood $\mathrm{Spec}(R')$ of s such that $M_* \otimes_R R'$ is a finitely generated $\mathrm{Sym}(V) \otimes_R R'$-module. Putting $Y := \mathrm{Supp}(\mathscr{F})$ (defined, as a subscheme of X, by the ideal $\mathrm{Ann}_{\mathscr{O}_X}(\mathscr{F})$), we proceed by induction on

$$d(\mathscr{F}) := \dim_{\kappa(s)} \mathrm{Im}\left(V \to \mathrm{H}^0(Y_s, \mathscr{L}_{|Y_s})\right).$$

If $d(\mathscr{F}) = 0$, then $Y_s \subset B$. Therefore $\mathscr{L}$ is ample on Y_s. By [3, (4.7.1)] we may assume that $\mathscr{L}$ is ample on Y by restricting to a neighborhood of s. But then the result is trivial since $M_t = 0$ for large t. (The condition $q > 0$ is used here).

Now assume $d(\mathscr{F}) > 0$, and the result proved for all sheaves with smaller d. We have $Y_s \not\subset B$. Fix a point $y \in Y_s \smallsetminus B$ and apply Proposition 1.5.3 with $\Sigma = \{y\}$. With our assumption on the residue fields, we see that there exists $\delta \in V$ such that

(i) $\rho(\delta) \in \mathrm{H}^0(X, \mathscr{L})$ does not vanish identically on Y_s, and

(ii) $\varphi : \mathscr{F} \otimes \mathscr{L}^{-1} \to \mathscr{F}$ given by $\rho(\delta)$ is injective on $X \smallsetminus B$.

We now apply Lemma 2.1.2. We have an exact sequence

$$0 \longrightarrow \mathscr{K} \longrightarrow \mathscr{F} \otimes \mathscr{L}^{-1} \xrightarrow{\varphi} \mathscr{F} \xrightarrow{\pi} \mathscr{C} \longrightarrow 0$$

where $\mathrm{Supp}\,\mathscr{K} \subset B$ because of (ii). In particular, $\mathscr{L}$ is ample on $\mathrm{Supp}\,\mathscr{K}$, whence $\mathrm{H}^{q+1}(\mathscr{K} \otimes \mathscr{L}^{\otimes t}) = 0$ for large t, which is condition (i) of 2.1.2.

Consider $N_* := \mathrm{Im}(M_* \to \mathrm{H}^q_*(\mathscr{C}))$. We have a Λ-module $(\mathscr{C}, q, N_*)$, and we shall use the induction hypothesis to prove that N_* is finitely generated: this will complete the proof by 2.1.2. For this, it suffices to prove that $d(\mathscr{C}) < d(\mathscr{F})$. Putting $Z = \mathrm{Supp}(\mathscr{C})$, we have a surjection of $\kappa(s)$-vector spaces

$$\mathrm{Im}\left(V \to \mathrm{H}^0(Y_s, L_{|Y_s})\right) \twoheadrightarrow \mathrm{Im}\left(V \to \mathrm{H}^0(Z_s, L_{|Z_s})\right)$$

of dimensions $d(\mathscr{F})$ and $d(\mathscr{C})$ respectively. The image of $\delta \in V$ in the first space is nonzero by the above condition (i), but it vanishes in the second space by definition of $\mathscr{C}$. Thus, $d(\mathscr{C}) < d(\mathscr{F})$, as claimed. □

2.2 Proof of Theorem 1.2

We keep the notation and assumptions of 1.2. As in the proof of 2.1.2, we assume that all residue fields of R are infinite. In addition, we may assume $V = \mathrm{H}^0(X, \mathscr{L})$: this does not change the conclusion, and can only make the base locus smaller.

We procced by noetherian induction on X, assuming that for all proper closed subschemes $Y \subsetneq X$, $\mathscr{L}_{|Y}^{\otimes t}$ is globally generated for large t. We also assume $B \subsetneq X$ set-theoretically: otherwise, $\mathscr{L}$ is ample. In particular, $V \neq 0$.

We shall apply the induction to the subscheme $Y \subsetneq X$ defined by $\rho(\delta)$, for a suitable $0 \neq \delta \in V$. The idea is to find δ such $\mathrm{H}^0(\mathscr{L}^{\otimes t}) \longrightarrow \mathrm{H}^0(\mathscr{L}_{|Y}^{\otimes t})$ is surjective for large t. This clearly suffices because then $\mathscr{L}^{\otimes t}$ will have no base points on Y (by induction) and no base points off Y since Y clearly contains B. The surjectivity requirement translates into an injectivity property on H^1, which motivates the choices below.

I claim that there is a fat $E \subset V$ (not containing 0) such that, for all $\delta \in E$:

(i) multiplication by δ on $\mathrm{H}^1_*(\mathscr{O}_X)$ is injective on components of large degree;

(ii) $\rho(\delta)$ is regular off B, i.e. $\varphi : \mathscr{L}^{-1} \to \mathscr{O}_X$ given by $\rho(\delta)$ is injective on $X \smallsetminus B$.

Indeed, (ii) is the same as in the proof of 2.1.1, and for (i) we see by 2.1.1 that $\mathrm{H}^1_*(\mathscr{O}_X)$ is a finitely generated $\mathrm{Sym}(V)$-module, so Corollary 1.5.4 applies. Putting $\mathscr{I} = \mathrm{Im}(\varphi) \subset \mathscr{O}_X$, we have exact sequences (where $Y \subsetneq X$ is the zero locus of δ)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{K} & \longrightarrow & \mathscr{L}^{-1} & \longrightarrow & \mathscr{I} & \longrightarrow & 0 \\ 0 & \longrightarrow & \mathscr{I} & \longrightarrow & \mathscr{O}_X & \longrightarrow & \mathscr{O}_Y & \longrightarrow & 0. \end{array}$$

and the same sequences twisted by $\mathscr{L}^{\otimes t}$ for all t. By (i), $\mathscr{K}$ has support in B on which $\mathscr{L}$ is ample, so the first sequence induces $\mathrm{H}^1(\mathscr{L}^{\otimes(t-1)}) \xrightarrow{\sim} \mathrm{H}^1(\mathscr{I} \otimes \mathscr{L}^{\otimes t})$ for large t. The composition $\mathrm{H}^1(\mathscr{L}^{\otimes(t-1)}) \xrightarrow{\sim} \mathrm{H}^1(\mathscr{I} \otimes \mathscr{L}^{\otimes t}) \longrightarrow \mathrm{H}^1(\mathscr{L}^{\otimes t})$ is given by multiplication by δ on $\mathrm{H}^1_*(\mathscr{O}_X)$, thus injective for large t by (i). In short, for $t \gg 0$ the second exact sequence gives rise to injections $\mathrm{H}^1(\mathscr{I} \otimes \mathscr{L}^{\otimes t}) \longrightarrow \mathrm{H}^1(\mathscr{L}^{\otimes t})$, hence the restriction maps $\mathrm{H}^0(\mathscr{L}^{\otimes t}) \longrightarrow \mathrm{H}^0(\mathscr{L}_{|Y}^{\otimes t})$ are surjective. As explained above, this completes the proof. □

References

[1] Takao Fujita. Semipositive line bundles. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 30(2):353–378, 1983.

[2] A. Grothendieck. Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes. *Inst. Hautes Études Sci. Publ. Math.*, (8):5–222, 1961.

[3] A. Grothendieck. Éléments de géométrie algébrique. III: Étude cohomologique des faisceaux cohérents. (Première partie). *Publ. Math., Inst. Hautes Étud. Sci.*, 11:81–167, 1961.

[4] Stefan Schröer. A characterization of semiampleness and contractions of relative curves. *Kodai Math. J.*, 24(2):207–213, 2001.

[5] Oscar Zariski. The theorem of Riemann-Roch for high multiples of an effective divisor on an algebraic surface. *Ann. of Math. (2)*, 76:560–615, 1962.