Descent varieties for algebraic covers

Pierre Dèbes†, Jean-Claude Douai†

and Laurent Moret-Bailly‡

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Abstract. This paper is about descent theory for algebraic covers. Typical questions concern fields of definition, models, moduli spaces, families, etc. of covers. Here we construct descent varieties. Associated to any given cover f, they have the property that whether they have rational points on a given field k is the obstruction to descending the field of definition of f to k. Our constructions have a global version above moduli spaces of covers (Hurwitz spaces): here descent varieties are parameter spaces for Hurwitz families with some versal property. Descent varieties provide a new diophantine viewpoint on descent theory by reducing the questions to that of finding points on varieties. There are concrete applications. We answer a question raised in [DeHa] about totally p-adic models of covers. We also show that the subset of a given moduli space of covers of $\mathbb{P}^1$ where the field of moduli is a field of definition is Zariski-dense.

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Introduction

The main theme of the paper is the construction of *descent varieties for algebraic covers*. Given a cover $f : X \to B$ over a field, these are parameter spaces V for families of models of f satisfying the versal property: that is, each model of f is a fiber of the family, and so corresponds to points of V. The parameter spaces we construct are algebraic varieties that are smooth, geometrically irreducible and defined over the *field of moduli* of the cover f. Fields of definition of models and of their corresponding representative points correspond to one another. Thus these varieties provide a new approach to descent theory for fields of definition of covers: finding a model defined over a given field k is tantamount to finding k-rational points on these varieties. In particular, this gives a new description of the classical obstruction to the field of moduli being a field of definition, which was as yet mostly described in cohomological terms [DeDo1].

A first application, which was the original motivation of this work, concerns covers with field of moduli a number field, say $\mathbb{Q}$ for simplicity. It is known that, although $\mathbb{Q}$ may not be a field of definition, the cover must be defined over all $\mathbb{Q}_p$, except possibly for finitely

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many "bad" primes p, *viz.* those which divide the order of the group or are such that the branch points coalesce modulo p [DeHa], [Em]. We answer here (corollary 1.4) a question raised in [DeHa] (Question 5.3) to show that for the same primes p, the cover can even be defined over the field $\mathbb{Q}^{\mathrm{t}p}$ of all totally p-adic algebraic numbers (*i.e.*, the maximal Galois extension of $\mathbb{Q}$ contained in $\mathbb{Q}_p$). This result conjoins our Main Theorem with the following

Local-global principle on varieties — *Let V be a geometrically irreducible smooth $\mathbb{Q}^{\mathrm{t}p}$-variety. If $V(\mathbb{Q}_p) \neq \emptyset$ for each embedding $\mathbb{Q}^{\mathrm{t}p} \hookrightarrow \mathbb{Q}_p$, then $V(\mathbb{Q}^{\mathrm{t}p}) \neq \emptyset$.*

This is a special case of a result of Moret-Bailly [Mo1], proved in 1989; other proofs appeared later in papers of Green-Pop-Roquette [GrPoRo] and Pop [Po].

Our descent varieties V are also unirational. A natural question is whether these varieties are rational and also whether the Hasse principle holds for these varieties. We hope to provide some answers which would yield some information about the Hasse principle for covers: if a $\overline{\mathbb{Q}}$-cover is defined over all completions of $\mathbb{Q}$, is it then necessarily defined over $\mathbb{Q}$? Only partial results are known [DeDo2], although this principle is known to hold in general for G-covers [DeDo1] (*i.e.* Galois covers given with their automorphisms).

Instead of dealing with one single cover, one can work above a whole moduli space of covers: our Main Theorem has a part B that *globalizes* our notion of descent varieties over *Hurwitz spaces*. More precisely, we construct a $\mathbb{Q}$-variety $\mathcal{V}$, and a *Hurwitz family* $f_{\mathcal{V}} : X \to \mathbb{P}^1_{\mathcal{V}}$ parametrized by $\mathcal{V}$ which has this versal property: for every field k of characteristic zero, every cover of $\mathbb{P}^1_k$ (of the type corresponding to the Hurwitz space in question) appears as the fiber of the Hurwitz family $f_{\mathcal{V}}$ at some k-point of $\mathcal{V}$. This second part provides a new set of applications. In particular, corollary 1.6 gives some information on the subset of closed points h on the Hurwitz space such that the corresponding cover $f_h : X_h \to \mathbb{P}^1_{\underline{k}}$ is defined over its field of moduli: this set is Zariski-dense. No such general result was known before in this direction. Stronger conclusions can be drawn when the base field is fixed and assumed to be *large* (corollaries 1.5 and 1.8).

We offer several types of constructions of our descent varieties, which correspond to several "cultures".

We begin with an elementary function field theoretic approach to Main Theorem A. In §2 we establish some representation results for covers of $\mathbb{P}^1$ over fields of characteristic 0. A first one (lemma 2.1) is that every cover has an affine equation $P(T, Y) = 0$ with bounded degree (in terms of the degree and the genus of the cover). Combining this with Liouville's inequality (lemma 2.2) and a certain non-Galois form of the normal basis theorem (lemma 2.3), we then obtain that all models can be parametrized by the points of an open subset of an affine space $\mathbb{A}^M$ (for some integer M) with some compatibility property between the

Galois action over the ground field and the action of GL_M. A precise statement is given in theorem 2.5. These representation results are the main tools for the first construction of descent varieties (performed in §3). This first construction, which uses classical results from the theory of covers, is quite explicit: equations of the descent variety V can theoretically be derived from the method.

The second construction, given in §4, involves different ideas. A notion of *marking* of a model of the given cover f is introduced, which basically consists in the choice of a basis of the fiber (regarded as a module over the structural scheaf of the base space) above a fixed closed point Q. This leads to some representation of the appropriate category of models of f (over some K-scheme U) by a smooth affine K-scheme, which is a homogeneous space under $\mathrm{GL}_{N,K}$ (for some integer N); here K is the base field. The techniques used also are different from those of §3. Proposition 4.2, which yields Main Theorem A, is phrased in terms of representability of functors and proved along these lines. Also, proposition 4.2 covers the general case of Main Theorem A: the base space of f is an arbitrary regular projective variety (and no longer $\mathbb{P}^1$) and the base field is of arbitrary characteristic.

A more general construction is given in §5. The method used there applies to any K-*gerbe* G that is a Deligne-Mumford stack, and in particular to the gerbe MOD_f of models of f (§5.2). Proposition 5.1, which also contains Main Theorem A, uses [LaMo] to show that such a gerbe G is a *linear quotient stack* (§5.1). And in fact, proposition 4.2 discussed above, when combined with a criterion from [EdHaKrVi], is actually an alternate proof of that (see §5.2). The second part of proposition 5.1 shows that these linear quotient stacks are isomorphic to quotient stacks $[V/\mathrm{GL}_{n,K}]$, where V is a smooth affine K-*scheme.*

The final section §6 is concerned with Main Theorem B, that is, the global form of descent varieties above Hurwitz spaces. The first part consists in showing that the stack $\mathsf{COV}^{d,g}$ of degree d covers of $\mathbb{P}^1$ by curves of genus g is a linear quotient stack (theorem 6.1). This uses again the criterion from [EdHaKrVi], conjoined with other previous ideas. The rest of the proof of Main Theorem B is then similar to the second part of the proof of Main Theorem A (see theorem 6.3).

1. Main results

In this section, we state the two parts of our Main Theorem and derive the main applications. The next sections provide proofs of the Main Theorem, along with generalizations.

1.1. Statement of Main Theorem A. We will freely use basics from the theory of algebraic branched covers; see for example [DeDo1]. Recall however that, given a field K (the base field) and a regular projective geometrically irreducible K-variety B (the base space), by *mere cover of* B *over* K, we mean a finite and generically unramified morphism $f : X \to B$ defined over K with X a normal and geometrically irreducible K-variety. The term "mere" is meant to distinguish mere covers from G-covers. A G-cover of group G is a Galois cover given with an isomorphism between its automorphism group and the group G. We use the phrase "[G-]cover" for "mere cover [resp. G-cover]".

Fix a base field K and a base space B as above. Denote the separable [resp. algebraic] closure of K by K^{s} [resp. $\overline{K}$] and the absolute Galois group $\mathrm{Gal}(K^{\mathrm{s}}/K)$ by G_K. Suppose given a [G-]cover $f : X \to B_{K^{\mathrm{s}}}$ of $B_{K^{\mathrm{s}}} = B \otimes_K K^{\mathrm{s}}$ (the K^{s}-variety obtained from B by extension of scalars). Covers $f : X \to \mathbb{P}^1_{\overline{\mathbb{Q}}}$ are typical examples of the situation we consider. The (monodromy) group of the cover f, which we denote by G, is the Galois group of the Galois closure of the field extension $K^{\mathrm{s}}(X)/K^{\mathrm{s}}(B)$; it is anti-isomorphic to the automorphism group of the Galois closure $\widehat{f} : \widehat{X} \to B_{K^{\mathrm{s}}}$ of f.

For a field k with $K \subset k$, a k-*model* of f is a k-[G-]cover $\widetilde{f} : \widetilde{X} \to B_k$ such that $f \otimes_{K^{\mathrm{s}}} \overline{k}$ and $\widetilde{f} \otimes_k \overline{k}$ are isomorphic as [G-]covers of $B_{\overline{k}}$ for every K-embedding $K^{\mathrm{s}} \to \overline{k}$ [1]; if such a model exists, f is said to be definable over k. Recall that the *field of moduli* of f (relative to the extension K^{s}/K) is defined to be the fixed field in K^{s} of the subgroup of G_K of all $\tau \in \mathrm{G}_K$ such that the conjugate [G-]cover $f^\tau : X^\tau \to B_{K^{\mathrm{s}}}$ is isomorphic to f over K^{s} [DeDo1] §2.7. The field of moduli is a finite extension of K. With no loss of generality, we may and will assume that the field of moduli is K itself (*e.g.* [DeEm] Prop.2.1).

Main Theorem A — *There exists an affine variety* V *with the following properties:*
(1) V *is geometrically irreducible and is defined over the field of moduli* K.
(2) *There exists a* [G-]*cover* $\mathcal{F} : \mathcal{X} \to V \times B$, *or in other words, a* K*-family of* [G-]*covers of* B *parametrized by* V, *such that*

(i) *For each* $v \in V$, *the fiber cover* $\mathcal{F}_v : \mathcal{X}_v \to B_{K(v)}$ *is a* $K(v)$*-model of* f.

(ii) *If* k *is an extension of* K *and* $\widetilde{f} : \widetilde{X} \to B_k$ *a* k*-model of* f, *there exists* $v \in V(k)$ *such that* $\widetilde{f}$ *is isomorphic to the fiber cover* $\mathcal{F}_v : \mathcal{X}_v \to B_k$ (*as* [G-]*covers of* B_k).

(3) V *is smooth.*
(4) *For every extension* k *of* K *for which* $V(k) \neq \emptyset$, V *is unirational over* k.

Remark 1.1. Statement above is what the stack theoretic approaches give (§§4–5). They produce V as a homogeneous space under GL_N for some integer N (*i.e.*, there is a transitive action of GL_N on V); properties (3), (4) and first part of (1) follow from this description.

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[1] If it is assumed as below that K is the field of moduli of f, then it suffices that $f \otimes_{K^{\mathrm{s}}} \overline{k}$ and $\widetilde{f} \otimes_k \overline{k}$ be isomorphic for *some* K-embedding $K^{\mathrm{s}} \to \overline{k}$ for them to be isomorphic for *all* K-embeddings $K^{\mathrm{s}} \to \overline{k}$ (see also remark 4.1.).

The first approach (§§2–3) is more specifically concerned with covers of $\mathbb{P}^1$ in characteristic 0; it is more explicit and "elementary" but leads to slightly weaker conclusions. Namely, assertion (4) should be replaced by this weaker assertion:

(4') V *is unirational over* $\overline{K}$. *Furthermore, if* k_o *is an algebraic extension of* K, *over which* f *has a model* $\widetilde{f}_o : \widetilde{X}_o \to \mathbb{P}^1_{k_o}$, *then the variety* V *can be constructed in such a way that* $V(k_o)$ *is Zariski-dense.*

Also the first approach does not provide directly a smooth variety[2]. We note however that the weaker version of Main Theorem A (that is, with conclusions (1), (2) and (4')) is sufficient to derive Corollaries 1.2–1.4 below (with $B = \mathbb{P}^1$ in characteristic 0).

1.2. Applications. Recall that a field k is said to be *existentially closed* in a regular extension Ω (Ω/k separable, $\Omega \cap \overline{k} = k$) [resp. k is PAC] if for each smooth geometrically irreducible K-variety V, we have $V(\Omega) \neq \emptyset \Rightarrow V(K) \neq \emptyset$ [resp. we have $V(K) \neq \emptyset$].

Corollary 1.2 — *Keep the same notation as above. Assume further that the field of moduli* K *is existentially closed in some field of definition* k_o *of the cover* f. *Then* f *is defined over* K.

For example, if the field of moduli K is PAC then it is a field of definition; this was already known, but as a consequence of the fact that PAC fields are of cohomological dimension ≤ 1 [DeDo1;Cor.3.3]. Corollary 1.2 is an immediate consequence of Main Theorem A. From condition (2)(ii), $V(k_o)$ is nonempty. As V is smooth (condition (3)), it follows from the definition of "existentially closed" that $V(K)$ is nonempty too. Therefore, from condition (2)(i), there is at least one K-model of f. The weaker version of Main Theorem A (see remark 1.1 above) is sufficient to prove corollary 1.2 (and similarly the next corollaries): from condition (4'), V can be constructed so that $V(k_o)$ is Zariski-dense, which ensures that there is at least one *smooth* k_o-point. The rest of the argument is unchanged.

The next corollaries generalize the result alluded to in the introduction about totally p-adic models of covers. Instead of $\mathbb{Q}$, consider a global field K (*i.e.*, either a number field or a one-variable function field over a finite field). Let Σ be a nonempty finite set of places of K; Σ replaces p. Denote the maximal extension of K in a fixed separable closure K^{s} which is totally split at each $v \in \Sigma$, by K^Σ (this replaces $\mathbb{Q}^{\mathrm{t}p}$).

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2 However by removing a Zariski closed subset of V, one can obtain a variety $V' \subset V$ that is smooth and still affine (but then condition (2) (ii) may fail: for some fields k, representative k-points of some k-models may be in $V \setminus V'$).

Corollary 1.3 — *Let $f : X \to B_{K^{\mathrm{s}}}$ be a* [G-]*cover over K^{s}. Assume that the field of moduli of f is contained in K^{Σ} and that for each $v \in \Sigma$, f has a K_v-model f_v (for each K-embedding $K^{\Sigma} \hookrightarrow K_v$). Then f is defined over K^{Σ}; more precisely f has a K^{Σ}-model that induces each f_v by extension of scalars ($v \in \Sigma$).*

Corollary 1.3 is a straightforward consequence of Main Theorem A and the *local-global principle on varieties*, a special case of which is mentioned in the introduction: the assumptions yield K_v-rational points x_v on the descent variety V associated to the [G-]cover f; it follows then from [Mo1] that there are K^{Σ}-points on V, which correspond to K^{Σ}-models of f. Furthermore, these K^{Σ}-points can be found arbitrarily v-adically close to each point x_v ($v \in \Sigma$). As a consequence, the K^{Σ}-model can indeed be required to induce each f_v by extension of scalars ($v \in \Sigma$).

The last corollary conjoins corollary 1.3 above and results of Dèbes-Harbater [DeHa] and Emsalem [Em]. We assume here that the base space B is a curve. A finite place v of K is said to be *good* (relative to the cover $f : X \to B$) if

- the residue characteristic p does not divide the order of the group G of the cover f,

- the base space B has good reduction at v,

- the branch locus of the cover f is smooth at v (that is, the geometric branch points of the cover do not coalesce in the residue field of v),

It can be shown then that if K is the field of moduli of the [G-]cover f, then K_v is a field of definition of f for all good places v of K; the result was proved for G-covers of $\mathbb{P}^1$ in [DeHa] and generalized in [Em]. We obtain

Corollary 1.4 — *Let $f : X \to B_{K^{\mathrm{s}}}$ be a* [G-]*cover of curves. Assume that the field of moduli of f is K. Let Σ be a finite set of good places of K. Then f is defined over K^{Σ}.*

1.3. Revisiting corollary 1.3 via stacks. Corollary 1.3 could also be obtained as a consequence of the main result of [Mo2], which is in fact a stack-theoretic version of [Mo1].

Namely, assuming as we may that the field of moduli K_1 of f is K [3], define then a Skolem datum, in the sense of [Mo2], 0.6. This consists in the following data:

• a "ground ring" R in a global field: this will be K itself;

• a finite set Σ of places of K, disjoint from the set of height one primes of R (this condition is empty here!): this will be our Σ;

[3] Otherwise replace K by K_1 and Σ by the set of places of K_1 above Σ: this does not change K^{Σ} because $K_1 \subset K^{\Sigma}$.

• for each $v \in \Sigma$, a Galois extension L_v of K_v: this will be K_v itself. These data define a subring $R^{\mathbb{L}}$ ([Mo2], 0.3) of K^{s} which is just K^{Σ} here;

• an algebraic stack X of finite type over R: this will be the K-gerbe MOD_f of models of f (introduced in [DeDo3] where it is denoted by $\mathcal{G}(f)$ and discussed further in §5.2). It is subject to certain conditions ([Mo2], 0.4) which hold in particular if $\mathsf{X} \to \mathrm{Spec}\,(R)$ is surjective and flat with geometrically irreducible fibers, so they are satisfied here;

• for each $v \in \Sigma$, a set Ω_v of objects of $\mathsf{X}(K_v)$: we take $\Omega_v = \mathsf{X}(K_v)$. Each Ω_v is required to be v-adically open in $\mathsf{X}(K_v)$ (obvious) and Zariski-dense in X_{K_v}: since our X is a K-gerbe, the latter condition just means $\Omega_v \neq \emptyset$, which is true by the assumption that f can be defined over K_v.

We then apply Theorem 0.7 of [Mo2] and conclude in particular that $\mathsf{MOD}_f(K^{\Sigma}) \neq \emptyset$, which means that f has a K^{Σ}-model.

The method of proof of [Mo2] consists in reducing to the case when the stack X is in fact a scheme, by finding a scheme X (to which [Mo1] applies) and a map $X \to \mathsf{X}$ with good properties. This is precisely what our Main Theorem A does, when X is the gerbe MOD_f: the variety V plays the role of X. Indeed, consider the family $\mathcal{F}$ in part (2) of Main Theorem A: by (2)(i) this is an object of $\mathsf{MOD}_f(V)$ (a *model* of f over V (§4.1)), or, equivalently, a morphism $\pi : V \to \mathsf{MOD}_f$. Next, condition (2)(ii) asserts that for every extension k of K, every point of $\mathsf{MOD}_f(k)$ lifts to $V(k)$, and condition (1) (geometric irreducibility of V) is one of the assumptions of [Mo1].

Our Main Theorem A can thus be regarded as a result about existence of affine presentations with good properties of the gerbe MOD_f. The stack-theoretic proof of Main Theorem A (proposition 5.1) goes along these lines. As explained there, conditions (1), (2) and (3) of Main Theorem A can be obtained by applying corollary (6.1.1) of [LaMo] to the stack MOD_f; condition (4) is more involved. The techniques used are quite general and can in fact be applied to other gerbes. In particular, it is by using them that the global version of the Main Theorem, for moduli spaces of covers instead of one single cover, can be obtained. We shall now state this global version and give some applications.

1.4. Descent varieties and Hurwitz spaces. Here we fix an integer $d > 0$, a finite subgroup $G \subset S_d$, and an integer $r \geq 3$. We consider covers $f : X \to \mathbb{P}^1$ over fields k of characteristic 0 with the following geometric invariants:

• the monodromy group (*i.e.*, the Galois group of the Galois closure of $f \otimes_k \overline{k}$) is (isomorphic to) G and the monodromy action on an unramified fiber is then, up to equivalence, given by the embedding $G \subset S_d$ ($d = \deg(f)$),

• the number of branch points is r.

To these invariants are classically associated moduli spaces of covers — Hurwitz spaces. Depending on whether the covers are regarded as mere covers or G-covers, and, whether the branch points are labeled or not, the moduli space is denoted by $\mathcal{H}_G^{\mathrm{ab}}$, $\mathcal{H}_G^{\mathrm{in}}$, $\mathcal{H}_G'^{\mathrm{ab}}$, $\mathcal{H}_G'^{\mathrm{in}}$ (see [DeDoEm], [Wew] and [De] for more references).

Each of these spaces is a coarse moduli scheme for the corresponding moduli stack, denoted by $\mathsf{H}_G^{\mathrm{ab}}$, $\mathsf{H}_G^{\mathrm{in}}$, $\mathsf{H}_G'^{\mathrm{ab}}$, $\mathsf{H}_G'^{\mathrm{in}}$ respectively. For instance, for any $\mathbb{Q}$-scheme T, an object of the category $\mathsf{H}_G^{\mathrm{ab}}(T)$ is a T-morphism $f : X \to \mathbb{P}^1_T$, which is finite and locally free and such that for every geometric point $\xi : \mathrm{Spec}\,(k) \to T$, the induced map $f_\xi : X_\xi \to \mathbb{P}^1_k$ is a cover of the corresponding type. These objects are called "Hurwitz families with parameter space T" in [DeDoEm], except that here no condition on T is required. Associated to each such object f there is the *classifying moduli map* $\gamma_f : T \to \mathcal{H}_G^{\mathrm{ab}}$ (called "structural morphism" in [DeDoEm]): for each $\xi : \mathrm{Spec}\,(\overline{k}) \to T$, $\gamma_f(\xi)$ is the representative point on $\mathcal{H}_G^{\mathrm{ab}}$ of the isomorphism class of the induced cover $f_\xi : X_\xi \to \mathbb{P}^1_{\overline{k}}$ (see [DeDoEm;§3.1.5]).

In the statement below, the situation is any of the four situations above. We denote by H (resp. $\mathcal{H}$) the corresponding stack (resp. moduli space). The moduli space $\mathcal{H}$ is known to be smooth and quasi-projective over $\mathbb{Q}$, and purely r-dimensional.

Main Theorem B — *There is a smooth quasi-projective $\mathbb{Q}$-scheme $\mathcal{V}$ and a Hurwitz family $f : X \to \mathbb{P}^1_{\mathcal{V}}$ over $\mathcal{V}$ (i.e., an object of $\mathsf{H}(\mathcal{V})$) with the following properties:*
(1) *The classifying moduli map $\gamma_f : \mathcal{V} \to \mathcal{H}$ is smooth, with geometrically irreducible fibers.*
(2) *For every field k of characteristic zero, every cover of $\mathbb{P}^1_k$ (of the type corresponding to $\mathcal{H}$) appears as the fiber of the Hurwitz family f at some k-point of $\mathcal{V}$.*
(3) (local description of $\mathcal{V}$ over $\mathcal{H}$). *There exists an integer $n \geq 0$, a right action of $\mathrm{GL}_{n,\mathbb{Q}}$ on $\mathcal{V}$ with finite stabilizers such that there is an étale surjective morphism $\rho : U \to \mathcal{H}$ such that $U \times_{\mathcal{H}} \mathcal{V}$ is U-isomorphic to $\Gamma \backslash \mathrm{GL}_{n,U}$ (with the natural projection on U and the natural right action of $\mathrm{GL}_{n,U}$), where Γ is a subgroup of $\mathrm{GL}_{n,U}$.*

A more precise version is given in §6 (theorem 6.3).

1.5. Application to fields of definition of covers. A long-standing question in descent theory for covers has been whether a [G-]cover is "often" defined over its field of moduli. To make the question precise, for any field k of characteristic 0, define the subset $\mathcal{H}(k)^{\mathrm{no\,ob}} \subset \mathcal{H}(k)$ to be the set of points $h \in \mathcal{H}(k)$ such that the corresponding cover $f_h : X_h \to \mathbb{P}^1_{\overline{k}}$ can be defined over k (*i.e.*, for which there is no ob(struction) to the field of moduli being a field of definition). From Main Theorem B, we deduce the following description of $\mathcal{H}(k)^{\mathrm{no\,ob}}$.

Corollary 1.5 — *With notation as in Main Theorem B, the set $\mathcal{H}(k)^{\rm no\,ob}$ is the image of $\mathcal{V}(k)$ into $\mathcal{H}(k)$. Consequently, if k is large, then for every connected component $\mathcal{Z}$ of $\mathcal{H}_k := \mathcal{H} \times_{\mathrm{Spec}\,(\mathbb{Q})} \mathrm{Spec}\,(k)$, the set $\mathcal{H}(k)^{\rm no\,ob} \cap \mathcal{Z}$ is either empty or Zariski-dense in $\mathcal{Z}$.*

Recall a field k is *large* if for every smooth geometrically connected k-variety X, then $X(k)$ is either empty or Zariski-dense in X [4]. In the last 10 years, many significant conjectures from geometric inverse Galois theory have been shown to hold when the base field is large. Essentially the reason is that one could reduce these conjectures to finding rational points on varieties. Corollary 1.5 is a new illustration of that. We have an exact description of the subset $\mathcal{H}(k)^{\rm no\,ob}$, which yields precise information if the base field is large; however, in general and in particular over $\mathbb{Q}$, giving a precise description of $\mathcal{H}(k)^{\rm no\,ob}$ remains a difficult problem (as other problems from geometric inverse Galois theory).

Proof. The first part readily follows from Main Theorem B.

For the second part, suppose $\mathcal{H}(k)^{\rm no\,ob} \cap \mathcal{Z}(k) \neq \emptyset$. Then $\mathcal{Z}$ is geometrically connected (it is smooth and connected and has a rational point), hence so is the inverse image $\mathcal{V}_{\mathcal{Z}}$ of $\mathcal{Z}$ in $\mathcal{V}$. Moreover the assumption means that $\mathcal{V}_{\mathcal{Z}}(k)$ is nonempty, hence dense in $\mathcal{V}_{\mathcal{Z}}$. The conclusion follows since the projection $\mathcal{V}_{\mathcal{Z}} \to \mathcal{Z}$ is open and surjective. □

We can play a slightly different game and, instead of considering k-valued points of $\mathcal{H}$ for fixed k, look at all *closed points* of $\mathcal{H}$. Precisely, let us define $\mathcal{H}^{\rm no\,ob} \subset \mathcal{H}$ to be the set of closed points $h \in \mathcal{H}$ such that the corresponding cover $f_h : X_h \to \mathbb{P}^1_{\overline{k}}$ can be defined over the residue field $\kappa(h)$ of h. Equivalently, $h \in \mathcal{H}^{\rm no\,ob}$ if and only if it is in $\mathcal{H}(\kappa(h))^{\rm no\,ob}$ when viewed as a $\kappa(h)$-valued point of $\mathcal{H}$ in the canonical way.

Clearly, as before, $h \in \mathcal{H}^{\rm no\,ob}$ if and only if h is the image of a closed point of $\mathcal{V}$ with the same residue field. From this we deduce:

Corollary 1.6 — *$\mathcal{H}^{\rm no\,ob}$ is Zariski-dense in $\mathcal{H}$.*

Proof. This follows from the preceding discussion and the following result of Poonen [Poo]. (Note that all components of $\mathcal{H}$ have positive dimension.) □

[4] It suffices to check this property when X is a curve.

Theorem 1.7 — *Let X and Y be schemes of finite type over a field, with Y irreducible of positive dimension, and let $f : X \to Y$ be a smooth and surjective morphism. Then the set of closed points $x \in X$ such that $[\kappa(x) : \kappa(f(x))] = 1$ is dense in X.* □

Let us finally return to covers of $\mathbb{P}^1$ over "totally Σ-adic" fields:

Corollary 1.8 — *Let K be a number field. Fix a nonempty finite set Σ of places of K and denote by K^{Σ} the maximal extension of K in a fixed separable closure K^{s} which is totally split at each $v \in \Sigma$. Further, let $\mathcal{H}_0$ be a connected component of $\mathcal{H}_K = \mathcal{H} \times_{\mathrm{Spec}\,(\mathbb{Q})}$* Spec (K), geometrically connected *over K. Assume that for each $v \in \Sigma$, there exists a K_v-*[G-]*cover $f_v : X_v \to \mathbb{P}^1_{K_v}$ corresponding to a point $h_v \in \mathcal{H}_0(K_v)$. Then there exists a K^{Σ}-*[G-]*cover $f : X \to \mathbb{P}^1_{K^{\Sigma}}$ corresponding to a point of $\mathcal{H}_0(K^{\Sigma})$.*

A first idea to prove this result would be to apply the local-global principle [Mo1] on the *variety* $\mathcal{H}_0$. But one then only obtains this weaker conclusion: "there exists a [G-]cover $\widetilde{f} : \widetilde{X} \to \mathbb{P}^1_{\overline{K}}$ with *field of moduli* K^{Σ}". This strategy works though but one needs to use the *stack-theoretic* form of the local-global principle [Mo2] (2nd proof below). Corollary 1.8 can be alternatively obtained as a consequence of Main Theorem B (1st proof below) (just as corollary 1.3 was proved both as a corollary of Main Theorem A (in §1.2) and as a corollary of [Mo2] (in §1.3).

1st proof. Consider the variety $\mathcal{V}$ of Main Theorem B, and denote by $\mathcal{V}_0$ the inverse image of $\mathcal{H}_0$ in $\mathcal{V}_K$. The assumption on $\mathcal{H}_0$ implies that $\mathcal{V}_0$ is smooth and geometrically irreducible over K. For each $v \in \Sigma$, the [G-]cover f_v yields a point in $\mathcal{V}_0(K_v)$ by Main Theorem B (2). It follows from the local-global principle on varieties [Mo1] that there are K^{Σ}-points on $\mathcal{V}_0$, whence K^{Σ}-covers of $\mathbb{P}^1$ with moduli points in $\mathcal{H}_0(K^{\Sigma})$, as desired. □

2nd proof. Let H_0 be the inverse image of $\mathcal{H}_0$ in H_K. It is smooth and geometrically irreducible over K, and has K_v-points for each $v \in \Sigma$ by assumption. So H_0 has K^{Σ}-points by the stack-theoretic version of the local-global principle proved in [Mo2]. □

Remark 1.9. In the same vein, the following statement can be proved. Let k be a large field, and let $\mathcal{Z}$ be a subscheme of $\mathcal{H}_k$; assume that there exists a $k((x))$-[G-]cover $f : X \to \mathbb{P}^1_{k((x))}$ corresponding to a point of $\mathcal{Z}(k((x)))$. Then there exists a k-[G-]cover $f_0 : X_0 \to \mathbb{P}^1_k$ corresponding to a point of $\mathcal{Z}(k)$.

2. Representation results for models of covers of $\mathbb{P}^1$

As in §1 fix a finite cover $f : X \to B_{K^s}$ with field of moduli K. In this section and in the next one we assume in addition that $B = \mathbb{P}^1_K$ and K is of characteristic 0; we denote the parameter of $\mathbb{P}^1$ by T. We also suppose given, as in condition (4') from Remark 1.1, a field of definition k_o of f with $K \subset k_o \subset \overline{K}$ along with a k_o-model $\widetilde{f}_o : \widetilde{X}_o \to \mathbb{P}^1_{k_o}$. Denote the genus of $X_{\overline{K}}$ by g and the degree of the cover f by d.

The various models $\widetilde{f}$ of f over subfields $k \subset \overline{K}$ correspond to function field extensions $E/k(T)$, regular over k and inducing $\overline{K}(X)/\overline{K}(T)$ by extension of scalars. If a $\overline{K}(T)$-basis of $\overline{K}(X)$ is fixed, one obtains a parametrization of all such models $\widetilde{f}$ by points from an open subset of the affine space $\mathbb{A}^d$ over $\overline{K}(T)$: namely take for representing points the d-tuples of components in the fixed basis of primitive elements of all field extensions $E/k(T)$ as above. Moreover, in this description, if K is the field of moduli, then action of automorphisms $\tau \in \mathrm{G}_K$ on the isomorphism class of f can be represented by elements of $\mathrm{GL}_d(\overline{K}(T))$.

In this section, we prove similar representation results but with $\overline{K}(T)$ replaced by $\overline{K}$. We will show models of f can be parametrized by points from an open subset of an affine space over $\overline{K}$ (of some dimension M). Next we prove that this parametrization can be performed in such a way it is as above "compatible" with action of G_K: action of automorphisms $\tau \in \mathrm{G}_K$ on the isomorphism class of f can be represented by elements of $\mathrm{GL}_M(\overline{K})$. A precise statement is given in theorem 2.5, which recapitulates lemmas 2.1–2.3. These representation results (over $\overline{K}$) will be used in §3 to provide a construction of a descent variety V as in Main Theorem A.

2.1. Normalizing the models.

2.1.1. Models with equation of bounded degree. The following result is proved in [Sa].

Lemma 2.1 — *Let k be an extension of K and $\widetilde{f} : \widetilde{X} \to \mathbb{P}^1_k$ be a k-model of f. Then there exists a primitive element $\mathcal{Y}$ of the extension $k(\widetilde{X})/k(T)$ such that*

(1) *the irreducible monic polynomial $P(T, Y)$ of $\mathcal{Y}$ over $k(T)$ lies in $k[T, Y]$ and satisfies* $\deg_T(P(T, Y)) < 2(2g + 1)d \log(d)$.

2.1.2. Primitive elements with bounded degree. Here we suppose given a $\overline{K}(T)$-basis $\varepsilon = [\varepsilon_1, \ldots, \varepsilon_d]$ of the extension $\overline{K}(X)/\overline{K}(T)$. For later use, we note that ε can be chosen to be a $k_o(T)$-basis of the extension $k_o(\widetilde{X}_o)/k_o(T)$ (attached to the fixed k_o-model $\widetilde{f}_o$). Let $\check{\Delta}(T) \in \overline{K}[T]$ be the discriminant of the basis ε and let $\Delta(T)$ be a polynomial in $K[T]$ such that $\check{\Delta}(T)$ divides $\Delta(T)$ in $\overline{K}[T]$: for example take the "norm" of $\check{\Delta}(T)$. Denote also the set of all $\overline{K}(T)$-isomorphisms $\overline{K}(X) \to \overline{K(T)}$ by Σ.

Suppose given an extension k of K and a k-model $\widetilde{f} : \widetilde{X} \to \mathbb{P}^1_k$ of f. Both extensions $k(\widetilde{X})/k(T)$ and $\overline{K}(X)/\overline{K}(T)$ are of degree d and induce the extension $\overline{k}(X)/\overline{k}(T)$ by extension of scalars. It follows that the d-tuple $\boldsymbol{\varepsilon} = [\varepsilon_1, \dots, \varepsilon_d]$ is a $\overline{k}(T)$-basis of the extension $\overline{k}(X)/\overline{k}(T)$ and that the set of all $k(T)$-isomorphisms $k(\widetilde{X}) \to \overline{k(T)}$ [resp. the set of all $\overline{k}(T)$-isomorphisms $\overline{k}(X) \to \overline{k(T)}$] canonically identifies to Σ. In particular, traces of extensions $\overline{K}(X)/\overline{K}(T)$, $k(\widetilde{X})/k(T)$ and $\overline{k}(X)/\overline{k}(T)$ naturally correspond to one another by restriction. For simplicity, we denote this trace function by Tr. Denote then the dual basis of $[\varepsilon_1, \dots, \varepsilon_d]$ by $[\varepsilon_1^*, \dots, \varepsilon_d^*]$: that is, $\mathrm{Tr}(\varepsilon_i^* \varepsilon_j) = \delta_{ij}$, $1 \le i, j \le d$.

Any element $z \in k(\widetilde{X})$ can be written in a unique way as a $\overline{k}(T)$-linear combination of $\varepsilon_1, \dots, \varepsilon_d$; furthermore if $\overline{K} \subset k$, this linear combination has coefficients in $k(T)$. If in addition z is integral over $k[T]$, then the decomposition can be written in the form

$$z = \frac{1}{\Delta(T)} \left(z_1(T)\varepsilon_1 + \cdots z_d(T)\varepsilon_d\right) \tag{1}$$

with $z_i(T) \in \overline{k}[T]$ in general and $z_i(T) \in k[T]$ if $\overline{K} \subset k$, $i = 1, \dots, d$.

Denote the $1/T$-adic valuation on $\overline{k}((1/T))$ and its unique extension to an algebraic closure of $\overline{k}((1/T))$ by v_∞. For polynomials $p(T) \in \overline{k}(T)$, we have $v_\infty(p) = -\deg(p)$.

The following result is a function field form of Liouville's inequality.

Lemma 2.2 — *Given $z \in k(\widetilde{X})$ integral over $k[T]$ and written as in* (1) *above, denote the irreducible monic polynomial of z over $k(T)$ by*

$$P_z(T, Y) = Y^{d_z} + a_{d-1}(T)Y^{d_z-1} + \cdots + a_1(T)Y + a_o(T)$$

where d_z is the degree of z over $k(T)$ and $a_i(T) \in k[T]$, $i = 0, 1, \dots, d_z - 1$. Then we have

(a) $v_\infty(z) \ge -\deg_T(P_z(T, Y))$

(b) $\deg(z_i) \le \deg_T(P_z(T, Y)) + \deg(\Delta) - \min\limits_{\substack{1 \le j \le d \\ \sigma \in \Sigma}} \{v_\infty((\varepsilon_j^*)^\sigma)\} \qquad (i = 1, \dots, d)$

(*This result is independent of the extension of v_∞ to $k(\widetilde{X})$*).

Proof. Set $v = v_\infty(z)$ and $\alpha = \min\limits_{0 \le i < d_z} \{v_\infty(a_i)\} = -\deg_T(P_z(T, Y))$.

Inequality (a) is obvious is $v \ge 0$. So assume $v < 0$. Then deduce from

$$z^{d_z} = -a_{d-1}(T)z^{d_z-1} - \cdots - a_1(T)z - a_o(T)$$

that $d_z v \ge \alpha + (d_z - 1)v$, thus $v \ge \alpha$, which is the desired inequality (a).

For $i = 1, \dots, d$, the polynomial $z_i(T)$ is given by

$$z_i(T) = \mathrm{Tr}(\varepsilon_i^*(\Delta(T)z)) = \Delta(T)\sum_{\sigma\in\Sigma}(\varepsilon_i^*)^\sigma z^\sigma$$

Thus we obtain

$$\begin{aligned} v_\infty(z_i(T)) &\geq \min_{\sigma\,\in\,\Sigma}\{v_\infty((\varepsilon_i^*)^\sigma z^\sigma)\} + v_\infty(\Delta(T)) \\ &\geq \min_{\substack{1\,\leq\, j\,\leq\, d \\ \sigma\,\in\,\Sigma}}\{v_\infty((\varepsilon_j^*)^\sigma)\} + \min_{\sigma\,\in\,\Sigma}\{v_\infty(z^\sigma)\} + v_\infty(\Delta(T)) \end{aligned}$$

Apply then inequality (a) to each of the z_σs (which, as z, satisfies $P_z(T, z^\sigma) = 0$) to get inequality (b). $\square$

2.2. Bases compatible with Galois action. Instead of the $\overline{K}$-[G-]cover f and its k_o-model $\widetilde{f}_o$, we will work here with models f' and $\widetilde{f}'_o$ over *finite* extensions of K. We explain this reduction.

In any case, there exists a finite extension k'_o of K contained in k_o over which $\widetilde{f}_o$ is defined. Denote the corresponding k'_o-model by $\widetilde{f}'_o : \widetilde{X}'_o \rightarrow \mathbb{P}^1_{k'_o}$. Then we claim that there exists a finite Galois extension F/K with $k'_o \subset F$ such that the F-model $f' = \widetilde{f}'_o \otimes F$ of f is *stable*, that is, the field of moduli of f' relative to the extension F/K is K (this new field of moduli is *a priori* bigger than K). Indeed, consider the Galois closure $\widehat{k'_o}$ of k'_o over K. As K is the field of moduli of the $\overline{K}$-[G-]cover f, for each $\tau \in \mathrm{Gal}(\widehat{k'_o}/K)$, there exists a $\overline{K}$-isomorphism χ_τ between $\widetilde{f}'_o \otimes \overline{K}$ and $(\widetilde{f}'_o \otimes \overline{K})^\tau$. Then take for F the Galois closure over K of the field generated by $\widehat{k'_o}$ and the coefficients of the isomorphisms χ_τ (for all $\tau \in \mathrm{Gal}(\widehat{k'_o}/K)$). For this field F, the model $\widetilde{f}'_o \otimes F$ is indeed stable; denote it by $f' : X' \rightarrow \mathbb{P}^1_F$. In terms of function fields, we have:

(2) for each $\tau \in \mathrm{Gal}(F/K)$, there exists a $F(T)$-isomorphism $\chi_\tau : F(X')^\tau \rightarrow F(X')$. Or, equivalently $\chi_\tau\tau$ is a $K(T)$-automorphism of $F(X')$ inducing τ on F.

The action of τ on $F(X')$ depends on the choice of an extension of τ to $F(X')$ but condition (2) itself does not. Condition (2) can also be rephrased to say that the following sequence is exact (more exactly, it is equivalent to the 3rd map being surjective):

$$1 \rightarrow \mathrm{Aut}(F(X')/F(T)) \rightarrow \mathrm{Aut}(F(X')/K(T)) \rightarrow \mathrm{Gal}(F/K) \rightarrow 1$$

Suppose given a $K(T)$-basis $\mathbf{e} = [e_1, \ldots, e_D]$ of the function field extension $F(X')/K(T)$ (with $D = d[F : K]$). For each $\tau \in \mathrm{Gal}(F/K)$, let $A_\tau \in \mathrm{GL}_D(K(T))$ be the $D \times D$-matrix such that ${}^t[\mathbf{e}^{\chi_\tau\tau}] = A_\tau {}^t[\mathbf{e}]$. Here $[\mathbf{e}]$ and $[\mathbf{e}^{\chi_\tau\tau}]$ respectively denote the tuples $[e_1, \ldots, e_d]$ and $[e_1^{\chi_\tau\tau}, \ldots, e_d^{\chi_\tau\tau}]$ and ${}^t[-]$ is the transposition operation.

In the following result, statement (*) says there exist bases $\mathbf{e}$ for which action of $\tau \in \mathrm{Gal}(F/K)$ can be described by "constant" matrices A_τ, *i.e.*, lying in $\mathrm{GL}_D(K)$ (instead of $\mathrm{GL}_D(K(T))$). The second point of statement (**) explains how to deduce a basis $\varepsilon^{(1)}$ of the sub-extension $k'_o(\widetilde{X}'_o)/K(T)$. The construction of the bases $\mathbf{e}$ and $\varepsilon^{(1)}$ can be viewed as a *non-Galois version of the normal basis theorem.* The more technical first point from (**) will be only used in §3.4.

Lemma 2.3 — *Assume as above that K is the field of moduli of the cover f' relative to the extension F/K. Then there exists a basis $\mathbf{e}$ of the extension $F(X')/K(T)$ with $e_1, \ldots, e_D$ integral over $K[T]$ such that the following assertions hold:*

(*) *For each $\tau \in \mathrm{Gal}(F/K)$, there exists a $F(T)$-isomorphism $\chi_\tau : F(X')^\tau \to F(X')$ such that the associated matrix A_τ (defined above) is a permutation matrix.*

(**) *Let $D_o = [k'_o(\widetilde{X}'_o) : K(T)]$ and $\delta = [F : k'_o]$. Then the elements of the basis $\mathbf{e}$ can be relabeled e_{ij}, $i = 1, \ldots, D_o$, $j = 1, \ldots, \delta$, so that the following holds. There exists an invertible $\delta \times \delta$-matrix $B = [b_{\ell m}]_{\ell,m}$ (depending on the choice of a basis $\mathbf{a} = [a_m]_{1 \leq m \leq \delta}$ of F/k'_o) with coefficients in F, with only "1" on first column and with these properties:*

- *The elements* $\varepsilon_i^m = \sum_{\ell=1}^{\delta} b_{\ell m}\, e_{i\ell}$ *lie in $k'_o(\widetilde{X}'_o)$, $i = 1, \ldots, D_o$, $m = 1, \ldots, \delta$*
- *$\varepsilon_1^1, \ldots, \varepsilon_{D_o}^1$ form a $K(T)$-basis of the extension $k'_o(\widetilde{X}'_o)/K(T)$.*

Proof. Let $N/K(T)$ be the Galois closure of $F(X')/K(T)$ and Γ be its Galois group. From the normal basis theorem, there exists $\omega \in N$ such that the elements $\gamma(\omega)$ ($\gamma \in \Gamma$) form a $K(T)$-basis of N; furthermore ω can be taken to be integral over $K[T]$.

Set $H = \mathrm{Gal}(N/F(X'))$ and $H_o = \mathrm{Gal}(N/k'_o(\widetilde{X}'_o))$. Let $\gamma_1, \ldots, \gamma_{D_o}$ be representatives of the right cosets $H_o\gamma$ of Γ modulo H_o. Let $h_1, \ldots, h_\delta$ be representatives of the right cosets Hh of H_o modulo H (with $\delta = D/D_o$). The elements $h_j\gamma_i$ ($i = 1, \ldots, D_o$, $j = 1, \ldots, \delta$) are then representatives of the right cosets of Γ modulo H. The extension $F(X')/k'_o(\widetilde{X}'_o)$ is Galois and its Galois group is isomorphic to $\mathrm{Gal}(F/k'_o)$; in fact $\mathrm{Gal}(F(X')/k'_o(\widetilde{X}'_o))$ consists of the δ restrictions of $h_1, \ldots, h_\delta$ to $F(X')$. This diagram summarizes the situation:

$$\begin{tikzcd}
N & \\
F(X') \arrow[u, "H"] & k'_o(\widetilde{X}'_o) \arrow[l] \arrow[ul, dash, "H_o"'] \\
F(T) \arrow[u] & k'_o(T) \arrow[l] \arrow[u] \\
K(T) \arrow[u] \arrow[uuu, dash, bend left=60, "\Gamma"] & K(T) \arrow[l, "\mathrm{Id}"'] \arrow[u]
\end{tikzcd}$$

For $i = 1\ldots, D_o$ and $j = 1,\ldots,\delta$, let e_{ij} be the trace of $(h_j\gamma_i)(\omega)$ relative to the extension $N/F(X')$:

$$e_{ij} = \sum_{h\in H} (hh_j\gamma_i)(\omega)$$

It is readily checked that these D elements $e_{ij} \in F(X')$ are linearly independent over $K(T)$, thus forming a $K(T)$-basis $\mathbf{e}$ of $F(X')$. Furthermore, they are integral over $K[T]$.

Let θ be a $K(T)$-automorphism of $F(X')$, for example $\theta = \chi_\tau\tau$ where $\tau \in \mathrm{G}_K$ and $\chi_\tau : F(X')^\tau \to F(X')$ is an $F(T)$-isomorphism. The automorphism θ is the restriction of an automorphism $\widetilde{\theta} \in \Gamma$ that normalizes H. The following calculation shows that θ permutes the e_{ij}s. Namely, for $i = 1\ldots, D_o$ and $j = 1,\ldots,\delta$, we have:

$$\begin{aligned}
\theta(e_{ij}) &= \sum_{h\in H} (\widetilde{\theta}hh_j\gamma_i)(\omega) \\
&= \sum_{h\in H} (\widetilde{\theta}h\widetilde{\theta}^{-1}\widetilde{\theta}h_j\gamma_i)(\omega) \\
&= \sum_{h\in H} (h\widetilde{\theta}h_j\gamma_i)(\omega) \quad = \quad e_{i'j'} \qquad \text{where } H\widetilde{\theta}h_j\gamma_i = Hh_{j'}\gamma_{i'}
\end{aligned}$$

This proves assertion (*).

For $i = 1, \ldots, D_o$ and $a \in F$, $a \neq 0$, let $\varepsilon_i^{(a)}$ be the trace of $a\gamma_i(\omega)$ relative to the extension $N/k'_o(\widetilde{X}'_o)$. We have

$$\begin{aligned}\varepsilon_i^{(a)} &= \sum_{h_o \in H_o} h_o(a\gamma_i(\omega)) = \sum_{j=1}^{\delta} \sum_{h \in H} (hh_j)(a\gamma_i(\omega)) \\ &= \sum_{j=1}^{\delta} \sum_{h \in H} h_j(a)(hh_j\gamma_i)(\omega) = \sum_{j=1}^{\delta} h_j(a) \sum_{h \in H} (hh_j\gamma_i)(\omega) \ = \sum_{j=1}^{\delta} h_j(a)\, e_{ij}\end{aligned}$$

By construction, these elements lie in $k'_o(\widetilde{X}'_o)$. Furthermore it is straightforwardly checked that for $a = 1$, the D_o elements $\varepsilon_1^{(1)}, \ldots \varepsilon_{D_o}^{(1)} \in k'_o(\widetilde{X}'_o)$ are linearly independent over $K(T)$, thus forming a $K(T)$-basis $\boldsymbol{\varepsilon}^{(1)}$ of $k'_o(\widetilde{X}'_o)$.

Suppose given a basis $\mathbf{a} = [a_1, \ldots, a_\delta]$ of F/k'_o with $a_1 = 1$. It is also a basis of the extension $F(X')/k'_o(\widetilde{X}'_o)$. For $i = 1, \ldots, D_o$ and $m = 1, \ldots, \delta$, set

$$\varepsilon_i^m = \varepsilon_i^{(a_m)} = \sum_{j=1}^{\delta} h_j(a_m) e_{ij}$$

Set $b_{\ell m} = h_\ell(a_m)$, $\ell = 1, \ldots, \delta$, $m = 1, \ldots, \delta$. Then assertion (**) follows directly from above, except the statement that the $\delta \times \delta$-matrix matrix $B = [b_{\ell m}]_{\ell, m}$ (with coefficients in F) is an invertible matrix. This follows from Dedekind's lemma: indeed, otherwise the rows of the matrix, and so the automorphisms $h_1, \ldots, h_\delta$, would be linearly dependent. This completes the proof of lemma 2.3. $\square$

Remark 2.4. In our situation, Weil's descent theory [We] reformulates to show that the field of moduli K is a field of definition if and only if there exists a basis $\mathbf{e}$ for which all matrices A_τ equal the identity matrix ($\tau \in \mathrm{Gal}(F/K)$)[5]. Thus whether the permutation matrices A_τ can be taken to be the identity matrix in lemma 2.3 (*) is the obstruction to K being a field of definition. If the cover f has no automorphisms, the 1-cochain $(A_\tau)_{\tau \in \mathrm{Gal}(F/K)}$ induces a 1-cocycle in $H^1(\mathrm{Gal}(F/K), \mathrm{GL}_D(F))$. One recovers the classical fact that in that case K is a field of definition as a consequence of Hilbert's theorem 90.

[5] Namely, from [We], K is a field of definition of the cover if and only if for each $\tau \in \mathrm{Gal}(F/K)$, there exists a $F(T)$-isomorphism $\chi_\tau : F(X')^\tau \to F(X')$ such that the correspondence $\tau \to \chi_\tau \tau$ yields a continuous section s to the map $\mathrm{Aut}(F(X')/K(T)) \to \mathrm{Gal}(F/K)$; and the associated K-model $\widetilde{f} : \widetilde{X} \to \mathbb{P}^1_K$ of f' is then obtained in the following way: its function field $K(\widetilde{X})$ is the fixed field in $F(X')$ of the image group $s(\mathrm{Gal}(F/K))$; by construction, every element of $K(\widetilde{X})$ is fixed by all $\chi_\tau \tau$, $\tau \in \mathrm{Gal}(F/K)$ (see also [Sa]).

2.3. Conclusion of §2. Theorem 2.5 combines preceding results to provide the representation result alluded to in the beginning of the section: the points of the K-variety $\mathcal{U}_N$ below parametrize the models of f, with some compatibility of action of G_K. In §3, we will use theorem 2.5 to construct a descent variety V as in Main Theorem A; the variety $\mathcal{U}_N$ is not yet one itself because the fields of definition of models and their corresponding representing points do not *a priori* correspond to one another.

Fix a $K(T)$-basis $\mathbf{e} = [e_1, \dots, e_D]$ of $F(X')$ satisfying the conclusions from lemma 2.3 above. The elements ε_i^1, $i = 1, \dots, D_o$, given in condition (**), form a basis of the extension $k_o'(\widetilde{X}_o')/K(T)$. There exists a subset $J \subset \{1, \dots, D_o\}$ with cardinality d such that the elements ε_i^1 ($i \in J$) form a basis of the extension $k_o'(\widetilde{X}_o')/k_o'(T)$. Relabel the elements of this basis so as to denote it by $\boldsymbol{\varepsilon} = [\varepsilon_1, \dots, \varepsilon_d]$. For $i = 1 \dots, d$, ε_i is a sum of elements e_i. In particular, $\varepsilon_1, \dots, \varepsilon_d$ are integral over $K[T]$. Since the extension $k_o'(\widetilde{X}_o')/k_o'$ is regular, $\boldsymbol{\varepsilon}$ is also a basis of the extension $\overline{K}(X)/\overline{K}(T)$.

Given an integer $N \geq 0$ and an overfield k of K, let $\mathcal{U}_N(k)$ be the set of all $D(N+1)$-tuples $\mathbf{y} = [y_{ij}]_{i,j} = [y_{1o}, \dots, y_{1N}, \dots, y_{Do}, \dots, y_{DN}] \in \mathbb{A}^{D(N+1)}(k)$ such that the element

$$\mathcal{Y}_{\mathbf{y}} = \frac{1}{\Delta(T)} \sum_{i=1}^{D} \left(\sum_{j=0}^{N} y_{ij} T^j \right) e_i$$

is a primitive element of the extension $\overline{k}(X)/\overline{k}(T)$. Next set

$$\mathcal{N} = \mathcal{N}(d, g, \boldsymbol{\varepsilon}) = 2(2g+1) d \log(d) + \deg(\Delta) - \min_{\substack{1 \leq j \leq d \\ \sigma \in \Sigma}} \{ v_\infty((\varepsilon_j^*)^\sigma) \}$$

Theorem 2.5 — *The following statements hold true.*
(a) $\mathcal{U}_N(k)$ *consists of the k-points of a nonempty open subset* $\mathcal{U}_N$ *of affine space* $\mathbb{A}^{D(N+1)}$.
(b) *If $N \geq \mathcal{N}$, for every field $k \supset K$ and every k-model $\widetilde{f} : \widetilde{X} \to \mathbb{P}^1_k$ of f, there exists* $\mathbf{y} \in \mathcal{U}_N(\overline{k})$ *such that the element* $\mathcal{Y}_{\mathbf{y}}$ *is a primitive element of the extension* $k(\widetilde{X})/k(T)$.
(c) *Assume K is the field of moduli of f relative to the extension $\overline{K}/K$. Then for each* $\mathbf{y} \in \mathcal{U}_N(\overline{K})$ *and each* $\tau \in \mathrm{G}_K$, *there exists* $\mathbf{z}_\tau \in \mathcal{U}_N(\overline{K})$ *such that the elements* $\mathcal{Y}_{\mathbf{y}}^\tau$ *and* $\mathcal{Y}_{\mathbf{z}_\tau}$ *are* $\overline{K}(T)$*-conjugate.*

Proof. (a) follows from standard constructive forms of the primitive element theorem.
(b) Let k be an extension of K and $\widetilde{f} : \widetilde{X} \to \mathbb{P}^1_k$ be a k-model of $f : X \to \mathbb{P}^1_{\overline{K}}$. From lemma 2.1, there exists a primitive element $\mathcal{Y}$ of the extension $k(\widetilde{X})/k(T)$ such that the irreducible monic polynomial $P(T, Y)$ of $\mathcal{Y}$ over $k(T)$ lies in $k[T, Y]$ and satifies $\deg_T(P(T, Y)) < 2(2g+1) d \log(d)$. From lemma 2.2, $\mathcal{Y}$ can be (uniquely) written in the

form $\mathcal{Y} = \frac{1}{\Delta(T)} \sum_{i=1}^{d} z_i(T)\varepsilon_i$ with $z_i(T) \in \overline{k}[T]$ and $\deg(z_i) \leq N$, $i = 1, \ldots, d$. In view of the form of the ε_is, $\mathcal{Y}$ can also be written $\mathcal{Y} = \frac{1}{\Delta(T)} \sum_{i=1}^{D} y_i(T) e_i$ where $y_i(T) \in \overline{k}[T]$ and $\deg(y_i) \leq N$, $i = 1, \ldots, D$. That is, $\mathcal{Y} = \mathcal{Y}_{\mathbf{y}}$ for some $\mathbf{y} \in \mathcal{U}_N(\overline{k})$.
(c) Let $\mathbf{y} \in \mathcal{U}_N(\overline{K})$ and $\tau \in \mathrm{G}_K$. Consider the conjugate of $\mathcal{Y}_{\mathbf{y}}$ under τ:

$$\mathcal{Y}_{\mathbf{y}}^{\tau} = \frac{1}{\Delta(T)} \sum_{i=1}^{D} \left(\sum_{j=0}^{N} y_{ij}^{\tau} T^j \right) e_i^{\tau}$$

Let $\tau' \in \mathrm{Gal}(F/K)$ be the restriction of τ to F. By the choice of the basis $\mathbf{e}$ (lemma 2.3), there exists a $F(T)$-isomorphism $\chi_{\tau'} : F(X')^{\tau} \to F(X')$ such that ${}^t[\mathbf{e}^{\chi_{\tau'}\tau}] = A_{\tau'}{}^t[\mathbf{e}]$ where $A_{\tau'} \in \mathrm{GL}_D(K)$ is a permutation matrix. Let $\chi_{\tau} : \overline{K}(X) \to \overline{K}(X)$ be the $\overline{K}(T)$-isomorphism induced by $\chi_{\tau'}$ (by extension of scalars). The element $\chi_{\tau}(\mathcal{Y}_{\mathbf{y}}^{\tau}) \in \overline{K}(X)$, which is $\overline{K}(T)$-conjugate to $\mathcal{Y}_{\mathbf{y}}^{\tau}$, can be written

$$\chi_{\tau}(\mathcal{Y}_{\mathbf{y}}^{\tau}) = \frac{1}{\Delta(T)} \sum_{i=1}^{D} \left(\sum_{j=0}^{N} z_{ij} T^j \right) e_i$$

with $z_{ij} \in \overline{K}$, $1 \leq i \leq D$, $0 \leq j \leq N$. Then $\mathbf{z}_{\tau} = [z_{ij}]_{i,j} \in \mathcal{U}_N(\overline{K})$ and $\chi_{\tau}(\mathcal{Y}_{\mathbf{y}}^{\tau}) = \mathcal{Y}_{\mathbf{z}_{\tau}}$. $\square$

3. First construction of descent varieties for covers

In this section we use §2 to give a first proof of Main Theorem A. As in §2, the base space is $B = \mathbb{P}^1_K$ and K is of characteristic 0. The given cover is a cover $f : X \to \mathbb{P}^1_{\overline{K}}$. We will only prove the weak version of Main Theorem A with conditions (1), (2) and (4') (from Remark 1.1). So we suppose also given a field of definition k_o with $K \subset k_o \subset \overline{K}$ along with a k_o-model $\widetilde{f}_o : \widetilde{X}_o \to \mathbb{P}^1_{k_o}$. The general case of Main Theorem A will be proved in §§4-5 (with different approaches). We retain the notation introduced in §2.

3.1. Construction of the variety V. Fix a $K(T)$-basis $\mathbf{e}$ of $F(X)$ satisfying conclusions from lemma 2.3 and let $\boldsymbol{\varepsilon} = [\varepsilon_1, \ldots, \varepsilon_d]$ be then the basis of $k_o(\widetilde{X}_o)/k_o(T)$ constructed in §2.3. Let $N \geq 0$ be an integer. We refer the reader to the definition (in §2.3) of the open subset $\mathcal{U}_N \subset \mathbb{A}^{D(N+1)}$, which is the starting point of the construction.

Let $\mathbf{Y} = [Y_{ij}]_{\substack{i=1,\ldots,D \\ j=o,\ldots,N}}$ be $D(N+1)$ indeterminates, algebraically independent over $\overline{K(T)}$; $\mathbf{Y}$ is a generic point of $\mathcal{U}_N$. Also set, for $\mathbf{y} = [y_{ij}]_{i,j} \in \mathbb{A}^{D(N+1)}$,

$$\begin{cases} y_i(T) = \sum_{j=0}^{N} y_{ij} T^j \\ \text{so that:} \\ \mathcal{Y}_{\mathbf{y}} = \frac{1}{\Delta(T)} \sum_{i=1}^{D} y_i(T) e_i \end{cases} \qquad \text{resp.} \qquad \begin{cases} Y_i(T) = \sum_{j=0}^{N} Y_{ij} T^j \\ \text{so that:} \\ \mathcal{Y}_{\mathbf{Y}} = \frac{1}{\Delta(T)} \sum_{i=1}^{D} Y_i(T) e_i \end{cases}$$

For every expression $\mathcal{Y} = \frac{1}{\Delta(T)}(\mathcal{Y}_1(T)e_1 + \cdots + \mathcal{Y}_D(T)e_D)$ with $\mathcal{Y}_1(T), \ldots, \mathcal{Y}_D(T)$ polynomials in T with coefficients $(\mathcal{Y}_{ij})_{(i,j)\in S}$ in an overfield of K, denote, for each geometric automorphism $\sigma \in \Sigma$, the conjugate $\frac{1}{\Delta(T)}(\mathcal{Y}_1(T)e_1^\sigma + \cdots + \mathcal{Y}_D(T)e_D^\sigma)$ of $\mathcal{Y}$ under σ by $\mathcal{Y}^\sigma$. Expand the product $\prod_{\sigma\in\Sigma}(Y - \mathcal{Y}^\sigma)$ to get a polynomial $P_{\mathcal{Y}}(T,Y)$ satisfying

$$\begin{cases} P_{\mathcal{Y}}(T,Y) \in \overline{K}\left[\frac{\mathcal{Y}_{ij}}{\Delta(T)} \middle| (i,j) \in S \right][T,Y] \\ P_{\mathcal{Y}}(T,Y) \text{ is monic in } Y \text{ and } \deg_Y(P_{\mathcal{Y}}(T,Y)) = d \\ P_{\mathcal{Y}}(T,\mathcal{Y}^\sigma) = 0 \ \text{ for all } \sigma \in \Sigma \end{cases}$$

Given any point $\mathbf{y} \in \mathcal{U}_N$ (*i.e.*, a $\overline{K}(\mathbf{y})$-point on $\mathcal{U}_N$), the polynomial $P_{\mathcal{Y}_{\mathbf{y}}}(T,Y)$ lies in the ring $\overline{K}\left[\frac{y_{ij}}{\Delta(T)} \middle| (i,j) \in S \right][T,Y]$ and is irreducible in $\overline{K(\mathbf{y})}(T)[Y]$: indeed this polynomial is the irreducible polynomial of $\mathcal{Y}_{\mathbf{y}}$, which is a primitive element of the extension $\overline{K(\mathbf{y})}(X)$, over $\overline{K(\mathbf{y})}(T)$. Let $\Delta(T)^\nu$ be the biggest power of $\Delta(T)$ appearing in the denominator of coefficients of the "generic" polynomial $P_{\mathcal{Y}_{\mathbf{Y}}}(T,Y)$ (equivalently ν is the degree of $P_{\mathcal{Y}_{\mathbf{Y}}}(T,Y)$ regarded as a polynomial in the indeterminates Y_{ij}). Set then

$$\tilde{P}_{\mathcal{Y}_{\mathbf{y}}}(T,Y) = \Delta(T)^\nu P_{\mathcal{Y}_{\mathbf{y}}}(T,Y) \qquad (\mathbf{y} \in \mathcal{U}_N)$$

The polynomial $\tilde{P}_{\mathcal{Y}_{\mathbf{y}}}(T,Y)$, which we will sometimes call the *normalized* irreducible polynomial of $\mathcal{Y}_{\mathbf{y}}$ (over $\overline{K}(\mathbf{y})$), lies in the ring $\overline{K}[\mathbf{y}][T,Y]$ and is irreducible in $\overline{K(\mathbf{y})}[T,Y]$. Set $C = \deg_T(\tilde{P}_{\mathcal{Y}_{\mathbf{Y}}}(T,Y))$. Then for all $\mathbf{y} \in \mathcal{U}_N$, we have $\deg_T(\tilde{P}_{\mathcal{Y}_{\mathbf{y}}}(T,Y)) \le C$ and so $\tilde{P}_{\mathcal{Y}_{\mathbf{y}}}(T,Y)$ can be written in the form

$$P(T,Y) = (p_{d,C}T^C + \cdots + p_{d,o})Y^d + \cdots + (p_{o,C}T^C + \cdots + p_{o,o})$$

For every polynomial with this shape, denote the $(d+1)(C+1)$-tuple of its coefficients $\mathbf{p} = [p_{o,o}, \ldots, p_{o,C}, \ldots, p_{d,o}, \ldots, p_{d,C}]$ by $[P(T,Y)]$. The $(d+1)(C+1)$-tuple $[\tilde{P}_{\mathcal{Y}_{\mathbf{Y}}}(T,Y)]$ defines an algebraic morphism over $\overline{K}$

$$[\tilde{P}_{\mathcal{Y}}] : \mathcal{U}_N \to \mathbb{A}^{(d+1)(C+1)}$$

Denote the image by V; it is a quasi-affine subvariety of $\mathbb{A}^{(d+1)(C+1)}$ *a priori* defined over $\overline{K}$. We will show that if $N \geq \mathcal{N} = \mathcal{N}(d, g, \varepsilon)$, Main Theorem A holds for this variety V.

3.2. Proof of Main Theorem A (1)**.** The variety V is absolutely irreducible (as $\mathcal{U}_N$ is) and is *a priori* defined over $\overline{K}$. To prove that V is defined over K, we will show that $V(\overline{K})^\tau = V(\overline{K})$ for each $\tau \in \mathrm{G}_K$ [6] . This classically guarantees that the variety V is defined over K; this follows for example from Weil's descent criterion [We]. Let $\tau \in \mathrm{G}_K$ and $v \in V(\overline{K})$. By construction, there exists $\mathbf{y} \in \mathcal{U}_N(\overline{K})$ such that $v = [\tilde{P}_{\mathcal{Y}_\mathbf{y}}(T, Y)]$. We have $v^\tau = [\tilde{P}_{\mathcal{Y}_\mathbf{y}}(T, Y)^\tau]$ (where τ acts on the coefficients in $\overline{K}$ of the polynomial). Clearly, $\tilde{P}_{\mathcal{Y}_\mathbf{y}}(T, Y)^\tau$ is the irreducible polynomial over $\overline{K}(T)$ of

$$\mathcal{Y}_\mathbf{y}^\tau = \frac{1}{\Delta(T)} \sum_{i=1}^{D} y_i(T)^\tau e_i^\tau$$

From theorem 2.5 (c), there exists $\mathbf{z}_\tau \in \mathcal{U}_N(\overline{K})$ such that the elements $\mathcal{Y}_\mathbf{y}^\tau$ and $\mathcal{Y}_{\mathbf{z}_\tau}$ are $\overline{K}(T)$-conjugate. Conclude that $v^\tau = [\tilde{P}_{\mathcal{Y}_\mathbf{y}}(T, Y)^\tau] = [\tilde{P}_{\mathcal{Y}_{\mathbf{z}_\tau}}(T, Y)] \in V(\overline{K})$. $\square$

3.3. Proof of Main Theorem A (2)**.** The "generic" polynomial $\tilde{P}_{\mathcal{Y}_\mathbf{Y}}(T, Y)$ determines a family $\mathcal{F} : \mathcal{X} \to V \times \mathbb{P}^1$ of covers of $\mathbb{P}^1$ parametrized by V. The generic cover $\mathcal{F}_{\mathrm{gen}}$ is the cover of $\mathbb{P}^1_{K(V)}$ induced by the polynomial $\tilde{P}_{\mathcal{Y}_\mathbf{Y}}(T, Y)$, *i.e.*, the $K(V)$-cover corresponding to the finite extension $K(V)(T, \mathcal{Y}_\mathbf{Y})/K(V)(T)$. The family $\mathcal{F}$ is defined over K since V is and the coefficients of $\tilde{P}_{\mathcal{Y}_\mathbf{Y}}(T, Y)$ (as a polynomial in T, Y) lie in $K(V)$.

Let $v \in V$. That is, v is the ordered $(d+1)(C+1)$-tuple of coefficients of the normalized irreducible polynomial over $\overline{K(v)}(T)$ of some element

$$\mathcal{Y}_\mathbf{y} = \frac{1}{\Delta(T)} \sum_{i=1}^{D} \left(\sum_{j=0}^{N} y_{ij} T^j \right) e_i$$

where $\mathbf{y} = [y_{ij}]_{i,j} \in \mathcal{U}_N(\overline{K(v)})$ and $\mathcal{Y}_\mathbf{y}$ is *a priori* an element of $\overline{K(v)}(X)$. Using previous notation, we have $v = [\tilde{P}_{\mathcal{Y}_\mathbf{y}}(T, Y)] \in K(v)[T, Y]$. The fiber cover $\mathcal{F}_v : \mathcal{X}_v \to \mathbb{P}^1_{K(v)}$ corresponds to the field extension obtained by specializing the coefficients of the "generic"

[6] For $\overline{\mathbb{Q}}$-covers, it is tempting to use the following alternate argument: the field of moduli K is the intersection of all fields of definition ([DeDo1;§3.4.1],[CoHa]); therefore, from assertion (2) of Main Theorem A (whose proof given below does not use assertion (1)), K contains the intersection of all fields of definitions of points in V. However one cannot conclude that K is a field of definition of V: in the preceding sentence, fields of definition of points should be understood as fields of definition of coordinates in an affine model; for points in a proper Zariski closed subset, this may not coincide with the intrinsic notion of fields of definition of points on a scheme (think of the affine line $y=\sqrt{2}x$ which is defined over $\mathbb{Q}(\sqrt{2})$ but passes through (0,0)).

polynomial $\tilde{P}_{\mathcal{Y}_{\mathbf{Y}}}(T,Y)$ to those of the polynomial $\tilde{P}_{\mathcal{Y}_{\mathbf{y}}}(T,Y)$ (*i.e.*, the entries of the tuple v). This field extension is the extension $K(v)(T,\mathcal{Y}_{\mathbf{y}})/K(v)(T)$. By definition of "$\mathbf{y}\in\mathcal{U}_N$", this extension yields the extension $\overline{K(v)}(X)/\overline{K(v)}(T)$ by extension of scalars. Conclude that the fiber cover $\mathcal{F}_v:\mathcal{X}_v\rightarrow\mathbb{P}^1_{K(v)}$ is indeed a $K(v)$-model of f. Whence assertion (i).

We prove now assertion (ii). From theorem 2.5 (b), if $N\geq\mathcal{N}$, for every overfield k of K and every k-model $\widetilde{f}:\widetilde{X}\rightarrow\mathbb{P}^1_k$ of f, there exist $\mathbf{y}\in\mathcal{U}_N(\overline{k})$ such that the corresponding element $\mathcal{Y}_{\mathbf{y}}$ is a primitive element of the extension $k(\widetilde{X})/k(T)$. Set $v=[\tilde{P}_{\mathcal{Y}_{\mathbf{y}}}(T,Y)]$. By construction $v\in V(k)$ and the fiber cover $\mathcal{F}_v:\mathcal{X}_v\rightarrow\mathbb{P}^1_k$ is isomorphic to $\widetilde{f}$ over k. □

3.4. Proof of Main Theorem A (4'). Consider the morphism $[P_{\mathcal{Y}}]:\mathcal{U}_N\rightarrow V$. By construction, this morphism is dominant and is defined over $\overline{K}$. The space $\mathcal{U}_N$ is a rational variety (it is an open subset of an affine space). Therefore, by definition, V is a $\overline{K}$-unirational variety.

It remains to prove that $V(k_o)$ is Zariski-dense. We claim that for each $\mathbf{y}\in\mathcal{U}_N(\overline{K})$, the corresponding element

$$\mathcal{Y}_{\mathbf{y}}=\frac{1}{\Delta(T)}\sum_{i=1}^{D}y_i(T)\,e_i\quad\text{is of the form}\quad\mathcal{Z}=\frac{1}{\Delta(T)}\sum_{i=1}^{D_o}\sum_{m=1}^{\delta}z_i^m(T)\,\varepsilon_i^m$$

for some polynomials $z_i^m(T)\in\overline{K}(T)$ with $\deg(z_i)\leq N$, $i=1,\ldots,D_o$ and $m=1,\ldots,\delta$, and *vice versa*, that is, every $\mathcal{Z}$ is of the form $\mathcal{Y}_{\mathbf{y}}$. Here the elements ε_i^m are those defined in lemma 2.3.

The Zariski-density of $V(k_o)$ will easily follow. Indeed consider the subset $\mathcal{V}_{k_o}$ of $\mathcal{U}_N(\overline{K})$ consisting of all points $\mathbf{y}$ such that $\mathcal{Y}_{\mathbf{y}}$ is of the form $\mathcal{Z}$ above with $z_i^m(T)\in k_o[T]$, $i=1,\ldots,D_o$ and $m=1,\ldots,\delta$. We have the following points:

- $\mathcal{V}_{k_o}$ is a Zariski-dense subset of $\mathcal{U}_N(\overline{K})$: this comes from the claim.
- $[\tilde{P}_{\mathcal{Y}}](\mathcal{V}_{k_o})\subset V(k_o)$: indeed, if $\mathbf{y}\in\mathcal{V}_{k_o}$, *i.e.*, with notation as above, $z_i^m(T)\in k_o[T]$, $i=1,\ldots,D_o$ and $m=1,\ldots,\delta$, then $\mathcal{Z}\in k_o(\widetilde{X}_o)$ and its irreducible polynomial $P_{\mathcal{Y}_{\mathbf{y}}}(T,Y)$ has coefficients in k_o, *i.e.*, $v=[\tilde{P}_{\mathcal{Y}}](\mathbf{y})\in V(k_o)$.

It readily follows that $V(k_o)$ is Zariski-dense in $V(\overline{K})=[\tilde{P}_{\mathcal{Y}}](\mathcal{U}_N(\overline{K}))$.

To prove the claim, we use the calculation below (see lemma 2.3 for notation; in particular we use the relabelling $[e_{ij}]_{\substack{i=1,\ldots,D_o\\ j=1,\ldots,\delta}}$ of $\mathbf{e}$ from assertion (**)).

$$\begin{aligned}
\mathcal{Z} &= \frac{1}{\Delta(T)} \sum_{i=1}^{D_o} \sum_{m=1}^{\delta} z_i^m(T)\, \varepsilon_i^m \\
&= \frac{1}{\Delta(T)} \sum_{i=1}^{D_o} \sum_{m=1}^{\delta} z_i^m(T) \sum_{\ell=1}^{\delta} b_{\ell m}\, e_{i\ell} \\
&= \frac{1}{\Delta(T)} \sum_{i=1}^{D_o} \sum_{\ell=1}^{\delta} \left[\sum_{m=1}^{\delta} b_{\ell m}\, z_i^m(T) \right] e_{i\ell}
\end{aligned}$$

The converse part of the claim readily follows. For the direct part, it suffices to prove that for $i = 1, \ldots, D_o$, there exist polynomials $z_i^m(T) \in \overline{K}(T)$, $m = 1, \ldots, \delta$ with $\deg(z_i^m) \leq N$ such that $\sum_{m=1}^{\delta} b_{\ell m}\, z_i^m(T)/\Delta(T)$ equals the $e_{i\ell}$-component of $\mathcal{Y}_{\mathbf{y}}$ ($\ell = 1, \ldots, \delta$). Each of these $\delta \times \delta$-linear systems has a (unique) solution since from lemma 2.3, the matrix $B = [b_{\ell m}]_{\ell, m}$ is invertible. $\square$

3.4.1. The G-cover situation. We briefly explain how to modify the construction of V in the situation that $f : X \to \mathbb{P}^1_{\overline{K}}$ is a G-cover. We omit the details. Other more general proofs are given in §4 and §5.

The idea is to enlarge the $(d+1)(C+1)$-tuple of coefficients of $\tilde{P}_{\mathcal{Y}_{\mathbf{y}}}(T, Y)$ so as to also contain the "coefficients" of the components of the geometric conjugates $\mathcal{Y}_{\mathbf{y}}^{\sigma}$ of $\mathcal{Y}_{\mathbf{y}}$ in the basis $1, \mathcal{Y}_{\mathbf{y}}, \ldots, \mathcal{Y}_{\mathbf{y}}^{d-1}$ (where σ ranges over Σ). These components lie (*a priori*) in $\overline{K}(\mathbf{y})(T)$. For the generic point $\mathbf{y} = \mathbf{Y}$, write these components in the form $P(\mathbf{Y}, T)/Q(\mathbf{Y}, T)$ where $P(\mathbf{Y}, T), Q(\mathbf{Y}, T) \in \overline{K}[\mathbf{Y}][T]$ are relatively prime and normalized by requiring that some monomial in $(\mathbf{Y}, T)$ in $P(\mathbf{Y}, T)$ has coefficient 1. Then the "coefficients" alluded to are all coefficients in $\overline{K}[\mathbf{Y}]$ of all numerators $P(\mathbf{Y}, T)$ and denominators $Q(\mathbf{Y}, T)$ of all components of $\mathcal{Y}_{\mathbf{y}}^{\sigma}$ in the basis $1, \mathcal{Y}_{\mathbf{y}}, \ldots, \mathcal{Y}_{\mathbf{y}}^{d-1}$ (where σ ranges over Σ).

4. Descent varieties for covers: using marked models

In this section, we consider the general situation of Main Theorem A, except that we can in fact relax the assumptions on the 'cover' considered:

• The base space B is a projective geometrically irreducible K-variety (and no longer $\mathbb{P}^1$ as in §2 and §3).

• The given [G-]cover $f : X \to B^{\mathrm{s}} := B_{K^{\mathrm{s}}}$ is a projective morphism, with X reduced; we only assume that every irreducible component of X maps surjectively on B^{s}, and that f is generically étale.

• In the G-cover case, we also fix an action of the finite group G on X, compatible with f and such that, for some nonempty open $B^0 \subset B^{\mathrm{s}}$, $f^{-1}(B^0)$ is a G-torsor over B^0. (If X and B are irreducible and normal, f is then Galois with group G in the usual sense).

Except for the above changes, we adhere to the notation of §1. Moreover we denote by Γ the (finite) group of [G-]B^{s}-automorphisms of X. With our assumptions, Γ, when viewed as a K^{s}-group scheme, is also the automorphism scheme of f ("f has no infinitesimal automorphisms"). More precisely, Γ represents the functor $\underline{\Gamma}$ from K^{s}-schemes to groups sending U to the group of $U \times_{K^{\mathrm{s}}} B^{\mathrm{s}}$-automorphisms of $U \times_{K^{\mathrm{s}}} X$.

(To see this, observe first that $\underline{\Gamma}$ is representable by a K^{s}-group scheme locally of finite type, by general results of Grothendieck; to conclude, it suffices to note that $\underline{\Gamma}(K^{\mathrm{s}}) = \Gamma$ and the Lie algebra of $\underline{\Gamma}$ is the space of $\mathcal{O}_{B^{\mathrm{s}}}$-derivations of $\mathcal{O}_X$ which is clearly zero with our assumptions.)

4.1. Models. The notion of a model of f can be generalized to arbitrary "parameter" K-schemes. Namely, if U is a K-scheme, a *U-model of f over K* (or "U/K-model" for short) is a U-morphism $g : Y \to B_U := B \times_K U$ [with a compatible G-action on Y], which is locally isomorphic to f in the following sense: over $U_{K^{\mathrm{s}}} := K^{\mathrm{s}} \otimes_K U$ we have two covers f' and g' of $B^{\mathrm{s}} \times_K U = B \times_K U_{K^{\mathrm{s}}}$ deduced from f and g by base change, and we require these to be isomorphic over some $U' \to U_{K^{\mathrm{s}}}$, étale and surjective over $U_{K^{\mathrm{s}}}$. (In this case, the $U_{K^{\mathrm{s}}}$-scheme $\underline{\mathrm{Isom}}(f', g')$ will in fact be a Γ-torsor: as a consequence, we can take U' finite étale over $U_{K^{\mathrm{s}}}$). A *morphism* of U/K-models is an isomorphism of [G-]covers of B_U. We denote by $\mathsf{MOD}_{f,K}(U)$ the category of U/K-models of f. For every K-morphism $U' \to U$, we have an obvious base change functor $\mathsf{MOD}_{f,K}(U) \to \mathsf{MOD}_{f,K}(U')$.

Observe that any two U/K-models of f are locally isomorphic for the étale topology on U.

Of course, if $U = \mathrm{Spec}\,(A)$ is affine, we shall speak of an A/K-model of f.

Remark 4.1. To compare with the definition of a model given in §1, let us see what an L/K-model is when L is a finite separable extension of K. Putting $U = \mathrm{Spec}\,(L)$ in the above definition, we note that the scheme $U_{K^{\mathrm{s}}} := \mathrm{Spec}\,(L \otimes_K K^{\mathrm{s}})$ is a disjoint sum of copies of $\mathrm{Spec}\,(K^{\mathrm{s}})$, indexed by the finite set of K-embeddings of L into K^{s}. Given a [G-]cover $g : Y \to B \times_K L$, the cover g' above corresponds to the finite family of covers of $B_{K^{\mathrm{s}}}$ deduced from g via these embeddings. To say that g is an L/K-model of f therefore means that *each* of these covers of $B_{K^{\mathrm{s}}}$ is isomorphic to f.

One can observe that the above definitions do not make use of the assumption that K is the field of moduli of f; on the other hand, K is indeed present in the definition of a

U/K-model via the fibre products such as $B \times_K U$. What would happen if K were not the field of moduli is simply that f would have no model over any nonempty K-scheme; in particular (equivalently, in fact) f would not be a K^{s}-model of f!

From now on we shall drop K from the notations and speak, for instance, of the category $\mathsf{MOD}_f(U)$ of U-models of f, whenever it can be done safely.

4.2. Markings. Since we have assumed f generically étale, we can choose a closed point Q of B such that f is étale over the finite subset $Q_{K^{\mathrm{s}}}$ of $B_{K^{\mathrm{s}}}$. The restriction of $X \to B_{K^{\mathrm{s}}}$ above $Q_{K^{\mathrm{s}}}$ is the spectrum of a finite-dimensional K^{s}-algebra: specifically, its dimension is $N = (\deg f)(\deg_K Q)$.

More generally, every model $g : Y \to B_U$ over some K-scheme U induces by restriction a finite étale Q_U-scheme which is (when viewed as a U-scheme) the spectrum of a locally free $\mathcal{O}_U$-algebra of rank N. We denote the underlying $\mathcal{O}_U$-module by $W(g)$, and we define a *Q-marking* of g to be a basis of $W(g)$. We view such a basis as an isomorphism $\alpha : \mathcal{O}_U^N \to W(g)$ of $\mathcal{O}_U$-modules. For given g we have a "scheme of Q-markings" of g which is a right torsor over U, for the natural action of $\mathrm{GL}_{N,U}$ on markings. In particular, every U-model admits a Q-marking locally on U for the Zariski topology.

A *Q-marked model* of f over U is a U-model g together with a Q-marking of g; a morphism of such Q-marked models is a morphism of models which is compatible with the markings, in the obvious sense. In this way we obtain a category $\mathsf{MOD}_f^Q(U)$ which depends functorially on U (i.e. there are base change functors), and we have obvious forgetful functors $\mathsf{MOD}_f^Q(U) \to \mathsf{MOD}_f(U)$, compatible with base change.

Note that by construction both $\mathsf{MOD}_f(U)$ and $\mathsf{MOD}_f^Q(U)$ are groupoids (all their morphisms are isomorphisms). But the nice feature of $\mathsf{MOD}_f^Q(U)$ (as compared with $\mathsf{MOD}_f(U)$) is that its objects have no nontrivial automorphisms. Indeed, an automorphism of a marked model (g, α) is an automorphism of the cover g which must fix a basis of $W(g)$, hence induce the identity above Q_U, so that it must be the identity. Since $\mathsf{MOD}_f^Q(U)$ is a groupoid, it is equivalent to the discrete category defined by the set

$$M_f^Q(U) := \{ \text{ isomorphism classes of } Q\text{-marked models of } f \text{ over } U \ \}$$

which is functorial in U. Now Main Theorem A follows from the following facts:

Proposition 4.2 — *Assume that $f : X \to B^{\mathrm{s}}$ is finite.*
(1) *The functor $U \mapsto M_f^Q(U)$ is representable by a smooth affine K-scheme V, which is a homogeneous space under $\mathrm{GL}_{N,K}$ for the natural action of this group on markings.*

(2) *Moreover, if R is a semi-local K-algebra (e.g. an extension of K), every R-model of f is obtained, up to isomorphism, from the universal marked model over V, by base change via some $v \in V(R)$.*

Proof. (2) follows from the fact that markings exist Zariski-locally, and even semilocally. Let us prove (1).

First, the functor M_f^Q in question is clearly a sheaf for the étale topology (even for the fpqc topology) on the category of K-schemes. This is because models of f over U are finite B_U-schemes, and finite (and, more generally, affine) morphisms satisfy étale descent ([SGA 1], VIII, 2.1); further, being a U-model of f is clearly a local condition on U for a given morphism $g : Y \to B_U$; finally, markings can be descended, since they can be seen as morphisms of quasi-coherent sheaves.

Now assume, temporarily, that f has a K-model $g_0 : Y_0 \to B$, and fix a Q-marking $\alpha_0 : K^N \xrightarrow{\sim} W(g_0)$ of g_0. The action of $\mathrm{GL}_{N,K}$ defines an "orbit" morphism of functors $\pi : \mathrm{GL}_{N,K} \to M_f^Q$ sending γ to $(g_0, \alpha_0 \circ \gamma)$. We claim that π is an *epimorphism* of étale sheaves: indeed, every U-model of f is locally isomorphic to g_0 for the étale topology on U, and any two markings of a given model "differ" by an element of GL_N. So, M_f^Q is isomorphic to the quotient (in the category of étale sheaves) of $\mathrm{GL}_{N,K}$ by the equivalence relation determined by π, which is easy to describe: γ and $\gamma' \in \mathrm{GL}_N(U)$ determine isomorphic marked models if and only if $\gamma\gamma'^{-1}$ is induced by an automorphism of $g_{0,U}$, under the natural map $\mathrm{Aut}\,(g_{0,U}) \to \mathrm{GL}_N(U)$ deduced from the marking α_0. This map comes from an injection $\underline{\mathrm{Aut}}\,(g_0) \hookrightarrow \mathrm{GL}_{N,K}$ of affine algebraic K-groups, and our discussion shows that $M^Q(f)$ is isomorphic to $\mathrm{GL}_{N,K}/\underline{\mathrm{Aut}}\,(g_0)$ which is well known to be representable, affine and smooth over K since $\underline{\mathrm{Aut}}\,(g_0)$ is finite.

Finally, in the general case, we observe that f admits a model over a finite separable extension of K, so the result follows from the above discussion by étale (or Galois) descent of affine schemes. □

Remark 4.3. The assumptions on f and B can be relaxed. In fact, the projectivity of B is not used (any K-scheme of finite type will do), and the properties of f that we have really used are étale descent (for which finiteness of f is enough), and finiteness of the automorphism group scheme of f. We leave it to the reader to work out variants of the above results under weaker assumptions.

5. Generalization to gerbes

5.1. Linear quotient stacks. We shall now freely use the language of algebraic stacks [LaMo]. If X is an algebraic stack (over a scheme S) and U is an S-scheme, we shall denote by $\mathsf{X}(U)$ the fiber category of X over U (denoted by X_U in [LaMo]). Recall that the stack X is determined by the data $\mathsf{X}(U)$ for affine $U \to S$, plus the base change functors $\mathsf{X}(U) \to \mathsf{X}(U')$ for all S-morphisms $U' \to U$.

If X is an algebraic stack over some Noetherian base scheme S, we shall say that X is a *linear quotient stack* if it is isomorphic to the quotient stack $[X/\mathrm{GL}_{n,S}]$, for some positive integer n and some algebraic space X of finite type over S with an action of $\mathrm{GL}_{n,S}$. We list some elementary facts:

(1) If H is a subgroup scheme of $\mathrm{GL}_{n,S}$, flat over S, acting on an algebraic S-space Y of finite type, then $[Y/H]$ is a linear quotient stack; in fact, it is isomorphic to $[X/\mathrm{GL}_{n,S}]$ where X is the "contracted product" $Y \times^H \mathrm{GL}_{n,S}$, the quotient of $Y \times \mathrm{GL}_{n,S}$ by H (where $h \in H$ acts on Y as itself and on $\mathrm{GL}_{n,S}$ by h^{-1}).

In particular, if Γ is a finite group acting on an S-scheme X of finite type, then $[X/\Gamma]$ is a linear quotient stack.

(2) ([EdHaKrVi], lemma 2.13) Let X be an algebraic stack of finite type over S. Then X is a linear quotient stack if and only if there is a vector bundle E on X with the property that, for every geometric point ξ of X (of the form $\mathrm{Spec}\,(k) \to \mathsf{X}$), the natural action of the algebraic k-group $\underline{\mathrm{Aut}}\,(\xi)$ on the k-vector space E_ξ is faithful.

Specifically, from such a vector bundle E (of rank N, say) we can construct the GL_N-torsor P over X parametrizing bases of sections of E; the condition on E means that P (*a priori* just an algebraic stack) is in fact an algebraic space: recall ([LaMo], 8.1.1) that this is equivalent to the condition that for any U, the objects of the groupoid P_U have no nontrivial automorphisms.

5.2. Example: the gerbe of models of a cover. Let us return to the situation of the preceding section. We have defined a stack over K, namely the "gerbe of models" MOD_f (the assignment to each K-scheme U of the groupoid $\mathsf{MOD}_f(U)$). It is an algebraic stack of finite type over K, and moreover it is a gerbe, meaning that:
(1) for some finite separable extension L of K, $\mathsf{MOD}_f(L)$ is nonempty;
(2) any two U-models of f are locally isomorphic (for the étale topology on U).

What we have actually shown in §4.2 is that MOD_f *is a linear quotient stack*, by finding a vector bundle as in (2) of §5.1. Namely, we have assigned to every U-model g of f a vector bundle $W(g)$ over U, in a way which is compatible with base change functors. This means that W is a vector bundle on MOD_f, and the faithfulness property of §5.1 (2) boils down to the fact, observed in §4.2, that an automorphism of g inducing the identity on $W(g)$ must be the identity.

So MOD_f "is" the quotient by GL_N of the "space of bases of W" which is nothing else than the variety V constructed in proposition 4.2 above. In fact, we can replace the proof of §4.2 by the following arguments:

- the "faithfulness property" considered above shows that the GL_N-torsor $V \to \mathsf{MOD}_f$ of bases of W (a priori an algebraic stack) is an algebraic space;
- the fact that MOD_f is a gerbe implies that the action of GL_N on V is transitive; as a consequence V is a smooth quasiprojective K-variety, and even an affine one since the action of GL_N has finite stabilizers (because $\mathrm{Aut}_{K^s}(f)$ is finite).

We can generalize this to other gerbes:

Proposition 5.1 — *Let K be a field and let G be a K-gerbe* (*for the étale topology*) *which is a Deligne-Mumford stack. Then:*

(1) *there is a finite separable K-algebra L with a left action of a finite group Γ, with ring of invariants K, such that G is isomorphic to the quotient stack* $[\mathrm{Spec}\,(L)/\Gamma]$*;*

(2) *there is a smooth affine K-scheme V, an integer $n \geq 0$, a right action of $\mathrm{GL}_{n,K}$ on V, and a 1-morphism $\pi : V \to \mathsf{G}$ (an object of the category $\mathsf{G}(V)$) with the following properties:*

(i) *π induces an isomorphism of the quotient stack $[V/\mathrm{GL}_{n,K}]$ with G;*

(ii) *V is* (*smooth and*) *geometrically irreducible;*

(iii) *the action of $\mathrm{GL}_{n,K}$ on V is transitive, with finite stabilizers;*

(iv) *for every extension k of K, every object of $\mathsf{G}(k)$ lifts to a point of $V(k)$ via π.*

In particular, because of (iii) *and* (iv)*, if k is an extension of K such that $\mathsf{G}(k) \neq \emptyset$, the k-variety $V \otimes_K k$ is isomorphic to the quotient of $\mathrm{GL}_{n,k}$ by a finite subgroup.*

Proof. (1) By [LaMo], (6.1.1), there is a nonempty open substack of G which is isomorphic to a quotient $[X/\Gamma]$ for some affine scheme X with an action of a finite group Γ. But since G is a K-gerbe, the only nonempty open substack of G is G, so we have an isomorphism $[X/\Gamma] \cong \mathsf{G}$ of K-stacks. Now, it is immediately seen that $[X/\Gamma]$ is a K-gerbe if and only if $X \to \mathrm{Spec}\,(K)$ is epimorphic (*i.e.* $X \neq \emptyset$) and the action of Γ on X is transitive (in the sense that the map $(x,\gamma) \mapsto (x, x\gamma)$ from $\Gamma \times X$ to $X \times X$ is epimorphic for the étale topology). We also know that, as a Deligne-Mumford gerbe, G must be étale over K. Since $X \to [X/\Gamma]$ is obviously étale, this implies that X must be the spectrum of a finite separable K-algebra, and the transitivity condition for the group action means that the ring of invariants is K.

(2) Since G is a quotient by a finite group, it is also a linear quotient stack by 5.1 (1) above. Now if $\mathsf{G} \cong [V/\mathrm{GL}_{n,K}]$, we deduce the other properties of V by much the same arguments as in proposition 4.2 (1): first, the action is transitive since G is a gerbe; next, properties (ii) and (iii) can be checked after a base field extension, so we may assume that V has a rational point x, with stabilizer H. In this case, $\mathsf{G} \cong [\mathrm{Spec}\,(K)/H]$ so H is finite étale over K, and on the other hand $V \cong \mathrm{GL}_{n,K}/H$ which is smooth, affine and geometrically irreducible. Finally, for property (iv) we may assume $k = K$, and we just observe that liftings of an object of $\mathsf{G}(k)$ to $V(k)$ are classified by a GL_n-torsor over K, and any such torsor is trivial. $\square$

Remark 5.2. (a) In (1), one may assume that L is a field (necessarily a Galois extension of K). Indeed, L must be isomorphic to a product of finitely many copies of a Galois extension M of K; the finite group Γ permutes the factors transitively, and the stabilizer Γ_0 of a given factor M_0 maps surjectively to $\mathrm{Gal}\,(M/K)$. Replacing L by M_0 and Γ by Γ_0 does not change the quotient stack.

(b) Let us briefly indicate how one can construct L and Γ in (1), without using the general method of [LaMo]. First choose a finite Galois extension L_0 of K such that G_{L_0} is trivial, and pick an object x of $\mathsf{G}(L_0)$. The automorphism scheme of x is a finite étale L_0-group scheme, so it becomes constant over a finite extension L_1 of L_0, which we may assume Galois over K (in classical terms, "all automorphisms of x are defined over L_1"). Finally, all conjugates of x under $\mathrm{Gal}\,(L_0/K)$ become isomorphic over a finite extension L_2 which we may assume Galois over K and containing L_1. Now take $L = L_2$: we have an object of $\mathsf{G}(L)$ deduced from x, which we shall still denote by x; for each $\sigma \in \mathrm{Gal}\,(L/K)$ we denote by σx the object of $\mathsf{G}(L)$ obtained from x by the extension $\sigma : L \to L$. (Note that we view $\mathrm{Gal}\,(L/K)$ as acting on the left on L, hence on the right on $\mathrm{Spec}\,(L)$ and on the left on the category $\mathsf{G}(L)$). Let Γ be the set of all pairs (σ, φ) where $\sigma \in \mathrm{Gal}\,(L/K)$ and $\varphi : \sigma x \xrightarrow{\sim} x$ is an isomorphism in $\mathsf{G}(L)$. There is an obvious group structure on Γ, for which the natural projection $\Gamma \to \mathrm{Gal}\,(L/K)$ is a morphism, which is surjective by the assumption that all σx's are isomorphic. So Γ acts transitively on $\mathrm{Spec}\,(L)$, and one then checks that the gerbes G and $[\mathrm{Spec}\,(L)/\Gamma]$ are equivalent.

(c) The choice of GL_n in the above results is motivated by the following properties: (i) $\mathrm{GL}_{n,K}$ is a smooth connected linear algebraic K-group; (ii) every given finite group can be embedded in it, for some n; (iii) for every field (and in fact every semi-local ring) k, every $\mathrm{GL}_{n,k}$-torsor is trivial; in other words, $H^1(k, \mathrm{GL}_n) = \{1\}$. We see in particular that GL_n could have been replaced by SL_n throughout. There are certainly variants using other groups, meeting various needs.

(d) Examples of K-gerbes to which proposition 5.1 (hence also Main Theorem A) applies are:

• the gerbe of models of a given projective K^{s}-variety X with field of moduli K, provided the automorphism scheme $\underline{\mathrm{Aut}}_{K^{\mathrm{s}}}(X)$ is a finite étale group scheme over K^{s}; this is the case in particular if X is a smooth (or more generally stable) curve of genus ≥ 2;

• the gerbe of polarized models of a given *polarized* abelian variety (X, λ) over K^{s}. Here of course, in the definition of a model, we require isomorphisms to respect the polarizations, and again we assume that the field of moduli of (X, λ) is K.

Here are examples where 5.1 does not apply:

• the gerbe of models of a given smooth curve of genus 1;

• the gerbe of models of a given abelian variety over K^{s} (without polarization);

In these cases, there is in general no "descent variety" V (affine or not) having property (iv) of 5.1, even if the given curve or abelian variety can be defined over K.

6. Descent varieties and moduli spaces of covers

6.1. General moduli for covers of $\mathbb{P}^1$. Fix integers $d > 0$, $g \geq 0$. For every scheme T, let $\mathsf{COV}^{d,g}(T)$ be the category whose objects are T-morphisms $\pi : X \to \mathbb{P}^1_T$, where:

• X is a smooth proper T-scheme whose geometric fibers are connected genus g curves;

• π is a finite flat T-morphism of degree d;

• for every geometric point ξ of T, the induced morphism $\pi_\xi : X_\xi \to \mathbb{P}^1_{\kappa(\xi)}$ is separable. (In other words, the étale locus of π in X is surjective over T).

We define morphisms in $\mathsf{COV}^{d,g}(T)$ to be $\mathbb{P}^1_T$-isomorphisms. Clearly, we get a fibered category $\mathsf{COV}^{d,g}$ over the category of schemes. Standard arguments (see for instance [Mo3], §7) show that $\mathsf{COV}^{d,g}$ is an algebraic stack of finite type over $\mathbb{Z}$. Moreover, it is a Deligne-Mumford stack since the automorphism group of a (finite separable) cover is unramified.[7]

[7] Remark for the experts: we don't have to worry about effectivity of descent for $\mathsf{COV}^{d,g}$ because we deal with curves which are finite over $\mathbb{P}^1$, hence satisfy effective descent. This is why we can safely take our curves to be schemes instead of algebraic spaces. In fact, for any $\pi : X \to \mathbb{P}^1_T$ in $\mathsf{COV}^{d,g}(T)$, the curve X carries a natural ample sheaf — namely $\pi^*(\mathcal{O}(1))$ — and is therefore projective over T.

Theorem 6.1 — $\mathsf{COV}^{d,g}$ *is a linear quotient stack.*

We give two proofs, both using the criterion (2) of §5.1, but with different vector bundles.

First proof. For every scheme T and object $\pi : X \to \mathbb{P}^1_T$ of $\mathsf{COV}^{d,g}(T)$, denote by $\mathcal{L}(\pi)$ the line bundle $\pi^*(\mathcal{O}(1))$ on X, and by f_π the structural morphism $X \to T$. There exists an integer N, depending only on g (in fact $N = 2g+1$ will do) such that $\mathcal{L}(\pi)^{\otimes N}$ is very ample relative to T and the direct image $f_{\pi *}\mathcal{L}(\pi)^{\otimes N}$ is locally free and commutes with every base change $T' \to T$. Put $W(\pi) = f_{\pi *}\mathcal{L}(\pi)^{\otimes N}$: this defines a vector bundle W on $\mathsf{COV}^{d,g}$ (of rank $Nd+1-g$), and the fact that $\mathcal{L}(\pi)^{\otimes N}$ is very ample implies that $\underline{\mathrm{Aut}}(\pi)$ injects into $\mathrm{GL}(W(\pi))$. We conclude by §5.1 (2).

Second proof. If k is an algebraically closed field, and $\pi : X \to \mathbb{P}^1_k$ is an object of $\mathsf{COV}^{d,g}(k)$, then the number of branch points of π in $\mathbb{P}^1(k)$ is bounded in terms of g and d alone (by $2g-2+2d$, in fact). As a consequence, we can choose a finite flat subscheme $Z \subset \mathbb{P}^1_{\mathbb{Z}}$ in such a way that any π as before is étale above at least one point of $Z(k)$. This implies that any automorphism of π inducing the identity over Z must be the identity.

Now, for any scheme T and object $\pi : X \to \mathbb{P}^1_T$ of $\mathsf{COV}^{d,g}(T)$, put $X_Z = X \times_{\mathbb{P}^1_T} Z_T$: this is a finite locally free T-scheme of degree md where m is the degree of Z over $\mathrm{Spec}\,(\mathbb{Z})$. So it is the spectrum of a finite locally free $\mathcal{O}_T$-algebra, whose underlying module we denote by $W'(\pi)$. Now it is clear that this construction defines a vector bundle W' on $\mathsf{COV}^{d,g}$, and the preceding discussion shows that $\underline{\mathrm{Aut}}(\pi)$ injects into $\mathrm{GL}(W'(\pi))$ for every π. □

Remark 6.2. The first proof is clearly related to the constructions in §2 and §3. On the other hand, the second proof is a variant of the idea of "markings" used in §4.

In the second proof, if one restricts to *tame* covers (which form an open substack of $\mathsf{COV}^{d,g}$) one can simply take $Z \subset \mathbb{P}^1_{\mathbb{Z}}$ to be the image of a section (for instance ∞). The point is that a nontrivial automorphism of a tame cover cannot induce the identity on a (scheme-theoretic) fiber, even at a branch point.

6.2. Hurwitz spaces and stacks. Here, as in §1.4, we fix an integer $d > 0$, a finite subgroup $G \subset S_d$, and an integer $r \geq 3$. We consider the moduli spaces $\mathcal{H}_G^{\mathrm{ab}}$, $\mathcal{H}_G^{\mathrm{in}}$, $\mathcal{H}_G'^{\mathrm{ab}}$, $\mathcal{H}_G'^{\mathrm{in}}$ and the corresponding moduli stacks $\mathsf{H}_G^{\mathrm{ab}}$, $\mathsf{H}_G^{\mathrm{in}}$, $\mathsf{H}_G'^{\mathrm{ab}}$, $\mathsf{H}_G'^{\mathrm{in}}$.

In the statements below, the situation is any of these four situations. We denote by H (resp. $\mathcal{H}$) the corresponding stack (resp. moduli space). Note that the choice of H determines a genus (the genus of X, for any cover $X \to \mathbb{P}^1$ corresponding to a point of H), which we denote by g. We have an obvious "forgetful" morphism

$$q : \mathsf{H} \to \mathsf{COV}^{d,g}_{\mathbb{Q}} = \mathsf{COV}^{d,g} \times_{\mathrm{Spec}\,(\mathbb{Z})} \mathrm{Spec}\,(\mathbb{Q})$$

which is easily seen to be representable and quasi-finite. In particular, H is a Deligne-Mumford stack of finite type over $\mathbb{Q}$, and in fact a *linear quotient stack* (because $\mathsf{COV}^{d,g}$ is, and q is representable; alternatively one can mimic the proof of theorem 6.1). The automorphism group of any object of $\mathsf{H}(T)$, for a $\mathbb{Q}$-scheme T, is a finite étale T-group scheme, locally isomorphic to

$$C := \begin{cases} \text{center of } G & \text{(G-cover case)} \\ \text{centralizer of } G \text{ in } S_d & \text{(mere cover case).} \end{cases}$$

We have a canonical morphism $\alpha : \mathsf{H} \to \mathcal{H}$ which is the "Hurwitz gerbe" of [DeDoEm]. It is known that H is an étale gerbe over $\mathcal{H}$, locally bound by C.

The following result is a more precise form of Main Theorem B.

Theorem 6.3 — *There is a smooth quasi-projective $\mathbb{Q}$-scheme $\mathcal{V}$, an integer $n \geq 0$, a right action of $\mathrm{GL}_{n,\mathbb{Q}}$ on $\mathcal{V}$ with finite stabilizers, and a 1-morphism $\pi : \mathcal{V} \to \mathsf{H}$ (an object of the category $\mathsf{H}(\mathcal{V})$) with the following properties:*

(i) *π induces an isomorphism of the quotient stack $[\mathcal{V}/\mathrm{GL}_{n,\mathbb{Q}}]$ with H;*

(ii) *the composite map $\mathcal{V} \xrightarrow{\pi} \mathsf{H} \xrightarrow{\alpha} \mathcal{H}$ is smooth, with geometrically irreducible fibers, and identifies $\mathcal{H}$ with the quotient scheme $\mathcal{V}/\mathrm{GL}_{n,\mathbb{Q}}$;*

(iii) *for every field k of characteristic zero (more generally, for every semi-local ring with residue field of characteristic zero), every object of $\mathsf{H}(k)$ lifts to a point of $\mathcal{V}(k)$ via π;*

(iv) (local description of $\mathcal{V}$ over $\mathcal{H}$) *there is an étale surjective morphism $\rho : U \to \mathcal{H}$ such that $U \times_{\mathcal{H}} \mathcal{V}$ is U-isomorphic to $\Gamma \backslash \mathrm{GL}_{n,U}$ (with the natural projection on U and the natural right action of $\mathrm{GL}_{n,U}$), where Γ is a subgroup of $\mathrm{GL}_{n,U}$ isomorphic to the group C defined above.*

Remark 6.4. The 1-morphism π corresponds to the Hurwitz family $f : X \to \mathbb{P}^1_{\mathcal{V}}$ of Main Theorem B. Condition (iii) corresponds to condition (2). Conditions (ii) and (iv) are properties of the composite map $\mathcal{V} \xrightarrow{\pi} \mathsf{H} \xrightarrow{\alpha} \mathcal{H}$, which is the classifying moduli map of f.

Condition (i), which implies the others, is somewhat less simple. Obviously, any morphism $h : T \to \mathcal{V}$ determines by pullback a Hurwitz family over T; conversely, given a Hurwitz family over T, one may ask whether it can be obtained in this way (for instance, if T is a point the answer is yes, according to (iii)). Condition (i) says that for such a Hurwitz family, there is a GL_n-torsor over T whose sections correspond to "maps $h : T \to \mathcal{V}$, plus isomorphisms of the given family with the pullback of f by h". In particular, the obstruction to finding h lives in $H^1(T, \mathrm{GL}_n)$.

Proof. We have just seen that H is a linear quotient stack, so there is an algebraic space $\mathcal{V}$ with an action of $\mathrm{GL}_{n,\mathbb{Q}}$ for some n, and an isomorphism $[\mathcal{V}/\mathrm{GL}_{n,\mathbb{Q}}] \xrightarrow{\sim} \mathsf{H}$. Define π

to be the composite of this isomorphism with the canonical projection $\mathcal{V} \to [\mathcal{V}/\mathrm{GL}_{n,\mathbb{Q}}]$. Property (i) is now obvious, and so is (iii) because every GL_n-torsor over a field is trivial. Property (ii) clearly follows from (iv), which also implies that the map $\mathcal{V} \to \mathcal{H}$ is an affine morphism, hence $\mathcal{V}$ is quasi-projective since $\mathcal{H}$ is.

So there remains to prove (iv). But we know that $[\mathcal{V}/\mathrm{GL}_{n,\mathbb{Q}}]$ is a gerbe over $\mathcal{H}$, which is equivalent to the assertions that $\mathcal{V} \to \mathcal{H}$ is epimorphic and the action of $\mathrm{GL}_{n,\mathcal{H}}$ on $\mathcal{V}$ is transitive (relative to $\mathcal{H}$). So, locally for the étale topology on $\mathcal{H}$, $\mathcal{V}$ is isomorphic to a quotient $\Gamma\backslash\mathrm{GL}_{n,\mathcal{H}}$ for some subgroup scheme Γ of $\mathrm{GL}_{n,\mathcal{H}}$ (the stabilizer of a local section of $\mathcal{V}$). But then the quotient gerbe is locally isomorphic to the trivial gerbe $\mathrm{B}\Gamma$ of Γ-torsors, which implies that Γ is locally isomorphic to C. $\square$

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Pierre.Debes@univ-lille1.fr douai@gat.univ-lille1.fr
Univ. Lille 1, Mathématiques, 59655 Villeneuve d'Ascq Cedex, France.

laurent.moret-bailly@univ-rennes1.fr http://name.math.univ-rennes1.fr/laurent.moret-bailly/
IRMAR, Univ. Rennes 1, Campus de Beaulieu, 35042 Rennes Cedex, France.