Constrained tensor models

Applications to MIMO wireless communication systems

Gérard Favier
I3S Laboratory
Sophia Antipolis

GDR ISIS

16th January 2013

Doctorate thesis (2005-2014)

- Alain Kibangou (2005) University of Marrakech, Maroc
- Andre de Almeida (2007) UFC, Fortaleza, Brasil
- Estevao Fernandes (2008) UFC, Fortaleza, Brasil
- Alexandre Fernandes (2009) UFC, Fortaleza, Brasil
- Thomas Bouilloc (20011) DGA
- Tristan Porges (2012) THALES
- Michele da Costa (2013) UNICAMP, Campinas, Brasil
- Leandro Ximenes (2014) UFC, Fortaleza, Brasil

Research subjects

- Constrained tensor models (de Almeida, da Costa)
  - CONFAC (IEEE TSP 2008)
  - Generalized Paratuck (Elsevier SP 2012)
- System identification (Kibangou, E. Fernandes, Bouilloc)
  - HOS-based linear system identification (Elsevier SP 2008,2010)
  - NL system identification
    - Block structured NL systems (Wiener, Hammerstein, ...)
    - Volterra systems (IEEE SPL 2006, 2007, 2009; IJ-STA 2009; Elsevier SP 2009, 2010, 2011, 2012; TS 2010; IEEE JSTSP 2010; IJACSP 2012)
- SAR image processing (Porges)
  - HOSVD-based object recognition/classification (EuRAD'2010, ICASSP'2011)
- Wireless communications (de Almeida, A. Fernandes, Bouilloc, da Costa, Ximenes):3 book ch., 15 j. papers, >30 conf. papers

Research subjects

- Constrained tensor models (de Almeida, da Costa)
  - CONFAC (IEEE TSP 2008)
  - Generalized Paratuck (Elsevier SP 2012)
- System identification (Kibangou, E. Fernandes, Bouilloc)
  - HOS-based linear system identification (Elsevier SP 2008,2010)
  - NL system identification
    - Block structured NL systems (Wiener, Hammerstein, ...)
    - Volterra systems (IEEE SPL 2006, 2007, 2009; IJ-STA 2009; Elsevier SP 2009, 2010, 2011, 2012; TS 2010; IEEE JSTSP 2010; IJACSP 2012)
- SAR image processing (Porges)
  - HOSVD-based object recognition/classification (EuRAD'2010, ICASSP'2011)
- Wireless communications (de Almeida, A. Fernandes, Bouilloc, da Costa, Ximenes):3 book ch., 15 j. papers, >30 conf. papers

Research subjects

- Constrained tensor models (de Almeida, da Costa)
  - CONFAC (IEEE TSP 2008)
  - Generalized Paratuck (Elsevier SP 2012)
- System identification (Kibangou, E. Fernandes, Bouilloc)
  - HOS-based linear system identification (Elsevier SP 2008,2010)
  - NL system identification
    - Block structured NL systems (Wiener, Hammerstein, ...)
    - Volterra systems
      (IEEE SPL 2006, 2007, 2009; IJ-STA 2009; Elsevier SP 2009, 2010, 2011, 2012; TS 2010; IEEE JSTSP 2010; IJACSP 2012)
- SAR image processing (Porges)
  - HOSVD-based object recognition/classification (EuRAD'2010, ICASSP'2011)
- Wireless communications (de Almeida, A. Fernandes, Bouilloc, da Costa, Ximenes):3 book ch., 15 j. papers, >30 conf. papers

Research subjects

- Constrained tensor models (de Almeida, da Costa)
  - CONFAC (IEEE TSP 2008)
  - Generalized Paratuck (Elsevier SP 2012)
- System identification (Kibangou, E. Fernandes, Bouilloc)
  - HOS-based linear system identification (Elsevier SP 2008,2010)
  - NL system identification
    - Block structured NL systems (Wiener, Hammerstein, ...)
    - Volterra systems
      (IEEE SPL 2006, 2007, 2009; IJ-STA 2009; Elsevier SP 2009, 2010, 2011, 2012; TS 2010; IEEE JSTSP 2010; IJACSP 2012)
- SAR image processing (Porges)
  - HOSVD-based object recognition/classification (EuRAD'2010, ICASSP'2011)
- Wireless communications (de Almeida, A. Fernandes, Bouilloc, da Costa, Ximenes): 3 book ch., 15 j. papers, >30 conf. papers

Basic tensor operations

Vector/Matrix/Tensor unfoldings

Partition of the set $\{1,\ldots,N\}$ into N_1 subsets S_{n_1}, constituted of $p(n_1)$ elements with $\sum_{n_1=1}^{N_1} p(n_1) = N$. Each subset S_{n_1} is associated with a combined mode of dimension $J_{n_1} = \prod_{n \in S_{n_1}} I_n$.

$$\mathcal{X} \in \mathbb{C}^{I_1 \times \cdots \times I_N} \rightarrow \mathcal{Y} \in \mathbb{C}^{J_1 \times \cdots \times J_{N_1}}$$

$$\mathcal{Y} = \sum_{j_1=1}^{J_1} \cdots \sum_{j_{N_1}=1}^{J_{N_1}} x_{j_1,\cdots,j_{N_1}} \mathop{\circ}_{n_1=1}^{N_1} \mathbf{e}_{j_{n_1}}^{(J_{n_1})} \text{ with } \mathbf{e}_{j_{n_1}}^{(J_{n_1})} = \bigotimes_{n \in S_{n_1}} \mathbf{e}_{i_n}^{(I_n)} \qquad (1)$$

Two particular mode combinations correspond to vectorization and matricization operations

Basic tensor operations

Case of third-order tensors

For $\mathcal{X} \in \mathbb{C}^{I \times J \times K}$, there are two different forms of matricization, called flat and tall unfoldings of $\mathcal{X}$, respectively. For each form, there are six different matrix representations:

Flat unfoldings: $\mathbf{X}_{I \times JK}, \mathbf{X}_{I \times KJ}, \mathbf{X}_{J \times KI}, \mathbf{X}_{J \times IK}, \mathbf{X}_{K \times IJ}, \mathbf{X}_{K \times JI}$

Tall unfoldings: $\mathbf{X}_{JK \times I}, \mathbf{X}_{KJ \times I}, \mathbf{X}_{KI \times J}, \mathbf{X}_{IK \times J}, \mathbf{X}_{IJ \times K}, \mathbf{X}_{JI \times K}$

$$\mathbf{X}_{I \times JK} = \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} x_{i,j,k} \, \mathbf{e}_i^{(I)} (\mathbf{e}_j^{(J)} \otimes \mathbf{e}_k^{(K)})^T \in \mathbb{C}^{I \times JK} \quad (2)$$

$$\mathbf{X}_{JK \times I} = \mathbf{X}_{I \times JK}^T \quad (3)$$

Basic tensor operations

Mode-n product of a tensor with a matrix

The mode-n product of $\mathcal{X} \in \mathbb{C}^{I_1 \times \cdots \times I_N}$ with $\mathbf{A} \in \mathbb{C}^{J_n \times I_n}$ along the n^{th} mode gives the tensor $\mathcal{Y}$ of order N and dimensions $I_1 \times \cdots \times I_{n-1} \times J_n \times I_{n+1} \times \cdots \times I_N$, such as

$$\mathcal{Y} = \mathcal{X} \times_n \mathbf{A} \tag{4}$$

$$\Updownarrow \tag{5}$$

$$y_{i_1,\cdots,i_{n-1},j_n,i_{n+1},\cdots,i_N} = \sum_{i_n=1}^{I_n} a_{j_n,i_n} x_{i_1,\cdots,i_{n-1},i_n,i_{n+1},\cdots,i_N} \tag{6}$$

or in terms of mode-n matrix unfoldings of $\mathcal{X}$ and $\mathcal{Y}$

$$\mathbf{Y}_n = \mathbf{A}\mathbf{X}_n \tag{7}$$

Interpretation as the linear transformation from the mode-n space of $\mathcal{X}$ to the mode-n space of $\mathcal{Y}$, associated to the matrix $\mathbf{A}$

Basic tensor operations

Mode-n product of a tensor with a matrix

Properties

$$\mathcal{X} \in \mathbb{C}^{I_1 \times \cdots \times I_N}$$

- For any permutation $\pi(.)$ of P distinct indices $m_p \in \{1, \cdots, N\}$ such as $q_p = \pi(m_p)$, $p \in \{1, \ldots, P\}$, with $P \leq N$, we have

$$\mathcal{X} \times_{q=q_1}^{q_P} \mathbf{A}^{(q)} = \mathcal{X} \times_{m=m_1}^{m_P} \mathbf{A}^{(m)} \tag{8}$$

  $\Rightarrow$ the order of the mode-m_p products is irrelevant when the indices m_p are all distinct.

- For two products along the same mode-n, with $\mathbf{A} \in \mathbb{C}^{J_n \times I_n}$ and $\mathbf{B} \in \mathbb{C}^{K_n \times J_n}$, we have

$$\mathcal{Y} = \mathcal{X} \times_n \mathbf{A} \times_n \mathbf{B} = \mathcal{X} \times_n (\mathbf{BA}) \tag{9}$$

$$\in \mathbb{C}^{I_1 \times \cdots \times I_{n-1} \times K_n \times I_{n+1} \times \cdots \times I_N} \tag{10}$$

Tucker models

Tucker equation

For a N^{th}-order tensor $\mathcal{X} \in \mathbb{C}^{I_1 \times \cdots \times I_N}$:

$$x_{i_1,\cdots,i_N} = \sum_{r_1=1}^{R_1} \cdots \sum_{r_N=1}^{R_N} g_{r_1,\cdots,r_N} \prod_{n=1}^{N} a_{i_n,r_n}^{(n)} \tag{11}$$

with $i_n = 1, \cdots, I_n$ for $n = 1, \cdots, N$, where $g_{r_1,\cdots,r_N}$ is an element of the core/input tensor $\mathcal{G} \in \mathbb{C}^{R_1 \times \cdots \times R_N}$ and $a_{i_n,r_n}^{(n)}$ is an element of the matrix factor $\mathbf{A}^{(n)} \in \mathbb{C}^{I_n \times R_n}$

Tucker models

Other writings

In terms of vector outer products:

$$\mathcal{X} = \sum_{r_1=1}^{R_1} \cdots \sum_{r_N=1}^{R_N} g_{r_1,\cdots,r_N} \overset{N}{\underset{n=1}{\circ}} \mathbf{A}^{(n)}_{.r_n} \qquad (12)$$

$\Rightarrow \mathcal{X}$ is decomposed into a weighted sum of $\prod_{n=1}^{N} R_n$ outer products.

In terms of mode-n products:

$$\begin{aligned} \mathcal{X} &= \mathcal{G} \times_1 \mathbf{A}^{(1)} \times_2 \mathbf{A}^{(2)} \times_3 \cdots \times_N \mathbf{A}^{(N)} \\ &= \mathcal{G} \times_{n=1}^{N} \mathbf{A}^{(n)} \end{aligned} \qquad (13)$$

$\Rightarrow$ The Tucker model can be interpreted as mode-n product-based transformations of the core tensor, i.e. linear transformations defined by the matrices $\mathbf{A}^{(n)}$ applied to each mode-n vector space of $\mathcal{G}$.

Tucker models

Case of third-order tensors (Tucker, 1966)

- Tucker-3 models
  Third-order tensor $\mathcal{X} \in \mathbb{C}^{I \times J \times K}$: core tensor $\mathcal{G} \in \mathbb{C}^{P \times Q \times S}$ and matrix factors $\mathbf{A} \in \mathbb{C}^{I \times P}, \mathbf{B} \in \mathbb{C}^{J \times Q}, \mathbf{C} \in \mathbb{C}^{K \times S}$

$$x_{ijk} = \sum_{p=1}^{P} \sum_{q=1}^{Q} \sum_{s=1}^{S} g_{pqs} a_{ip} b_{jq} c_{ks} \quad (14)$$

$$\mathcal{X} = \mathcal{G} \times_1 \mathbf{A} \times_2 \mathbf{B} \times_3 \mathbf{C} \quad (15)$$

$$\mathbf{X}_{IJ \times K} = (\mathbf{A} \otimes \mathbf{B}) \mathbf{G}_{PQ \times S} \mathbf{C}^T \quad (16)$$

$$\mathrm{vec}(\mathcal{X}) = (\mathbf{C} \otimes \mathbf{A} \otimes \mathbf{B}) \mathrm{vec}(\mathcal{G}) \quad (17)$$

Tucker models

Uniqueness issue

The Tucker model is not unique. Its matrix factors can be determined only up to invertible transformations, i.e. nonsingular matrices.

If the matrix factors $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ and the core tensor $\mathcal{G}$ are replaced by $(\mathbf{AT}_a, \mathbf{BT}_b, \mathbf{CT}_c)$ and $\mathcal{G} \times_1 \mathbf{T}_a^{-1} \times_2 \mathbf{T}_b^{-1} \times_3 \mathbf{T}_c^{-1}$, respectively, use of Property P2 of mode-n product gives:

$$\mathcal{G} \times_1 \mathbf{T}_a^{-1} \times_2 \mathbf{T}_b^{-1} \times_3 \mathbf{T}_c^{-1} \times_1 \mathbf{AT}_a \times_2 \mathbf{BT}_b \times_3 \mathbf{CT}_c$$

$$\begin{aligned} &= \mathcal{G} \times_1 \left(\mathbf{AT}_a \mathbf{T}_a^{-1}\right) \times_2 \left(\mathbf{BT}_b \mathbf{T}_b^{-1}\right) \times_3 \left(\mathbf{CT}_c \mathbf{T}_c^{-1}\right) \\ &= \mathcal{G} \times_1 \mathbf{A} \times_2 \mathbf{B} \times_3 \mathbf{C} \\ &= \mathcal{X} \end{aligned}$$

Constrained tensor models

CONFAC models (de Almeida, Favier, Mota, IEEE TSP'2008)

Tucker model: $\mathcal{X} = \mathcal{G} \times_1 \mathbf{A} \times_2 \mathbf{B} \times_3 \mathbf{C}$
with

$$\mathcal{G} = \mathcal{I} \times_1 \mathbf{\Psi} \times_2 \mathbf{\Phi} \times_3 \mathbf{\Omega} \tag{18}$$

$$\Downarrow$$

$$\mathcal{X} = \mathcal{I} \times_1 (\mathbf{A\Psi}) \times_2 (\mathbf{B\Phi}) \times_3 (\mathbf{C\Omega}) \tag{19}$$

$$\Downarrow$$

Constrained PARAFAC model (PARAFAC with Constrained Factors)

Constraint matrices $\mathbf{\Psi} \in \mathbb{R}^{P \times R}$, $\mathbf{\Phi} \in \mathbb{R}^{Q \times R}$ and $\mathbf{\Omega} \in \mathbb{R}^{S \times R}$ whose columns are chosen as canonical vectors of the Euclidean spaces $\mathbb{R}^P$, $\mathbb{R}^Q$ and $\mathbb{R}^S$, respectively, with $R \geq max(P, Q, S)$.

Constrained tensor models

CONFAC models

- In a telecommunications context, such constraint matrices $(\mathbf{\Psi} \in \mathbb{R}^{P\times R}, \mathbf{\Phi} \in \mathbb{R}^{Q\times R}, \mathbf{\Omega} \in \mathbb{R}^{S\times R})$ can be interpreted as allocation matrices allowing to allocate (P, Q, S) resources, like data streams, codes, and transmit antennas, to the R components that form the signal to be transmitted.

- In this case, the core tensor $\mathcal{G}$ is called an allocation tensor.

- By assumption, each column of an allocation matrix is a canonical vector, which means that there is only one value of p, q, and s such that $\psi_{p,r}\phi_{q,r}\omega_{s,r} = 1$, and these values of p, q, and s correspond to the resources allocated to the r^{th} component of $\mathcal{X}$.

Constrained tensor models

CONFAC-3 models

$$x_{i,j,k} = \sum_{r=1}^{R} \sum_{p=1}^{P} \sum_{q=1}^{Q} \sum_{s=1}^{S} (\psi_{p,r}\phi_{q,r}\omega_{s,r}) a_{i,p} b_{j,q} c_{k,s}$$

- Each element $x_{i,j,k}$ of the received signal tensor $\mathcal{X}$ is equal to the sum of R components, each component r resulting from the combination of three resources, each resource being associated with a column of each matrix factor $(\mathbf{A}, \mathbf{B}, \mathbf{C})$. This combination, determined by the allocation tensor, is defined by a set of three indices p, q, s such that $\psi_{p,r}\phi_{q,r}\omega_{s,r} = 1$.

- $R \geq max(P, Q, S) \Rightarrow$ Each resource can be allocated several times.
- Extension to order N: $R \geq max(R_1, ..., R_N)$.

Constrained tensor models

PARATUCK models

PARATUCK-2 model (Harshman, Lundy; 1996)

$$x_{i,j,k} = \sum_{p=1}^{P}\sum_{q=1}^{Q} w_{p,q} a_{i,p} b_{j,q} \psi_{p,k} \phi_{q,k}$$

$$\Downarrow$$

$$\mathbf{X}_{..k} = \mathbf{A}\mathbf{D}_k(\mathbf{\Psi})\mathbf{W}\mathbf{D}_k(\mathbf{\Phi})\mathbf{B}^T = \underbrace{\mathbf{A}\mathbf{G}_{..k}\mathbf{B}^T}_{\textbf{Tucker-2}}$$

$$\mathbf{G}_{..k} = \mathbf{D}_k(\mathbf{\Psi})\mathbf{W}\mathbf{D}_k(\mathbf{\Phi}) = \mathbf{\Psi}_{.k}\mathbf{\Phi}_{.k}^T \odot \mathbf{W} = \underbrace{(\mathbf{\Psi}\mathbf{D}_k(\mathbf{I}_K)\mathbf{\Phi}^T)}_{\textbf{PARAFAC}(\mathbf{\Psi},\mathbf{\Phi},\mathbf{I}_K)} \odot \mathbf{W}$$

Constrained tensor models

PARATUCK-2 models

$$x_{i,j,k} = \sum_{p=1}^{P} \sum_{q=1}^{Q} w_{p,q} a_{i,p} b_{j,q} \psi_{p,k} \phi_{q,k}$$

Two interpretations of $\mathbf{\Psi}$ and $\mathbf{\Phi}$: Interaction or allocation matrices:

- Interaction between the columns p and q of the matrix factors **A** and **B** along the mode-k of $\mathcal{X}$, with the weights $w_{p,q}\psi_{p,k}\phi_{q,k}$.

- Allocation of resources p and q to the mode-k of $\mathcal{X}$, with the weight (code) $w_{p,q}$.

Constrained tensor models

Generalized PARATUCK models

PARATUCK-(2,4) model (Favier et al., EUSIPCO'2011)

$$x_{i,j,k,l} = \sum_{p=1}^{P} \sum_{q=1}^{Q} w_{p,q,l} a_{i,p} b_{j,q} \psi_{p,k} \phi_{q,k}$$

PARATUCK-(N_1, N) (Favier et al., SP 2012).
$\mathcal{X} \in \mathbb{C}^{I_1 \times \cdots \times I_N}$, with $N > N_1$

$$x_{i_1,\cdots,i_{N_1+1},\cdots,i_N} = \sum_{r_1=1}^{R_1} \cdots \sum_{r_{N_1}=1}^{R_{N_1}} c_{r_1,\cdots,r_{N_1},i_{N_1+2},\cdots,i_N} \prod_{n=1}^{N_1} a^{(n)}_{i_n,r_n} \phi^{(n)}_{r_n,i_{N_1+1}}$$

$a^{(n)}_{i_n,r_n}$, and $\phi^{(n)}_{r_n,i_{N_1+1}}$ are entries of the factor matrix $\mathbf{A}^{(n)} \in \mathbb{C}^{I_n \times R_n}$ and the allocation matrix $\mathbf{\Phi}^{(n)} \in \mathbb{C}^{R_n \times I_{N_1+1}}$, $\forall n = 1, \cdots, N_1$, respectively.

Constrained tensor models

Generalized PARATUCK models (Favier, de Almeida; 2013)

$$x_{i_1,\cdots,i_N} = \sum_{r_1=1}^{R_1} \cdots \sum_{r_{N_1}=1}^{R_{N_1}} w_{r_1,\cdots,r_{N_1},S} \prod_{n=1}^{N_1} a^{(n)}_{i_n,r_n,S_n} c_{r_1,\cdots,r_{N_1},T}$$

$\{r_1, \cdots, r_{N_1}\}$: input (or resource) modes,
$\{i_1, \cdots, i_N\}$: output (or diversity) modes,

S, T, and $S_n \subseteq S \bigcup T$ (for $n = 1, \cdots, N_1$): subsets of $\{i_{N_1+1}, \cdots, i_N\}$,

$a^{(n)}_{i_n,r_n,S_n}$, $c_{r_1,\cdots,r_{N_1},T}$ (equal to 0 or 1), and $w_{r_1,\cdots,r_{N_1},S}$: entries of the tensor factor $\mathcal{A}^{(n)}$, of the allocation tensor $\mathcal{C}$, and of the input tensor $\mathcal{W}$.

Tensor modeling of MIMO communication systems

Motivations

Future wireless communication systems

- Aim: Best tradeoff between error performance (SER or BER), transmission rate (in symbols or bits per channel use), energy consumption, and receiver complexity for symbol recovery.

- Performance improvement by jointly exploiting several diversities.

  $\Updownarrow$

  To exploit redundancy into the information-bearing signals at the receiver.

- Blind joint channel/symbol estimation.

Tensor modeling of MIMO communication systems

Motivations

Redundancy can be provided by:

- Channels:
  Frequency-selective / Time-selective channels ⇔ Multipath / Doppler diversities
- Spreading/Coding operations at the transmitter in space, time and/or frequency domains

Tensor modeling of MIMO communication systems

Space/Time/Frequency spreadings

- **Space/Time/frequency diversities** by:
  - Transmitting the same symbols (or data streams) using several Tx antennas, and using several Rx antennas at the receiver.

  $\Rightarrow$ **MIMO systems**

  - Repeating the same symbols during several chip periods or/and multiple time blocks or/and over several subcarriers.

  $\Rightarrow$ **Reliability improvement**

- **Space multiplexing** by transmitting independent data streams in parallel on multiple-transmit antennas.

  $\Rightarrow$ **Transmission rate increase**

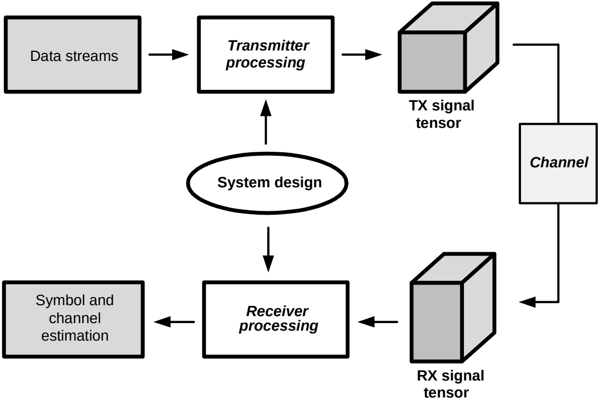
Tensor modeling of MIMO communication systems

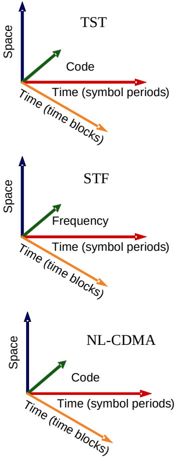
Motivations

- Multidimensional data $\Rightarrow$ Third- to fifth-order tensors for transmitted and received signals
- Structure of tensor model results from system design
- Structure parameters (rank, mode dimensions) are design parameters (code lengths, nb of Tx/Rx antennas, data streams, subcarriers, time slots, ...)
- Uniqueness properties of tensor models
- Possibility of tensor coding
- Possibility of resource allocation
- Blind/semi-blind receivers (channel/symbol/code joint estimation)
- Deterministic approach

Tensor modeling of MIMO communication systems

Block-diagram of tensor-based systems





Tensor modeling of MIMO communication systems

Two basic tensor-based systems

PARAFAC-CDMA (code division multiple access) system

(Sidiropoulos, Giannakis, Bro, IEEE TSP 2000)

M users, K Rx antennas, N symbol periods, J chips (spreading length)

n-th coded/spread symbol of user m

$$u_{m,n,j} = s_{n,m} w_{j,m}$$

Signal received by antenna k

$$x_{k,n,j} = \sum_{m=1}^{M} h_{k,m} u_{m,n,j} = \sum_{m=1}^{M} h_{k,m} s_{n,m} w_{j,m} \Rightarrow \mathcal{X} \in \mathbb{C}^{K \times N \times J}$$

$$\Downarrow$$

Three different diversities: space (K), time (N), code (J).

Tensor modeling of MIMO communication systems

PARAFAC-CDMA system

Assumptions

- Flat Rayleigh fading channel time-invariant over N symbol periods.
- K, M, N, J are known.
- Symbol-level synchronization.

Advantages

- Essential uniqueness (up to column permutation and scaling).
- Possibility of more users than Rx antennas (underdetermined systems).
- Deterministic approach (ALS) without space/time statistical independence constraint.
- Blind receiver.
- Joint channel/symbol/code estimation.

Tensor modeling of MIMO communication systems

PARAFAC-CDMA system

Kruskal's condition

Under the assumptions **H**, **S**, and **C** full k-rank

$$min(K, M) + min(N, M) + min(J, M) \geq 2M + 2$$

$$\Downarrow$$

- If N and $J \geq M$, then $K \geq 2$ antennas are sufficient for M users.
- If K and $J \geq M$, then $N \geq 2$ symbol periods are sufficient.

Tensor modeling of MIMO communication systems

PARAFAC-KRST coding system

PARAFAC-KRST coding system
(Sidiropoulos, Budampati, IEEE TSP 2002)
T time blocks of J time slots

KRST coding

- $\mathbf{s}_t \in \mathbb{C}^{M \times 1}$: symbol vector transmitted during block t.
- $\mathbf{v}_t = \mathbf{W}\mathbf{s}_t \in \mathbb{C}^{M \times 1}$: precoding (space spreading), with $\mathbf{W} \in \mathbb{C}^{M \times M}$.
- $D(\mathbf{W}\mathbf{s}_t) \in \mathbb{C}^{M \times M}$: diagonal matrix containing information-bearing signals to be transmitted by each Tx antenna.
- $\mathbf{U}_t = D(\mathbf{W}\mathbf{s}_t)\mathbf{C}^T \in \mathbb{C}^{M \times J}$: postcoding (time spreading onto J time slots) for each block t, with $\mathbf{C} \in \mathbb{C}^{J \times M}$.

Tensor modeling of MIMO communication systems

PARAFAC-KRST coding system

Reception

$\mathbf{X}_t = \mathbf{H}\mathbf{U}_t \in \mathbb{C}^{K \times J}$: signals received during each time block t.

$$\Downarrow$$

Tensor of received signals: $\mathcal{X} \in \mathbb{C}^{K \times J \times T}$

$$\mathbf{s}_t = [s_{t,1}, \cdots, s_{t,M}]^T, \quad \mathbf{v}_t = [v_{t,1}, \cdots, v_{t,M}]^T$$

$$\mathbf{v}_t = \mathbf{W}\mathbf{s}_t \quad \Leftrightarrow \quad v_{t,m} = \sum_{l=1}^{M} s_{t,l} w_{m,l}$$

$$\mathbf{U}_t = D(\mathbf{W}\mathbf{s}_t)\mathbf{C}^T \quad \Leftrightarrow \quad u_{m,j,t} = v_{t,m} c_{j,m}$$

$$x_{k,j,t} = \sum_{m=1}^{R} h_{k,m} u_{m,j,t} \quad = \quad \sum_{m=1}^{M} h_{k,m} v_{t,m} c_{j,m}$$

$\Rightarrow$ PARAFAC model with matrix factors $(\mathbf{H}, \mathbf{V}, \mathbf{C})$

Tensor modeling of MIMO communication systems

PARAFAC-KRST coding system

Advantages

- ST coding $\Rightarrow$ Two diversities: space (K), time (J).
- $\mathcal{X} \in \mathbb{C}^{K \times J \times T} \Rightarrow$ Three diversities: space (K), time (J and T).
- Transmission rate: $\frac{M}{J} log_2(\mu)$ bits/channel use, where μ is the constellation cardinality.

$\Rightarrow$ **Performance/Rate tradeoff**

- Blind joint channel/symbol estimation $\Rightarrow$ No training is needed for acquiring CSI (channel state information).

Drawback

Decoding is needed (estimation of $\mathbf{s}_t$ from estimated $\mathbf{v}_t = \mathbf{W}\mathbf{s}_t$

Tensor modeling of MIMO communication systems

PARATUCK-TST coding system (Favier, da Costa, de Almeida, J. Romano, SP 2012)

R data streams of N symbols

$$\underset{\substack{\text{Data streams}\\ \in\mathbb{C}^{N\times R}}}{\mathbf{S}} \overset{\text{Coding + Allocation}}{\longrightarrow} \underset{\substack{\text{Tx signal tensor}\\ \in\mathbb{C}^{M\times N\times P\times J}}}{\mathcal{U}} \overset{\text{Channel}}{\longrightarrow} \underset{\substack{\text{Rx signal tensor}\\ \in\mathbb{C}^{K\times N\times P\times J}}}{\mathcal{X} = \mathcal{U} \times_1 \mathbf{H}}$$

Tensor modeling of MIMO communication systems

PARATUCK-TST coding system

- MIMO communication system with M transmit antennas and K receive antennas.
- Transmission of R data streams composed of N symbols each.
- Transmission decomposed into P data blocks formed of N time slots each.

Tensor of transmitted signals

Coded signal transmitted from the transmit antenna m, during the time slot n of block p, and associated with the chip j:

$$u_{m,n,p,j} = \sum_{r=1}^{R} \underbrace{w_{m,r,j}}_{\textbf{code}} \quad \underbrace{s_{n,r}}_{\textbf{symbol}} \quad \underbrace{\phi_{p,m}\,\psi_{p,r}}_{\textbf{allocations}}$$

$$\mathcal{W} \in \mathbb{C}^{M\times R\times J} \quad \mathbf{S} \in \mathbb{C}^{N\times R} \quad \mathbf{\Phi} \in \mathbb{R}^{P\times M}, \mathbf{\Psi} \in \mathbb{R}^{P\times R}$$

$$\begin{cases} s_{n,r} = n^{\text{th}} \text{ symbol of } r^{\text{th}} \text{ data stream.} \\ \psi_{p,r} = 1 \Leftrightarrow \text{ data stream } r \text{ allocated to block } p. \\ \phi_{p,m} = 1 \Leftrightarrow \text{ transmit antenna } m \text{ allocated to block } p. \end{cases}$$

Tensor modeling of MIMO communication systems

PARATUCK-TST coding system

Tensor of received signals

- **Flat Rayleigh fading** propagation channel $\mathbf{H} \in \mathbb{C}^{K \times M}$ with i.i.d. CN(0,1) entries.
- Channel assumed to be **constant during at least** P **blocks**.

$$x_{k,n,p,j} = \sum_{m=1}^{M} h_{k,m}\, u_{m,n,p,j} \;\Leftrightarrow\; \mathcal{X} = \mathcal{U} \times_1 \mathbf{H}$$
$$= \sum_{m=1}^{M} \sum_{r=1}^{R} w_{m,r,j}\, h_{k,m}\, s_{n,r}\, \phi_{p,m}\, \psi_{p,r}$$

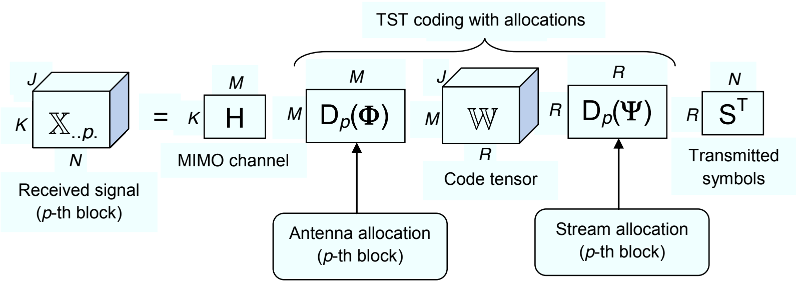
$\Rightarrow \mathcal{X} \in \mathbb{C}^{K \times N \times P \times J}$ satisfies a **PARATUCK-(2,4) model**

Tensor modeling of MIMO communication systems

Visualization of the tensor slice $\mathcal{X}_{..p.}$

Matrix slice of the received signal tensor obtained by slicing it along the plane (p, j), i.e. by fixing the two last indices:

$$\mathbf{X}_{..p,j} = \mathbf{H}\,\mathbf{G}_{..p,j}\,\mathbf{S}^T \;\text{ with }\; \mathbf{G}_{..p,j} = D_p(\mathbf{\Phi})\,\mathbf{W}_{..j}\,D_p(\mathbf{\Psi})$$



Tensor modeling of MIMO communication systems

PARATUCK-TST coding system

Blind joint symbol and channel estimation

Matrix representations of the received signal tensor

$$\mathbf{X}_2 = \begin{bmatrix} \mathbf{X}_{\cdot\cdot 1,1} \\ \vdots \\ \mathbf{X}_{\cdot\cdot P,1} \\ \vdots \\ \mathbf{X}_{\cdot\cdot 1,J} \\ \vdots \\ \mathbf{X}_{\cdot\cdot P,J} \end{bmatrix} \in \mathcal{C}^{JPK \times N}, \quad \mathbf{G}_2 = \begin{bmatrix} \mathbf{G}_{\cdot\cdot 1,1} \\ \vdots \\ \mathbf{G}_{\cdot\cdot P,1} \\ \vdots \\ \mathbf{G}_{\cdot\cdot 1,J} \\ \vdots \\ \mathbf{G}_{\cdot\cdot P,J} \end{bmatrix} \in \mathbb{C}^{JPM \times R}, \quad \mathbf{X}_3 = \begin{bmatrix} \mathbf{X}^T_{\cdot\cdot 1,1} \\ \vdots \\ \mathbf{X}^T_{\cdot\cdot P,1} \\ \vdots \\ \mathbf{X}^T_{\cdot\cdot 1,J} \\ \vdots \\ \mathbf{X}^T_{\cdot\cdot P,J} \end{bmatrix} \in \mathcal{C}^{JPN \times K}, \quad \mathbf{G}_3 = \begin{bmatrix} \mathbf{G}^T_{\cdot\cdot 1,1} \\ \vdots \\ \mathbf{G}^T_{\cdot\cdot P,1} \\ \vdots \\ \mathbf{G}^T_{\cdot\cdot 1,J} \\ \vdots \\ \mathbf{G}^T_{\cdot\cdot P,J} \end{bmatrix} \in \mathbb{C}^{JPR \times M}$$

$$\mathbf{X}_2 = (\mathbf{I}_{JP} \otimes \mathbf{H})\,\mathbf{G}_2\mathbf{S}^T \quad \Rightarrow \quad \mathbf{S}^T = [(\mathbf{I}_{JP} \otimes \mathbf{H})\,\mathbf{G}_2]^{\dagger}\mathbf{X}_2 \tag{20}$$

$$\mathbf{X}_3 = (\mathbf{I}_{JP} \otimes \mathbf{S})\,\mathbf{G}_3\mathbf{H}^T \quad \Rightarrow \quad \mathbf{H}^T = [(\mathbf{I}_{JP} \otimes \mathbf{S})\,\mathbf{G}_3]^{\dagger}\mathbf{X}_3 \tag{21}$$

S and **H** are estimated **by alternately solving** (20)-(21) **in the LS sense** w.r.t. one matrix conditionally to the knowledge of previously estimated value of the other matrix.

Tensor modeling of MIMO communication systems

PARATUCK-TST coding system

Identifiability

If **S** and **H** are **full column-rank**, the uniqueness of their conditional LS estimates requires that $\mathbf{G}_2 \in \mathcal{C}^{JPM \times R}$ and $\mathbf{G}_3 \in \mathcal{C}^{JPR \times M}$ be also **full column-rank**.

Theorem (Necessary condition)
Assuming that **S** and **H** are **full column-rank**, a necessary condition for LS identifiability is given by:

$$PJ \geq \max\left(\left\lceil \frac{R}{M}, \frac{M}{R} \right\rceil\right). \quad (22)$$

where $\lceil x \rceil$ denotes the smallest integer number greater than or equal to x.

Tensor modeling of MIMO communication systems

Identifiability

Code tensor: third-order Vandermonde tensor

$$w_{m,r,j} = e^{i2\pi mrj/MRJ}, \quad i^2 = -1.$$

Theorem (Sufficient condition)

Assuming that $\mathbf{S}$ and $\mathbf{H}$ are full column-rank, and $(\mathbf{\Phi}_{p\cdot} = \mathbf{1}_M^T, \mathbf{\Psi}_{p\cdot} = \mathbf{1}_R^T)$ for a given $p \in \{1, ..., P\}$, the minimum value of the spreading length J ensuring the LS identifiability of $\mathbf{S}$ and $\mathbf{H}$ is given by:

M \ R	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	1	2	3	3	4	5	5
3	3	2	1	2	2	3	4	4
4	4	3	2	1	2	2	3	3
5	5	3	2	2	1	2	2	2
6	6	4	3	2	2	1	2	2
7	7	5	4	3	2	2	1	2
8	8	5	4	3	2	2	2	1

Tensor modeling of MIMO communication systems

Uniqueness

Uniqueness

Theorem
Assuming a **Vandermonde structure** for the code tensor, if $\boldsymbol{\Psi}^T \diamond \boldsymbol{\Phi}^T$ is full row-rank, which implies $P \geq RM$, then **S** and **H** are unique up to a scalar factor, i.e.

$$\mathbf{S} = \alpha \hat{\mathbf{S}}, \quad \mathbf{H} = \frac{1}{\alpha} \hat{\mathbf{H}}. \tag{23}$$

and the scaling ambiguity factor α can be eliminated by simply transmitting a known symbol $s_{1,1}$: $\alpha = s_{1,1}/\hat{s}_{1,1}(\infty)$, where $\hat{s}_{1,1}(\infty)$ is the estimated value of $s_{1,1}$ at convergence.

Tensor modeling of MIMO communication systems

PARATUCK-TST coding system

Advantages

- Tensor coding and resource allocation (Tx antennas and data streams to time blocks).
- Three diversities are exploited: space (K), time (J and P) $\Rightarrow$ Improved performance w.r.t. KRST coding.
- Transmission rate: $\frac{R}{P} log_2(\mu)$ bits/channel use, where μ is the constellation cardinality, independent of M (unlike KRST-coding)

$\Rightarrow$ **Performance/Rate tradeoff**

- Blind joint channel/symbol estimation $\Rightarrow$ No training is needed for acquiring CSI (channel state information).

PARATUCK-TST coding system

Simulation results

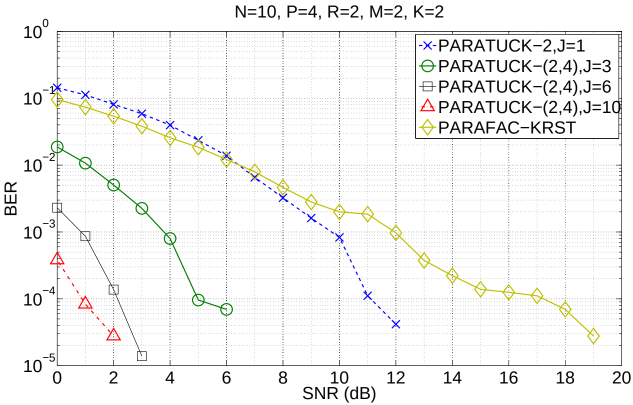


Figure : Influence of the spreading length: BER versus SNR.

PARATUCK-TST coding system

Simulation results

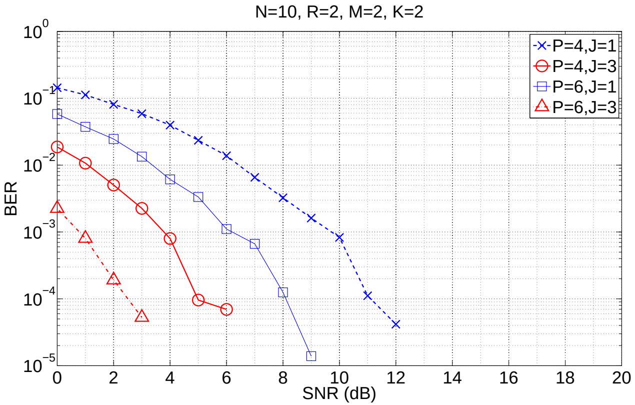


Figure : Influence of the block number: BER versus SNR.

PARATUCK-TST coding system

Simulation results

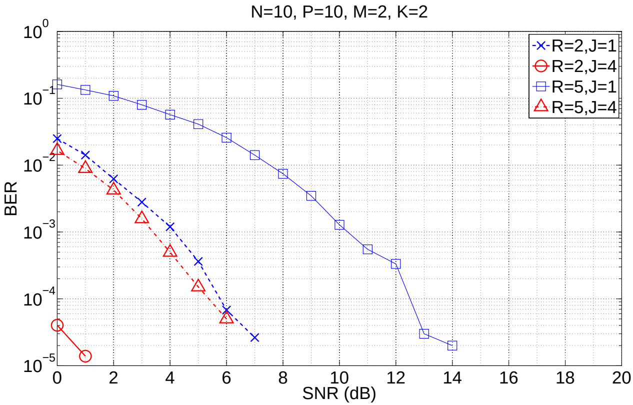


Figure : Influence of the data stream number: BER versus SNR.

PARATUCK-TST coding system

Simulation results

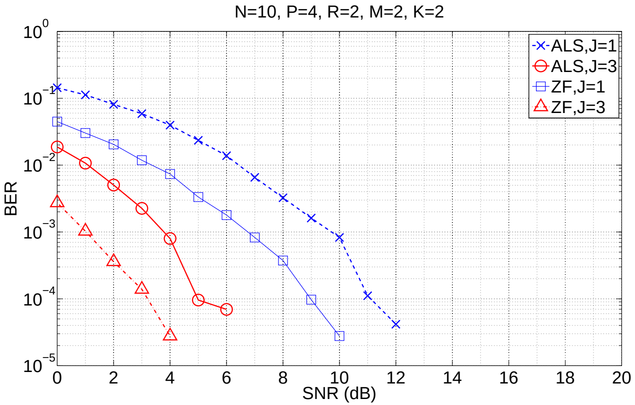


Figure : TST-ALS vs. TST-ZF.

Tensor modeling of MIMO communication systems

Comparison of tensor-based systems

Systems	Core tensors	Received signals	τ
TST PARATUCK-(2,4) SP 2012	$g_{m,r,p,j} = w_{m,r,j}\phi_{p,m}\psi_{p,r}$	$x_{k,n,p,j} = \sum_{m=1}^{M}\sum_{r=1}^{R} g_{m,r,p,j} h_{k,m} s_{n,r}$	$\frac{R}{P}$
STF Gen. PARATUCK-(2,4) IEEE TSP 2013	$g_{m,r,f,p} = w_{m,r} c^{(\mathcal{H})}_{f,p,m} c^{(\mathbf{S})}_{f,p,r}$	$x_{k,n,f,p} = \sum_{m=1}^{M}\sum_{r=1}^{R} g_{m,r,f,p} h_{k,m,f} s_{n,r}$	$\frac{R}{PF}$
ST PARATUCK-2 SP 2009	$g_{m,r,p} = w_{m,r}\phi_{p,m}\psi_{p,r}$	$x_{k,n,p} = \sum_{m=1}^{M}\sum_{r=1}^{R} g_{m,r,p} h_{k,m} s_{n,r}$	$\frac{R}{P}$
STF PARAFAC SPAWC 2007	$g_{m,r,f,j} = w_{m,r,f,j}$	$x_{k,n,f,j} = \sum_{m=1}^{M}\sum_{r=1}^{R} g_{m,r,f,j} h_{k,m,f} s_{n,r}$	$\frac{R}{F}$
ST Tucker-(2,3) SPAWC 2006	$g_{m,r,p} = w_{m,r,p}$	$x_{k,n,p} = \sum_{m=1}^{M}\sum_{r=1}^{R} g_{m,r,p} h_{k,m} s_{n,r}$	$\frac{R}{P}$

The transmission rate (in bits per channel use) for each system is equal to $\tau \log_2(\mu)$ where μ is the cardinality of the information symbol constellation.

Tensor modeling of MIMO communication systems

Two other tensor-based systems

NL-CDMA systems

Blind constrained block-Tucker2 receiver for multiuser SIMO NL-CDMA communication systems. Signal Processing 92, 1624-1636, July 2012.

Assumptions

- Multiuser.
- Nonlinear coding with column-orthonormal and mutually orthogonal code matrices.
- FIR channels.

Tensor modeling of MIMO communication systems

NL-CDMA systems

System properties

- Block-Tucker2 model for the fourth-order tensor of received signals, with constrained core tensor (matrix slices having a Vandermonde or an Hankel structure) implying uniqueness of the model.
- Simultaneous users separation and decoding.
- Blind joint channel estimation and symbol recovery for each user using an ALS type algorithm that takes the constrained structure of the core tensor into account.

Tensor modeling of MIMO communication systems

Two other tensor-based systems

Cooperative systems

PARAFAC-PARATUCK based blind receivers for dual-hop cooperative MIMO relay systems. To be published.

Relay-based MIMO systems with Amplify-and-Forward (AF) protocol, for augmenting signal power and diversity at reception, with KRST coding at the transmission that allows blind joint channel/symbol estimation using the direct link (PARAFAC model) and the relay-assisted link (PARATUCK model).

Conclusion

Advantages of tensor models

Tensor models are very useful for:

- Representing and analysing multidimensional signals
- Modeling and designing MIMO communication systems
- Joint semi-blind estimation of symbols and channels

Future works

- Uniqueness of constrained tensor models
- Estimation algorithms for structured/constrained tensor models
- Optimization/Estimation of resource allocation matrices/tensors
- Tensor-based cooperative communication systems

End

Thank you for your attention

List of bibliographical references sent upon request to

favier@i3s.unice.fr