Décompositions tensorielles et applications

Pierre Comon

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14 janvier 2013
Representation of multivariate functions

\[ x \overset{\text{def}}{=} [x_1, \ldots, x_D] \]

Discrete case:

\[ F(i, j, \ldots, k) = \sum_{r=1}^{R} \lambda_r g_{r1}(i) g_{r2}(j) \cdots g_{rD}(k) \]
Representation of multivariate functions

\( x \overset{\text{def}}{=} [x_1, \ldots, x_D] \)

- Parsimony:

\[
\begin{align*}
  f(x) &= \sum_{r=1}^{R} \lambda_r g_r(x) \\
  F(i, j, \ldots, k) &= \sum_{r=1}^{R} \lambda_r g_{r1}(i) g_{r2}(j) \cdots g_{rD}(k)
\end{align*}
\]

- Discrete case:

- Approximation
Representation of multivariate functions

\[ x \overset{\text{def}}{=} [x_1, \ldots, x_D] \]

- Separability:
  \[ f(x) = \prod_{d=1}^{D} h_d(x_d) \]

- Discrete case:
  \[ F(i, j, \ldots, k) = \sum_{r=1}^{R} \lambda_r g_{r1}(i) g_{r2}(j) \cdots g_{rD}(k) \]

- Approximation
Representation of multivariate functions

\[ \mathbf{x} \overset{\text{def}}{=} [x_1, \ldots, x_D] \]

- **Parsimony and separability:**

  \[
  f(\mathbf{x}) = \sum_{r=1}^{R} \lambda_r \prod_{d=1}^{D} g_{rd}(x_d)
  \]

- **Discrete case:**

  \[
  F(i, j, \ldots, k) \sum_{r=1}^{R} \lambda_r g_{r1}(i) g_{r2}(j) \cdots g_{rD}(k)
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F(i,j,\ldots,k) \approx \sum_{r=1}^{R} \lambda_r g_{r1}(i) g_{r2}(j) \cdots g_{rD}(k)
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- Approximation
Definition (1/2)

- **Order** $D$ tensor: *multi-linear* map $S_1 \times \ldots S_m \rightarrow S_{m+1} \times \ldots S_D$

- In this talk, $T$ is an *array* with $D$ indices, of dimension $I_2 \times I_2 \times \ldots \times I_D$

- **Examples:**
  - bilinear form: $S_1 \times S_2 \rightarrow K$, represented by a matrix
  - linear operator: $S_1 \rightarrow S_2$, represented by a matrix
  - bilinear operator: $S_1 \times S_2 \rightarrow S_3$, represented by a 3-way array
  - trilinear form: $S_1 \times S_2 \times S_3 \rightarrow K$, represented by a 3-way array
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Definition (2/2)

- **Decomposable tensor**: \( E_{i j...k} = u_i v_j \ldots w_k \)

\[
E = u \otimes v \otimes \ldots \otimes w
\]

- \( E \) may be seen as the discretization of a function in \( D \) variables, with *separated variables*:

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e(x,y,..z) = u(x) v(y) \ldots w(z)
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Tensor vs array

If bases of linear spaces $\mathbb{S}_\ell$ are fixed, $\{e_p^{(\ell)}, 1 \leq p \leq l_{\ell}\}$, then

$$\mathcal{T} = \sum_{i,j,,k} T_{ij..k} e_i^{(1)} \otimes e_j^{(2)} \otimes \ldots \otimes e_k^{(D)}$$

where $T_{ij..k}$ are the coordinates of $\mathcal{T} \Rightarrow \mathcal{T}$ can be arranged in a $D$-way array.

*Usual abuse of notation:* assimilate $\mathcal{T}$ with $\mathbf{T}$.

- Array of coordinates changes multi-linearly under linear change of basis: $\mathbf{T}' = (A,B,\ldots,C) \cdot \mathbf{T}$.
- Bases of $\mathbb{S}_\ell$ induce decomposable tensors.
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Kronecker product $\otimes$

- product defined only between 1-way or 2-way arrays:

$$A \otimes B = \begin{bmatrix}
a_{11}B & \ldots & a_{1n}B \\
\vdots & & \vdots \\
a_{m1}B & \ldots & a_{mn}B
\end{bmatrix}$$

- It should not be confused with tensor product $\otimes$.

- Usefulness: for convenience, an array with $D$ indices may be stored in larger vectors or matrices. For instance:
  - tensor product $a \otimes b$ can be stored in vector $a \otimes b$ or in matrix $ab^T$. Usual abuse of notation.
  - tensor $a \otimes b \otimes c$ can be stored in matrix $(a \otimes b)c^T$, or in vector $a \otimes b \otimes c$, or in matrix $a(b \otimes c)^T$, etc...
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Terminology

Let the array \( \{ T_{ij..k} \} \) associated with tensor \( T \).

- **Order** \( D \) of \( T \) : \# of its **ways** or **modes**
- There are \( D \) **dimensions**
- **Cubic** if all dimensions equal
- **Symmetric**: \( T_{ij..k} = T_{\sigma(ij..k)} \), \( \forall \sigma \) permutation
e.g. derivatives of a multivariate scalar function
- **Diagonal**: \( T_{ij..k} = 0 \) unless \( i = j = \cdots = k \).

**NB**: avoid words such as “super-symmetric” or “super-diagonal”.
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Decomposable tensor of order $D = 3$

- Take $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$
- Then $\mathbf{v} \otimes^3 \overset{\text{def}}{=} \mathbf{v} \otimes \mathbf{v} \otimes \mathbf{v} \rightarrow$ can be represented by

$$
\mathbf{v} (\mathbf{v} \otimes \mathbf{v})^T = 
\begin{bmatrix}
1 & -1 & -1 & 1 \\
-1 & 1 & 1 & -1
\end{bmatrix}
$$

- It is symmetric, and decomposable

blue bullets $= 1$, red bullets $= -1$. 
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Claude Elwood Shannon (1916-2001)

Pierre Comon

GdR ISIS – 16 Janv 2013
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Frank Lauren Hitchcock (1875-1957)
Every tensor $T$ can be written as a sum of decomposable tensors \textit{[Hitchcock'27]}:

$$T = \sum_{r=1}^{R(T)} \lambda_r E(r)$$

- \textit{Rank} of $T$ is the minimal $R(T)$ of necessary terms.
- Decomposable $\iff$ rank 1.
- $E(r)$ normalised (scale uniqueness), $\lambda_r \in \mathbb{R}^+$.
- Special tensors (positive, symmetric..) $\Rightarrow$ different rank definition!
Canonical Polyadic decomposition (CP) 1/2

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Canonical Polyadic decomposition (CP) 2/2

- Explicit writing of CP:

\[
T = \sum_{r=1}^{R} \lambda_r \, u^{(1)}(r) \otimes u^{(2)}(r) \otimes \ldots \otimes u^{(D)}(r)\tag{1}
\]

- CP is one possible extension of SVD
- CP is unique if pairs \{\lambda_r, E(r)\} are uniquely determined, even if \(E(r)\) can be written in different manners (unimodulus scale indeterminacy).
Explicit writing of CP:

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\mathbf{T} = \sum_{r=1}^{R} \lambda_r \mathbf{u}^{(1)}(r) \otimes \mathbf{u}^{(2)}(r) \otimes \ldots \otimes \mathbf{u}^{(D)}(r) \quad (1)
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Towards a unique terminology?

- Minimal *Polyadic Form* [Hitchcock’27]
- *Canonical decomposition* [Weinstein’84, Carroll’70, Chiantini-Ciliberto’06, Comon’00, Khoromskij, Tyrtyshnikov]
- *Parafac* [Harshman’70, Sidiropoulos’00]
- *Optimal computation* [Strassen’83]
- Minimum-length *additive decomposition* (AD) [Iarrobino’96]

**Suggestion:**

- *Canonical Polyadic decomposition (CP)* [Comon’08, Grasedyk, Espig...]
- *CP* does also already stand for Candecomp/Parafac [Bro’97, Kiers’98, tenBerge’04...]
Psychometrics

Richard A. Harshman (1970)

J. Douglas Carroll (1970)
Tucker decomposition

- **Tucker (1963)** Any tensor $\mathcal{T}$ can be decomposed into

$$T_{ijk} = \sum_{pqr} A_{ip} B_{jq} C_{kr} G_{pqr}$$

where tensor $\mathcal{G}$ is at most as big as $\mathcal{T}$

- This decomposition is not unique → it can be used for compression, but *not* for identification

- **HOSVD (2000)** One can impose matrices $A$, $B$ and $C$ to be unitary, and core tensor $\mathcal{G}$ to be *all orthogonal*, i.e. its matrix slices to be orthogonal.
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Link with SVD

Matrix SVD, $M = (U,V) \cdot \Sigma$, may be extended in at least two ways to tensors

- **Keep orthogonality:** Tucker, HOSVD
  
  $$T = (U,V,W) \cdot C$$

  $C$ is $R_1 \times R_2 \times R_3$: multilinear rank = $(R_1,R_2,R_3)$

- **Keep diagonality:** Canonical Polyadic decomposition (CP)
  
  $$T = (A,B,C) \cdot L$$

  $L$ is $R \times R \times R$ diagonal: rank = $R$. 
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2 extensions of SVD

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  $\mathbf{L}$ is $R \times R \times R$ diagonal: **rank** $= R$. 
Link with SVD

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- **Keep orthogonality:** Tucker, HOSVD
  \[ T = (U, V, W) \cdot C \]
  
  \( C \) is \( R_1 \times R_2 \times R_3 \): **multilinear rank** = \( (R_1, R_2, R_3) \)

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  \[ T = (A, B, C) \cdot L \]
  
  \( L \) is \( R \times R \times R \) diagonal: **rank** = \( R \).
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Matrix SVD, \( \mathbf{M} = (\mathbf{U}, \mathbf{V}) \cdot \Sigma \), may be extended in at least two ways to tensors

- **Keep orthogonality:** Tucker, HOSVD

  \[ \mathcal{T} = (\mathbf{U}, \mathbf{V}, \mathbf{W}) \cdot \mathcal{C} \]

  \( \mathcal{C} \) is \( R_1 \times R_2 \times R_3 \): *multilinear rank* = \( (R_1, R_2, R_3) \)

- **Keep diagonality:** Canonical Polyadic decomposition (CP)

  \[ \mathcal{T} = (\mathbf{A}, \mathbf{B}, \mathbf{C}) \cdot \mathcal{L} \]

  \( \mathcal{L} \) is \( R \times R \times R \) diagonal: *rank* = \( R \).
The **Kruskal rank** of a matrix $A$ is the largest number $k_A$, such that any subset of $k_A$ columns is linearly independent.
**CP uniqueness: Kruskal 2/2**

*Sufficiency condition* ensuring uniqueness of CP

[Kruskal’77, Sidiropoulos-Bro’00, Landsberg’09, Rhodes’10]:

**Theorem** CP unique up to scale factors if \( R \) is smaller than the so-called *de Kruskal’s bound*:

\[
2R + D - 1 \leq \sum_{d=1}^{D} \text{rank}\{U^{(d)}\}
\]  

\( (2) \)

→ Bound smaller than *expected rank*  \( \Rightarrow \)  \( \exists \) better bound, in almost sure sense...
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[Kruskal’77, Sidiropoulos-Bro’00, Landsberg’09, Rhodes’10]:

**Theorem**  CP unique up to scale factors if $R$ is smaller than the so-called de Kruskal’s bound:

$$2R + D - 1 \leq \sum_{d=1}^{D} \text{krank}\{U^{(d)}\}$$  (2)

▶ Bound smaller than *expected rank*  ⇒  ∃ *better bound*, in almost sure sense...
Approximation

- Presence of additive noise with continuous pdf (rank is generic)
- Choose a smaller rank: reduces noise and restores uniqueness

\[
\inf_{\lambda_r, E(r)} \left\| T - \sum_{r=1}^{R} \lambda_r E(r) \right\|^2
\]

But *ill posed* problem: the set of tensors with rank at most \( R \) is closed only if \( R = 1 \) or \( R = R_{\text{max}} \)!
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$$\inf_{\lambda_r, E(r)} \left\| T - \sum_{r=1}^{R} \lambda_r E(r) \right\|^2$$

$decomposables$

- But *ill posed* problem: the set of tensors with rank at most $R$ is closed only if $R = 1$ or $R = R_{\text{max}}$!
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$$\inf_{\lambda_r, E(r)} \left\| T - \sum_{r=1}^{R} \lambda_r E(r) \right\|^2$$ (3) 

- But *ill posed* problem: the set of tensors with rank at most $R$ is closed only if $R = 1$ or $R = R_{\text{max}}$!
Rang frontière (Border rank)

$T$ a un *rang frontière* $R = \text{rank}\{T\}$ ssi il est limite de tenseurs de rang $R$, et pas limite de tenseurs de rang plus faible

*Bini’79, Schönhage’81, Strassen’83, Likteig’85*
Exemple dans le cas symétrique

Soient $u$ et $v$ deux vecteurs non colinéaires.
On pose $T_0$ [Comon et al.'08]:

$$T_0 = u \otimes u \otimes u \otimes v + u \otimes u \otimes v \otimes u + u \otimes v \otimes u \otimes u + v \otimes u \otimes u \otimes u$$

Alors la suite $T_\varepsilon = \frac{1}{\varepsilon} \left[ (u + \varepsilon v)^{\otimes 4} - u^{\otimes 4} \right]$ est de rang 2,
Mais $T_\varepsilon \to T_0$ quand $\varepsilon \to 0$

$\Rightarrow$ Donc $\text{rank}\{T_0\} = 4$, et $\text{rank}\{T_0\} = 2$
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Une idée simple

Dans notre exemple, 2 causes:

- certains \( \{\lambda_p, \lambda_q\} \to \infty \)
- certaines colonnes \( \{u^{(d)}(p), u^{(d)}(q)\} \) tendent à être colinéaires dans la CP (1):

\[
T = \sum_{r=1}^{R} \lambda_r u^{(1)}(r) \otimes u^{(2)}(r) \otimes \ldots \otimes u^{(D)}(r)
\]

Idée: empêcher la colinéarité, et voir si on peut alors démontrer l'existence?
Une idée simple

Dans notre exemple, 2 causes:

- certains \( \{\lambda_p, \lambda_q\} \to \infty \)
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\]

▷ Idée: empêcher la colinéarité, et voir si on peut alors démontrer l'existence?
Approches du problème mal posé

1. Imposer l'orthogonalité entre les vecteurs \( u^{(d)}(r) \) [Comon’92]
2. Imposer l'orthogonalité entre les tenseurs \( E(r) \) [Kolda’01]
3. Empêcher la divergence en imposant une contrainte de norme [Paatero’00] ⇔ borner les \( \lambda_r \)
4. Tenseurs non négatifs: imposer la non négativité à chaque terme de la CP [Lim-Comon’09] → “rang non négatif”
5. Pour chaque \( d \), imposer un angle minimal entre les vecteurs \( u^{(d)}(r) \) [Lim-Comon’10, Comon-Lim’11] → lien avec l’échantillonnage compressé
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Cohérence

- Pour chaque matrice $\mathbf{U}^{(d)}$, on définit

$$\mu_d \stackrel{\text{def}}{=} \max_{p \neq q} \left| \mathbf{u}(p)^{\text{(d)}}^T \mathbf{u}(q)^{\text{(d)}} \right|$$

- Alors

$$\text{krank}\{\mathbf{U}^{(d)}\} \geq \frac{1}{\mu_d} \quad \text{(4)}$$
Cohérence

Pour chaque matrice $\mathbf{U}^{(d)}$, on définit

$$\mu_d \overset{\text{def}}{=} \max_{p \neq q} \left| \mathbf{u}(p)^{(d)T} \mathbf{u}(q)^{(d)} \right|$$

Alors

$$\kappa \text{rank}\{\mathbf{U}^{(d)}\} \geq \frac{1}{\mu_d}$$  \hspace{1cm} (4)
Condition suffisante d’existence et d’unicité (1/2)

■ Théorème 1 (Existence) Si

\[
\text{rank}\{ \mathbf{T} \} < \prod_{d=1}^{D} \mu_d^{-1}
\]

alors l’\textit{Inf} est atteint.
Condition suffisante d’existence et d’unicité (2/2)

- Théorème 2 (Existence et unicité) Si

\[
\left( \prod_{d=1}^{D} \mu_d \right)^{1/D} \leq \frac{D}{2R + D - 1}
\]

alors la meilleure approximation de rang \( R \) existe et est unique, à des facteurs d’échelle de module 1 près.

Référence: arxiv:1212.6663
Applications

1. Analyse de facteurs
2. Sonar, Radar, Télécommunications, Traitement d’antenne
3. Imagerie EEG, ECG
4. Environnement: tenseurs à éléments positifs
5. Complexité arithmétique
6. Compression
7. Codage espace-temps (CDMA)

Tenseurs symétriques

8. Statistiques: cumulants d’ordre supérieur à 2
9. Filtrage de Volterra, compression
Spectrométrie fluorescente

*Mélange inconnu de $R$ solutés*

Loi de Beer-Lambert pour de faibles concentrations:

$$I(\lambda_f, \lambda_e, k) = I_o \sum_{\ell=1}^{R} \gamma_{\ell}(\lambda_f) \epsilon_{\ell}(\lambda_e) c_{k,\ell}$$

$\gamma_{\ell}(\lambda_f)$: spectre de fluorescence

$\epsilon_{\ell}(\lambda_e)$: spectre d’absorbance

$c_{k,\ell}$: concentration
Thèse Jean-Philip Royer (soutenance avril 2013)

Mélange de 4 solutés (une concentration est représentée)
Thèse Jean-Philip Royer (soutenance avril 2013)

Résultat obtenu avec $R = 4$
Antennes structurées (1/3)

Diversité d’espace: invariance par translation
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Diversité d’espace: invariance par translation
Antennes structurées (2/3)

■ Exemple 1

\[
\begin{align*}
\ & = \begin{array}{cccc}
\ & \ & \ & \\
\ & \ & \ & \\
\ & \ & \ & \\
\ & \ & \ & \\
\end{array} \times \begin{array}{ccc}
\ & \ & \\
\ & \ & \\
\ & \ & \\
\end{array} = \begin{array}{c}
\ & \\
\ & \\
\ & \\
\end{array} \times \begin{array}{ccc}
\ & \\
\ & \\
\ & \\
\end{array} = \begin{array}{cccc}
\ & \ & \ & \\
\ & \ & \ & \\
\ & \ & \ & \\
\ & \ & \ & \\
\end{array}
\end{align*}
\]

■ Exemple 2

■ Exemple 3

■ Identification et égalisation conjointes déterministes.
■ Signification de la contrainte angulaire: séparation spatiale minimale ou corrélation maximale
Antennes structurées (2/3)

■ Exemple 1

\[
\begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\end{array}
= \begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\end{array} \otimes \begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\end{array} \otimes \begin{array}{c}
\begin{array}{c}
\bullet \\
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\bullet \\
\bullet \\
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\end{array} \otimes \begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\end{array}
\end{array}
\]

■ Exemple 2

\[
\begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\end{array}
= \begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\end{array}
\]

■ Exemple 3

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■ Signification de la contrainte angulaire:
  séparation spatiale minimale ou corrélation maximale.
Antennes structurées (2/3)

■ Exemple 1

\[ \begin{array}{c}
\bigotimes \\
= \bigotimes \\
= \bigotimes \\
\end{array} \]

■ Exemple 2

\[ \begin{array}{c}
\bigotimes \\
= \bigotimes \\
\end{array} \]

■ Exemple 3

\[ \begin{array}{c}
\bigotimes \\
= \bigotimes \\
\end{array} \]

■ Identification et égalisation conjointes déterministes.

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Antennes structurées (2/3)

■ Exemple 1

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix} \otimes \begin{bmatrix}
1 \\
0 \\
0 \\
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix} \otimes \begin{bmatrix}
1 \\
0 \\
0 \\
\end{bmatrix} = \begin{bmatrix}
1 \\
0 \\
0 \\
\end{bmatrix} \otimes \begin{bmatrix}
1 & 0 \\
0 \\
0 \\
\end{bmatrix}
\]

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\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
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■ Identification et égalisation conjointes *déterministes*.

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Antennes structurées (2/3)

- **Exemple 1**

\[
\begin{array}{c}
\begin{array}{c}
\cdots \\
\end{array} \\
\begin{array}{c}
\cdots \\
\end{array} \\
\begin{array}{c}
\cdots \\
\end{array} \\
\begin{array}{c}
\cdots \\
\end{array}
\end{array}
\]

- **Exemple 2**

\[
\begin{array}{c}
\begin{array}{c}
\cdots \\
\end{array} \\
\begin{array}{c}
\cdots \\
\end{array} \\
\begin{array}{c}
\cdots \\
\end{array} \\
\begin{array}{c}
\cdots \\
\end{array}
\end{array}
\]

- **Exemple 3**

\[
\begin{array}{c}
\begin{array}{c}
\cdots \\
\end{array} \\
\begin{array}{c}
\cdots \\
\end{array} \\
\begin{array}{c}
\cdots \\
\end{array} \\
\begin{array}{c}
\cdots \\
\end{array}
\end{array}
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- Identification et égalisation conjointes *déterministes*.
- Signification de la contrainte angulaire: séparation spatiale minimale *ou* corrélation maximale.
For narrow-band 3-dimensional transmission in the far field, we have

**Definition** A sensor array is resolvent w.r.t. direction \( \mathbf{v} \) if two sensors are less than \( \lambda/2 \) apart in direction \( \mathbf{v} \).

**Theorem 4** If a subarray is resolvent w.r.t. 3 linearly independent directions, then \( \mu_1 = 1 \) if and only if two source paths have the same direction of arrival.
Polarization diversity

For narrow-band 3-dimensional transmission in the far field of polarized waves, we can build a $N \times 6 \times T$ tensor, where the second mode represents polarization.

**Theorem 5** \( \mu_2 = 1 \) if and only if two source paths have the same orientation and ellipticity polarization angles, and the same direction of arrival.
Conclusion

One can localize and estimate source paths signals if at least one on the conditions below is satisfied:

- low correlation between paths
- minimal angular separation between DoA’s
- minimal angular separation between polarization angles
Conclusion

One can localize *and* estimate source paths signals if *at least one* on the conditions below is satisfied:

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One can localize \textit{and} estimate source paths signals if \textit{at least one} on the conditions below is satisfied:

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Matrices vs. Tensors (1)

For matrices:
1. Rank cannot exceed dimensions
2. CP is not unique (SVD imposes orthogonality)
3. Rank is the same in real and complex fields
4. Rank of symmetric matrix is the same if CanD is symmetric

For tensors:
1. Rank can largely exceed dimensions
2. CP is almost surely unique for subgeneric ranks
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4. Imposing symmetry in the CanD has not yet been proved to lead to same rank
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Matrices vs. Tensors (2)

For matrices:
5. Matrices with entries drawn randomly have maximal rank.
6. Maximal rank equals generic rank.
7. The set of matrices of rank at most $r$ is closed, $\forall r$.
8. Subtracting the best rank-1 approximate decreases rank by 1.

For tensors:
5. Generic rank is not maximal in most cases.
6. Maximal rank still unknown for most order/dimensions.
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   ▶ Approximation problem ill-posed!
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Matrices vs. Tensors (2)

For matrices:
1. Matrices with entries drawn randomly have maximal rank
2. Maximal rank equals generic rank
3. The set of matrices of rank at most \( r \) is closed, \( \forall r \)
4. Subtracting the best rank-1 approximate decreases rank by 1

For tensors:
1. Generic rank is not maximal in most cases
2. Maximal rank still unknown for most order/dimensions
3. The set of tensors of rank at most \( r \) is not closed, except for \( r = 1 \) and \( r \) maximal
   - Approximation problem ill-posed!
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Matrices vs. Tensors (2)

For matrices:

5. Matrices with entries drawn randomly have maximal rank
6. Maximal rank equals generic rank
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Further readings

Free-access references:

1. “Tensors, usefulness and unexpected properties,” Keynote SSP’09, Cardiff. *hal-00417258*


*L’ignorance est la condition nécessaire du bonheur des hommes, et il faut reconnaître que le plus souvent, ils la remplissent bien.*

Anatole France (1844-1924)