FOURTH ORDER BLIND IDENTIFICATION OF UNDERDETERMINED MIXTURES OF SOURCES (FOBIUM)

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ABSTRACT

Most of the current Second Order (SO)[1] and Fourth Order (FO)[3][6] Blind Sources Identification (BSI) methods aim at blindly identifying the steering vectors of statistically independent sources, provided the number of sources is not greater than the number of sensors. However in practical situations, the probability of receiving more sources than sensors increases with the reception bandwidth. In this context the purpose of this paper is to propose a new attractive FO BSI method, able to identify the steering vector of more sources than sensors, jointly with a new pertinent performance criterion for the quality evaluation of the BSI process. The new method implements a FO pre-whitening step and exploits the tri-spectrum diversities of the sources.

1. INTRODUCTION

For more than a decade, SO [1] and FO [3][6] methods have been developed to blindly identify the steering vectors of several statistically independent sources, provided the number of sources remains lower than or equal to the number of sensors. However, in practical situations, such as in the HF context, the reception of more sources than sensors is possible and its probability increases with the reception bandwidth. To process such situations, several BSI methods have been developed this last decade, among which we find the methods [2] [7-8] [10]. The methods proposed in [2] and [7-8] only exploit the information contained in the FO statistics of the data whereas the one proposed in [10] exploits the information contained in one of the characteristic function of the observations. However, all these methods suffer from severe drawbacks in operational contexts. Indeed, the method [2] is still very difficult to implement and does not ensure the BSI of the sources steering vectors when the sources have the same kurtosis. The methods [7-8] assume non circular sources and fail in separating circular sources, omnipresent in practice. Finally, the method [10] has been developed only for real mixtures of real-valued sources and is probably not robust to an over estimation of the source number. To overcome these drawbacks we propose in this paper a new FO method, corresponding to the FO extension of the SOBI algorithm [1], able to blindly identify the steering vectors of up to $N^2 - N + 1$ sources with N sensors, without the previously mentioned drawbacks but assuming the sources have different tri-spectrum and have non zero kurtosis with the same sign (the latter assumption is generally verified in radiocommunications contexts). This method implements a FO pre-whitening step and fully exploits the assumed non whiteness property of the sources. Finally a new performance criterion, able to quantify the identification quality of the steering vector of each source and allowing the quantitative comparison of two methods for the blind identification (BI) of each source is proposed.

2. PROBLEM FORMULATION

A noisy mixture of P Narrow-Band (NB) statistically independent sources is assumed to be received by an array of N sensors. The vector, $\boldsymbol{x}(t)$, of the complex envelopes of the signals at the output of the sensors is thus given by

$$\boldsymbol{x}(t) = \sum_{p=1}^{P} m_p(t)\, \boldsymbol{a}_p + \boldsymbol{n}(t) = A\, \boldsymbol{m}(t) + \boldsymbol{n}(t) \qquad (1)$$

where $m_p(t)$ is the p-th component of the vector $\boldsymbol{m}(t)$, assumed zero-mean and stationary, $\boldsymbol{n}(t)$ is the noise vector, assumed zero-mean, stationary, Gaussian, spatially and temporally white in the reception band, $\boldsymbol{a}_p$ corresponds to the steering vector of the source p and A is the (NxP) matrix whose columns are the vectors $\boldsymbol{a}_p$.

The problem adressed in this paper is the BI of the steering vectors $\boldsymbol{a}_p$ from the FO statistics of the data.

3. THE FOBIUM METHOD

The purpose of the FOBIUM method is to extend the SOBI method [1] at the FO. It firstly implements a FO pre-whitening step aiming at orthonormalizing the so-called *virtual steering vector* [5] of the sources in some data quadricovariance matrices and secondly it jointly diagonalizes several well chosen pre-whitened quadricovariance matrices in order to identify the steering vectors of more sources than sensors. The number of sources able to be processed by this method is addressed in section 4.

3.1 FO statistics of the data

Under the assumption of zero-mean stationary sources, the FO statistics of the observations are characterized by the $(N^2 \mathrm{x} N^2)$ quadricovariance matrices $Q_x(\tau_1,\tau_2,\tau_3)$, whose elements, $Q_x(\tau_1,\tau_2,\tau_3)[i, j, k, l]$, are defined by

$$Q_x(\tau_1,\tau_2,\tau_3)[i, j, k, l] \stackrel{\Delta}{=} \mathrm{Cum}(x_i(t), x_j(t\text{-}\tau_1)^*, x_k(t\text{-}\tau_2)^*, x_l(t\text{-}\tau_3)) \quad (2)$$

where * means complex conjugate and $x_i(t)$ is the i^{th} component of the vector $\boldsymbol{x}(t)$. Using (1) into (2) and assuming that $Q_x(\tau_1,\tau_2,\tau_3)[i, j, k, l]$ is the element $[N(i-1) + j,\ N(k-1) + l]$ of the matrix $Q_x(\tau_1,\tau_2,\tau_3)$, we obtain the expression of the latter, given, under a Gaussian noise assumption, by

$$Q_x(\tau_1,\tau_2,\tau_3) \ = \ (A \otimes A^*)\, Q_m(\tau_1,\tau_2,\tau_3)\, (A \otimes A^*)^{\mathrm{H}} \qquad (3)$$

where $Q_m(\tau_1,\tau_2,\tau_3)$ is the $(P^2 \mathrm{x} P^2)$ quadricovariance matrix of $\boldsymbol{m}(t)$, $\otimes$ is the Kronecker product and $^{\mathrm{H}}$ means transpose and complex conjugate.

Under the assumption of statistically independent sources, the matrix $Q_m(\tau_1,\tau_2,\tau_3)$ contains at least $P^4 - P$ zeros and the expression (3) degenerates in a simpler one given by

$$Q_x(\tau_1,\tau_2,\tau_3) \ = \sum_{p=1}^{P} c_p(\tau_1,\tau_2,\tau_3)\, (\boldsymbol{a}_p \otimes \boldsymbol{a}_p^*)\, (\boldsymbol{a}_p \otimes \boldsymbol{a}_p^*)^{\mathrm{H}} \qquad (4a)$$

$$= \ A_Q \ C_m(\tau_1,\tau_2,\tau_3)\ A_Q^{\ \mathrm{H}} \qquad (4b)$$

where A_Q is the $(N^2 \mathrm{x}\ P)$ matrix defined by $A_Q \stackrel{\Delta}{=} [(\boldsymbol{a}_1 \otimes \boldsymbol{a}_1^*), \ldots, (\boldsymbol{a}_p \otimes \boldsymbol{a}_p^*)]$, $C_m(\tau_1,\tau_2,\tau_3)$ is the $(P \times P)$ diagonal matrix defined by $C_m(\tau_1,\tau_2,\tau_3) \stackrel{\Delta}{=} \mathrm{diag}[c_1(\tau_1,\tau_2,\tau_3), \ldots, c_p(\tau_1,\tau_2,\tau_3)]$ and $c_p(\tau_1,\tau_2,\tau_3)$ is defined by

$$c_p(\tau_1,\tau_2,\tau_3) \stackrel{\Delta}{=} \mathrm{Cum}(m_p(t), m_p(t\text{-}\tau_1)^*, m_p(t\text{-}\tau_2)^*, m_p(t\text{-}\tau_3)) \quad (5)$$

The expression (4b), which has an algebraic structure similar to that of data correlation matrices [1], is at the basis of the FOBIUM method as it is shown in the next sections.

We note in the following $Q_x \stackrel{\Delta}{=} Q_x(0, 0, 0)$, $c_p \stackrel{\Delta}{=} c_p(0, 0, 0)$, $C_m \stackrel{\Delta}{=} C_m(0, 0, 0)$ and we obtain

$$Q_x \ = \ A_Q\, C_m\, A_Q^{\ \mathrm{H}} \qquad (6)$$

We also assume in the following that $P \leq N^2$, the matrix A_Q is full rank, the c_p, $1 \leq p \leq P$, are non zero (non Gaussian sources) and have the same sign and whatever the couple (i, j) of sources, it exists at least three delays (τ_1,τ_2,τ_3) such that $|\tau_1|+|\tau_2|+|\tau_3| \neq 0$ and

$$c_i(\tau_1,\tau_2,\tau_3)\ /\ |c_i| \neq c_j(\tau_1,\tau_2,\tau_3)\ /\ |c_j| \qquad (7)$$

Note that the condition (7) requires in particular that the sources have different tri-spectrum.

3.2 FO Pre-whitening step

The first step of the FOBIUM method is to orthonormalize, in the Q_x matrix (6), the columns of A_Q, which can be considered as *virtual steering vectors* of the sources for the considered array of sensors [5]. For this purpose, let us consider the eigen decomposition of the Hermitian matrix Q_x, whose rank is P under the previous assumptions, given by

$$Q_x \ = \ E_x\, \Lambda_x\, E_x^{\ \mathrm{H}} \qquad (8)$$

where Λ_x is the $(P \times P)$ real-valued diagonal matrix of the P non zero eigen-values of Q_x and E_x is the $(N^2 \mathrm{x} P)$ matrix of the associated orthonormalized eigen-vectors. For a full rank A_Q matrix, it is possible to verify that assuming P sources with non zero kurtosis having the same sign ε ($\varepsilon = \pm 1$) is equivalent to assume that the diagonal elements of Λ_x are not zero and have also the same sign corresponding to ε. In this context, considering the $(P \times N^2)$ whitening matrix T defined by

$$T \stackrel{\Delta}{=} (\Lambda_x)^{-1/2}\, E_x^{\ \mathrm{H}} \qquad (9)$$

where $(\Lambda_x)^{-1/2}$ is the inverse of a square root of Λ_x, we obtain, from (6) and (8)

$$\varepsilon\, T\, Q_x\, T^{\mathrm{H}} \ = \ T\, A_Q\, (\varepsilon\, C_m)\, A_Q^{\ \mathrm{H}}\, T^{\mathrm{H}} \ = \ \mathrm{I}_P \qquad (10)$$

where I_P is the $(P \times P)$ identity matrix and where $\varepsilon C_m = \mathrm{diag}[|c_1|, \ldots, |c_p|]$. This last expression shows that the $(P \mathrm{x} P)$ matrix $TA_Q\, (\varepsilon C_m)^{1/2}$ is an unitary matrix U and we obtain

$$T\, A_Q \ = \ U\, (\varepsilon\, C_m)^{-1/2} \qquad (11)$$

3.3 FO Blind identification step

We deduce from expressions (4b) and (11) that

$$T\, Q_x(\tau_1,\tau_2,\tau_3) T^{\mathrm{H}} = U\, (\varepsilon C_m)^{-1/2} C_m(\tau_1,\tau_2,\tau_3)\, (\varepsilon C_m)^{-1/2}\, U^{\mathrm{H}} \quad (12)$$

which shows that the unitary matrix U diagonalizes the matrices $T\ Q_x(\tau_1,\tau_2,\tau_3)\ T^{\mathrm{H}}$ whatever the set of delays (τ_1,τ_2,τ_3) and the associated eigen-values correspond to the diagonal terms of the diagonal matrix $(\varepsilon C_m)^{-1/2} C_m(\tau_1,\tau_2,\tau_3)\, (\varepsilon C_m)^{-1/2}$.

For a given set (τ_1,τ_2,τ_3), U is unique to within a permutation and an unitary diagonal matrix if and only if the eigen-values of the matrix $(\varepsilon C_m)^{-1/2} C_m(\tau_1,\tau_2,\tau_3)\, (\varepsilon C_m)^{-1/2}$ are all different. If it is not the case, we have to consider several sets $(\tau_1^{\,k},\tau_2^{\,k},\tau_3^{\,k})$, $1 \leq k \leq K$, such that for each couple of sources (i, j), it exists at least a set $(\tau_1^{\,k},\tau_2^{\,k},\tau_3^{\,k})$ such that the condition (7) is verified. In these conditions, the unitary matrix U becomes, to within a permutation and an unitary diagonal matrix, the only one which jointly diagonalizes the K matrices $T\ Q_x(\tau_1^{\,k},\tau_2^{\,k},\tau_3^{\,k})\ T^{\mathrm{H}}$. In other words, the unitary matrix, U_{sol}, solution to the previous problem of joint diagonalization can be written as

$$U_{sol} = U \Lambda \Pi \quad (13)$$

where Λ and Π are unitary diagonal and permutation matrices respectively. Using (11) and (13), the matrix A_Q can be deduced from U_{sol} and T, to within unitary diagonal and permutation matrices, by

$$T^{\#} U_{sol} \triangleq E_x \Lambda_x^{1/2} U_{sol} = A_Q (\varepsilon C_m)^{1/2} \Lambda \Pi \quad (14)$$

where $T^{\#}$ corresponds to the pseudo-inverse of T. Each column, $\boldsymbol{b}_l$ ($1 \le l \le P$), of $T^{\#} U_{sol}$ corresponds to one of the vectors $\mu_q |c_q|^{1/2} (\boldsymbol{a}_q \otimes \boldsymbol{a}_q^*)$, $1 \le q \le P$, where μ_q is a complex scalar such that $|\mu_q| = 1$. Thus, mapping the components of each column $\boldsymbol{b}_l$ of $T^{\#} U_{sol}$ into a (N x N) matrix B_l such that $B_l[i, j] = \boldsymbol{b}_l((i-1)N + j)$ ($1 \le i, j \le N$) consists to built the matrices $\mu_q |c_q|^{1/2} \boldsymbol{a}_q \boldsymbol{a}_q^{\mathrm{H}}$ ($1 \le q \le P$). In this context, the source steering vector $\boldsymbol{a}_q$ corresponds to the eigen-vector of B_l associated to the strongest eigen-value.

3.4 Implementation of the FOBIUM method

The different steps of the FOBIUM method are summarized hereafter when L snapshots of the observations, $\boldsymbol{x}(l)$ ($1 \le l \le L$), are available.

Step1: Estimation, $\hat{Q}_x$, of the Q_x matrix from the L snapshots $\boldsymbol{x}(l)$ using the empirical estimator of the FO cumulants [9]. Note that the FOBIUM method can also be applied for zero-mean cyclo-stationary sources provided that the previous empirical estimator is replaced by the unbiased FO statistics estimator proposed in [9].

Step2: Eigen Value Decomposition (EVD) of the matrix $\hat{Q}_X$, estimation of the number of sources P and restriction of this EVD to the P principal components : $\hat{Q}_X \approx \hat{E}_x \hat{\Lambda}_x \hat{E}_x^{\mathrm{H}}$, where $\hat{\Lambda}_x$ is the diagonal matrix of the P eigen-values with the strongest modulus and $\hat{E}_x$ is the matrix of the associated eigen-vectors.

Step3: Computation of the pre-whitening matrix : $\hat{T} \triangleq (\hat{\Lambda}_x)^{-1/2} \hat{E}_x^{\mathrm{H}}$.

Step4: Selection of K sets of delays $(\tau_1^k, \tau_2^k, \tau_3^k)$ where $|\tau_1^k| + |\tau_2^k| + |\tau_3^k| \neq 0$.

Step5: Estimation, $\hat{Q}_X(\tau_1^k, \tau_2^k, \tau_3^k)$, of $Q_x(\tau_1^k, \tau_2^k, \tau_3^k)$, for the K delay sets, using the empirical estimator of the FO statistics [9] (or, for zero-mean cyclo-stationary sources, the unbiased estimators similar to that presented in [9]).

Step6: Computation of the matrices $\hat{T} \hat{Q}_X(\tau_1^k, \tau_2^k, \tau_3^k) \hat{T}^{\mathrm{H}}$ and estimation, $\hat{U}_{sol}$, of the unitary matrix U_{sol} from the joint diagonalization of the K matrices $\hat{T} \hat{Q}_X(\tau_1^k, \tau_2^k, \tau_3^k) \hat{T}^{\mathrm{H}}$.

Step7: Computation of $\hat{T}^{\#} \hat{U}_{sol}$ and mapping each column $\hat{\boldsymbol{b}}_l$ into a (N x N) matrix $\hat{B}_l$

Step8: Estimation, $\hat{\boldsymbol{a}}_q$ ($1 \le q \le P$), of the P source steering vectors by EVD of the P matrices $\hat{B}_l$

4. IDENTIFIABILITY

Following the development of the previous sections, we deduce that the FOBIUM method is able to identify, from an array of N sensors, the steering vectors of P ($P \le N^2$) non Gaussian sources having different tri-spectrum and kurtosis with the same sign provided that the A_Q matrix has full rank P, i.e that the *virtual steering vectors* $\boldsymbol{a}_q \otimes \boldsymbol{a}_q^*$ ($1 \le q \le P$) for the considered array of N sensors remain linearly independent. Besides, it has been shown in [5] that the vector $\boldsymbol{a}_q \otimes \boldsymbol{a}_q^*$ can also be considered as a *true steering vector* but for a *virtual array* of N_{e} different sensors, where N_{e} is directly related to the geometry of the true array of N sensors. This means in particular that $N^2 - N_{\mathrm{e}}$ components of each vector $\boldsymbol{a}_q \otimes \boldsymbol{a}_q^*$ are redundant components which bring no information. As a consequence, $N^2 - N_{\mathrm{e}}$ rows of the A_Q matrix bring no information and are linear combinations of the others, which means that the rank of A_Q cannot be greater than N_{e} and is equal to $\mathrm{Inf}(N_{\mathrm{e}}, P)$ when the A matrix is full rank. In these conditions, the A_Q matrix is full rank if and only if $\mathrm{Inf}(N_{\mathrm{e}}, P) = P$, i.e if and only if $P \le N_{\mathrm{e}}$. Thus the FOBIUM method is able to process N_{e} sources, where N_{e} is the number of sensors of the virtual array associated to the chosen array of N sensors. For an Uniform linear array $N_{\mathrm{e}} = 2N + 1$ whereas for most of other arrays $N_{\mathrm{e}} = N^2 - N + 1$ [5].

5. PERFORMANCE CRITERION

Most of the existing performance criterions used to evaluate the quality of the BI process [6-7] [10] are *global criterions* which evaluate a distance between the true mixing matrix A and its blind estimate $\hat{A}$. Although practice, a global performance criterion necessarily contains a part of arbitrary considerations in the manner of combining all the distances between the vectors $\boldsymbol{a}_q$ and $\hat{\boldsymbol{a}}_q$. Moreover, it is possible to find that an estimate $\hat{A}_1$ of A is better than an estimate $\hat{A}_2$, with respect to the global criterion, while some columns of $\hat{A}_2$ estimate the associated true steering vectors in a better way than $\hat{A}_1$.

For these reasons, we propose in this section a new performance criterion for the evaluation of the BI process. This new criterion is not global and allows both the evaluation quality of the BI of each source and the quantitative comparison of two methods for the BI of a given source. It corresponds, for the BI problem, to the one proposed in [4] for the extraction problem. It is defined by the P-uplet

$$\mathrm{D}(A, \hat{A}) \triangleq (\alpha_1, \alpha_2, \ldots, \alpha_P) \quad (15)$$

where

$$\alpha_p \triangleq \min_{1 \le i \le P} [\mathrm{d}(\boldsymbol{a}_p, \hat{\boldsymbol{a}}_i)] \quad (16)$$

and where $d(\boldsymbol{u},\boldsymbol{v})$ is the pseudo-distance between the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, defined by

$$d(\boldsymbol{u},\boldsymbol{v}) \triangleq 1 - \frac{\left|\boldsymbol{u}^{H}\boldsymbol{v}\right|^{2}}{\left(\boldsymbol{u}^{H}\boldsymbol{u}\right)\left(\boldsymbol{v}^{H}\boldsymbol{v}\right)} \qquad (17)$$

6. SIMULATIONS

To illustrate the previous results, we assume that P=6 statistically independent non filtered QPSK sources are received by a circular array of N=3 sensors of radius r such that r/λ=0.55 (λ: wavelength). The 6 sources, assumed synchronized, have the same input SNR (Signal to Noise Ratio) of 20 dB with a symbol period $T = 4T_e$, where T_e is the sample period.

The direction of arrival of the sources are such that θ_1=2.16°, θ_2=25.2°, θ_3=50°, θ_4=272.16°, θ_5=315.36°, θ_6=336.96° and the associated carrier frequencies verify $\Delta f_1\, T_e$=0, $\Delta f_2\, T_e$=1/2, $\Delta f_3\, T_e$=1/3, $\Delta f_4\, T_e$=1/5, $\Delta f_5\, T_e$=1/7 and $\Delta f_6\, T_e$=1/11. We apply the JADE [3] , SOBI [1] and FOBIUM methods, and the performance α_q for q=1…6 is computed and averaged over 1000 realizations. For the FOBIUM method we choose K=4 sets of delays $(\tau_1^k, \tau_2^k, \tau_3^k)$ where $\tau_1^k = kT_e$ and $\tau_2^k = \tau_3^k = 0$.

Under the previous assumptions, the figure 1 shows the variations of α_2 (source 2 performance) at the output of the JADE, SOBI and FOBIUM separators as a function of the number of snapshots L. We verify the difficulty of the JADE and SOBI methods to well identify the steering vector of the source 2 in an underdetermined context and the very good performance of the FOBIUM method in the same context. Note the complete convergence of the FOBIUM method as soon as L is in the area of 2000.

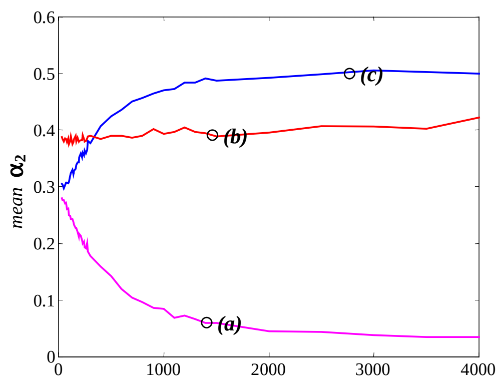

Fig.1 - *α_2 as a function of L, (a) (FOBIUM), (b) (JADE), (c) (SOBI)*

The Figure 2 shows, in the same context, the variations of all the α_p ($1 \le p \le 6$) at the output of the FOBIUM method as a function of L. Note the decreasing values toward zero of all the previous coefficients as L increases.

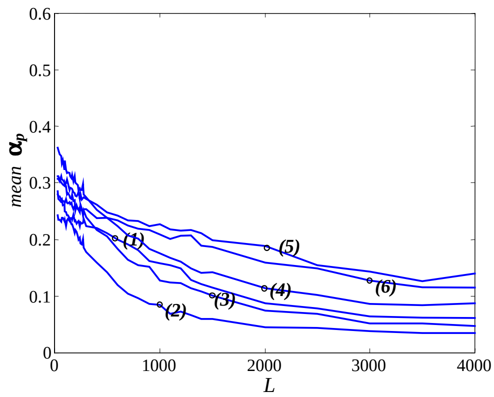

Fig.2 - *α_p of FOBIUM as a function of L, (p) performance criterion of the pth source*

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