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Higher Order Direction Finding From Arrays With Diversely Polarized Antennas: The PD-2q-MUSIC Algorithms

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***Abstract*—Fourth-order (FO) and, a short while ago, $2q$th-order, $q \geq 2$, high-resolution methods exploiting the information contained in the FO and the $2q$th-order, $q \geq 2$, statistics of the data, respectively, are now available for direction finding of non-Gaussian signals. Among these methods, the $2q$-MUSIC methods, $q \geq 2$, are the most popular. These methods are asymptotically robust to a Gaussian background noise whose spatial coherence is unknown and offer increasing resolution and robustness to modeling errors jointly with an increasing processing capacity as q increases. However, these methods have been mainly developed for arrays with identical sensors only and cannot put up with arrays of diversely polarized sensors in the presence of diversely polarized sources. In this context, the purpose of this paper is to introduce, for arbitrary values of q, $q \geq 1$, three extensions of the $2q$-MUSIC method, able to put up with arrays having diversely polarized sensors for diversely polarized sources. This gives rise to the so-called polarization diversity $2q$-MUSIC (PD-$2q$-MUSIC) algorithms. For a given value of q, these algorithms are shown to increase the resolution, the robustness to modeling errors, and the processing capacity of the $2q$-MUSIC method in the presence of diversely polarized sources. Besides, some PD-$2q$-MUSIC algorithms are shown to offer increasing performances with q when resolution in both direction of arrival and polarization is required.**

***Index Terms*—Direction finding (DF), direction of arrival (DOA), higher order, identifiability, polarization diversity, underdetermined mixtures, virtual array (VA), 2q-MUSIC.**

I. Introduction

During the last two decades, fourth-order (FO) direction finding (DF) methods [1], [8], [28], [31], exploiting the information contained in the FO statistics of the observations, have been developed for non-Gaussian signals. Among these methods, the FO extension of the well-known MUSIC method [30], called 4-MUSIC [28], is the most popular. These methods are asymptotically robust to a Gaussian noise whose spatial coherence is unknown and generate a virtual increase of both the number of sensors and the effective aperture of the considered array [4], [10]. This introduces the FO virtual array (VA) concept presented in [10] and [4]. A consequence of this property is that, despite of their higher variance [2], FO DF methods allow for the processing of more sources than sensors and an increase of both the resolution and, at least for several poorly angularly separated sources, the robustness to modeling errors of second-order (SO) methods [6]. To still increase the resolution power of DF methods, their robustness to modeling errors and the number of sources to be processed from a given array of sensors, while keeping their robustness to a Gaussian background noise whose spatial coherence is unknown, the MUSIC method has been extended recently [6] to an arbitrary even order $2q$, $q \geq 2$. This gives rise to the so-called $2q$-MUSIC method, which exploits the information contained in the $2q$th-order statistics of the observations. This method is shown in [6] to have resolution, robustness to modeling errors (for several poorly angularly separated sources), and processing capacity increasing with q. These results are directly related to the higher order extension, presented in [3], of the FO VA concept. This concept allows to explain why, despite of their higher variance, $2q$-MUSIC methods with $q > 2$ may offer better performances than 2-MUSIC or 4-MUSIC methods when some resolution is required. This is, in particular, the case in the presence of several sources, when the latter are poorly angularly separated or in the presence of modeling errors inherent in operational contexts.

However, both 4-MUSIC [28] and $2q$-MUSIC, $q \geq 2$ [6] algorithms have been mainly developed for arrays with identical sensors, and cannot put up, in the presence of arbitrary polarized sources, with arrays of diversely polarized sensors. The exploitation of arrays with diversely polarized sensors is very advantageous since for such arrays, multiple signals may be resolved on the basis of polarization as well as direction of arrival (DOA). This added information improves both DOA accuracy and resolution in general [14], [36], [38] and also increases robustness to modeling errors [15]. However, most of methods which are currently available for DF from arrays with diversely polarized sensors exploit only the information contained in the SO statistics of the observations. Among these SO methods, we find extensions to array with diversely polarized, and possibly collocated, sensors of SO methods such as MUSIC [11], [41], pencil-MUSIC [20], root-MUSIC [13], [37], [42], ESPRIT [21]–[25], [39], [40], [44], subspace fitting [32], MODE [26], and maximum likelihood [29], [43] methods, respectively. Note that a comparative performance analysis of MUSIC and pencil-MUSIC methods for such arrays

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is presented in [7]. Nevertheless, HO DF methods available for arrays with diversely polarized sensors are very scarce, among which we find an FO ESPRIT-like algorithm developed for very specific array configurations [16]. In this context, in order to increase the performance of the $2q$-MUSIC algorithm in the presence of sources having different polarizations, the purpose of this paper is to introduce, for arbitrary values of q, three extensions of the $2q$-MUSIC method able to put up with arrays having diversely polarized sensors. This gives rise to the polarization diversity $2q$-MUSIC (PD-$2q$-MUSIC) algorithms. For a given value of q, these algorithms are shown in this paper to increase the resolution, the robustness to modeling errors (at least for poorly angularly separated sources), and the processing capacity of the $2q$-MUSIC method in the presence of diversely polarized sources. Besides, despite a higher variance of HO DF methods, some PD-$2q$-MUSIC algorithms are shown in this paper to offer increasing performances with q when resolution in DOA and polarization is required.

After an introduction of some notations, hypotheses, and data statistics in Section II, three versions of the PD-$2q$-MUSIC method are presented in Section III for particular arrangements of the $2q$th-order data statistics in a $2q$th-order statistical matrix. Identifiability issues for several kinds of array configurations are addressed in Section IV. Considerations about resolution of PD-$2q$-MUSIC methods are investigated in Section V. Some simulations about the behavior of PD-$2q$-MUSIC algorithms for both overdetermined and underdetermined mixtures of sources are presented in Section VI, showing off, in particular, the great interest of PD-$2q$-MUSIC methods for $q > 2$. Finally, Section VII concludes this paper. Note that the content of this paper has been patented in [5].

II. Hypotheses, Notations, and Statistics of the Data

A. Hypotheses and Notations

We consider an array of N narrowband (NB) potentially different sensors and we call $\boldsymbol{x}(t)$ the vector of complex amplitudes of the signals at the output of these sensors. Each sensor is assumed to receive the contribution of P zero-mean stationary NB sources, which may be statistically independent or not, corrupted by a noise. We assume that the P sources can be divided into G groups, with P_g sources in the group g, such that the sources in each group are assumed to be statistically dependent, but not perfectly coherent, while sources belonging to different groups are assumed to be statistically independent. Of course, P is the sum of all the parameters P_g, $1 \le g \le G$. Under these assumptions, the observation vector can be written as follows:

$$\begin{aligned}\boldsymbol{x}(t) &= \sum_{i=1}^{P} m_i(t)\boldsymbol{a}(\boldsymbol{\theta}_i,\boldsymbol{\beta}_i) + \boldsymbol{\eta}(t)\\ &\triangleq \sum_{g=1}^{G} A_g\boldsymbol{m}_g(t) + \boldsymbol{\eta}(t) = \sum_{g=1}^{G}\boldsymbol{x}_g(t) + \boldsymbol{\eta}(t)\end{aligned} \tag{1}$$

where $\boldsymbol{\eta}(t)$ is the noise vector, assumed zero-mean, stationary, and Gaussian, whereas $m_i(t)$, independent of $\boldsymbol{\eta}(t)$, is the complex envelope of the source i. Couple $\boldsymbol{\theta}_i \triangleq (\theta_i, \varphi_i)$ defines the azimuth θ_i and elevation angle φ_i of source i (Fig. 1). Vector $\boldsymbol{\beta}_i$ is a 2×1 vector characterizing the state of polarization of source i and whose components will be defined hereafter. Vector $\boldsymbol{a}(\boldsymbol{\theta}_i,\boldsymbol{\beta}_i)$, $1 \le i \le P$, is the steering vector of the source i, which contains, in particular, the information about the DOA and the polarization of the latter jointly with the characteristics of the sensors and array. Matrix A_g is the $N \times P_g$ matrix containing the steering vectors of the sources belonging to the gth group of sources, whereas $\boldsymbol{m}_g(t)$ is the corresponding $P_g \times 1$ vector of complex envelopes and $\boldsymbol{x}_g(t) \triangleq A_g\boldsymbol{m}_g(t)$. Methods developed in this paper may be implemented in the presence of coupling between sensors. Nevertheless, in the absence of mutual coupling between sensors, assuming a plane-wave propagation, component n of vector $\boldsymbol{a}(\boldsymbol{\theta}_i,\boldsymbol{\beta}_i)$, denoted $a_n(\boldsymbol{\theta}_i,\boldsymbol{\beta}_i)$, can be written, in the general case of an array with space and polarization diversity, as [9]

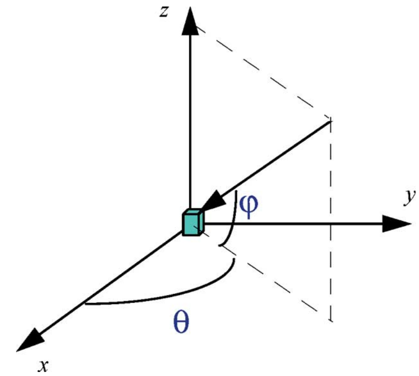


Fig. 1. Incoming signal in 3-D.

$$\begin{aligned}a_n(\boldsymbol{\theta}_i,\boldsymbol{\beta}_i) = f_n(\boldsymbol{\theta}_i,\boldsymbol{\beta}_i)\exp\{\mathrm{j}2\pi[x_n\cos(\theta_i)\cos(\varphi_i)\\ + y_n\sin(\theta_i)\cos(\varphi_i)\\ + z_n\sin(\varphi_i)]/\lambda\}\end{aligned} \tag{2}$$

where λ is the wavelength, $x_n, y_n,$ and z_n are the coordinates of sensor n of the array, and $f_n(\boldsymbol{\theta}_i,\boldsymbol{\beta}_i)$ is a complex number corresponding to the response of sensor n to a unit electric field coming from the direction $\boldsymbol{\theta}_i$ and having the state of polarization $\boldsymbol{\beta}_i$ [9]. Let us recall that an array of sensors has space diversity if all the sensors do not have the same phase center. The array has polarization diversity if all the sensors do not have the same polarization.

Let $\boldsymbol{\beta}_{i1}$ and $\boldsymbol{\beta}_{i2}$ be two distinct polarizations for the source i (for example, vertical and horizontal) and $\boldsymbol{a}_1(\boldsymbol{\theta}_i) \triangleq \boldsymbol{a}(\boldsymbol{\theta}_i,\boldsymbol{\beta}_{i1})$ and $\boldsymbol{a}_2(\boldsymbol{\theta}_i) \triangleq \boldsymbol{a}(\boldsymbol{\theta}_i,\boldsymbol{\beta}_{i2})$ be the corresponding steering vectors for DOA $\boldsymbol{\theta}_i$. We assume that the vectors $\boldsymbol{a}_1(\boldsymbol{\theta})$ and $\boldsymbol{a}_2(\boldsymbol{\theta})$ can be calculated analytically or measured by calibration whatever the value of $\boldsymbol{\theta}$. Considering an arbitrary polarization $\boldsymbol{\beta}_i$ for the source i, the complex electric field of the latter can be broken down into the sum of two complex fields, each arriving from the same direction, and having the polarizations $\boldsymbol{\beta}_{i1}$ and $\boldsymbol{\beta}_{i2}$ [9]. The steering vector $\boldsymbol{a}(\boldsymbol{\theta}_i,\boldsymbol{\beta}_i)$ of the source i is then the weighted sum of the steering vectors $\boldsymbol{a}_1(\boldsymbol{\theta}_i)$ and $\boldsymbol{a}_2(\boldsymbol{\theta}_i)$ given by

$$\boldsymbol{a}(\boldsymbol{\theta}_i,\boldsymbol{\beta}_i) = \beta_{i1}\boldsymbol{a}_1(\boldsymbol{\theta}_i) + \beta_{i2}\boldsymbol{a}_2(\boldsymbol{\theta}_i) \triangleq A_{12}(\boldsymbol{\theta}_i)\boldsymbol{\beta}_i \tag{3}$$

where $A_{12}(\boldsymbol{\theta}_i)$ is the $N \times 2$ matrix of vectors $\boldsymbol{a}_1(\boldsymbol{\theta}_i)$ and $\boldsymbol{a}_2(\boldsymbol{\theta}_i)$, whereas β_{i1} and β_{i2} are complex numbers such that $|\beta_{i1}|^2 +$

$|\beta_{i2}|^2 = 1$. Vector $\boldsymbol{\beta}_i$ is the unit norm 2×1 vector with components β_{i1} and β_{i2}. This vector can be written, to within a phase term, as $\boldsymbol{\beta}_i = [\cos\gamma_i, \mathrm{e}^{\mathrm{j}\phi_i} \sin\gamma_i]^{\mathrm{T}}$ where γ_i and ϕ_i are two angles characterizing the polarization of source i and such that $0 \leq \gamma_i \leq \pi/2$ and $-\pi \leq \phi_i < \pi$. Note that for an array with space diversity only, $\boldsymbol{a}_1(\boldsymbol{\theta}_i)$, $\boldsymbol{a}_2(\boldsymbol{\theta}_i)$, and $\boldsymbol{a}(\boldsymbol{\theta}_i, \boldsymbol{\beta}_i)$ are colinear, which means that, to within a constant, $\boldsymbol{a}(\boldsymbol{\theta}_i, \boldsymbol{\beta}_i)$ does not depend on the polarization of the source i.

B. Statistics of the Data

1) Presentation: The $2q$th-order, $q \geq 1$, DF methods considered in this paper exploit the information contained in the $N^q \times N^q$ $2q$th-order covariance matrix $C_{2q,x}$, whose entries are the $2q$th-order cumulants of the data $\mathrm{Cum}[x_{i_1}(t), \ldots, x_{i_q}(t), x_{i_{q+1}}(t)^*, \ldots, x_{i_{2q}}(t)^*]$, $1 \leq i_j \leq N$, $1 \leq j \leq 2q$, where $*$ corresponds to the complex conjugation. However, the previous entries can be arranged in the $C_{2q,x}$ matrix in different ways, indexed by an integer l such that $0 \leq l \leq q$, as it is explained in [6]. This gives rise, under hypotheses of Section II-A, to the $C_{2q,x}(l)$ matrix given by [6]

$$C_{2q,x}(l) = \sum_{g=1}^{G} C_{2q,x_g}(l) + \eta_2 V(l)\delta(q-1) \tag{4}$$

where η_2 is the mean power of the noise per sensor, $V(l)$ is the $N \times N$ spatial coherence matrix of the noise for the arrangement indexed by l, such that $\mathrm{Tr}[V(l)] = N$, $\mathrm{Tr}[.]$ means trace, and $\delta(.)$ is the Kronecker symbol. The $N^q \times N^q$ matrix $C_{2q,x_g}(l)$ contains the $2q$th-order cumulants of $\boldsymbol{x}_g(t)$ for the arrangement indexed by l and can be written as

$$C_{2q,x_g}(l) = \left[A_g^{\otimes l} \otimes A_g^{*\otimes(q-l)}\right] C_{2q,m_g}(l) \left[A_g^{\otimes l} \otimes A_g^{*\otimes(q-l)}\right]^{\dagger} \tag{5}$$

where $C_{2q,m_g}(l)$ is the $P_g^q \times P_g^q$ matrix of the $2q$th-order cumulants of $\boldsymbol{m}_g(t)$ for the arrangement indexed by l, $\dagger$ corresponds to the conjugate transposition, $\otimes$ is the Kronecker product, and $A_g^{\otimes l}$ is the $N^l \times P_g^l$ matrix defined by $A_g^{\otimes l} \triangleq A_g \otimes A_g \otimes \ldots\ldots \otimes A_g$ with a number of Kronecker product equal to $l-1$. Note that it is shown in [3] and verified in this paper that the parameter l determines, in particular, the maximal processing power of PD-$2q$-MUSIC algorithms.

2) Estimation: In situations of practical interests, the $2q$th-order statistics of the data $\mathrm{Cum}[x_{i_1}(t), \ldots, x_{i_q}(t), x_{i_{q+1}}(t)^*, \ldots, x_{i_{2q}}(t)^*]$ are not known *a priori* and have to be estimated from L samples of data, $\boldsymbol{x}(k) \triangleq \boldsymbol{x}(kT_e)$, $1 \leq k \leq L$, where T_e is the sample period, in a way that is completely described in [6] and which is not recalled here.

III. PD-2q-MUSIC Algorithms

In this section, we analyze the properties of matrix $C_{2q,x}(l)$ and we deduce from the latter three versions, depending on the *a priori* information about the polarization of the sources, of the PD-$2q$-MUSIC algorithm for the arrangement indexed by l.

A. Hypotheses

To develop the PD-$2q$-MUSIC algorithms for the arrangement indexed by l, we have the following hypotheses:

H1) $P_g < N$, $1 \leq g \leq G$;
H2) matrix $A_g^{\otimes l} \otimes A_g^{*\otimes(q-l)}$ has full rank P_g^q, $1 \leq g \leq G$;
H3) $P(G,q) \triangleq \sum_{g=1}^{G} P_g^q < N^q$;
H4) matrix $\overline{A}_{q,l} \triangleq [A_1^{\otimes l} \otimes A_1^{*\otimes(q-l)}, \ldots\ldots, A_G^{\otimes l} \otimes A_G^{*\otimes(q-l)}]$ has full rank $P(G,q)$.

B. Properties of $C_{2q,x}(l)$

Although components of $\boldsymbol{m}_g(t)$ are statistically dependent, the $P_g^q \times P_g^q$ matrix $C_{2q,m_g}(l)$, which contains the $2q$th-order cumulants of $\boldsymbol{m}_g(t)$ for the arrangement indexed by l, may not be full rank for some couples (q,l). This result was unknown before the publication of [6]. Indeed, assuming, for example, that $(q,l) = (2,2)$, it is easy to verify that the maximal rank of $C_{4,m_g}(2)$ is 3 (and not 4) for $P_g = 2$ and 6 (and not 9) for $P_g = 3$. In this context, noting $r_{2q,m_g}(l)$, the rank of $C_{2q,m_g}(l)$, $r_{2q,m_g}(l) \leq P_g^q$, we deduce from H1) and H2) that matrix $C_{2q,xg}(l)$ for $q > 1$ has also rank $r_{2q,m_g}(l)$. Hence, using H4) and for $q > 1$, matrix $C_{2q,x}(l)$ has a rank $r_{2q,x}(l)$ equal to

$$r_{2q,x}(l) = \sum_{g=1}^{G} r_{2q,m_g}(l) \tag{6}$$

and such that $r_{2q,x}(l) < N^q$ from H3). As matrix $C_{2q,x}(l)$ is Hermitian, we deduce that $C_{2q,x}(l)$ has $r_{2q,x}(l)$ real-valued nonzero eigenvalues and $N^q - r_{2q,x}(l)$ zero eigenvalues for $q > 1$.

C. PD-2q-MUSIC Algorithms

1) Case of Sources With Known Polarization (KP-PD-$2q$-MUSIC Algorithm): To built a MUSIC-like algorithm from matrix $C_{2q,x}(l)$, for $q > 1$, we first compute the eigendecomposition of the latter, given by

$$C_{2q,x}(l) = U_{2q,s}(l)\Lambda_{2q,s}(l)U_{2q,s}(l)^{\dagger} + U_{2q,n}(l)\Lambda_{2q,n}(l)U_{2q,n}(l)^{\dagger} \tag{7}$$

where $\Lambda_{2q,s}(l)$ is the $r_{2q,x}(l) \times r_{2q,x}(l)$ diagonal matrix of the nonzero eigenvalues of $C_{2q,x}(l)$, $U_{2q,s}(l)$ is the $N^q \times r_{2q,x}(l)$ unitary matrix of the associated eigenvectors, $\Lambda_{2q,n}(l)$ is the $(N^q - r_{2q,x}(l)) \times (N^q - r_{2q,x}(l))$ diagonal matrix of the zero eigenvalues of $C_{2q,x}(l)$, and $U_{2q,n}(l)$ is the $N^q \times (N^q - r_{2q,x}(l))$ unitary matrix of the associated eigenvectors. As $C_{2q,x}(l)$ is Hermitian, all the columns of $U_{2q,s}(l)$ are orthogonal to all the columns of $U_{2q,n}(l)$. Moreover, $\mathrm{Span}\{U_{2q,s}(l)\} = \mathrm{Span}\{\overline{A}_{q,l}\}$ when matrices $C_{2q,mg}(l)$, $1 \leq g \leq G$, are full rank whereas $\mathrm{Span}\{U_{2q,s}(l)\} \subset \mathrm{Span}\{\overline{A}_{q,l}\}$, otherwise. We define the $N^q \times 1$ vector $\boldsymbol{a}_{q,l}(\boldsymbol{\theta}, \boldsymbol{\beta})$ by

$$\boldsymbol{a}_{q,l}(\boldsymbol{\theta}, \boldsymbol{\beta}) \triangleq \boldsymbol{a}(\boldsymbol{\theta}, \boldsymbol{\beta})^{\otimes l} \otimes \boldsymbol{a}(\boldsymbol{\theta}, \boldsymbol{\beta})^{*\otimes(q-l)}. \tag{8}$$

Then, noting $\boldsymbol{\theta}_{ig}, \boldsymbol{\beta}_{ig}$, the DOA and polarization parameters of the ith source in the gth group, it can be easily verified that, in all cases, the vector $\boldsymbol{a}_{q,l}(\boldsymbol{\theta}_{ig}, \boldsymbol{\beta}_{ig})$ always belongs to $\mathrm{Span}\{U_{2q,s}(l)\}$. Consequently, all vectors $\{\boldsymbol{a}_{q,l}(\boldsymbol{\theta}_{ig}, \boldsymbol{\beta}_{ig}), 1 \leq i \leq P_g, 1 \leq g \leq G\}$ are orthogonal to the columns of $U_{2q,n}(l)$ and are solutions of the following:

$$\boldsymbol{a}_{q,l}(\boldsymbol{\theta}, \boldsymbol{\beta})^{\dagger}\Pi_{2q,n}(l)\boldsymbol{a}_{q,l}(\boldsymbol{\theta}, \boldsymbol{\beta}) = 0 \tag{9}$$

where $\Pi_{2q,n}(l) \triangleq U_{2q,n}(l)U_{2q,n}(l)^\dagger$. Equation (9) corresponds to the heart of the PD-$2q$-MUSIC algorithms for the arrangement l and can also be written, using (3) and (8), as

$$\boldsymbol{\beta}_{q,l}^\dagger A_{12,q,l}(\boldsymbol{\theta})^\dagger \Pi_{2q,n}(l) A_{12,q,l}(\boldsymbol{\theta})\boldsymbol{\beta}_{q,l} = 0 \tag{10}$$

where $\boldsymbol{\beta}_{q,l}$ and $A_{12,q,l}(\boldsymbol{\theta})$ are the $2^q \times 1$ vector and $N^q \times 2^q$ matrix defined by

$$\boldsymbol{\beta}_{q,l} \triangleq \left[\boldsymbol{\beta}^{\otimes l} \otimes \boldsymbol{\beta}^{*\otimes(q-l)}\right] \tag{11}$$

$$A_{12,q,l}(\boldsymbol{\theta}) \triangleq \left[A_{12}(\boldsymbol{\theta})^{\otimes l} \otimes A_{12}(\boldsymbol{\theta})^{*\otimes(q-l)}\right] \tag{12}$$

respectively. For some values of (q,l), some components of $\boldsymbol{\beta}_{q,l}$ are equal and it may be useful to at least reduce the complexity of the computation of the left-hand side of (10), but also to improve the performance of the algorithms presented in Section III-C2, to remove the redundant components of $\boldsymbol{\beta}_{q,l}$. This can be done by removing the redundant components of both $\boldsymbol{\beta}^{\otimes l}$ and $\boldsymbol{\beta}^{*\otimes(q-l)}$. It is straightforward to show that $\boldsymbol{\beta}^{\otimes l}$ can be written as

$$\boldsymbol{\beta}^{\otimes l} = B_l \widetilde{\boldsymbol{\beta}}_l \tag{13}$$

where B_l is the $2^l \times (l+1)$ real matrix such that

$$B_1 = \mathrm{I}_2 \tag{14a}$$

$$B_{l+1} = [B_l \otimes \mathrm{I}_2]\left\{[\mathrm{I}_{l+1} \otimes \boldsymbol{c}_1][\mathrm{I}_{l+1}\ \mathbf{0}_{l+1}] + [\mathrm{I}_{l+1} \otimes \boldsymbol{c}_2][\mathbf{0}_{l+1}\ \mathrm{I}_{l+1}]\right\}, \qquad l > 1 \tag{14b}$$

where I_r is the $r \times r$ identity matrix, $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ are the 2×1 vectors defined by $\boldsymbol{c}_1 \triangleq [1\ 0]^{\mathrm{T}}$ and $\boldsymbol{c}_2 \triangleq [0\ 1]^{\mathrm{T}}$, $\mathbf{0}_{l+1}$ is the $(l+1) \times 1$ zero vector, and $\widetilde{\boldsymbol{\beta}}_l$ is the $(l+1) \times 1$ vector with components $\widetilde{\beta}_l[j]$ defined by

$$\widetilde{\beta}_l[j] = \beta_1^{l-j+1}\beta_2^{j-1}, \qquad 1 \le j \le l+1 \tag{15}$$

where β_1 and β_2 are the components of the polarization vector $\boldsymbol{\beta}$. From (11) and (13), we deduce that

$$\begin{aligned}\boldsymbol{\beta}_{q,l} &= (B_l\widetilde{\boldsymbol{\beta}}_l) \otimes (B_{q-l}\widetilde{\boldsymbol{\beta}}_{q-l})^* \\ &= [B_l \otimes B_{q-l}]\left[\widetilde{\boldsymbol{\beta}}_l \otimes \widetilde{\boldsymbol{\beta}}_{q-l}^*\right] \triangleq [B_l \otimes B_{q-l}]\widetilde{\boldsymbol{\beta}}_{q,l}\end{aligned} \tag{16}$$

where $\widetilde{\boldsymbol{\beta}}_{q,l} \triangleq [\widetilde{\boldsymbol{\beta}}_l \otimes \widetilde{\boldsymbol{\beta}}_{q-l}^*]$ is a $(l+1)(q-l+1) \times 1$ vector. Note a dimension reduction of $\widetilde{\boldsymbol{\beta}}_{q,l}$ with respect to $\boldsymbol{\beta}_{q,l}$ for most values of (q,l). To ensure, in the absence of sources, i.e., when $\Pi_{2q,n}(l) = \mathrm{I}_{N^q}$, a constant value, independent of parameters $\boldsymbol{\theta}$ and $\boldsymbol{\beta}$, of the left-hand side of (10), it is necessary to normalize the latter by the quantity $\boldsymbol{\beta}_{q,l}^\dagger A_{12,q,l}(\boldsymbol{\theta})^\dagger A_{12,q,l}(\boldsymbol{\theta})\boldsymbol{\beta}_{q,l}$. Using (16) into (10), the problem of sources DOA estimation by the PD-$2q$-MUSIC algorithm for the arrangement l then consists of finding the P sets of parameters $(\hat{\boldsymbol{\theta}}_i, \hat{\boldsymbol{\beta}}_i) = (\hat{\theta}_i, \hat{\varphi}_i, \hat{\gamma}_i, \hat{\phi}_i)$, $1 \le i \le P$, which are solutions of (17) or which minimize the left-hand side of the latter, defined by

$$\frac{\widetilde{\boldsymbol{\beta}}_{q,l}^\dagger Q_{q,l,1}(\boldsymbol{\theta})\widetilde{\boldsymbol{\beta}}_{q,l}}{\widetilde{\boldsymbol{\beta}}_{q,l}^\dagger Q_{q,l,2}(\boldsymbol{\theta})\widetilde{\boldsymbol{\beta}}_{q,l}} = 0 \tag{17}$$

where the $(l+1)(q-l+1) \times (l+1)(q-l+1)$ matrices $Q_{q,l,1}(\boldsymbol{\theta})$ and $Q_{q,l,2}(\boldsymbol{\theta})$ are defined by

$$\begin{aligned}Q_{q,l,1}(\boldsymbol{\theta}) \triangleq\ & [B_l \otimes B_{q-l}]^\dagger A_{12,q,l}(\boldsymbol{\theta})^\dagger \Pi_{2q,n}(l) \\ & \times A_{12,q,l}(\boldsymbol{\theta})[B_l \otimes B_{q-l}]\end{aligned} \tag{18}$$

$$\begin{aligned}Q_{q,l,2}(\boldsymbol{\theta}) \triangleq\ & [B_l \otimes B_{q-l}]^\dagger A_{12,q,l}(\boldsymbol{\theta})^\dagger \\ & \times A_{12,q,l}(\boldsymbol{\theta})[B_l \otimes B_{q-l}].\end{aligned} \tag{19}$$

For sources with known polarization, the set of parameters for a given source reduces to the set of its DOA and the complexity of the PD-$2q$-MUSIC algorithm, called in this case known polarization PD-$2q$-MUSIC (KP-PD-$2q$-MUSIC) algorithm, corresponds to that of the $2q$-MUSIC algorithm. However, for sources with unknown polarization, the set of parameters for a given source has to take into account polarization parameters in addition to DOA parameters and the complexity of the searching procedure of the PD-$2q$-MUSIC algorithm dramatically increases beyond what is generally practically reasonable. For this reason, our choice in this paper is to limit the use of the previous algorithm to the case where sources' polarization is known. Otherwise, we consider alternative algorithms which do not require the searching procedure with respect to the polarization parameters and which are presented in Section III-C2. Note that for unknown polarizations, despite the fact that it is not our choice in this paper, solutions of (17) may also be found from a searching procedure in both polarization and DOA parameters. Removing the redundancy of $\boldsymbol{\beta}_{q,l}$ by (16) then allows in this case to decrease the complexity of the searching procedure.

In practical situations, matrices $\Pi_{2q,n}(l)$ and $U_{2q,n}(l)$ have to be estimated from the observations and assuming sources with known polarization, the DOA of the sources may be found by searching for the minima of the estimated left-hand side of (17). The different steps of the KP-PD-$2q$-MUSIC algorithm for the arrangement l are summarized as follows.

1) Estimation $\hat{C}_{2q,x}(l)$ of the matrix $C_{2q,x}(l)$ from L snapshots $\boldsymbol{x}(k)$, $1 \le k \le L$, using a suitable estimator of the $2q$th-order cumulants of observations.
2) Eigenvalue decomposition of the matrix $\hat{C}_{2q,x}(l)$ and extraction of an estimate $\hat{U}_{2q,n}(l)$ of the $U_{2q,n}(l)$ matrix. This step may involve rank determination in cases where the number of sources and/or their mutual statistical dependence are not known *a priori*.
3) Computation, for each known vector $\boldsymbol{\beta} = \boldsymbol{\beta}_i$, $1 \le i \le P$, of the estimated pseudospectrum

$$\hat{P}_{\mathrm{KP-PD-}2q\mathrm{-MUSIC}(l)}(\boldsymbol{\theta}, \boldsymbol{\beta}) \triangleq \frac{\widetilde{\boldsymbol{\beta}}_{q,l}^\dagger \hat{Q}_{q,l,1}(\boldsymbol{\theta})\widetilde{\boldsymbol{\beta}}_{q,l}}{\widetilde{\boldsymbol{\beta}}_{q,l}^\dagger Q_{q,l,2}(\boldsymbol{\theta})\widetilde{\boldsymbol{\beta}}_{q,l}} \tag{20}$$

over a suitably chosen grid. Then, search for the local minima (including interpolation at each local minimum), where the $(l+1)(q-l+1) \times (l+1)(q-l+1)$ matrix $\hat{Q}_{q,l,1}(\boldsymbol{\theta})$ is defined by

$$\begin{aligned}\hat{Q}_{q,l,1}(\boldsymbol{\theta}) \triangleq\ & [B_l \otimes B_{q-l}]^\dagger A_{12,q,l}(\boldsymbol{\theta})^\dagger \hat{\Pi}_{2q,n}(l) \\ & \times A_{12,q,l}(\boldsymbol{\theta})[B_l \otimes B_{q-l}]\end{aligned} \tag{21}$$

where $\hat{\Pi}_{2q,n}(l) \triangleq \hat{U}_{2q,n}(l)\hat{U}_{2q,n}(l)^\dagger$.

In some cases, the number of sources P is known, such that $P < N$, but their statistical dependence is not known. In such a case, $r_{2q,x}(l) \leq P^q$ and a conservative approach may be to use only the $N^q - P^q$ eigenvectors associated with the smallest eigenvalues to built $\hat{U}_{2q,n}(l)$, which implicitly assumes the statistical dependence of all the sources and the full rank of $C_{2q,m_g}(l)$ for the associated group. Finally, note that similarly to $2q$-MUSIC algorithm, PD-$2q$-MUSIC algorithms cannot handle perfectly coherent sources.

2) Case of Sources With Unknown Polarization (UP-PD-$2q$-MUSIC Algorithms): For sources with unknown polarization, the complexity of the searching procedure, described in Section III-C1, of the PD-$2q$-MUSIC algorithm with respect to DOA and polarization parameters is dramatically high. A simple way to remove the searching procedure with respect to the polarization parameter consists, for any fixed DOA, of minimizing the left-hand side of (17) with respect to polarization parameter, as it is proposed in [11] for $q = 1$. This gives rise to the unknown polarization PD-$2q$-MUSIC (UP-PD-$2q$-MUSIC) algorithm whose pseudospectrum, for the arrangement indexed by l, is given by

$$P_{\mathrm{UP-PD-}2q\mathrm{-MUSIC}(l)}(\boldsymbol{\theta}) \triangleq \underset{\widetilde{\boldsymbol{\beta}}_{q,l}}{\mathrm{MIN}} \left\{ \frac{\widetilde{\boldsymbol{\beta}}_{q,l}^{\dagger} Q_{q,l,1}(\boldsymbol{\theta}) \widetilde{\boldsymbol{\beta}}_{q,l}}{\widetilde{\boldsymbol{\beta}}_{q,l}^{\dagger} Q_{q,l,2}(\boldsymbol{\theta}) \widetilde{\boldsymbol{\beta}}_{q,l}} \right\}. \tag{22}$$

It is well known [11] that the right-hand side of (22) corresponds to the minimum eigenvalue $\lambda_{q,l,\min}(\boldsymbol{\theta})$ of the $(l+1)(q-l+1) \times (l+1)(q-l+1)$ matrix $Q_{q,l,1}(\boldsymbol{\theta})$ in the metric $Q_{q,l,2}(\boldsymbol{\theta})$ and that the minimizing vector $\widetilde{\boldsymbol{\beta}}_{q,l}$, noted $\widetilde{\boldsymbol{\beta}}_{q,l,\min}(\boldsymbol{\theta})$, corresponds to the associated eigenvector. In other words, $\lambda_{q,l,\min}(\boldsymbol{\theta})$ and $\widetilde{\boldsymbol{\beta}}_{q,l,\min}(\boldsymbol{\theta})$ satisfy the following:

$$Q_{q,l,1}(\boldsymbol{\theta}) \widetilde{\boldsymbol{\beta}}_{q,l,\min}(\boldsymbol{\theta}) = \lambda_{q,l,\min}(\boldsymbol{\theta}) Q_{q,l,2}(\boldsymbol{\theta}) \widetilde{\boldsymbol{\beta}}_{q,l,\min}(\theta). \tag{23}$$

Thus, a first version of the UP-PD-$2q$-MUSIC algorithm for the arrangement indexed by l, called UP-PD-$2q$-MUSIC(l)-1, consists of finding the P sets of parameters $\hat{\boldsymbol{\theta}}_i = (\hat{\theta}_i, \hat{\varphi}_i)$, $1 \leq i \leq P$, for which the pseudospectrum

$$P_{\mathrm{UP-PD-}2q\mathrm{-MUSIC}(l)-1}(\boldsymbol{\theta}) \triangleq \lambda_{q,l,\min}(\boldsymbol{\theta}) \tag{24}$$

is zero. This algorithm corresponds to a $2q$th-order extension, for the arrangement indexed by l, of the algorithm proposed in [11] for $q = 1$. Then, it is shown in the Appendix that it becomes possible to estimate the polarization of each source i from the associated eigenvector $\widetilde{\boldsymbol{\beta}}_{q,l,\min}(\hat{\boldsymbol{\theta}}_i)$ which is solution of (23) for $\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}_i$. Note that one way in which the eigenvalue $\lambda_{q,l,\min}(\boldsymbol{\theta})$ can be computed is by determining the minimum root of the following:

$$\det\left[Q_{q,l,1}(\boldsymbol{\theta}) - \lambda Q_{q,l,2}(\boldsymbol{\theta})\right] = 0 \tag{25}$$

where $\det[X]$ means determinant of X. Thus, for each value of $\boldsymbol{\theta}$, searching in polarization space has been avoided by finding the roots of an equation of order $(l+1)(q-l+1)$, which corresponds to a substantial reduction in computation, at least for small values of q. We deduce from (25) and [12] that for invertible matrix $Q_{q,l,2}(\boldsymbol{\theta})$, finding $\boldsymbol{\theta}$ such that $\lambda = \lambda_{q,l,\min}(\boldsymbol{\theta})$ is zero is equivalent to finding $\boldsymbol{\theta}$ such that $\det[Q_{q,l,2}(\boldsymbol{\theta})^{-1} Q_{q,l,1}(\boldsymbol{\theta})] = \det[Q_{q,l,1}(\boldsymbol{\theta})]/\det[Q_{q,l,2}(\boldsymbol{\theta})] = 0$. A second version of the UP-PD-$2q$-MUSIC algorithm for the arrangement indexed by l, called UP-PD-$2q$-MUSIC(l)-2, then consists of finding the P sets of parameters $\hat{\boldsymbol{\theta}}_i = (\hat{\theta}_i, \hat{\varphi}_i)$, $1 \leq i \leq P$, for which the pseudospectrum

$$P_{\mathrm{UP-PD-}2q\mathrm{-MUSIC}(l)-2}(\boldsymbol{\theta}) \triangleq \frac{\det\left[Q_{q,l,1}(\boldsymbol{\theta})\right]}{\det\left[Q_{q,l,2}(\boldsymbol{\theta})\right]} \tag{26}$$

is zero, which allows a complexity decrease with respect to the computation of (24).

In practical situations, matrix $U_{2q,n}(l)$ has to be estimated from the observations and, assuming sources with unknown polarization, the DOA of the sources may be found by searching for the minima of the right-hand side of (24) or (26). The different steps of the two versions of the UP-PD-$2q$-MUSIC algorithm for the arrangement l are summarized as follows.

1) Estimation $\hat{C}_{2q,x}(l)$ of the matrix $C_{2q,x}(l)$ from L snapshots $\boldsymbol{x}(k)$, $1 \leq k \leq L$, using a suitable estimator of the $2q$th-order cumulants of observations.
2) Eigenvalue decomposition of the matrix $\hat{C}_{2q,x}(l)$ and extraction of an estimate $\hat{U}_{2q,n}(l)$ of the $U_{2q,n}(l)$ matrix. This step may involve rank determination in cases where the number of sources and/or their mutual statistical dependence are not known *a priori*.
3) Computation of matrices $Q_{q,l,2}(\boldsymbol{\theta})$, $\hat{Q}_{q,l,1}(\boldsymbol{\theta})$, and one of the two estimated pseudospectra
$$\hat{P}_{\mathrm{UP-PD-}2q\mathrm{-MUSIC}(l)-1}(\theta) \triangleq \hat{\lambda}_{q,l,\min}(\theta)$$
$$\hat{P}_{\mathrm{UP-PD-}2q\mathrm{-MUSIC}(l)-2}(\theta) \triangleq \frac{\det\left[\hat{Q}_{q,l,1}(\boldsymbol{\theta})\right]}{\det\left[Q_{q,l,2}(\boldsymbol{\theta})\right]}$$
over a suitably chosen grid. Then, search for the local minima of $\hat{P}_{\mathrm{UP-PD-}2q\mathrm{-MUSIC}(l)-1}(\boldsymbol{\theta})$ or $\hat{P}_{\mathrm{UP-PD-}2q\mathrm{-MUSIC}(l)-2}(\boldsymbol{\theta})$ (including interpolation at each local minimum), where $\hat{\lambda}_{q,l,\min}(\boldsymbol{\theta})$ is the minimum eigenvalue of $\hat{Q}_{q,l,1}(\boldsymbol{\theta})$ in the metric $Q_{q,l,2}(\boldsymbol{\theta})$.
4) If needed, computation of both the associated estimated vectors $\hat{\widetilde{\boldsymbol{\beta}}}_{q,l,\min}(\hat{\boldsymbol{\theta}}_i)$ and the polarization vector of the sources.

IV. Identifiability

Although algorithms presented in Section III may be used in the presence of coupling between sensors, provided that the matrix $A_{12}(\boldsymbol{\theta})$ is known or can be estimated by calibration, the identifiability analysis presented in this section assumes the absence of coupling between sensors. Moreover, as the maximal number of sources that can be processed by a given version of the PD-$2q$-MUSIC algorithm is obtained when all the sources are statistically independent, we limit the identifiability analysis of the three algorithms introduced in Section III to the latter case.

A. KP-PD-$2q$-MUSIC Algorithm

1) General Results: Following the developments of Section III, we deduce that the KP-PD-$2q$-MUSIC algorithm for the arrangement indexed by l is able to estimate the

DOA of P statistically independent sources from an array of N sensors provided that hypotheses H1)–H4) are verified and the DOA and the polarization of the sources are the only solutions of (9). It has been shown in [3] that, in the absence of coupling between sensors, the vector $\boldsymbol{a}_{q,l}(\boldsymbol{\theta},\boldsymbol{\beta})$ can be considered as a *true steering vector* but for a HO VA of N^q virtual sensors (VSs) with coordinates $(x^l_{k_1,k_2,\ldots,k_q}, y^l_{k_1,k_2,\ldots,k_q}, z^l_{k_1,k_2,\ldots,k_q})$ and complex amplitude patterns $f^l_{k_1,k_2,\ldots,k_q}(\boldsymbol{\theta},\boldsymbol{\beta})$, $1 \le k_j \le N$ for $1 \le j \le q$, given by

$$\begin{aligned}&\left(x^l_{k_1 k_2,\ldots,k_q}, y^l_{k_1 k_2,\ldots,k_q}, z^l_{k_1 k_2,\ldots,k_q}\right)\\&=\left(\sum_{j=1}^{l} x_{k_j} - \sum_{u=1}^{q-l} x_{k_{l+u}}, \sum_{j=1}^{l} y_{k_j} - \sum_{u=1}^{q-l} y_{k_{l+u}},\right.\\&\qquad\left.\sum_{j=1}^{l} z_{k_j} - \sum_{u=1}^{q-l} z_{k_{l+u}}\right)\end{aligned} \tag{27}$$

$$f^l_{k_1,k_2,\ldots,k_q}(\boldsymbol{\theta},\boldsymbol{\beta}) = \prod_{j=1}^{l}\prod_{u=1}^{q-l} f_{k_j}(\boldsymbol{\theta},\boldsymbol{\beta}) f_{k_{l+u}}(\boldsymbol{\theta},\boldsymbol{\beta})^*. \tag{28}$$

As some of these N^q VSs may coincide, we note N^l_{2q}, $N^l_{2q} \le N^q$, the number of different VSs of the associated VA. In these conditions, $N^q - N^l_{2q}$ components of all the vectors $\boldsymbol{a}_{q,l}(\boldsymbol{\theta},\boldsymbol{\beta})$ are redundant components that bring no information. As a consequence, the rank of $\overline{A}_{q,l}$ cannot be greater than N^l_{2q}. We then deduce that the $\overline{A}_{q,l}$ matrix may have a rank equal to P only if $P \le N^l_{2q}$. Conversely, for a $2q$th-order VA for the arrangement l without any ambiguities up to order $N^l_{2q} - 1$, P sources coming from P different directions with different polarizations generate an $\overline{A}_{q,l}$ matrix with a full rank P as long as $P \le N^l_{2q}$. Let us recall that the $2q$th-order VA for the arrangement indexed by l has no ambiguities of order $r-1$ if any set of r vectors $(\boldsymbol{a}_{q,l}(\boldsymbol{\theta}_1,\boldsymbol{\beta}_1),\ldots,\boldsymbol{a}_{q,l}(\boldsymbol{\theta}_r,\boldsymbol{\beta}_r))$ with distinct parameters $(\boldsymbol{\theta}_i,\boldsymbol{\beta}_i)$, $1 \le i \le r$, are linearly independent. Thus, provided the $2q$th-order VA for the arrangement indexed by l has no ambiguities up to the order $N^l_{2q} - 1$, the maximal number of statistically independent sources able to generate a matrix $\overline{A}_{q,l}$ with rank P is N^l_{2q}. However, when $P = N^l_{2q}$, an arbitrary vector $\boldsymbol{a}_{q,l}(\boldsymbol{\theta},\boldsymbol{\beta})$ associated with an arbitrary set of parameters $(\boldsymbol{\theta},\boldsymbol{\beta})$ is necessarily a linear combination of the source steering vectors $\boldsymbol{a}_{q,l}(\boldsymbol{\theta}_i,\boldsymbol{\beta}_i)$, $1 \le i \le N^l_{2q}$, since matrix $\overline{A}_{q,l}$ cannot have a rank greater than N^l_{2q}. Then, all the sets of parameters $(\boldsymbol{\theta},\boldsymbol{\beta})$ are solutions of (9), which does not allow the sources' DOA estimation. Thus, a necessary condition for the DOAs and polarizations of the sources to be the only solutions of (9) is that $P < N^l_{2q}$ and this condition becomes sufficient for HO VA with no ambiguities up to the order $N^l_{2q} - 1$. From the previous results, assuming a $2q$th-order VA for the arrangement indexed by l with N^l_{2q} different VSs and with no ambiguities up to order $N^l_{2q} - 1$, we deduce that the KP-PD-$2q$-MUSIC algorithm for the arrangement indexed by l is able to process up to

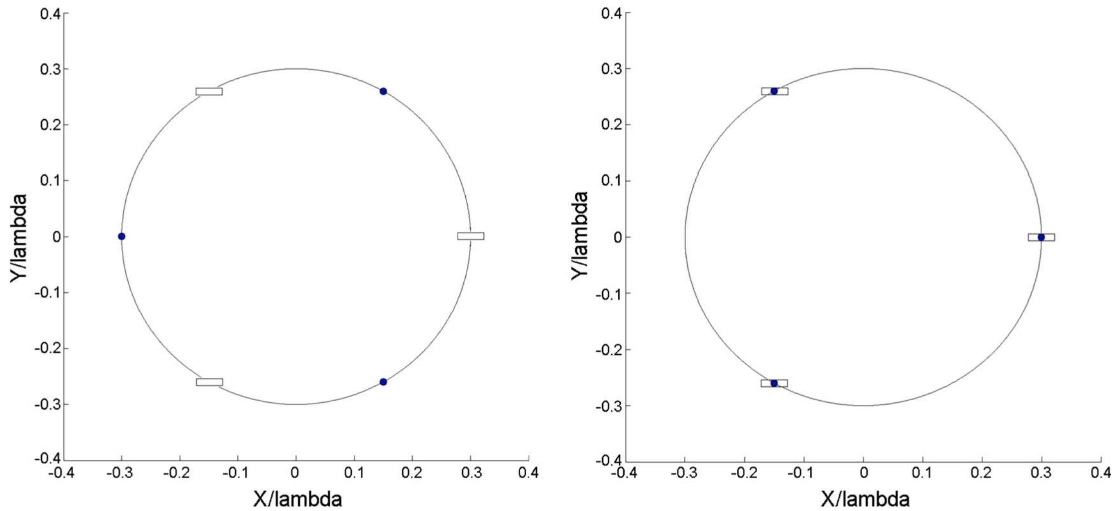
$$P_{\max} \triangleq P^l_{2q,K} = N^l_{2q} - 1 \tag{29}$$

sources. As, for a given array of N sensors, N^l_{2q} is a function of q and l [3]; the processing capacity of the KP-PD-$2q$-MUSIC algorithm is also a function of q and l. This shows off, in particular, the existence of an optimal arrangement of the $2q$th-order data statistics for a given value of q, which is discussed in [3].

Note that the problem of rth-order ambiguities of HO VA is an important open problem which deserves to be analyzed in detail but which is beyond the scope of this paper. For a VA associated with the parameters (q,l) and without colocalized or vector sensors, i.e., with scalar sensors only, all the N^l_{2q} different VSs bring information. Then, despite the potential existence of HO ambiguities of the VA for some sources' configurations, there always exist, in general, some sources configurations for which matrix $\overline{A}_{q,l}$ has a rank equal to N^l_{2q} and for which KP-PD-$2q$-MUSIC algorithm for the arrangement indexed by l is able to process $N^l_{2q} - 1$ sources. However, for a VA associated with the parameters (q,l) such that some of the N^l_{2q} different sensors are colocalized, some of the colocalized sensors may bring no information as it is shown and discussed, for $q = 1$, in [17]–[19], [27], [33]–[35] for electric and electromagnetic vector sensors, respectively. In this case, $\overline{A}_{q,l}$ matrix may not have a maximal rank equal to N^l_{2q} and the KP-PD-$2q$-MUSIC algorithm for the arrangement indexed by l may only be able to process ${N^l_{2q}}' - 1$ sources, at least in some situations, where ${N^l_{2q}}'$ is the maximal possible rank of $\overline{A}_{q,l}$. To quantify expression of N^l_{2q} as a function of N, some values of N^l_{2q} are presented in Sections IV-A2 and IV-A3 for different arrays' and sensors' configurations.

2) Case of an Array With Different Sensors: For given values of N, q, and l and for arrays of N sensors with both space and polarization diversities, it has been shown in [3] that N^l_{2q} is necessary upperbounded by a quantity, noted $N_{\max}[2q,l]$, such that $N_{\max}[2q,l] \le N^q$. Table I shows, for a general array with space and polarization diversities having sensors arbitrary located with different responses, the expression of $N_{\max}[2q,l]$ as a function of N for $2 \le q \le 4$ and several values of l. This upperbound corresponds to N^l_{2q} in most cases of sensors' responses and array geometry. Moreover, [3, Tables 1 and 3] show that for $(q,l) = (2,2), (3,3)$, or $(4,4)$ and for arrays with no particular symmetry, the associated VA has no vector sensors. In these cases, expression (29) generally holds. On the contrary, [3, Tables 1 and 3] show that for $(q,l) = (2,1), (3,2), (4,3)$, or $(4,2)$, the associated VA has at least one vector sensor with at least N different components. In these cases, ${N^l_{2q}}'$ may be strictly lower than N^l_{2q} and expression (29) may not hold, especially for high values of N.

3) Case of an Array With Two Subarrays of Sensors Having Orthogonal Polarizations: A particular case of practical interest corresponds to the case of an array of $N = 2M$ sensors composed of two subarrays of M sensors having orthogonal polarizations. Two kinds of such arrays are considered in this section and correspond to arrays for which the sensors of the two subarrays are either colocated or not. Examples of noncolocated and colocated subarrays of $M = 3$ sensors are presented in Fig. 2 for $N = 2M = 6$. Fig. 2(a) shows a circular array of six equispaced scalar sensors composed of two overlapped orthogonally polarized circular subarrays of three scalar sensors such that two adjacent sensors of the array have different polarizations. Fig. 2(b) shows a circular array of three equispaced and identical vectorial sensors such that each vectorial sensor is composed of two orthogonally polarized sensors having the



(a) (b)

Fig. 2. Circular array of six equispaced sensors composed of two overlapped orthogonally polarized subarrays of three sensors: (a) noncollocated subarrays and (b) collocated subarrays.

TABLE I
$N_{\max}[2q, l]$ AS A FUNCTION OF N FOR SEVERAL VALUES OF q AND l AND FOR ARRAYS WITH SPACE AND POLARIZATION DIVERSITY

$m = 2q$	l	$N_{max}[2q, l]$
4 ($q = 2$)	2	$N(N+1)/2$
	1	N^2
6 ($q = 3$)	3	$N!/[6(N-3)!] + N(N-1) + N$
	2	$N!/[2(N-3)!] + 2N(N-1) + N$
8 ($q = 4$)	4	$N!/[24(N-4)!] + N!/[2(N-3)!] + 1.5N(N-1) + N$
	3	$N!/[6(N-4)!] + 1.5N!/(N-3)! + 3N(N-1) + N$
	2	$N!/[4(N-4)!] + 2N!/(N-3)! + 3.5N(N-1) + N$

TABLE II
$N_{\max}[2q, l]$ AS A FUNCTION OF $N = 2M$ FOR $q = 2$ AND SEVERAL VALUES OF l AND FOR ARRAYS WITH TWO ORTHOGONALLY POLARIZED NONCOLOCATED SUBARRAYS

$m = 2q$	l	$N_{max}[2q,l]$
4 ($q = 2$)	2	$N(N+1)/2$
	1	$N^2 - N + 2$

TABLE III
$N_{\max}[2q, l]$ AS A FUNCTION OF $N = 2M$ FOR $q = 2$ AND SEVERAL VALUES OF l AND FOR ARRAYS WITH TWO ORTHOGONALLY POLARIZED COLLOCATED SUBARRAYS

$m = 2q$	l	$N_{max}[2q,l]$
4 ($q = 2$)	2	$3N(N+2)/8$
	1	$N^2 - 2N + 4$

same phase center. Tables II and III show, for noncolocated and colocated subarrays, respectively, and from (27) and (28), the expression of $N_{\max}[2q, l]$ as a function of N for $q = 2$ and several values of l. Note that this upperbound corresponds to N_{2q}^{l} in most cases of array geometry with no particular symmetry, which is, in particular, the case for uniform circular array (UCA) of M vectorial sensors with two components, when M is a prime number, as depicted in Fig. 2(b) for $M = 3$. However, while fourth-order VA associated with noncolocated subarrays contains no vector sensor for $(q, l) = (2, 2)$, it contains several scalar sensors and one vector sensor with two components for $(q, l) = (2, 1)$. Besides, the fourth-order VA associated with colocated subarrays contains only vector sensors with two or three components for $(q, l) = (2, 2)$ and with two or four components for $(q, l) = (2, 1)$. As a consequence, for noncolocated subarrays, (29) holds for $(q, l) = (2, 2)$ and probably for $(q, l) = (2, 1)$ but may not hold for $q = 2$ and colocated subarrays, but this potential result has to be verified. Finally, Table IV shows the expression of N_{2q}^{l} as a function of $N = 2M$ for $q = 2$ and several values of l, for an array composed of two colocalized and orthogonally polarized uniformly spaced linear array (ULA) of M identical sensors. Note that, in this case, fourth-order VA contains $2M - 1 = N - 1$ vector sensors with three and four components for $(q, l) = (2, 2)$ and $(q, l) = (2, 1)$, respectively.

B. UP-PD-2q-MUSIC Algorithms

The developments of Section IV-A are still valid for UP-PD-2q-MUSIC algorithms. In particular, the maximal number of statistically independent sources that may be processed by UP-PD-2q-MUSIC algorithms for the arrangement indexed by l cannot exceeds $N_{2q}^{l} - 1$ if the associated VA has no ambiguities up to the order $N_{2q}^{l} - 1$. Moreover, we deduce from the HO VA theory [3] that $N^q - N_{2q}^{l}$ components of

TABLE IV
N_{2q}^l AS A FUNCTION OF $N = 2M$ FOR $q = 2$ AND SEVERAL VALUES OF l AND FOR AN ARRAY WITH TWO ORTHOGONALLY POLARIZED AND COLOCATED UNIFORMLY SPACED LINEAR SUBARRAYS

$m = 2q$	l	N_{2q}^l
4	2	$3(N-1)$
($q = 2$)	1	$4(N-1)$

$\boldsymbol{a}_{q,l}(\boldsymbol{\theta},\boldsymbol{\beta})$, defined by (8), bring no information. This means that $\boldsymbol{a}_{q,l}(\boldsymbol{\theta},\boldsymbol{\beta})$ can be written as

$$\boldsymbol{a}_{q,l}(\boldsymbol{\theta},\boldsymbol{\beta}) \triangleq G\boldsymbol{a}_{q,l,nr}(\boldsymbol{\theta},\boldsymbol{\beta}) \tag{30}$$

where G is a full rank and constant $N^q \times N_{2q}^l$ matrix and $\boldsymbol{a}_{q,l,nr}(\boldsymbol{\theta},\boldsymbol{\beta})$ is the nonredundant $N_{2q}^l \times 1$ steering vector of a source coming from DOA $\boldsymbol{\theta}$ with polarization $\boldsymbol{\beta}$ for the VA associated with parameters (q,l). Expression (30) shows that an arbitrary steering vector $\boldsymbol{a}_{q,l}(\boldsymbol{\theta},\boldsymbol{\beta})$ necessarily belongs to the space spanned by the N_{2q}^l columns of G, noted $\mathrm{span}\{G\}$. Noting $G^\perp$, a $N^q \times (N^q - N_{2q}^l)$ full-rank matrix whose columns span the space orthogonal to $\mathrm{span}\{G\}$, we deduce that all the vectors $\boldsymbol{u} = G^\perp \boldsymbol{v}$ of $\mathrm{Im}\{G^\perp\}$, where $\boldsymbol{v}$ is an arbitrary $(N^q - N_{2q}^l) \times 1$ vector, are orthogonal to $\boldsymbol{a}_{q,l}(\boldsymbol{\theta},\boldsymbol{\beta})$ for arbitrary values of $(\boldsymbol{\theta},\boldsymbol{\beta})$. A direct consequence of this result is that whatever the number P of statistically independent sources such that $P \le N_{2q}^l - 1$, and whatever their DOA and polarization $\mathrm{Im}\{G^\perp\} \subset \mathrm{span}\{U_{2q,n}(l)\}$, this means that $N^q - N_{2q}^l$ columns of $U_{2q,n}(l)$ are not discriminant. In other words, we deduce from (30) and the previous results that only $N_{2q}^l - P$ columns of $U_{2q,n}(l)$ are discriminant, while expression (9) takes the form

$$\begin{aligned}\boldsymbol{a}_{q,l,nr}(\boldsymbol{\theta},\boldsymbol{\beta})^\dagger G^\dagger \Pi_{2q,n}(l) G \boldsymbol{a}_{q,l,nr}(\boldsymbol{\theta},\boldsymbol{\beta}) \\ = \boldsymbol{a}_{q,l}(\boldsymbol{\theta},\boldsymbol{\beta})^\dagger \Pi_{2q,n,G}(l)\boldsymbol{a}_{q,l}(\boldsymbol{\theta},\boldsymbol{\beta}) = 0\end{aligned} \tag{31}$$

where $\Pi_{2q,n,G}(l)$ is the $N^q \times (N_{2q}^l - P)$ orthogonal projector on $\mathrm{span}\{U_{2q,n}(l)\} \cap \mathrm{span}\{G\}$. Replacing (9) by (31) in the developments of Section III, we deduce that, for given values of (q,l), $\lambda_{q,l,\min}(\boldsymbol{\theta})$ and $\widetilde{\boldsymbol{\beta}}_{q,l,\min}(\boldsymbol{\theta})$ are also solution of (23) where $Q_{q,l,1}(\boldsymbol{\theta})$ has been replaced by $Q_{q,l,1,G}(\boldsymbol{\theta})$ defined by

$$\begin{aligned}Q_{q,l,1,G}(\boldsymbol{\theta}) \triangleq [B_l \otimes B_{q-l}]^\dagger A_{12,q,l}(\boldsymbol{\theta})^\dagger \Pi_{2q,n,G}(l) \\ \times A_{12,q,l}(\boldsymbol{\theta})[B_l \otimes B_{q-l}].\end{aligned} \tag{32}$$

As the quantity $\lambda_{q,l,\min}(\boldsymbol{\theta})$, defined by (23) with $Q_{q,l,1,G}(\boldsymbol{\theta})$ instead of $Q_{q,l,1}(\boldsymbol{\theta})$, has to be nulled only for the DOA of the sources and not for other DOAs, the $(l+1)(q-l+1) \times (l+1)(q-l+1)$ matrix $Q_{q,l,1,G}(\boldsymbol{\theta})$ has to be full rank when $\boldsymbol{\theta}$ does not correspond to a source's DOA. Using (32), this means that rank of $\Pi_{2q,n,G}(l)$ cannot be lower than $(l+1)(q-l+1)$. This means that the number of columns of $\Pi_{2q,n,G}(l)$ has to be greater than or equal to $(l+1)(q-l+1)$. Moreover, in the presence of P statistically independent sources such that $P \le N_{2q}^l - 1$, the number of columns of $\Pi_{2q,n,G}(l)$ is equal to $N_{2q}^l - P$ for associated VA with no ambiguities up to the order $N_{2q}^l - 1$. As a consequence, the maximal number of sources $P_{\max}$ that may be processed by UP-PD-$2q$-MUSIC algorithms for the arrangement indexed by l has to, for such VAs, verify $P_{\max} \le N_{2q}^l - (l+1)(q-l+1)$. Conversely, for a $2q$th-order VA without any ambiguities up to the order $N_{2q}^l - 1$, P sources coming from P different directions with different polarizations and such that $P \le N_{2q}^l - (l+1)(q-l+1)$ are such that their DOA are the only solutions of $\lambda_{q,l,\min}(\boldsymbol{\theta}) = 0$. From the previous results, assuming a $2q$th-order VA for the arrangement indexed by l with N_{2q}^l different VSs and with no ambiguities up to the order $N_{2q}^l - 1$, we deduce that UP-PD-$2q$-MUSIC algorithms for the arrangement l are able to process up to

$$P_{\max} \triangleq P_{2q,U}^l = N_{2q}^l - (l+1)(q-l+1) \tag{33}$$

sources. This is strictly lower than (29) and this gives $P_{\max} = N - 2$ for $q = 1$ and arrays with scalar sensors, result already obtained in [11]. Note that for VA with HO ambiguities, N_{2q}^l has to be replaced by $N_{2q}^{l\,\prime}$ in (33).

V. VIRTUAL ARRAY RESOLUTION

To get more insights into the gain in resolution obtained with HO VA with polarization diversity, let us compute the normalized inner product of the steering vectors $\boldsymbol{a}_{q,l}(\boldsymbol{\theta},\boldsymbol{\beta})$ and $\boldsymbol{a}_{q,l}(\boldsymbol{\theta}_0,\boldsymbol{\beta}_0)$ for two arbitrary couples $(\boldsymbol{\theta},\boldsymbol{\beta})$ and $(\boldsymbol{\theta}_0,\boldsymbol{\beta}_0)$. This quantity is denoted by $\alpha_{2q,l}(\boldsymbol{\theta},\boldsymbol{\beta},\boldsymbol{\theta}_0,\boldsymbol{\beta}_0)$ and is such that $|\alpha_{2q,l}(\boldsymbol{\theta},\boldsymbol{\beta},\boldsymbol{\theta}_0,\boldsymbol{\beta}_0)| \le 1$. Using the results of [3], we obtain

$$|\alpha_{2q,l}(\boldsymbol{\theta},\boldsymbol{\beta},\boldsymbol{\theta}_0,\boldsymbol{\beta}_0)| = |\alpha_{2,1}(\boldsymbol{\theta},\boldsymbol{\beta},\boldsymbol{\theta}_0,\boldsymbol{\beta}_0)|^q \tag{34}$$

which shows an increasing resolution as q increases. To show also the interest of exploiting polarization diversity, we consider an array of $N = 2M$ sensors composed of two colocalized and orthogonally polarized subarrays of M sensors. We define the 2×1 vector $\boldsymbol{\beta}_f \triangleq [\beta_1 f_1, \beta_2 f_2]^{\mathrm{T}}$, where β_1 and β_2 are the two components of $\boldsymbol{\beta}$ and where f_1 and f_2 are the complex responses of the two orthogonally polarized components of a vector sensor to a unitary source coming from DOA $\boldsymbol{\theta}$ with adapted polarizations, respectively. Under these assumptions, it is straightforward to show that (34) becomes

$$|\alpha_{2q,l}(\boldsymbol{\theta},\boldsymbol{\beta},\boldsymbol{\theta}_0,\boldsymbol{\beta}_0)| = |\alpha_{2,1,sd}(\boldsymbol{\theta},\boldsymbol{\theta}_0)|^q \, |\alpha_{\beta f}(\boldsymbol{\theta},\boldsymbol{\beta},\boldsymbol{\theta}_0,\boldsymbol{\beta}_0)|^q \tag{35}$$

where $\alpha_{2,1,sd}(\boldsymbol{\theta},\boldsymbol{\theta}_0)$, such that $|\alpha_{2,1,sd}(\boldsymbol{\theta},\boldsymbol{\theta}_0)| \le 1$, is the normalized inner product of the steering vectors of the two sources for the array, with space diversity only, composed of M omnidirectional sensors located at the positions of the vector sensors of the initial array. In addition, $\alpha_{\beta f}(\boldsymbol{\theta},\boldsymbol{\beta},\boldsymbol{\theta}_0,\boldsymbol{\beta}_0)$, such that $|\alpha_{\beta f}(\boldsymbol{\theta},\boldsymbol{\beta},\boldsymbol{\theta}_0,\boldsymbol{\beta}_0)| \le 1$, is the normalized inner product of the vectors $\boldsymbol{\beta}_f$ and $\boldsymbol{\beta}_{f0}$, where $\boldsymbol{\beta}_{f0}$ corresponds to $\boldsymbol{\beta}_f$ with $(\boldsymbol{\theta}_0,\boldsymbol{\beta}_0)$ replacing $(\boldsymbol{\theta},\boldsymbol{\beta})$. For elementary sensors such that $f_1 = f_2$, $|\alpha_{\beta f}(\boldsymbol{\theta},\boldsymbol{\beta},\boldsymbol{\theta}_0,\boldsymbol{\theta}_0)| < 1$ as soon as the two sources have different polarization. For elementary sensors such that $f_1 \ne f_2$, $|\alpha_{\beta f}(\boldsymbol{\theta},\boldsymbol{\beta},\boldsymbol{\theta}_0,\boldsymbol{\beta}_0)| < 1$ as soon as the two sources have either different polarization or different DOA. This shows an increasing resolution obtained with an array with polarization diversity, at least for sources with different polarization.

VI. Computer Simulations

The results of the previous sections are illustrated in this section through computer simulations. To do so, we first introduce a performance criterion in Section VI-A and describe the simulations in Sections VI-B and VI-C for overdetermined and underdetermined mixtures of sources, respectively. The sources are assumed to have a zero elevation angle φ and to be zero-mean stationary sources corresponding to quaternary phase-shift keying (QPSK) sources sampled at the symbol rate.

A. Performance Criterion

For each of the P considered sources and for a given DF method, two criteria are used in the following to quantify the quality of the associated DOA estimation. For a given source, the first criterion is a probability of aberrant results generated by a given method for this source. The second one is an averaged root mean square error (RMSE), computed from the nonaberrant results, generated by a given method for this source. These two criteria were precisely defined in [6] and are not recalled in this paper.

B. Overdetermined Mixtures of Sources

To show the interest of taking into account both the polarization of the sources and HO statistics for DF, we consider a UCA of $N = 6$ crossed dipoles with a radius r such that $r = 0.3\lambda$. One dipole is parallel to the x-axis whereas the other is parallel to the z-axis. Three of these crossed dipoles are combined to generate a right sense circular polarization in the y-axis while the three other dipoles are combined to generate a left sense circular polarization in the y-axis. The array is then composed of two orthogonally polarized overlapped (noncolocated) circular subarrays of $M = 3$ sensors so that adjacent sensors always have different polarizations as depicted in Fig. 2(a). Under these assumptions, the sensors of the first and second subarray have a complex response to a unit electric field coming from DOA $\boldsymbol{\theta}$ with polarization $\boldsymbol{\beta}$ equal to $f(\boldsymbol{\theta},\boldsymbol{\beta}) = g_1(\boldsymbol{\theta},\boldsymbol{\beta}) + \mathrm{j}g_2(\boldsymbol{\theta},\boldsymbol{\beta})$ and $f'(\boldsymbol{\theta},\boldsymbol{\beta}) = g_1(\boldsymbol{\theta},\boldsymbol{\beta}) - \mathrm{j}g_2(\boldsymbol{\theta},\boldsymbol{\beta})$, respectively. In these expressions, $g_1(\boldsymbol{\theta},\boldsymbol{\beta})$ and $g_2(\boldsymbol{\theta},\boldsymbol{\beta})$, which correspond to the complex responses of the two dipoles, are given by [4], [9]

$$\begin{aligned} g_1(\boldsymbol{\theta},\boldsymbol{\beta}) &= g_1(\theta,\varphi,\gamma,\phi) \\ &= \sin\gamma\sin\varphi\cos\theta \mathrm{e}^{\mathrm{j}\phi} - \cos\gamma\sin\theta \quad (36) \\ g_2(\boldsymbol{\theta},\boldsymbol{\beta}) &= g_2(\theta,\varphi,\gamma,\phi) = -\sin\gamma\cos\varphi \mathrm{e}^{\mathrm{j}\phi}. \quad (37) \end{aligned}$$

In other words, the complex response $f_n(\boldsymbol{\theta},\boldsymbol{\beta})$ of sensor n, $1 \le n \le N$, to a unit electric field coming from DOA $\boldsymbol{\theta}$ with polarization $\boldsymbol{\beta}$ is given by

$$f_n(\boldsymbol{\theta},\boldsymbol{\beta}) = (\sin\varphi\cos\theta + \mathrm{j}(-1)^n\cos\varphi)\sin\gamma \mathrm{e}^{\mathrm{j}\phi} - \cos\gamma\sin\theta. \quad (38)$$

We then deduce from (2), (3), and (38) that, in this case, the $2N$ coefficients of matrix $A_{12}(\boldsymbol{\theta})$ are defined by

$$A_{12}(\boldsymbol{\theta})[n,1] = -\sin\theta \mathrm{e}^{\mathrm{j}\zeta_n} \quad (39a)$$
$$A_{12}(\boldsymbol{\theta})[n,2] = (\sin\varphi\cos\theta + \mathrm{j}(-1)^n\cos\varphi)\,\mathrm{e}^{\mathrm{j}\zeta_n} \quad (39b)$$

where $1 \le n \le N$ and where $\zeta_n, 1 \le n \le N$, is defined by

$$\zeta_n \triangleq 2\pi\left[x_n\cos(\theta)\cos(\varphi) + y_n\sin(\theta)\cos(\varphi) + z_n\sin(\varphi)\right]/\lambda. \quad (40)$$

Note that this corresponds to choosing the vectors $\boldsymbol{a}_1(\boldsymbol{\theta}) \triangleq \boldsymbol{a}(\boldsymbol{\theta},\boldsymbol{\beta}_1)$ and $\boldsymbol{a}_2(\boldsymbol{\theta}) \triangleq \boldsymbol{a}(\boldsymbol{\theta},\boldsymbol{\beta}_2)$ such that $\boldsymbol{\beta}_1 = [1,0]^{\mathrm{T}}$ and $\boldsymbol{\beta}_2 = [0,1]^{\mathrm{T}}$. Note that the chosen array of sensors presents ambiguities for $\boldsymbol{\theta}_0 = (0, k\pi)$, where k is an integer, and thus prevents from estimating DOA of sources coming from $\boldsymbol{\theta}_0$. Indeed, $\boldsymbol{a}_1(\boldsymbol{\theta}_0) = 0$, $Q_{q,l,1}(\boldsymbol{\theta}_0)$ and $Q_{q,l,2}(\boldsymbol{\theta}_0)$ are not full rank, and $\boldsymbol{\theta}_0$ is always solution of (10), (23), and $\lambda_{q,l,\min}(\boldsymbol{\theta}_0) = 0$. In this context, two QPSK sources with the same symbol duration, the same raise cosine pulse shaped filter with a rolloff $= 0.3$, and the same input signal-to-noise ratio (SNR), which would be received by an omnidirectional sensor, equal to 5 dB, are assumed to be received by the array. The sources are first assumed to be weakly separated in both space and polarization and such that $(\theta_1,\gamma_1,\phi_1) = (50°, 45°, 0°)$ and $(\theta_2,\gamma_2,\phi_2) = (60°, 45°, 10°)$, respectively. Under these assumptions, Figs. 3 and 4 show the variations, as a function of the number of snapshots L, of the RMSE for the source 2, RMSE_2, and the associated probability of nonabberant results for DF methods with and without modeling errors, respectively. The DF methods correspond to 2-MUSIC, KP-PD-2-MUSIC, UP-PD-2-MUSIC-1, UP-PD-2-MUSIC-2, 4-MUSIC, KP-PD-4-MUSIC, UP-PD-4-MUSIC-1, UP-PD-4-MUSIC-2, 6-MUSIC, KP-PD-6-MUSIC, UP-PD-6-MUSIC-1, and UP-PD-6-MUSIC-2 algorithms for arrangement of the considered statistics indexed by $l = 1$. The performances are computed from 300 realizations and similar results are obtained for the source 1. In the presence of modeling errors, the steering vector of the source i at the output of the sensors becomes $\boldsymbol{a}(\boldsymbol{\theta}_i,\boldsymbol{\beta}_i) + \boldsymbol{e}_i$, where the vectors $\boldsymbol{e}_i$, $1 \le i \le 2$, are assumed to be zero-mean statistically independent circular Gaussian vectors such that $\mathrm{E}[\boldsymbol{e}_i\,\boldsymbol{e}_j^\dagger] = \sigma^2\delta_{ij}\mathrm{I}_N$. For the simulations, $\sigma^2 = 0.0003$, which corresponds, for example, to a phase error with a standard deviation of 0.76° plus an amplitude error with a standard deviation of 0.1 dB. For 2-MUSIC, 4-MUSIC, and 6-MUSIC algorithms, the six sensors of the UCA are assumed to be identical with complex responses $f_n(\boldsymbol{\theta},\boldsymbol{\beta}) = f(\boldsymbol{\theta},\boldsymbol{\beta}) = g_1(\boldsymbol{\theta},\boldsymbol{\beta}) + \mathrm{j}\,g_2(\boldsymbol{\theta},\boldsymbol{\beta})$, $1 \le n \le N$. Figs. 3 and 4 show, for sources which are weakly separated both in DOA and polarization, with and without modeling errors and for a given value of $q(= 1,2,3)$, the best behavior of DP-$2q$-MUSIC methods with respect to $2q$-MUSIC ones as soon as polarizations of the sources are different. This shows the better resolution and robustness to modeling errors, whatever the value of q, of methods exploiting both polarization and space diversity with respect to methods exploiting space diversity only. Moreover, we note, for a given value of q, similar performances of UP-PD-$2q$-MUSIC-1 and UP-PD-$2q$-MUSIC-2 algorithms, which seem to differ only from a complexity point of view. Besides, we note increasing performances with q of UP-PD-$2q$-MUSIC methods, for situations where resolution is required. This is due to an increasing resolution in both DOA and polarization of the associated $2q$th-order VA and this shows the interest of exploiting both polarization diversity and HO statistics for DF of poorly separated sources. Note also, for a

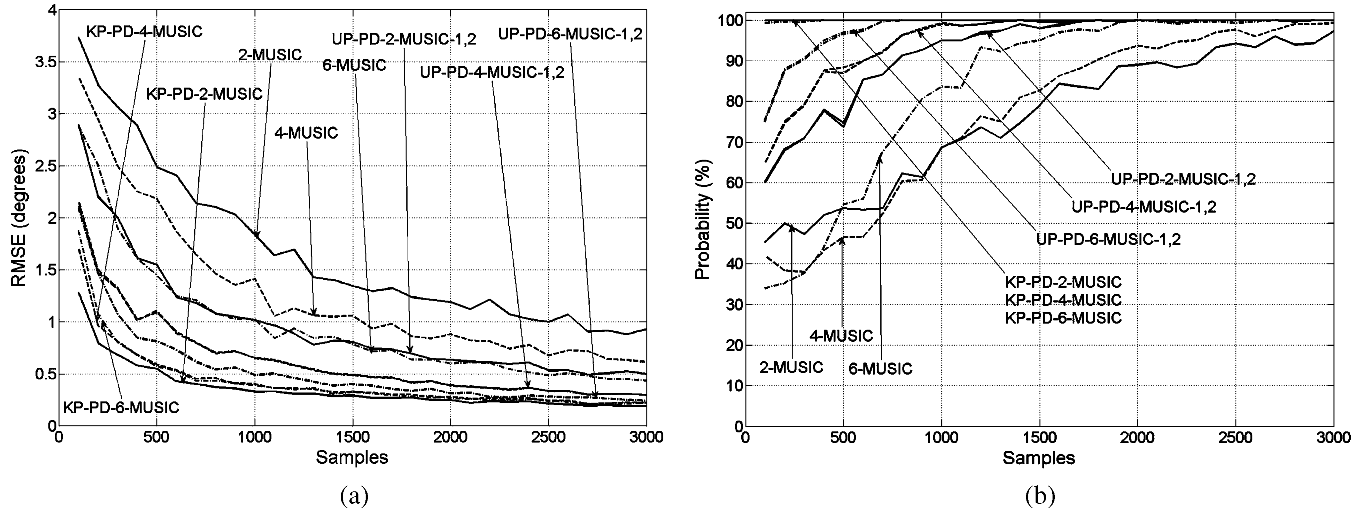


Fig. 3. (a) RMSE and (b) probability nonaberrant results of source 2 as a function of L: 1) 2-MUSIC, 2) KP-PD-2-MUSIC, 3) UP-PD-2-MUSIC-1, 4) UP-PD-2-MUSIC-2, 5) 4-MUSIC, 6) KP-PD-4-MUSIC, 7) UP-PD-4-MUSIC-1, 8) UP-PD-4-MUSIC-2, 9) 6-MUSIC, 10) KP-PD-6-MUSIC, 11) UP-PD-6-MUSIC-1, and 12) UP-PD-6-MUSIC-2,. $P = 2$, $N = 6$, UCA, SNR = 5 dB, $(\theta_1, \gamma_1, \phi_1) = (50°, 45°, 0°)$, and $(\theta_2, \gamma_2, \phi_2) = (60°, 45°, 10°)$, without modeling errors.

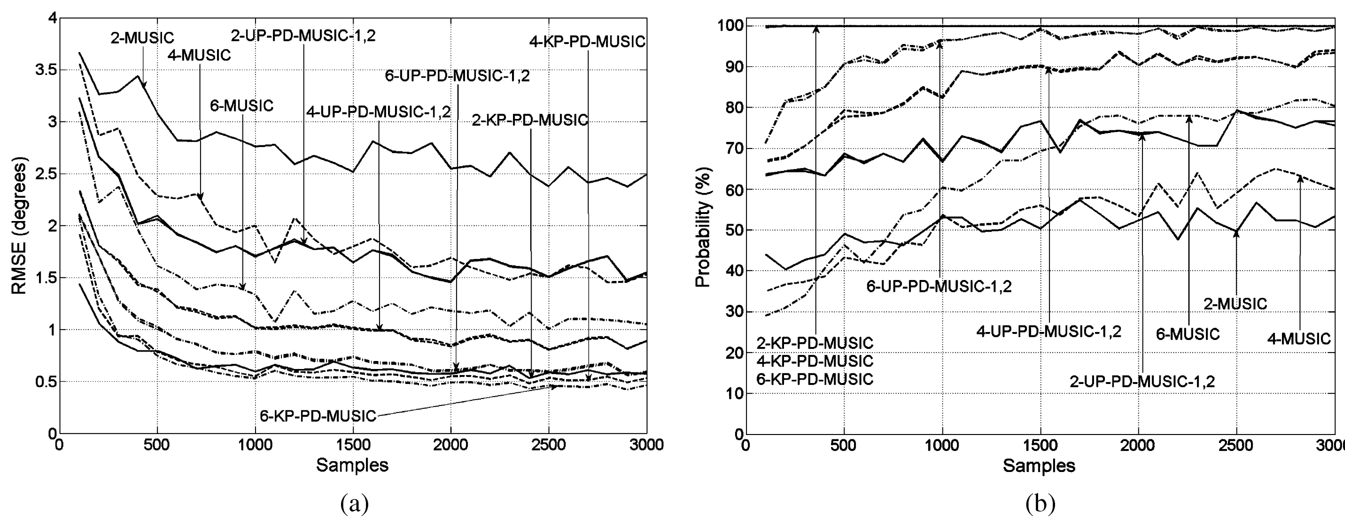
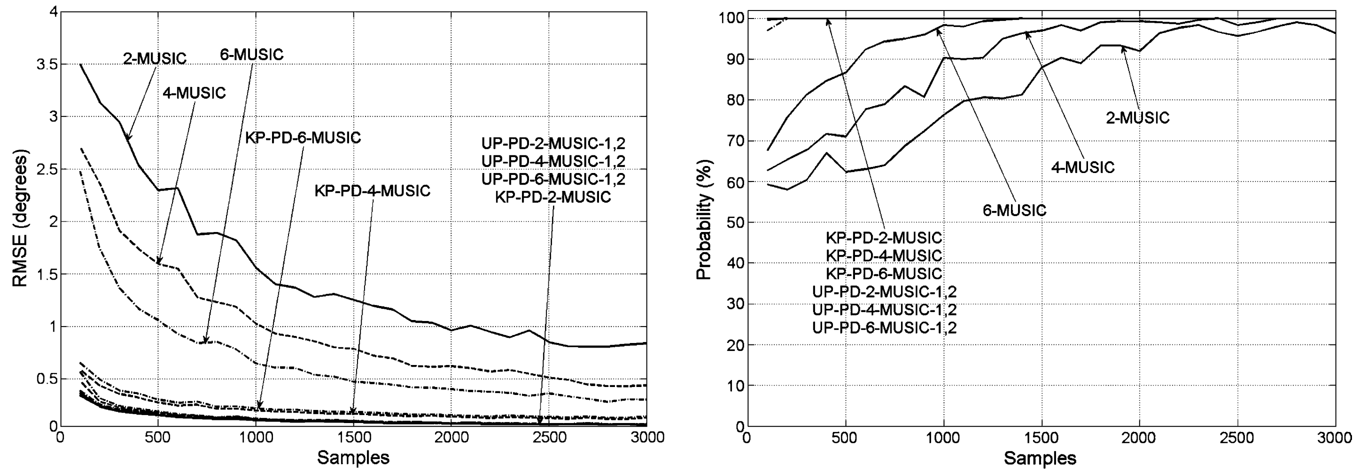


Fig. 4. (a) RMSE and (b) probability nonaberrant results of source 2 as a function of L: 1) 2-MUSIC, 2) KP-PD-2-MUSIC, 3) UP-PD-2-MUSIC-1, 4) UP-PD-2-MUSIC-2, 5) 4-MUSIC, 6) KP-PD-4-MUSIC, 7) UP-PD-4-MUSIC-1, 8) UP-PD-4-MUSIC-2, 9) 6-MUSIC, 10) KP-PD-6-MUSIC, 11) UP-PD-6-MUSIC-1, and 12) UP-PD-6-MUSIC-2. $P = 2$, $N = 6$, UCA, SNR = 5 dB, $(\theta_1, \gamma_1, \phi_1) = (50°, 45°, 0°)$, and $(\theta_2, \gamma_2, \phi_2) = (60°, 45°, 10°)$, with modeling errors.

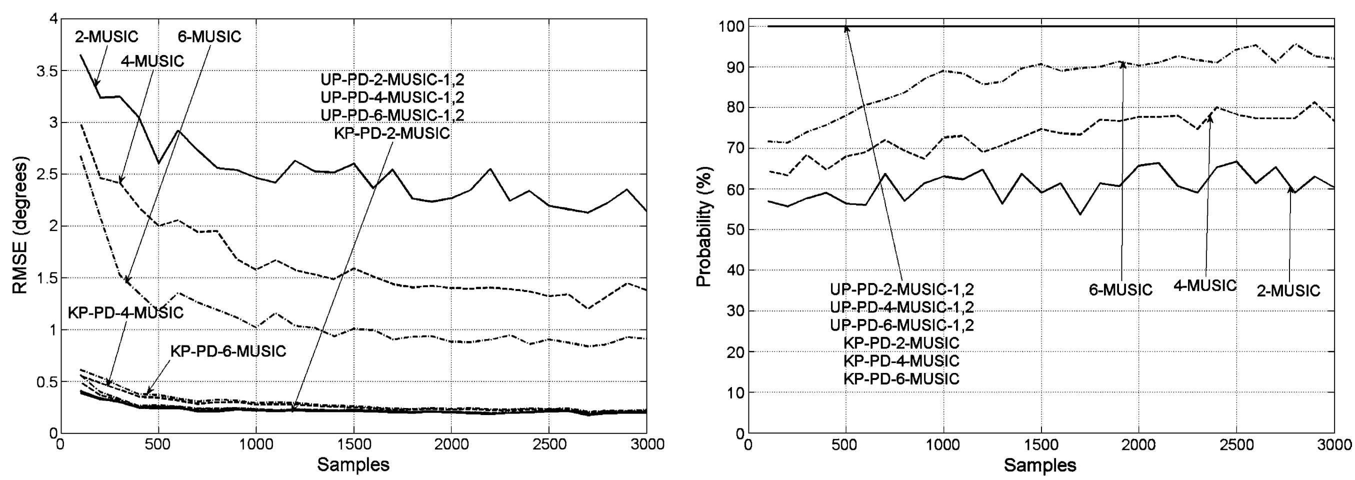
given value of q, the better performance of KP-PD-2q-MUSIC method with respect to UP-PD-2q-MUSIC methods, due to the exploitation of the true *a priori* knowledge of the sources polarization. Note finally, for two sources with known polarizations, increasing performances of KP-PD-2q-MUSIC methods as q increases in the presence of modeling errors as soon as the number of snapshots gets beyond 1300. This result seems to be directly related to the degree of coupling of the two estimated pseudospectra (one for each polarization), computed by a given KP-PD-2q-MUSIC method, which increases with modeling errors and when the polarization separation of the sources decreases. More precisely, when this coupling is high (weak polarization separation with modeling errors), the two sources interact in each of the two computed pseudospectra and resolution is required to separate them; hence, the increasing performance with q of KP-PD-2q-MUSIC methods. However, when this coupling is weak (strong polarization separation or absence of modeling errors), sources no longer interact in a given pseudospectrum. Then, only one source has to be found for a given pseudospectrum and no resolution is required, hence decreasing performance due to a higher variance in the statistics estimation as q increases.

We now consider the scenario of Figs. 3 and 4 but we now assume that the two sources are still poorly angularly separated but are well separated in polarization, such that $(\theta_1, \gamma_1, \phi_1) = (50°, 45°, 0°)$ and $(\theta_2, \gamma_2, \phi_2) = (60°, 45°, 180°)$, respectively. Under these assumptions, Figs. 5 and 6 show similar variations as for Figs. 3 and 4, respectively. We still note that whatever the value of q, there is a better performances of PD-2q-MUSIC methods with respect to 2q-MUSIC ones, due to the exploitation of polarization diversity in addition to space diversity. We still note very close performances of UP-PD-2q-MUSIC-1 and UP-PD-2q-MUSIC-2 algorithms. Moreover, we note the decreasing perfor-



(a) (b)

Fig. 5. (a) RMSE and (b) probability nonaberrant results of source 2 as a function of L: 1) 2-MUSIC, 2) KP-PD-2-MUSIC, 3) UP-PD-2-MUSIC-1, 4) UP-PD-2-MUSIC-2, 5) 4-MUSIC, 6) KP-PD-4-MUSIC, 7) UP-PD-4-MUSIC-1, 8) UP-PD-4-MUSIC-2, 9) 6-MUSIC, 10) KP-PD-6-MUSIC, 11) UP-PD-6-MUSIC-1, and 12) UP-PD-6-MUSIC-2. $P = 2$, $N = 6$, UCA, SNR = 5 dB, $(\theta_1, \gamma_1, \phi_1) = (50°, 45°, 0°)$, and $(\theta_2, \gamma_2, \phi_2) = (60°, 45°, 180°)$, without modeling errors.



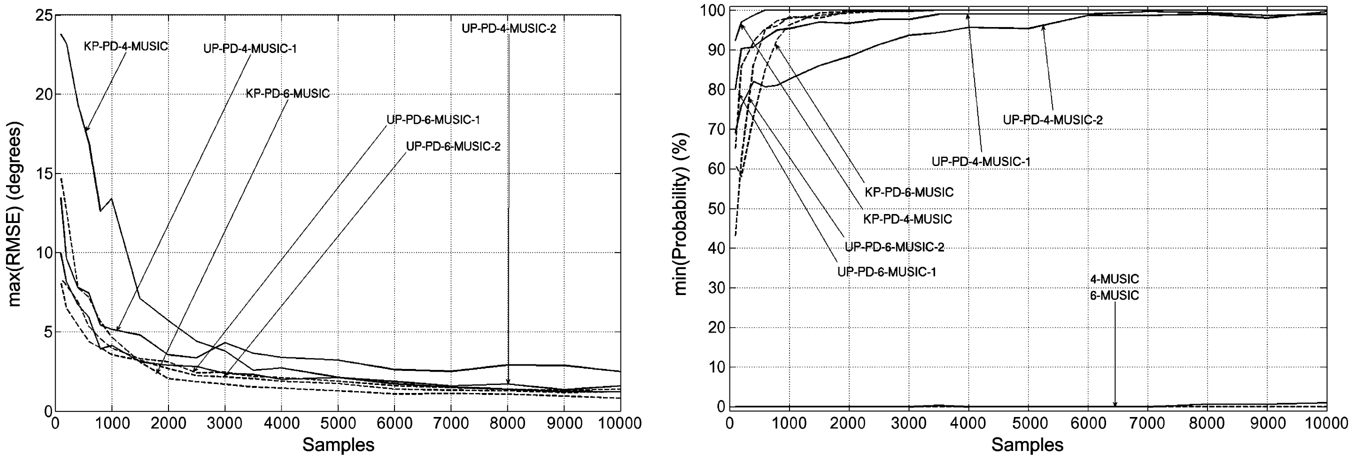
(a) (b)

Fig. 6. (a) RMSE and (b) probability nonaberrant results of source 2 as a function of L: 1) 2-MUSIC, 2) KP-PD-2-MUSIC, 3) UP-PD-2-MUSIC-1, 4) UP-PD-2-MUSIC-2, 5) 4-MUSIC, 6) KP-PD-4-MUSIC, 7) UP-PD-4-MUSIC-1, 8) UP-PD-4-MUSIC-2, 9) 6-MUSIC, 10) KP-PD-6-MUSIC, 11) UP-PD-6-MUSIC-1, and 12) UP-PD-6-MUSIC-2. $P = 2$, $N = 6$, UCA, SNR = 5 dB, $(\theta_1, \gamma_1, \phi_1) = (50°, 45°, 0°)$, and $(\theta_2, \gamma_2, \phi_2) = (60°, 45°, 180°)$, with modeling errors.

mance of UP-PD-2q-MUSIC-1, UP-PD-2q-MUSIC-2, and KP-PD-2q-MUSIC algorithms as q increases. This is due to a higher variance in the statistics estimation since no resolution is required for DF due to a high separation of sources in polarization. Finally, note that, for $q > 2$, KP-PD-2q-MUSIC algorithm may be surprisingly worse than UP-PD-2q-MUSIC algorithm.

C. *Underdetermined Mixtures of Sources*

To illustrate the capability of PD-4-MUSIC and PD-6-MUSIC algorithms to process underdetermined mixtures of sources, we limit the number of sensors of the previous circular array to $N = 3$ sensors. These sensors are such that $f_1(\boldsymbol{\theta}, \boldsymbol{\beta}) = f_3(\boldsymbol{\theta}, \boldsymbol{\beta}) = g_1(\boldsymbol{\theta}, \boldsymbol{\beta}) + \mathrm{j}g_2(\boldsymbol{\theta}, \boldsymbol{\beta})$ and $f_2(\boldsymbol{\theta}, \boldsymbol{\beta}) = g_1(\boldsymbol{\theta}, \boldsymbol{\beta}) - \mathrm{j}g_2(\boldsymbol{\theta}, \boldsymbol{\beta})$ for PD-2q-MUSIC methods and $f_1(\boldsymbol{\theta}, \boldsymbol{\beta}) = f_2(\boldsymbol{\theta}, \boldsymbol{\beta}) = f_3(\boldsymbol{\theta}, \boldsymbol{\beta}) = g_1(\boldsymbol{\theta}, \boldsymbol{\beta}) + \mathrm{j}g_2(\boldsymbol{\theta}, \boldsymbol{\beta})$ for 2q-MUSIC methods and we assume that $l = 1$. Under these assumptions, we deduce from [3, Tables 1 and 2], (29), and (33) that $(N_4^1, P_{4,K}^1, P_{4,U}^1) = (8, 7, 4)$ and $(N_6^1, P_{6,K}^1, P_{6,U}^1) = (15, 14, 9)$ for PD-2q-MUSIC methods and from [3, Tables 6 and 7] that $(N_4^1, P_{\max}) = (7, 6)$ and $(N_6^1, P_{\max}) = (12, 11)$ for 2q-MUSIC methods. We then assume that four statistically independent QPSK sources with a raise cosine pulse shaped filter are received by the array. The four QPSK sources have the same symbol duration, the same rolloff $\mu = 0.3$, the same input SNR equal to 15 dB, and DOA and polarization parameters equal to $(\theta_1, \gamma_1, \phi_1) = (15°, 45°, -75°)$, $(\theta_2, \gamma_2, \phi_2) = (45°, 45°, 0°)$, $(\theta_3, \gamma_3, \phi_3) = (95°, 22.5°, 75°)$, and $(\theta_4, \gamma_4, \phi_4) = (122.5°, 45°, 150°)$, respectively. Under these assumptions, Fig. 7 shows the variations, as a function of the number of snapshots L, of the highest RMSE and the lowest probability of nonabberant results, among all the sources, at the output of several DF methods without modeling errors. These methods correspond to 4-MUSIC,



(a) (b)

Fig. 7. (a) Maximal RMSE and (b) minimal probability of nonaberrant results of sources as a function of L: 1) 4-MUSIC, 2) KP-PD-4-MUSIC, 3) UP-PD-4-MUSIC-1, 4) UP-PD-4-MUSIC-2, 5) 6-MUSIC, 6) KP-PD-6-MUSIC, 7) UP-PD-6-MUSIC, and 8) UP-PD-6-MUSIC-2. $P = 4$, $N = 3$, UCA, SNR = 15 dB, $(\theta_1, \gamma_1, \phi_1) = (15°, 45°, -75°)$, $(\theta_2, \gamma_2, \phi_2) = (45°, 45°, 0°)$, $(\theta_3, \gamma_3, \phi_3) = (95°, 22.5°, 75°)$, and $(\theta_4, \gamma_4, \phi_4) = (122.5°, 45°, 150°)$, without modeling errors.

KP-PD-4-MUSIC, UP-PD-4-MUSIC-1, UP-PD-4-MUSIC-2, 6-MUSIC, KP-PD-6-MUSIC, UP-PD-6-MUSIC-1, and UP-PD-6-MUSIC-2 algorithms, respectively, and the performance are computed from 300 realizations. Note the capability of PD-2q-MUSIC methods to process underdetermined mixtures of sources provided that $P \leq P_{\max}$ given by (29) or (33). Note the poor performance of 2q-MUSIC methods for the considered scenario due to the low input power of the weakest source at the output of the sensors. Better performance would be obtained for higher values of L.

VII. Conclusion

In this paper, the 2q-MUSIC algorithm, $q \geq 1$, has been extended to put up with arrays having polarization diversity and receiving diversely polarized sources, which gives rise to PD-2q-MUSIC algorithms. Three PD-2q-MUSIC algorithms have been presented, depending on the *a priori* knowledge about the polarization of the sources. The first version, called KP-PD-2q-MUSIC, is well suited for sources with known polarization, while the two others, called UP-PD-2q-MUSIC-1 and UP-PD-2q-MUSIC-2, are able to estimate the sources' DOA without any knowledge about their polarization and allow to estimate the polarization of the sources if necessary. For a given value of q, these algorithms are shown in this paper to increase the resolution, the robustness to modeling errors (at least for several poorly angularly separated sources), and the processing capacity (at least for VA without any HO ambiguities) of the 2q-MUSIC method in the presence of diversely polarized sources and from an array with polarization diversity. Moreover, despite a higher variance in the statistics estimation, performance of UP-PD-2q-MUSIC algorithms have been shown to generally increase with q when some resolution is required. This occurs, in particular, for sources which are poorly separated in both DOA and polarization. This result shows off for these scenarios the interest to jointly exploit polarization diversity and HO statistics for DF. Identifiability results have been presented for each of the three PD-2q-MUSIC methods, for VA without HO ambiguities. Nevertheless, for VA with vectorial sensors, a deeper analysis of the identifiability of PD-2q-MUSIC algorithms is required.

Appendix

We show here that it is possible to obtain an estimate $\hat{\boldsymbol{\beta}}_i$ of the polarization vector $\boldsymbol{\beta}_i$ of the source i from the estimate $\hat{\tilde{\boldsymbol{\beta}}}_{q,l,\min}(\hat{\boldsymbol{\theta}}_i)$ of the vector $\tilde{\boldsymbol{\beta}}_{q,l,\min}(\hat{\boldsymbol{\theta}}_i)$. To do so, we implement the different following steps.

1) Compute, from (16), an estimate $\hat{\boldsymbol{\beta}}_{q,l,\min}(\hat{\boldsymbol{\theta}}_i)$ of the $2^q \times 1$ vector $\boldsymbol{\beta}_{q,l,\min}(\hat{\boldsymbol{\theta}}_i)$ by $\hat{\boldsymbol{\beta}}_{q,l,\min}(\hat{\boldsymbol{\theta}}_i) = [B_l \otimes B_{q-l}]\hat{\tilde{\boldsymbol{\beta}}}_{q,l,\min}(\hat{\boldsymbol{\theta}}_i)$.
2) Decompose the vector $\hat{\boldsymbol{\beta}}_{q,l,\min}(\hat{\boldsymbol{\theta}}_i)$ into 2^{q-2} 4×1 subvectors $\hat{\boldsymbol{\beta}}_{q,l,s,\min}(\hat{\boldsymbol{\theta}}_i)$, $1 \leq s \leq q-2$, such that $\hat{\boldsymbol{\beta}}_{q,l,\min}(\hat{\boldsymbol{\theta}}_i) = [\hat{\boldsymbol{\beta}}_{q,l,1,\min}(\hat{\boldsymbol{\theta}}_i)^{\mathrm{T}}, \ldots, \hat{\boldsymbol{\beta}}_{q,l,(q-2),\min}(\hat{\boldsymbol{\theta}}_i)^{\mathrm{T}}]^{\mathrm{T}}$.
3) Map the components of each subvector $\hat{\boldsymbol{\beta}}_{q,l,s,\min}(\hat{\boldsymbol{\theta}}_i)$ into a 2×2 matrix $\hat{\Gamma}_{q,l,s}(\hat{\boldsymbol{\theta}}_i)$ such that $\hat{\Gamma}_{q,l,s}(\hat{\boldsymbol{\theta}}_i)[k,j] = \hat{\boldsymbol{\beta}}_{q,l,s,\min}(\hat{\boldsymbol{\theta}}_i)[2(k-1)+j]$, where $\hat{\Gamma}_{q,l,s}(\hat{\boldsymbol{\theta}}_i)[k,j]$ and $\hat{\boldsymbol{\beta}}_{q,l,s,\min}(\hat{\boldsymbol{\theta}}_i)[k]$ are the elements $[k,j]$ and k of $\hat{\Gamma}_{q,l,s}(\hat{\boldsymbol{\theta}}_i)$ and $\hat{\boldsymbol{\beta}}_{q,l,s,\min}(\hat{\boldsymbol{\theta}}_i)$, respectively
4) Build the matrices $\hat{\Delta}_{q,l,s}(\hat{\boldsymbol{\theta}}_i)$, $1 \leq s \leq q-2$, defined by the following:
   a) $\hat{\Delta}_{q,l,s}(\hat{\boldsymbol{\theta}}_i) = \hat{\Gamma}_{q,l,s}(\hat{\boldsymbol{\theta}}_i)\hat{\Gamma}_{q,l,s}(\hat{\boldsymbol{\theta}}_i)^{\dagger}$ if $q-l=0$;
   b) $\hat{\Delta}_{q,l,s}(\hat{\boldsymbol{\theta}}_i) = \hat{\Gamma}_{q,l,s}(\hat{\boldsymbol{\theta}}_i)\hat{\Gamma}_{q,l,s}(\hat{\theta}_i)^{\dagger}$ if $q-l=1$;
   c) $\hat{\Delta}_{q,l,s}(\hat{\boldsymbol{\theta}}_i) = \hat{\Gamma}_{q,l,s}(\hat{\boldsymbol{\theta}}_i)^{*}\hat{\Gamma}_{q,l,s}(\hat{\boldsymbol{\theta}}_i)^{\mathrm{T}}$ if $q-l>1$.
5) Jointly diagonalize the matrices $\hat{\Delta}_{q,l,s}(\hat{\boldsymbol{\theta}}_i)$, $1 \leq s \leq q-2$.
6) $\hat{\boldsymbol{\beta}}_i$ is the associated eigenvector corresponding to the maximal eigenvalue.

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