High-Resolution Direction Finding From Higher Order Statistics: The $2q$-MUSIC Algorithm

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***Abstract*—From the beginning of the 1980s, many second-order (SO) high-resolution direction-finding methods, such as the MUSIC method (or 2-MUSIC), have been developed mainly to process efficiently the multisource environments. Despite of their great interests, these methods suffer from serious drawbacks such as a weak robustness to both modeling errors and the presence of a strong colored background noise whose spatial coherence is unknown, poor performance in the presence of several poorly angularly separated sources from a limited duration observation and a maximum of $N-1$ sources to be processed from an array of N sensors. Mainly to overcome these limitations and in particular to increase both the resolution and the number of sources to be processed from an array of N sensors, fourth-order (FO) high-resolution direction-finding methods have been developed, from the end of the 1980s, to process non-Gaussian sources, omnipresent in radio communications, among which the 4-MUSIC method is the most popular. To increase even more the resolution, the robustness to modeling errors, and the number of sources to be processed from a given array of sensors, and thus to minimize the number of sensors in operational contexts, we propose in this paper an extension of the MUSIC method to an arbitrary even order $2q$ ($q \geq 1$), giving rise to the $2q$-MUSIC methods. The performance analysis of these new methods show off new important results for direction-finding applications and in particular the best performances, with respect to 2-MUSIC and 4-MUSIC, of $2q$-MUSIC methods with $q > 2$, despite their higher variance, when some resolution is required.**

***Index Terms*—$2q$-MUSIC, direction finding, higher order, modeling errors, MUSIC, resolution, underdetermined mixtures, virtual array.**

I. Introduction

FROM the beginning of the 1980s, many second-order (SO) high-resolution direction-finding methods [2]–[4], [24], [31], [32], [37] have been developed mainly to overcome the limitations of the so-called superresolution methods [5], [6] in weak sources contexts. Among these high-resolution methods, subspace-based methods such as the MUSIC (or 2-MUSIC) method [2], [32] are the most popular. These high-resolution methods are very powerful in multisource environments since they are characterized, in the absence of modeling errors and for a background noise whose spatial coherence is known, by an asymptotic resolution that becomes infinite whatever the source signal-to-noise ratio (SNR) [22], [23], [29], [34], [35]. However, these high-resolution methods suffer from serious drawbacks. Indeed, they are not able to process more than $N-1$ noncoherent sources from an array of N sensors and are weakly robust both to modeling errors [20], [25], [36], always present in operational contexts, and to the presence of a strong background noise whose spatial coherence is unknown [27], such as in the high-frequency (HF) band [14]. Finally, their performance may be strongly affected when several poorly angularly separated sources with a low SNR have to be separated from a limited number of snapshots [23], [34], [35].

From the end of the 1980s, mainly to overcome the previous limitations, fourth-order (FO) high-resolution direction-finding methods [7], [11], [21], [30], [33] have been developed for non-Gaussian sources, omnipresent in radio communications contexts, among which the extension of the MUSIC method to FO [30], called 4-MUSIC, is the most popular. Indeed, FO methods are asymptotically robust to the presence of a Gaussian noise whose spatial coherence is unknown [30]. Moreover, despite of their higher variance [8], they generate a virtual increase of both the effective aperture and the number of sensors of the array, introducing the FO virtual array (VA) concept presented in [15] and [10] and allowing both an increasing resolution and the processing of more sources than sensors [7], [10], [15], [33]. In particular, it has been shown in [10] that, from an array of N sensors, the 4-MUSIC method may process up to N^2-N sources when the sensors are identical and up to N^2-1 sources for different sensors.

In order to still increase both the resolution power of high-resolution direction-finding methods and the number of sources to be processed from a given array of sensors, while keeping the asymptotic robustness to a strong background Gaussian noise whose spatial coherence is unknown, we propose in this paper an extension of the MUSIC algorithm to an arbitrary even order $2q$ ($q \geq 1$) giving rise to the so-called $2q$-MUSIC algorithms. The performance analysis of $2q$-MUSIC algorithms with $q > 1$ shows off new important results for direction-finding applications, opening new perspectives in array processing. More precisely, using the results of higher order (HO) VA concept presented recently in [9], it is verified in the paper that the way the $2q$th-order data statistics are arranged in the exploited $2q$th-order statistical matrix, controls the maximal number of sources that can be processed by the $2q$-MUSIC method, showing off the existence of an optimal arrangement of these statistics. Moreover, for a given array of sensors, we also verify from the results of [9] that the maximal number of sources that can be processed by the $2q$-MUSIC method is an increasing function of q. A consequence of this result is that for operational contexts characterized by a high source density, such as for example airborne surveillance over urban areas, the use of $2q$-MUSIC methods with $q > 1$ for direction finding allows the

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reduction or even the minimization of the number of sensors of the array and thus the number of reception chains, which finally drastically reduces the overall cost. Another important result of the paper is that both the resolution of $2q$-MUSIC method, implemented from a finite number of snapshots, and its robustness to modeling errors increase with q. This result jointly with the HO VA concept [9] allow to explain why, despite of their higher variance, $2q$-MUSIC methods with $q > 2$ may offer better performances than 2-MUSIC or 4-MUSIC methods when some resolution is required, i.e., in the presence of several sources, when the latter are poorly angularly separated or in the presence of modeling errors inherent in operational contexts. A consequence of this latter result is that, for given performances, $2q$-MUSIC method for $q > 1$ may allow to slacken some constraints about antennas' calibration or receivers'chains equalization.

After an introduction of some notations, hypotheses and data statistics in Section II, the $2q$-MUSIC method is presented in Section III for particular arrangements of the $2q$th-order data statistics in a $2q$th-order statistical matrix. Identifiability issue together with the problem of the optimal arrangement of the data statistics are addressed in Section IV. The performance of the $2q$-MUSIC method in the presence of modeling errors are computed analytically in Section V where the resolution of $2q$-MUSIC algorithm is also discussed. Some simulations about the behavior of $2q$-MUSIC algorithm for both overdetermined and underdetermined mixtures of sources are presented in Section VI, showing off in particular the great interest of $2q$-MUSIC methods for $q > 2$. Finally Section VII concludes this paper.

II. Hypotheses, Notations, and Statistics of the Data

A. Hypotheses and Notations

We consider an array of N narrowband (NB) identical sensors and we call $\boldsymbol{x}(t)$ the vector of complex amplitudes of the signals at the output of these sensors. Each sensor is assumed to receive the contribution of P zero-mean stationary NB sources, which may be statistically independent or not, corrupted by a noise. We assume that the P sources can be divided into G groups, with P_g sources in the group g, such that the sources in each group are assumed to be statistically dependent, but not perfectly coherent, while sources belonging to different groups are assumed to be statistically independent. In particular, $G = P$ corresponds to P statistically independent sources whereas $G = 1$ corresponds to the case where all the sources are dependent. Of course, the P_g parameters are such that

$$P = \sum_{g=1}^{G} P_g. \tag{1}$$

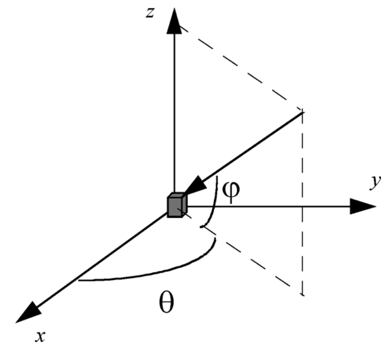


Fig. 1. Incoming signal in three dimensions.

Under these assumptions, the observation vector can approximately be written as follows:

$$\begin{aligned}\boldsymbol{x}(t) &\approx \sum_{i=1}^{P} m_i(t)\boldsymbol{a}(\theta_i,\varphi_i) + \boldsymbol{v}(t) \triangleq A\boldsymbol{m}(t) + \boldsymbol{v}(t)\\ &= \sum_{g=1}^{G} A_g \boldsymbol{m}_g(t) + \boldsymbol{v}(t) = \sum_{g=1}^{G} \boldsymbol{x}_g(t) + \boldsymbol{v}(t)\end{aligned} \tag{2}$$

where $\boldsymbol{v}(t)$ is the noise vector, assumed zero-mean and Gaussian, $\boldsymbol{m}(t)$, independent of $\boldsymbol{v}(t)$, is the vector whose components $m_i(t)$ are the complex amplitudes of the sources, θ_i and φ_i are the azimuth and the elevation angles of source i (Fig. 1), A is the $(N \times P)$ matrix of the source steering vectors $\boldsymbol{a}(\theta_i,\varphi_i)$ $(1 \le i \le P)$, which contains in particular the information about the direction of arrival of the sources, A_g is the $(N \times P_g)$ submatrix of A corresponding to the gth group of sources, $\boldsymbol{m}_g(t)$ is the corresponding $(P_g \times 1)$ subvector of $\boldsymbol{m}(t)$ and $\boldsymbol{x}_g(t) \triangleq A_g\boldsymbol{m}_g(t)$. In particular, in the absence of coupling between sensors, assuming a planewave propagation, component n of vector $\boldsymbol{a}(\theta_i,\varphi_i)$, denoted $a_n(\theta_i,\varphi_i)$, can be written, in the case of an array with space diversity only, as [12] (3), shown at the bottom of the page, where λ is the wavelength, and (x_n, y_n, z_n) are the coordinates of sensor n of the array.

B. Statistics of the Data

1) Presentation: The $2q$th $(q \ge 1)$-order direction-finding methods presented in this paper exploit the information contained in the $(N^q \times N^q)$ $2q$th-order circular covariance matrix, $C_{2q,x}$, whose entries are the $2q$th-order circular cumulants of the data, $\mathrm{Cum}[x_{i1}(t),\ldots,x_{iq}(t),x_{iq+1}(t)^*,\ldots,x_{i2q}(t)^*]$ $(1 \le i_j \le N)$ $(1 \le j \le 2q)$, where * corresponds to the complex conjugation. Note that the previous statistics are called *circular* since they are invariant by a phase rotation of the components $x_{i_j}(t)$ [1], [28]. However, the previous entries can be arranged in the $C_{2q,x}$ matrix in different ways, and it is shown in [9] and verified in this paper that the way these entries are arranged in the $C_{2q,x}$ matrix determines in particular

$$a_n(\theta_i,\varphi_i) = \exp\left\{\frac{\mathrm{j}2\pi[x_n\cos(\theta_i)\cos(\varphi_i) + y_n\sin(\theta_i)\cos(\varphi_i) + z_n\sin(\varphi_i)]}{\lambda}\right\} \tag{3}$$

both the resolution and the maximal processing power of the $2q$-MUSIC method.

In order to verify this important result in Sections IV and VI, let us introduce an arbitrary integer l such that $(0 \leq l \leq q)$ and let us arrange the $2q$-uplet $(i_1, \ldots, i_q, i_{q+1}, \ldots, i_{2q})$ of indexes i_j $(1 \leq j \leq 2q)$ into two q-uplets indexed by l and defined by $(i_1, i_2, \ldots\cdot, i_l, i_{q+1}, \ldots\cdot, i_{2q-l})$ and $(i_{2q-l+1}, \ldots, i_{2q}, i_{l+1}, \ldots\cdot, i_q)$ respectively. As the indexes i_j $(1 \leq j \leq 2q)$ vary from 1 to N, the two latter q-uplets take N^q values. Numbering, in a natural way, the N^q values of each of two latter q-uplets by the integers I_l and J_l respectively, such that $1 \leq I_l, J_l \leq N^q$, we obtain

$$I_l \triangleq \sum_{j=1}^{l} N^{q-j}(i_j - 1) + \sum_{j=1}^{q-l} N^{q-l-j}(i_{q+j} - 1) + 1 \tag{4a}$$

$$J_l \triangleq \sum_{j=1}^{l} N^{q-j}(i_{2q-l+j} - 1) + \sum_{j=1}^{q-l} N^{q-l-j}(i_{l+j} - 1) + 1. \tag{4b}$$

In particular, for $(q, l) = (2, 1)$, the integers I_l and J_l are given by $I_1 = N(i_1 - 1) + i_3$ and $J_1 = N(i_4 - 1) + i_2$ respectively whereas for $(q, l) = (2, 2)$ we obtain $I_2 = N(i_1 - 1) + i_2$ and $J_2 = N(i_3 - 1) + i_4$. Using the permutation invariance property of the cumulants, we deduce that

$$\begin{aligned}&\mathrm{Cum}[x_{i1}(t), \ldots, x_{iq}(t), x_{iq+1}(t)^*, \ldots, x_{i2q}(t)^*] \\ &\quad = \mathrm{Cum}[x_{i1}(t), \ldots, x_{il}(t), x_{iq+1}(t)^*, \ldots, \\ &\qquad\qquad x_{i2q-l}(t)^*, x_{i2q-l+1}(t)^*, \ldots, \\ &\qquad\qquad x_{i2q}(t)^*, x_{il+1}(t), \ldots, x_{iq}(t)]\end{aligned}$$

and assuming that the latter quantity is the element $[I_l, J_l]$ of the $C_{2q,x}$ matrix, thus noted $C_{2q,x}(l)$, it is straightforward to show, using (2), that the $(N^q \times N^q)$ $C_{2q,x}(l)$ matrix can be written as

$$C_{2q,x}(l) \approx \sum_{g=1}^{G} C_{2q,x_g}(l) + \eta_2 V(l)\delta(q-1) \tag{5}$$

where η_2 is the mean power of the noise per sensor, $V(l)$ is the $(N \times N)$ spatial coherence matrix of the noise for the arrangement indexed by l, such that $\mathrm{Tr}[V(l)] = N$, $\mathrm{Tr}[.]$ means Trace, $\delta(.)$ is the Kronecker symbol, and the $(N^q \times N^q)$ matrix $C_{2q,x_g}(l)$ corresponds to the $2q$th-order circular cumulants of $\boldsymbol{x}_g(t)$ for the arrangement indexed by l, which can be written as

$$C_{2q,x_g}(l) = [{A_g}^{\otimes l} \otimes {A_g}^{*\otimes(q-l)}]C_{2q,m_g}(l)[{A_g}^{\otimes l} \otimes {A_g}^{*\otimes(q-l)}]^{\dagger} \tag{6}$$

where $C_{2q,m_g}(l)$ is the $({P_g}^q \times {P_g}^q)$ matrix of the $2q$th-order circular cumulants of $\boldsymbol{m}_g(t)$ for the arrangement indexed by l, $\dagger$ corresponds to the conjugate transposition, $\otimes$ is the Kronecker product, and ${A_g}^{\otimes l}$ is the $(N^l \times {P_g}^l)$ matrix defined by ${A_g}^{\otimes l} \triangleq A_g \otimes A_g \otimes \cdots\cdot \otimes A_g$ with a number of Kronecker product equal to $l - 1$.

In particular, for $q = 1$ and $l = 1$, the $(N \times N)$ $C_{2q,x}(l)$ matrix corresponds to the well-known data covariance matrix (since the observations are zero-mean) defined by

$$\begin{aligned}R_x \triangleq C_{2,x}(1) &= \mathrm{E}[\boldsymbol{x}(t)\boldsymbol{x}(t)^{\dagger}] \\ &\approx \sum_{g=1}^{G} A_g C_{2,m_g}(1){A_g}^{\dagger} + \eta_2 V(1).\end{aligned} \tag{7}$$

For $q = 2$ and $l = 1$, the $(N^2 \times N^2)$ $C_{2q,x}(l)$ matrix corresponds to the classical expression of the data quadricovariance matrix, used in [7], [8], [15], and [30] and in most of the papers dealing with FO array processing problems, and defined by

$$Q_x \triangleq C_{4,x}(1) \approx \sum_{g=1}^{G}[A_g \otimes {A_g}^*]C_{4,m_g}(1)[A_g \otimes {A_g}^*]^{\dagger} \tag{8}$$

whereas for $q = 2$ and $l = 2$, the $(N^2 \times N^2)$ $C_{2q,x}(l)$ matrix corresponds to an alternative expression of the data quadricovariance matrix, not often used and defined by

$$\overline{Q}_x \triangleq C_{4,x}(2) \approx \sum_{g=1}^{G}[A_g \otimes A_g]C_{4,m_g}(2)[A_g \otimes A_g]^{\dagger}. \tag{9}$$

2) Estimation: In situations of practical interests, the $2q$th-order statistics of the data $\mathrm{Cum}[x_{i1}(t), \ldots, x_{iq}(t), x_{iq+1}(t)^*, \ldots, x_{i2q}(t)^*]$ are not known *a priori* and have to be estimated from L samples of data $\boldsymbol{x}(k) \triangleq \boldsymbol{x}(kT_e)$, $1 \leq k \leq L$, where T_e is the sample period.

For zero-mean stationary observations, using the ergodicity property, an empirical estimator of $\mathrm{Cum}[x_{i1}(t), \ldots, x_{iq}(t), x_{iq+1}(t)^*, \ldots, x_{i2q}(t)^*]$, asymptotically unbiased and consistent, may be built from the well-known Leonov–Shiryaev formula [26], giving the expression of a nth-order cumulant of $\boldsymbol{x}(t)$ as a function of its pth-order moments $(1 \leq p \leq n)$ by replacing in the latter all the moments by their empirical estimate. More precisely, the Leonov–Shiryaev formula is given by

$$\begin{aligned}&\mathrm{Cum}[x_{i_1}(t)^{\varepsilon 1}, x_{i_2}(t)^{\varepsilon 2}, \ldots, x_{i_n}(t)^{\varepsilon n}] \\ &\quad = \sum_{p=1}^{n}(-1)^{p-1}(p-1)!\mathrm{E}[\prod_{j\in \mathrm{S}1} x_{i_j}(t)^{\varepsilon j}]\mathrm{E}[\prod_{j\in \mathrm{S}2} x_{i_j}(t)^{\varepsilon j}]\ldots \\ &\qquad \mathrm{E}[\prod_{j\in \mathrm{S}p} x_{i_j}(t)^{\varepsilon j}]\end{aligned} \tag{10}$$

where $(\mathrm{S}1, \mathrm{S}2, \ldots, \mathrm{S}p)$ describes all the partitions in p sets of $(1, 2, \ldots, n)$, $\epsilon_j = \pm 1$ $(1 \leq p \leq n)$ with the convention $x^1 = x$ and $x^{-1} = x^*$ and an empirical estimate of (10) is obtained by replacing in (10) all the moments

$\mathrm{E}[x_{i_1}(t)^{\varepsilon 1}x_{i_2}(t)^{\varepsilon 2}\dots x_{i_p}(t)^{\varepsilon p}]$ $(1 \le p \le n)$ by their empirical estimate given by

$$\hat{\mathrm{E}}[x_{i_1}(t)^{\varepsilon 1}x_{i_2}(t)^{\varepsilon 2}\cdots x_{i_p}(t)^{\varepsilon p}](L) \triangleq \frac{1}{L}\sum_{k=1}^{L} x_{i_1}(k)^{\varepsilon 1}x_{i_2}(k)^{\varepsilon 2}\cdots x_{i_p}(k)^{\varepsilon p}. \quad (11)$$

Explicit expressions of (10) for $n = 2q$ with $1 \le q \le 3$ are given in [9].

However, in radio communications contexts, most of the sources are no longer stationary but become cyclostationary (digital modulations). For zero-mean cyclostationary observations, the statistical matrix defined by (5) become time dependent, noted $C_{2q,x}(l)(t)$, and the theory developed in the paper can be extended without any difficulties by considering that $C_{2q,x}(l)$ is, in this case, the temporal mean $\langle C_{2q,x}(l)(t)\rangle$ over an infinite interval duration, of the instantaneous statistics $C_{2q,x}(l)(t)$. In these conditions, using a cyclo-ergodicity property, the matrix $C_{2q,x}(l)$ has to be estimated from the sampled data by a nonempirical estimator such as that presented in [16] for $q = 2$, whose convergence is shown in [13]. Note finally that this extension can also be applied to nonzero-mean cyclostationary sources, such as some nonlinearly digitally modulated sources [18], provided that a nonempirical statistics estimators, such as that presented in [18] for $q = 1$ and in [17] for $q = 2$, is used.

III. $2q$-MUSIC ALGORITHM

In this section, we analyse the properties of the matrix $C_{2q,x}(l)$ and we deduce from the latter the $2q$-MUSIC algorithm for the arrangement indexed by l.

A. Hypotheses

To develop the $2q$-MUSIC algorithm for the arrangement l, we need some hypotheses that correspond to the following :

H1: $P_g < N, 1 \le g \le G$;

H2: matrix $A_g{}^{\otimes l} \otimes A_g{}^{*\otimes(q-l)}$ has full rank $P_g{}^q, 1 \le g \le G$;

H3: $P(G,q) \triangleq \sum_{g=1}^{G} P_g{}^q < N^q$;

H4: matrix $\overline{A}_{q,l} \triangleq [A_1{}^{\otimes l} \otimes A_1{}^{*\otimes(q-l)}, \dots\dots, A_G{}^{\otimes l} \otimes A_G{}^{*\otimes(q-l)}]$ as full rank $P(G,q)$.

For example, for $(q,l) = (2,1)$, hypothesis H2 reduces to $A_g \otimes A_g{}^*$ has full rank $P_g{}^2$, assumption made in [30] to develop the 4-MUSIC algorithm for correlated sources. In particular, for sources which are all statistically dependent $(G = 1)$, $P(G,q) = P^q$, matrix A_1 reduces to A and hypotheses H1 to H4 reduce to the following:

H1': $P < N$;

H2': matrix $A^{\otimes l} \otimes A^{*\otimes(q-l)}$ as full rank P^q.

For statistically independent sources $(G = P)$, $P(G,q) = P$, matrix A_g reduces to the vector $\boldsymbol{a}(\theta_g, \varphi_g)$ and hypotheses H1 to H4 reduce to the following:

H1'': $P < N^q$;

H2'': matrix

$$\overline{A}_{q,l} \triangleq [\boldsymbol{a}(\theta_1,\varphi_1)^{\otimes l} \otimes \boldsymbol{a}(\theta_1,\varphi_1)^{*\otimes(q-l)}, \dots\dots, \boldsymbol{a}(\theta_P,\varphi_P)^{\otimes l} \otimes \boldsymbol{a}(\theta_P,\varphi_P)^{*\otimes(q-l)}]$$

as full rank P.

B. Properties of $C_{2q,x}(l)$

The $(P_g{}^q \times P_g{}^q)$ matrix $C_{2q,m_g}(l)$, which contains the $2q$th-order circular cumulants of $\boldsymbol{m}_g(t)$ for the arrangement indexed by l, has full rank, $P_g{}^q$, in general since the components of $\boldsymbol{m}_g(t)$ are statistically dependent. Therefore, using H1 and H2, the matrix $C_{2q,xg}(l)$ for $q > 1$ has also rank $P_g{}^q$. Hence, using H4 and for $q > 1$, matrix $C_{2q,x}(l)$ has a rank, $r_{2q,x}(l)$, equal to $P(G,q)$, and such that $r_{2q,x}(l) < N^q$ from H3. In particular, for sources that are all statistically dependent, $r_{2q,x}(l) = P^q$ whereas for statistically independent sources, $r_{2q,x}(l) = P$. As matrix $C_{2q,x}(l)$ is Hermitian, but not positive definite, we deduce from the previous results that $C_{2q,x}(l)$ has $P(G,q)$ nonzero eigenvalues and $N^q - P(G,q)$ zero eigenvalues for $q > 1$.

C. $2q$-MUSIC Algorithm

To built a MUSIC-like algorithm from the matrix $C_{2q,x}(l)$, for $q > 1$, we first compute the eigendecomposition of the latter, given by

$$C_{2q,x}(l) = U_{2q,s}(l)\Lambda_{2q,s}(l)U_{2q,s}(l)^\dagger + U_{2q,n}(l)\Lambda_{2q,n}(l)U_{2q,n}(l)^\dagger \quad (12)$$

where $\Lambda_{2q,s}(l)$ is the $(P(G,q) \times P(G,q))$ diagonal matrix of the nonzero eigenvalues of $C_{2q,x}(l)$, $U_{2q,s}(l)$ is the $(N^q \times P(G,q))$ unitary matrix of the eigenvectors of $C_{2q,x}(l)$ associated with the $P(G,q)$ nonzero eigenvalues of $C_{2q,x}(l)$, $\Lambda_{2q,n}(l)$ is the $((N^q - P(G,q)) \times (N^q - P(G,q)))$ diagonal matrix of the zero eigenvalues of $C_{2q,x}(l)$, and $U_{2q,n}(l)$ is the $(N^q \times (N^q - P(G,q)))$ unitary matrix of the eigenvectors of $C_{2q,x}(l)$ associated with the $(N^q - P(G,q))$ zero eigenvalues of $C_{2q,x}(l)$. As $C_{2q,x}(l)$ is Hermitian, all the columns of $U_{2q,s}(l)$ are orthogonal to all the columns of $U_{2q,n}(l)$. Moreover, as $\mathrm{Span}\{U_{2q,s}(l)\} = \mathrm{Span}\{\overline{A}_{q,l}\}$, we deduce that all the columns of all the matrices $A_g{}^{\otimes l} \otimes A_g{}^{*\otimes(q-l)}$, $1 \le g \le G$ are orthogonal to all the columns of $U_{2q,n}(l)$. Let $(\theta_{ig}, \varphi_{ig})$ be the DOA of the ith source in the gth group. Then, the vector $\boldsymbol{a}(\theta_{ig},\varphi_{ig})^{\otimes l} \otimes \boldsymbol{a}(\theta_{ig},\varphi_{ig})^{*\otimes(q-l)}$ appears as the $[(1-P_g{}^q)(i-1)/(1-P_g)+1]$th column of $A_g{}^{\otimes l} \otimes A_g{}^{*\otimes(q-l)}$. Hence, all vectors $\{\boldsymbol{a}(\theta_{ig},\varphi_{ig})^{\otimes l} \otimes \boldsymbol{a}(\theta_{ig},\varphi_{ig})^{*\otimes(q-l)}, 1 \le i \le P_g, 1 \le g \le G\}$ are orthogonal to the columns of $U_{2q,n}(l)$ and are solutions of the following equation:

$$[\boldsymbol{a}(\theta,\varphi)^{\otimes l} \otimes \boldsymbol{a}(\theta,\varphi)^{*\otimes(q-l)}]^\dagger U_{2q,n}(l)U_{2q,n}(l)^\dagger \cdot [\boldsymbol{a}(\theta,\varphi)^{\otimes l} \otimes \boldsymbol{a}(\theta,\varphi)^{*\otimes(q-l)}] = 0 \quad (13)$$

which corresponds to the heart of the $2q$-MUSIC algorithm for the arrangement l. In practical situations, matrix $U_{2q,n}(l)$ has to be estimated from the observations and the direction of arrival (DOA) of the sources may be found by searching for the minima of the left-hand side of (13). The different steps of the $2q$-MUSIC algorithm for the arrangement l are summarized hereafter :

1. Estimation, $\hat{C}_{2q,x}(l)$, of the matrix $C_{2q,x}(l)$ from L snapshots $\boldsymbol{x}(k)$, $1 \leq k \leq L$, using a suitable estimator of the $2q$th-order cumulants of observations
2. Eigenvalue decomposition of the matrix, $\hat{C}_{2q,x}(l)$, and extraction of an estimate, $\hat{U}_{2q,n}(l)$, of the $U_{2q,n}(l)$ matrix. This step may involve rank determination in cases where the number of sources and/or their mutual statistical dependence are not known *a priori*.
3. Computation of the estimated pseudo-spectrum

$$\hat{P}_{2q\text{-Music}(l)}(\theta,\varphi) \triangleq [\boldsymbol{a}(\theta,\varphi)^{\otimes l} \otimes \boldsymbol{a}(\theta,\varphi)^{*\otimes(q-l)}]^{\dagger} \hat{U}_{2q,n}(l)\hat{U}_{2q,n}(l)^{\dagger}[\boldsymbol{a}(\theta,\varphi)^{\otimes l} \otimes \boldsymbol{a}(\theta,\varphi)^{*\otimes(q-l)}] \quad (14)$$

over a suitably chosen grid, and search for the local minima (including interpolation at each local minimum).

In some cases, the number of sources P is known, such that $P < N$, but their statistical dependence is not known. In such a case, $P(G,q) \leq P^q$ and a conservative approach is to use only the $(N^q - P^q)$ eigenvectors associated with the smallest eigenvalues to built $\hat{U}_{2q,n}(l)$, which implicitly assumes the statistical dependence of all the sources.

Similarly to 2-MUSIC algorithm, the $2q$-MUSIC algorithm cannot handle perfectly coherent sources. Indeed, in such a case, one or more of the matrices $C_{2q,m_g}(l)$ will have rank less than ${P_g}^q$ and the corresponding sources will become indistinguishable to the algorithm.

IV. Identifiability

Following the developments of the previous section, we deduce that the $2q$-MUSIC algorithm for the arrangement indexed by l is able to estimate the DOA of P noncoherent sources from an array of N sensors provided that hypotheses H1 to H4 are verified and the DOA of the sources are the only solutions of (13). As the maximal number of sources that can be processed by the $2q$-MUSIC algorithm is obtained when all the sources are statistically independent, we limit the analysis to the latter case. In such a situation, hypotheses H1 to H4 reduce to H1'' and H2'' respectively. It has been shown in [9] that the vector $\boldsymbol{a}_{q,l}(\theta,\varphi) \triangleq \boldsymbol{a}(\theta,\varphi)^{\otimes l} \otimes \boldsymbol{a}(\theta,\varphi)^{*\otimes(q-l)}$ can be considered as a *true steering vector* but for a HO *virtual array* of N^q virtual sensors (VSs), with coordinates, $(x^l_{k_1,k_2\ldots k_q}, y^l_{k_1,k_2\ldots k_q}, z^l_{k_1,k_2\ldots k_q})$, $1 \leq k_j \leq N$ for $1 \leq j \leq q$, given by

$$(x^l_{k_1,k_2\ldots k_q}, y^l_{k_1,k_2\ldots k_q}, z^l_{k_1,k_2\ldots k_q}) = \left(\sum_{j=1}^{l} x_{k_j} - \sum_{u=1}^{q-l} x_{k_{l+u}}, \sum_{j=1}^{l} y_{k_j} - \sum_{u=1}^{q-l} y_{k_{l+u}}, \sum_{j=1}^{l} z_{k_j} - \sum_{u=1}^{q-l} z_{k_{l+u}}\right). \quad (15)$$

As some of these N^q VSs may coincide, we note N^l_{2q} ($N^l_{2q} \leq N^q$) the number of different VSs of the VA associated with the $2q$th-order direction-finding problem for the arrangement $C_{2q,x}(l)$. This number, N^l_{2q}, is directly related to the geometry of the true array of N sensors. In these conditions, $N^q - N^l_{2q}$ components of all the vectors $\boldsymbol{a}_{q,l}(\theta,\varphi)$ are redundant components that bring no information. As a consequence, $N^q - N^l_{2q}$ rows of the $\overline{A}_{q,l}$ matrix bring no information and are linear combinations of the others, which means that the rank of $\overline{A}_{q,l}$ cannot be greater than N^l_{2q}. We then deduce that the $\overline{A}_{q,l}$ matrix may have a rank equal to P only if $P \leq N^l_{2q}$. Conversely, for a $2q$th-order virtual array without any ambiguities up to order N^l_{2q}, P sources coming from P different directions generate an $\overline{A}_{q,l}$ matrix with a full rank P as long as $P \leq N^l_{2q}$. Thus, the maximal number of statistically independent sources able to generate a matrix $\overline{A}_{q,l}$ with rank P is N^l_{2q}. However, when $P = N^l_{2q}$, an arbitrary vector $\boldsymbol{a}_{q,l}(\theta,\varphi)$ associated with an arbitrary DOA (θ,φ) is necessarily a linear combination of the source steering vectors $\boldsymbol{a}_{q,l}(\theta_i,\varphi_i)$, $1 \leq i \leq N^l_{2q}$, since matrix $\overline{A}_{q,l}$ cannot have a rank greater than N^l_{2q} and all the DOA (θ,φ) are then solutions of (13), which does not allow the sources' DOA estimation. Thus, a necessary condition for the DOA of the sources to be the only solutions of (13) is that $P < N^l_{2q}$ and this condition becomes sufficient for HO virtual arrays with no ambiguities. From the previous results we deduce that the $2q$-MUSIC algorithm is able to process up to $N^l_{2q} - 1$ sources, where N^l_{2q} is the number of different sensors of the $2q$th-order virtual array associated with the considered array of N sensors and the $2q$th-order statistics arrangement indexed by l. As N^l_{2q} is a function of q and l [9], we deduce that the processing capacity of the $2q$-MUSIC algorithm is also a function of q and l, which shows off in particular the existence of an optimal arrangement of the $2q$th-order data statistics for a given value of q. It is shown in [9] that for a given value of q, the optimal arrangement is associated with the integer l which minimizes the quantity $|2l - q|$, which finally generates steering vectors $\boldsymbol{a}_{q,l}(\theta_i,\varphi_i)$ for which the number of conjugate vectors is the least different from the number of nonconjugate vectors. In particular, for $q = 2$, it corresponds to $l = 1$, i.e., to steering vectors of the form $[\boldsymbol{a}(\theta_i,\varphi_i) \otimes \boldsymbol{a}(\theta_i,\varphi_i)^*]$.

For given values of N, q and l, it has been shown in [9] that N^l_{2q} is necessary upper-bounded by a quantity, noted $N_{\max}[2q,l]$, such that $N_{\max}[2q,l] < N^q$ for arrays with space diversity only. Table I shows, for arrays with space diversity only (i.e., with identical sensors), the expression of $N_{\max}[2q,l]$ as a function of N for $2 \leq q \leq 4$ and several values of l. This upper bound corresponds to N^l_{2q} in most cases of array geometries with no particular symmetries, such as the uniform circular array (UCA) with a number of sensors corresponding to a prime number [9], but cannot be reached by N^l_{2q} for arrays with particular symmetries. It is in particular the case for the uniform linear array (ULA) for which N^l_{2q} is shown in [9] to be independent of l and given by

$$N^l_{2q} = q(N-1) + 1 \quad (16)$$

which is an increasing function of N and q.

TABLE I
$N_{\max}[2q, l]$ AS A FUNCTION OF N FOR SEVERAL VALUES OF q AND l AND FOR ARRAYS WITH SPACE DIVERSITY ONLY

$m = 2q$	l	$N_{max}[2q, l]$
4 ($q = 2$)	2	$N(N+1)/2$
	1	$N^2 - N + 1$
6 ($q = 3$)	3	$N!/[6(N-3)!] + N(N-1) + N$
	2	$N!/[2(N-3)!] + N(N-1) + N$
8 ($q = 4$)	4	$N!/[24(N-4)!] + N!/[2(N-3)!] + 1.5N(N-1) + N$
	3	$N!/[6(N-4)!] + N!/(N-3)! + 1.5N(N-1) + N$
	2	$N!/[4(N-4)!] + N!/(N-3)! + 2N(N-1) + 1$

V. PERFORMANCE OF $2q$-MUSIC WITH MODELING ERRORS

In operational contexts, for given choices of array of sensors and algorithm, the performance of the latter is mainly controlled by modeling errors such as array calibration errors or phase and amplitude residual mismatches between reception chains. For this reason, it is important to compute the asymptotic performances of $2q$-MUSIC algorithms in the presence of modeling errors, showing off the influence of q and l on the robustness of the latter.

A. Modelization and Problem Formulation

In the presence of modeling errors, the observation vector defined by (2) becomes

$$\boldsymbol{x}(t) \approx \sum_{i=1}^{P} m_i(t)\tilde{\boldsymbol{a}}(\theta_i, \varphi_i) + \boldsymbol{v}(t) \triangleq \tilde{A}\boldsymbol{m}(t) + \boldsymbol{v}(t) \quad (17)$$

where $\tilde{A}$ is the $(N \times P)$ matrix of the corrupted source steering vectors $\tilde{\boldsymbol{a}}(\theta_i, \varphi_i)$, $(1 \leq i \leq P)$, such that

$$\tilde{\boldsymbol{a}}(\theta_i, \varphi_i) = \boldsymbol{a}(\theta_i, \varphi_i) + \boldsymbol{e}(\theta_i, \varphi_i) \quad (18)$$

where $\boldsymbol{e}(\theta_i, \varphi_i)$ is the modeling error vector of the source i. From (18), we deduce that

$$\tilde{A} = A + E \triangleq \tilde{A}(E) \quad (19)$$

where E is the $(N \times P)$ matrix of the error vectors $\boldsymbol{e}(\theta_i, \varphi_i)$, $(1 \leq i \leq P)$. Note that the model (18) is well suited for any kinds of distortion (mutual coupling between sensors, mismatches between reception chains, position errors of sensors, etc.) [19]. To simplify the notations, we note in the following $\boldsymbol{a}_i \triangleq \boldsymbol{a}(\theta_i, \varphi_i), \tilde{\boldsymbol{a}}_i \triangleq \tilde{\boldsymbol{a}}(\theta_i, \varphi_i), \boldsymbol{e}_i \triangleq \boldsymbol{e}(\theta_i, \varphi_i)$ and $\boldsymbol{\theta}_i \triangleq (\theta_i, \varphi_i)$. Besides, we assume that a very large number of sampled observation vectors $\boldsymbol{x}(kT_e)$ have been collected resulting in a perfect measurement of the matrix $C_{2q,x}(l)$, given by (5) but where $C_{2q,xg}(l)$ is now given by

$$C_{2q,x_g}(l) = [\tilde{A}_g^{\otimes l} \otimes \tilde{A}_g^{*\otimes(q-l)}]C_{2q,m_g}(l)[\tilde{A}_g^{\otimes l} \otimes \tilde{A}_g^{*\otimes(q-l)}]^\dagger \quad (20)$$

where $\tilde{A}_g$ is the $(N \times P_g)$ submatrix of $\tilde{A}$ corresponding to the gth group of sources. Assuming that the number of sources P and their correlation properties are known, the problem considered in this section is to find the P DOA $\hat{\boldsymbol{\theta}}_i$ $(1 \leq i \leq P)$, minimizing the left-hand side of (13), or the following criterion:

$$P_{2q,l}(\boldsymbol{\theta}, E) \triangleq \boldsymbol{a}_{q,l}(\boldsymbol{\theta})^\dagger \Pi_{2q,n}(l)(E)\boldsymbol{a}_{q,l}(\boldsymbol{\theta}) \quad (21)$$

where $\boldsymbol{a}_{q,l}(\boldsymbol{\theta}) \triangleq [\boldsymbol{a}(\boldsymbol{\theta})^{\otimes l} \otimes \boldsymbol{a}(\boldsymbol{\theta})^{*\otimes(q-l)}]$, $\Pi_{2q,n}(l)(E) \triangleq U_{2q,n}(l)U_{2q,n}(l)^\dagger$ and where $U_{2q,n}(l)$ is the $(N^q \times (N^q - P(G,q)))$ unitary matrix of the eigenvectors of $C_{2q,x}(l)$ associated with the $(N^q - P(G,q))$ zero eigenvalues of $C_{2q,x}(l)$. In the absence of modeling errors $(E = 0)$, the P minima of (21) are P zeros corresponding to the DOA of the sources, $\boldsymbol{\theta}_i$ $(1 \leq i \leq P)$. However, in the presence of modeling errors $(E \neq 0)$, the criterion (21) is no longer zero for the DOA of the sources but presents P local minima for directions $\hat{\boldsymbol{\theta}}_i$ $(1 \leq i \leq P)$, different from $\boldsymbol{\theta}_i$ $(1 \leq i \leq P)$. The variable $\Delta\boldsymbol{\theta}_i \triangleq \hat{\boldsymbol{\theta}}_i - \boldsymbol{\theta}_i$ defines the estimation error on the DOA of the source i. To simplify the mathematical developments, we limit the analysis to a monodimensional DOA estimation problem for which θ_i, $\hat{\theta}_i$ and $\Delta\theta_i$, $(1 \leq i \leq P)$, are scalar quantities. Using a first-order Taylor expansion of the first partial derivative $\dot{P}_{2q,l}(\theta, E)$ of $P_{2q,l}(\theta, E)$ with respect to θ around $\theta = \theta_i$ and the fact that $\dot{P}_{2q,l}(\hat{\theta}_i, E) = 0$, we obtain an approximated expression of $\Delta\theta_i$, given by

$$\Delta\theta_i \approx -\frac{\dot{P}_{2q,l}(\theta_i, E)}{\ddot{P}_{2q,l}(\theta_i E)} \quad (22)$$

where $\ddot{P}_{2q,l}(\theta, E)$ corresponds to the second derivative of $P_{2q,l}(\theta, E)$ with respect to θ at θ. Using (21), $\dot{P}_{2q,l}(\theta_i, E)$ and $\ddot{P}_{2q,l}(\theta_i, E)$ can be written as

$$\dot{P}_{2q,l}(\theta_i, E) = 2\mathrm{Re}[\dot{\boldsymbol{a}}_{q,l}(\theta_i)^\dagger \Pi_{2q,n}(l)(E)\boldsymbol{a}_{q,l}(\theta_i)] \quad (23)$$

$$\ddot{P}_{2q,l}(\theta_i, E) = 2\mathrm{Re}[\ddot{\boldsymbol{a}}_{q,l}(\theta_i)^\dagger \Pi_{2q,n}(l)(E)\boldsymbol{a}_{q,l}(\theta_i)] + 2\dot{\boldsymbol{a}}_{q,l}(\theta_i)^\dagger \Pi_{2q,n}(l)(E)\dot{\boldsymbol{a}}_{q,l}(\theta_i) \quad (24)$$

where Re[.] indicates the real part and $\dot{\boldsymbol{a}}_{q,l}(\theta_i)$ and $\ddot{\boldsymbol{a}}_{q,l}(\theta_i)$ indicate the first and second derivative of $\boldsymbol{a}_{q,l}(\theta)$ with respect to θ at $\theta = \theta_i$, respectively. Considering that E is a random matrix, the quantities $\Delta\theta_i$ $(1 \leq i \leq P)$ become random variables and the purpose of this section is to compute the root mean-square error (RMSE) of $\hat{\theta}_i$, defined by $\mathrm{RMSE}_i \triangleq (E[\Delta\theta_i{}^2])^{1/2}$, as a function of q, l and the statistics of E.

B. Solution for $q = 1$

The computation of the RMSE of the previous DOA estimates $\hat{\theta}_i$, $(1 \leq i \leq P)$, has already been considered in [20], [25],

[36] but for $q = 1$, i.e., for the 2-MUSIC algorithm, and from an expansion of $\Pi_{2,n}(1)(E) \triangleq \Pi_{2,n}(E)$ at the first-order in E around $E = 0$. Unfortunately, this first-order expansion generates, for each source, an RMSE that is unbounded in limit of resolution, which does not agree well with simulation results. To overcome this limitation, a second-order expansion of $\Pi_{2,n}(E)$ around $E = 0$ has been considered recently in [19], generating RMSE in complete agreement with simulations even for high spatial correlation between the sources. More precisely, it has been shown in [19] that $\Delta\theta_i$ and $E[\Delta\theta_i{}^2]$ are given by

$$\Delta\theta_i \approx -\frac{\boldsymbol{\varepsilon}^\dagger \dot{Q}_i \boldsymbol{\varepsilon}}{\boldsymbol{\varepsilon}^\dagger \ddot{Q}_i \boldsymbol{\varepsilon}} \quad (25)$$

$$E[\Delta\theta_i{}^2] \approx \frac{\mathrm{Tr}[\dot{Q}_i^{\otimes 2} R_\varepsilon{}^{(4)}]}{\mathrm{Tr}[\ddot{Q}_i^{\otimes 2} R_\varepsilon{}^{(4)}]} \quad (26)$$

where $\mathrm{Tr}[.]$ means Trace, $\boldsymbol{\varepsilon}$ is the $((2NP+1)\times 1)$ vector defined by $\boldsymbol{\varepsilon} \triangleq [1, \mathrm{vec}(E)^\mathrm{T}, \mathrm{vec}(E)^\dagger]^\mathrm{T}$, $\mathrm{vec}(E) \triangleq [\boldsymbol{e}_1{}^\mathrm{T}, \ldots, \boldsymbol{e}_P{}^\mathrm{T}]^\mathrm{T}$, $R_\varepsilon{}^{(4)}$ is the $((2NP+1)^2 \times (2NP+1)^2)$ matrix defined by $R_\varepsilon{}^{(4)} \triangleq \mathrm{E}[\boldsymbol{\varepsilon}^{\otimes 2}\boldsymbol{\varepsilon}^{\otimes 2\dagger}]^\mathrm{T}$, $\dot{Q}_i$ and $\ddot{Q}_i$ are the $((2NP+1)\times(2NP+1))$ matrices defined by

$$\dot{Q}_i \triangleq Q(\boldsymbol{a}_i, \dot{\boldsymbol{a}}_i) + Q(\dot{\boldsymbol{a}}_i, \boldsymbol{a}_i) \quad (27)$$

$$\ddot{Q}_i \triangleq Q(\boldsymbol{a}_i, \ddot{\boldsymbol{a}}_i) + Q(\ddot{\boldsymbol{a}}_i, \boldsymbol{a}_i) + 2Q(\dot{\boldsymbol{a}}_i, \dot{\boldsymbol{a}}_i) \quad (28)$$

where, for the $(N \times 1)$ vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, $Q(\boldsymbol{u}, \boldsymbol{v})$ is the $((2NP+1) \times (2NP+1))$ matrix defined by

$$Q(\boldsymbol{u}, \boldsymbol{v}) \triangleq \begin{pmatrix} \alpha & -\boldsymbol{q}_{12}{}^\dagger & \mathbf{0}^\mathrm{T} \\ -\boldsymbol{q}_{21} & Q_{22} & Q_{23} \\ \mathbf{0} & Q_{32} & Q_{33} \end{pmatrix} \quad (29)$$

where $\alpha \triangleq \boldsymbol{v}^\dagger \Pi_{2,n}(E=0)\boldsymbol{u}$, $\boldsymbol{q}_{12}$ and $\boldsymbol{q}_{21}$ are the $(NP\times 1)$ vectors defined by $\boldsymbol{q}_{12} \triangleq \boldsymbol{\phi}(\boldsymbol{u}, \boldsymbol{v})$ and $\boldsymbol{q}_{21} \triangleq \boldsymbol{\phi}(\boldsymbol{v}, \boldsymbol{u})$ respectively with $\boldsymbol{\phi}(\boldsymbol{u}, \boldsymbol{v}) \triangleq [(A^\dagger A)^{-1} A^\dagger \boldsymbol{u}]^* \otimes [\Pi_{2,n}(E=0)\boldsymbol{v}]$, $\mathbf{0}$ is the $(NP \times 1)$ zero vector, Q_{22}, Q_{23}, Q_{32} and Q_{33} are $(NP \times NP)$ matrices defined by

$$Q_{22} \triangleq \psi[(A^\dagger A)^{-1}A^\dagger, (A^\dagger A)^{-1}A^\dagger, \Pi_{2,n}(E=0)] \quad (30)$$

$$Q_{23} \triangleq \psi[(A^\dagger A)^{-1}A^\dagger, \Pi_{2,n}(E=0), A(A^\dagger A)^{-1}]\Pi_p \quad (31)$$

$$Q_{32} \triangleq \Pi_p{}^\dagger \psi[\Pi_{2,n}(E=0), (A^\dagger A)^{-1}A^\dagger, (A^\dagger A)^{-1}A^\dagger] \quad (32)$$

$$Q_{33} \triangleq -\Pi_p{}^\dagger \psi[\Pi_{2,n}(E=0), \Pi_{2,n}(E=0), (A^\dagger A)^{-1}]\Pi_p \quad (33)$$

respectively, where $\psi[\mathrm{X}, \mathrm{Y}, \mathrm{Z}] \triangleq [(\mathrm{X}\boldsymbol{v})^*(\mathrm{Y}\boldsymbol{u})^\mathrm{T}] \otimes \mathrm{Z}$ and Π_p is the $(NP \times NP)$ permutation matrix such that $\mathrm{vec}(E^\mathrm{T}) \triangleq \Pi_p \mathrm{vec}(E)$. These results are used in Section V-C to compute $E[\Delta\theta_i{}^2]$ for $q > 1$.

C. Solution for $q > 1$

The RMSE of the source DOA estimates have been computed in Section V-B from expressions (22) to (24), with $q = l = 1$, and a second-order Taylor expansion of

$$\Pi_{2q,n}(l)(E) = \Pi_{2,n}(E) \triangleq \mathrm{I}_N - \widetilde{A}(E)(\widetilde{A}(E)^\dagger \widetilde{A}(E))^{-1}\widetilde{A}(E)^\dagger$$

around $E = 0$, where I_N is the $(N \times N)$ identity matrix and $\widetilde{A}(E) \triangleq [\widetilde{a}_1, \ldots\ldots, \widetilde{a}_P] \triangleq A + E$. Assuming statistically

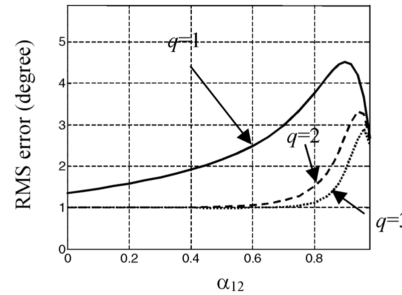


Fig. 2. RMSE of the source 1 as a function of $|\alpha_{12}|$, $q = 1, 2, 3$, $N = 5$, UCA, $r/\lambda = 0.5$, $P = 2$, i.i.d. Gaussian and circular errors, $\sigma^2 = 0.0149$. Theoretical results.

independent sources and considering arbitrary values of q and l, the projector on the noise subspace, $\Pi_{2q,n}(l)(E)$, takes the form $\Pi_{2q,n}(l)(E) \triangleq \mathrm{I}_N{}^q - \widetilde{\overline{A}}_{q,l}(E)(\widetilde{\overline{A}}_{q,l}(E)^\dagger \widetilde{\overline{A}}_{q,l}(E))^{-1}\widetilde{\overline{A}}_{q,l}(E)^\dagger$ where $\widetilde{\overline{A}}_{q,l}(E) \triangleq [\widetilde{\boldsymbol{a}}_{q,l}(\theta_1), \ldots\ldots, \widetilde{\boldsymbol{a}}_{q,l}(\theta_P)]$ and $\widetilde{\boldsymbol{a}}_{q,l}(\theta_i) \triangleq \widetilde{\boldsymbol{a}}_i^{\otimes l} \otimes_i{}^{*\otimes(q-l)}$. Considering a first-order expansion of $\widetilde{\overline{A}}_{q,l}(E)$ around $E = 0$, we obtain

$$\widetilde{\overline{A}}_{q,l}(E) \approx \overline{A}_{q,l} + E_{q,l} \triangleq \widetilde{\overline{A}}_{q,l}(E_{q,l}) \quad (34)$$

where $\overline{A}_{q,l}$ is defined by H2''and where it is easy to show that $E_{q,l}$ is the $(N^q \times P)$ matrix defined by $E_{q,l} \triangleq [\boldsymbol{e}_{q,l}(\theta_1), \ldots\ldots, \boldsymbol{e}_{q,l}(\theta_P)]$ where $\boldsymbol{e}_{q,l}(\theta_i)$ is defined by

$$\boldsymbol{e}_{q,l}(\theta_i) = \sum_{u=0}^{l-1} \boldsymbol{a}_i{}^{\otimes u} \otimes \boldsymbol{e}_i \otimes \boldsymbol{a}_i{}^{\otimes(l-u-1)} \otimes \boldsymbol{a}_i{}^{*\otimes(q-l)} + \sum_{u=0}^{q-l-1} \boldsymbol{a}_i{}^{\otimes l} \otimes \boldsymbol{a}_i{}^{*\otimes u} \otimes \boldsymbol{e}_i{}^* \otimes \boldsymbol{a}_i{}^{*\otimes(q-l-u-1)} \quad (35)$$

with the convention $\boldsymbol{a}_i{}^{\otimes 0} \triangleq 1$. In these conditions, the RMS error of the source DOA estimates can be computed for arbitrary values of q and l from expressions (22) to (24) and a second-order Taylor expansion of

$$\Pi_{2q,n}(l)(E) \approx \Pi_{2q,n}(l)(E_{q,l}) \triangleq \mathrm{I}_N{}^q - \widetilde{\overline{A}}_{q,l}(E_{q,l})(\widetilde{\overline{A}}_{q,l}(E_{q,l})^\dagger \widetilde{\overline{A}}_{q,l}(E_{q,l}))^{-1}\widetilde{\overline{A}}_{q,l}(E_{q,l})^\dagger$$

around $E_{q,l} = 0$, where $\widetilde{\overline{A}}_{q,l}(E_{q,l}) \triangleq \overline{A}_{q,l} + E_{q,l}$, by replacing N by N^q, E by $E_{q,l}$, A by $\overline{A}_{q,l}$ and $\boldsymbol{a}(\theta_i)$ by $\boldsymbol{a}_{q,l}(\theta_i)$ in expressions (25) to (33).

D. Illustration

To illustrate the previous results, we assume that two statistically independent sources are received by a UCA of $N = 5$ omnidirectional sensors with a radius equal to half the wavelenght. The DOA of the source 1 is $\theta_1 = 100°$. The modeling error vectors $\boldsymbol{e}_i$ $(1 \leq i \leq 2)$ are assumed to be zero-mean statistically independent circular Gaussian vectors such that $\mathrm{E}[\boldsymbol{e}_i \boldsymbol{e}_j{}^\dagger] = \sigma^2 \delta_{ij} \mathrm{I}_N$ where $\sigma^2 = 0.0149$, which correspond for example to a phase error with a standard deviation of $7°$ without any amplitude error. Under these assumptions, Fig. 2 shows, for $q = 1, 2, 3$ and for $l = 1$ (optimal arrangement), the variations of the RMSE of the source 1 (it is similar

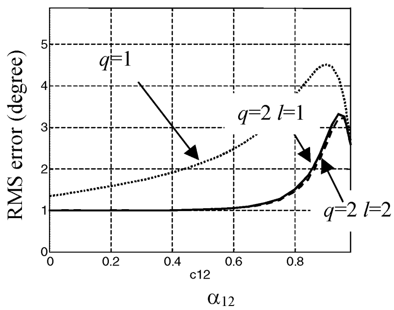


Fig. 3. RMSE of the source 1 as a function of $\mid \alpha_{12} \mid$, $q = 1, 2$, $l = 1, 2$, $N = 5$, UCA, $r/\lambda = 0.5$, $P = 2$, i.i.d. Gaussian and circular errors, $\sigma^2 = 0.0149$. Theoretical results.

for the source 2) as a function of the modulus of the spatial correlation coefficient between the two sources α_{12}, defined by

$$\alpha_{12} = \frac{\boldsymbol{a}_1{}^\dagger \boldsymbol{a}_2}{(\boldsymbol{a}_1{}^\dagger \boldsymbol{a}_1)^{1/2} (\boldsymbol{a}_2{}^\dagger \boldsymbol{a}_2)^{1/2}}. \tag{36}$$

Fig. 2 shows, for each value of q, an increasing RMSE with $\mid \alpha_{12} \mid$ as long as the two sources are resolved by the $2q$-MUSIC algorithm and a decreasing RMSE with $\mid \alpha_{12} \mid$ as soon as the two sources become unresolvable by the algorithm. Besides, for a given value of $\mid \alpha_{12} \mid$, Fig. 2 shows a decreasing value of the RMSE as q increases, showing off the better robustness of the $2q$-MUSIC algorithm to modeling errors as q increases, especially for poor angular separation between the sources, in the limit of resolvability of the latter. Moreover, the maximal value of $\mid \alpha_{12} \mid$ ensuring resolvability of the two sources increases with q, showing off the increasing resolution of $2q$-MUSIC algorithm as q increases. All these results may be physically interpreted through the HO virtual array concept introduced in [9]. Indeed, the increasing resolution of the $2q$-MUSIC algorithm as q increases is directly related to the decreasing value with q of the modulus of the spatial correlation coefficient of two sources for the associated virtual array, $\alpha_{12}(2q, l)$, defined by the normalized inner product of $\boldsymbol{a}_{q,l}(\theta_1)$ and $\boldsymbol{a}_{q,l}(\theta_2)$, since $\mid \alpha_{12}(2q, l) \mid = \mid \alpha_{12} \mid^q$ as shown in [9]. In a same way, the increasing robustness to modeling errors of the $2q$-MUSIC algorithm as q increases is also directly related to the increasing resolution, with q, of the associated virtual array. All these results definitely show off the great interest of $2q$-MUSIC methods for $q \geq 2$, even for overdetermined mixtures of sources $(P < N)$, despite their increasing variance and complexity with q.

The influence of the arrangement of the statistics on the robustness to modeling errors of $2q$-MUSIC algorithm is illustrated in Fig. 3, which shows the same things as Fig. 2 for the same scenario but for $q = 1, 2$ and for $l = 1, 2$. We note that performance of 4-MUSIC for $l = 1$ and $l = 2$ are very close to each other since the resolution of the associated virtual arrays is independent of l ($\mid \alpha_{12}(2q, l) \mid$ independent of l). Nevertheless, we note surprisingly lightly better performance for the suboptimal arrangement ($l = 2$), probably due to the lower number of virtual sensors of the associated virtual array.

Let us recall that the illustrations presented in this section are computed for exact statistics of the data. Estimated statistics are considered in the next section.

VI. Computer Simulations

The results of the previous sections are illustrated in this section through computer simulations. To do so, we first introduce a performance criterion in Section VI-A and describe the simulations in Sections VI-B and VI-C for overdetermined and underdetermined mixtures of sources respectively. The sources are assumed to have a zero elevation angle φ and to be zero-mean stationary sources corresponding to quaternary phase-shift keying (QPSK) sources sampled at the symbol rate.

A. Performance Criterion

For each of the P considered sources and for a given direction-finding method, two criteria are used in the following to quantify the quality of the associated DOA estimation. For a given source, the first criterion is a probability of aberrant results generated by a given method for this source and the second one is an averaged RMSE, computed from the nonaberrant results, generated by a given method for this source.

More precisely, for given values of q and l, a given number of snapshots, L, and a particular realization of the L observation vectors $\boldsymbol{x}(k)$ $(1 \leq k \leq L)$, the estimation, $\hat{\theta}_p$, of the DOA of the source p $(1 \leq p \leq P)$ from $2q$-MUSIC is defined by

$$\hat{\theta}_p \triangleq \operatorname*{Arg}_{\zeta_i}(\operatorname*{Min}_{i} |\zeta_i - \theta_p|) \tag{37}$$

where the quantities ζ_i $(1 \leq i \leq \hat{P})$ correspond to the $\hat{P}$ minima of the pseudo-spectrum $\hat{P}_{2q\text{-Music}(l)}(\theta)$ defined by (14) for $\varphi = 0$ and $\hat{P}$ is an estimate of the source number P. To each estimate $\hat{\theta}_p$ $(1 \leq p \leq P)$, we associate the corresponding value of the pseudo-spectrum, defined by $\eta_p = \hat{P}_{2q\text{-Music}(l)}(\hat{\theta}_p)$. In this context, the estimate $\hat{\theta}_p$ is considered to be aberrant if $\eta_p > \eta$, where η is a threshold to be defined. In the following $\eta = 0.1$.

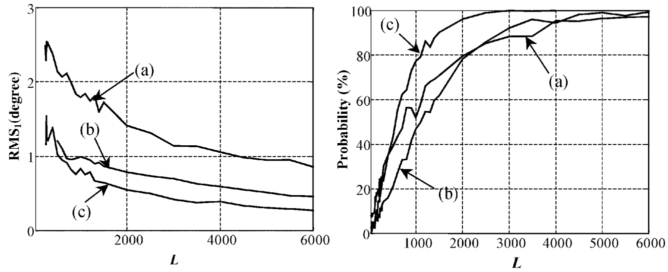
Let us now consider M realizations of the L observation vectors $\boldsymbol{x}(k)$ $(1 \leq k \leq L)$. For a given method, the probability of abberant results for a given source p, $p(\eta_p > \eta)$, is defined by the ratio between the number of realizations for which $\hat{\theta}_p$ is aberrant and the number of realizations M. From the nonaberrant realizations for the source p, we then define the averaged RMSE for the source p, RMSE_p, by the quantity

$$\text{RMSP}_p \triangleq \sqrt{\frac{1}{M_p} \sum_{m=1}^{M_p} \left| \hat{\theta}_{pm} - \theta_p \right|^2} \tag{38}$$

where M_p is the number of nonaberrant realizations for the source p and $\hat{\theta}_{pm}$ is the estimate of θ_p for the nonaberrant realization m.

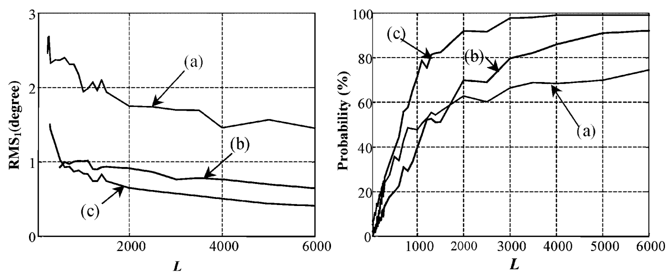
B. Overdetermined Mixtures of Sources

To quantify the influence of both the number of independent snapshots L and the parameter q on the performance of



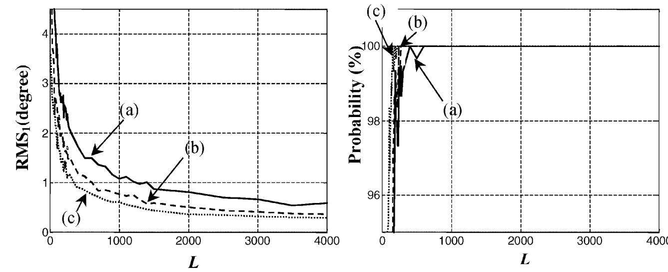
(a) (b)

Fig. 4. RMSE of the source 1 and $p(\eta_1 \leq \eta)$ as a function of L: (a) 2-MUSIC, (b) 4-MUSIC, (c) 6-MUSIC, $P = 2$, $N = 3$, ULA, SNR = 5 dB, $\theta_1 = 90°$, $\theta_2 = 82.7°$, no modeling errors.



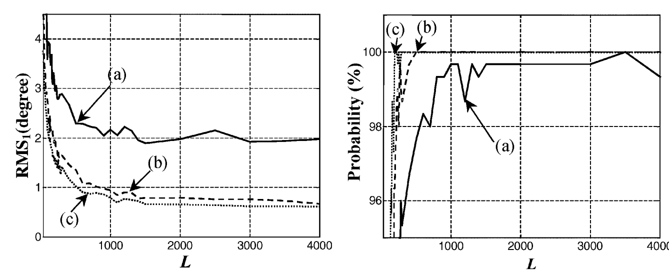
(a) (b)

Fig. 5. RMSE of the source 1 and $p(\eta_1 \leq \eta)$ as a function of L: (a) 2-MUSIC, (b) 4-MUSIC, (c) 6-MUSIC, $P = 2$, $N = 3$, ULA, SNR = 5 dB, $\theta_1 = 90°$, $\theta_2 = 82.7°$, with modeling errors.



(a) (b)

Fig. 6. RMSE of the source 1 and $p(\eta_1 \leq \eta)$ as a function of L: (a) 2-MUSIC, (b) 4-MUSIC ($l = 1$), (c) 4-MUSIC ($l = 2$), $P = 2$, $N = 3$, UCA, SNR = 10 dB, $\theta_1 = 90°$, $\theta_2 = 110°$, no modeling errors.



(a) (b)

Fig. 7. RMSE of the source 1 and $p(\eta_1 \leq \eta)$ as a function of L: (a) 2-MUSIC, (b) 4-MUSIC ($l = 1$), (c) 4-MUSIC ($l = 2$), $P = 2$, $N = 3$, UCA, SNR = 10 dB, $\theta_1 = 90°$, $\theta_2 = 110°$, with modeling errors.

$2q$-MUSIC algorithms, we assume that two statistically independent QPSK sources with a raise cosine pulse shape filter are received by a ULA of $N = 3$ omnidirectional sensors spaced half a wavelength apart. The two QPSK sources have the same symbol duration, the same roll-off $\mu = 0.3$, the same input SNR equal to 5 dB, and a DOA equal to $\theta_1 = 90°$ and $\theta_2 = 82.7°$, respectively. Note that the normalized autocumulant of the QPSK symbols is equal to -1 at the FO and $+4$ at the sixth order. Under these assumptions, Figs. 4 and 5 show the variations, as a function of the number of snapshots L, of the RMSE for the source 1, RMSE_1, and the associated probability of nonabberant results, $p(\eta_1 \leq \eta)$, (we obtain similar results for the source 2), estimated from $M = 300$ realizations, at the output of both 2-MUSIC, 4-MUSIC, and 6-MUSIC methods for optimal arrangements of the considered statistics, without and with modeling errors, respectively. In the latter case, the modeling error vectors $\boldsymbol{e}_i$ $(1 \leq i \leq 2)$, are assumed to be zero-mean statistically independent circular Gaussian vectors such that $E[\boldsymbol{e}_i \boldsymbol{e}_j \dagger] = \sigma^2 \delta_{ij} I_N$ where $\sigma^2 = 0.0003$, which corresponds, for example, to a phase error with a standard deviation of 1° with no amplitude error. Both in terms of probability of nonaberrant results and estimation precision, Figs. 4 and 5 show, for poorly angularly separated sources, the best behavior of the 6-MUSIC method with respect to 2-MUSIC and 4-MUSIC as soon as L becomes greater than 400 snapshots without modeling errors and 500 snapshots with modeling errors. For such values of L, the resolution gain and the better robustness to modeling errors obtained with 6-MUSIC with respect to 2-MUSIC and 4-MUSIC, due to the narrower 3-dB beamwidth of the associated sixth-order VA respectively, is higher than the loss due to a higher variance in the statistics estimates. A similar analysis can be done for 4-MUSIC with respect to 2-MUSIC as soon as L becomes greater than 2000 without modeling errors and 1700 snapshots with modeling errors. These results confirm that, despite of their higher variance and contrary to some generally accepted ideas, HO MUSIC methods may offer better performances than 2-MUSIC or 4-MUSIC methods when some resolution is required, i.e., in the presence of several sources, when the latter are weak, poorly angularly separated or in the presence of modeling errors inherent in operational contexts.

To quantify the influence of the arrangement of the HO statistics on the performance of $2q$-MUSIC algorithms, we consider now that the two statistically independent QPSK sources are received by a UCA of $N = 3$ omnidirectional sensors with a radius r such that $r = 0.3\,\lambda$. The two QPSK sources have the same symbol duration, the same roll-off $\mu = 0.3$, the same input SNR equal to 10 dB and a direction of arrival equal to $\theta_1 = 90°$ and $\theta_2 = 110°$, respectively. Under these assumptions, Figs. 6 and 7 show the variations, as a function of the number of snapshots L, of the RMSE for the source 1, RMSE_1, and the associated probability of nonabberant results, $p(\eta_1 \leq \eta)$, estimated from $M = 300$ realizations, at the output of both 2-MUSIC algorithm and 4-MUSIC algorithm with $l = 1$ and 2, without and with modeling errors similar to those of Fig. 5 respectively. We note, in both cases, the better behavior of 4-MUSIC with respect to 2-MUSIC whatever the chosen arrangement of the FO statistics jointly with the better behavior of 4-MUSIC ($l = 2$) with respect to 4-MUSIC ($l = 1$) despite the optimality of the arrangement indexed by $l = 1$. Although this result might seem to be in contradiction with results of Section IV and [9], it is not since the optimality of the arrangement indexed by $l = 1$ for the FO direction-finding problem is related to the number of virtual

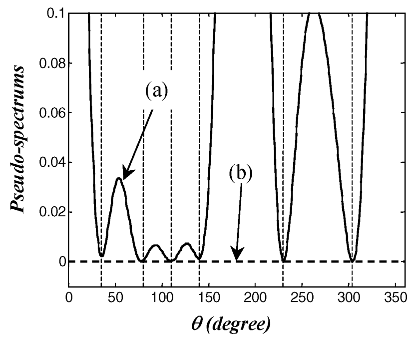


Fig. 8. Pseudo-spectrum of 4-MUSIC as a function of θ: (a) 4-MUSIC ($l = 1$) (b) 4-MUSIC ($l = 2$), $P = 6$, $N = 3$, UCA, SNR = 15 dB, $L = 10\,000$, $\theta_1 = 35°$, $\theta_2 = 80°$, $\theta_3 = 110°$, $\theta_4 = 140°$, $\theta_5 = 230°$, $\theta_6 = 304°$, no modeling errors.

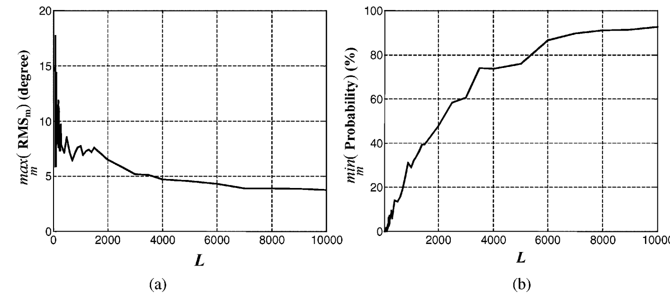


Fig. 9. Maximal RMSE and minimum probability of nonaberrant results as a function of L, 4-MUSIC ($l = 1$), $P = 6$, $N = 3$, UCA, SNR = 15 dB, $\theta_1 = 35°$, $\theta_2 = 80°$, $\theta_3 = 110°$, $\theta_4 = 140°$, $\theta_5 = 230°$, $\theta_6 = 304°$, no modeling errors.

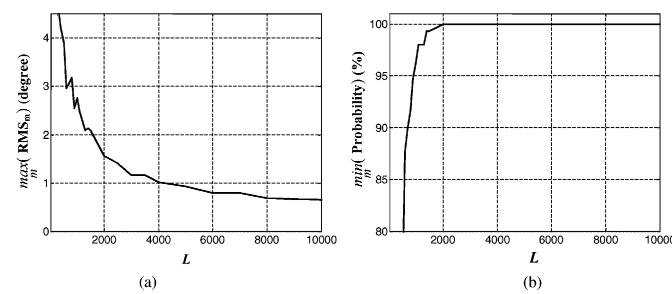


Fig. 10. Maximal RMSE and minimum probability of nonaberrant results as a function of L, 6-MUSIC, $P = 3$, $N = 2$, ULA, SNR = 10 dB, $\theta_1 = 60°$, $\theta_2 = 90°$, $\theta_3 = 120°$, no modeling errors.

sensors of the associated VA and thus to the number of sources that can be processed by 4-MUSIC, which is maximal for $l = 1$. However, as the modulus of the spatial correlation coefficient of two sources for the associated virtual array $\alpha_{12}(2q, l)$ is independent of the arrangement of the statistics [9], i.e., of the index l, the resolution of 4-MUSIC algorithm is independent of l whereas the number of virtual sensors of the associated VA is lower for $l = 2$, which generates less variance in the statistics estimation and which explains the result.

C. Underdetermined Mixtures of Sources

To illustrate the influence of the arrangement of the HO statistics on the number of sources that can be processed by the $2q$-MUSIC algorithm, we assume that six statistically independent QPSK sources with a raise cosine pulse shape filter are received by a UCA of $N = 3$ omnidirectional sensors with a radius r such that $r = 0.3\ \lambda$. The six QPSK sources have the same symbol duration, the same roll-off $\mu = 0.3$, the same input SNR equal to 15 dB and a direction of arrival equal to $\theta_1 = 35°$, $\theta_2 = 80°$, $\theta_3 = 110°$, $\theta_4 = 140°$, $\theta_5 = 230°$, $\theta_6 = 304°$, respectively. Under these assumptions, Fig. 8 shows the variations of the pseudo-spectrum of 4-MUSIC, for $l = 1$ and 2, as a function of θ for $L = 10\,000$ without modeling errors. Note the good behavior of 4-MUSIC for $l = 1$, which succeeds in estimating the DOA of the six sources, and the poor behavior of 4-MUSIC for $l = 2$, which fails in estimating the latter, since the number of virtual sensors of the associated VA is equal to $N_4^1 = 7$ for $l = 1$ and to $N_4^2 = 6$ for $l = 2$. To complete these results, Fig. 9 shows the variations, as a function of the number of snapshots L, of the highest RMSE and the lowest probability of nonabberant results, among all the sources, estimated from $M = 300$ realizations, at the output of the 4-MUSIC algorithm for $l = 1$ and without modeling errors. Note the increasing minimum probability of nonaberrant results and the decreasing maximal RMSE as L increases, showing off the capability of 4-MUSIC to efficiently estimate the DOA of all the sources for $l = 1$.

Finally, to illustrate the capability for the 6-MUSIC algorithm to process three sources from $N = 2$ sensors, we assume that three statistically independent QPSK sources with a raise cosine pulse shape filter are received by a ULA of $N = 2$ omnidirectional sensors equispaced half a wavelength apart. The three QPSK sources have the same symbol duration, the same roll-off $\mu = 0.3$, the same input SNR equal to 10 dB, and a DOA equal to $\theta_1 = 60°$, $\theta_2 = 90°$ and $\theta_3 = 120°$ respectively. Under these assumptions, Fig. 10 shows the variations, as a function of the number of snapshots L, of the highest RMSE and the lowest probability of nonabberant results, among all the sources, estimated from $M = 300$ realizations, at the output of the 6-MUSIC algorithm without modeling errors. Note the increasing minimum probability of nonaberrant results and the decreasing maximal RMSE as L increases, showing off the capability of 6-MUSIC to efficiently estimate the DOA of the three sources since in this case P is strictly lower than the number of virtual sensors, $N_6^1 = 4$, of the associated virtual array.

VII. CONCLUSION

In this paper, an extension of the MUSIC algorithm to an arbitrary even-order $2q$ ($q \geq 1$) and for several arrangements, indexed by an integer l, of the $2q$th-order data statistics has been introduced, giving rise to the so-called $2q$-MUSIC algorithms. The performance analysis of $2q$-MUSIC algorithms with $q > 1$, shows off new important results for direction-finding applications, opening new perspectives in array processing. Indeed, from the HO VA concept presented recently in [9], it has been verified in the paper that the way the $2q$th-order data statistics are arranged in the exploited $2q$th-order statistical matrix, controls the maximal number of sources that can be processed by the $2q$-MUSIC method, showing off the existence of an optimal arrangement of these statistics, which has been described in the paper. Besides, for a given array of identical sensors, it has been shown from [9] that the maximal number of sources that can be processed by the $2q$-MUSIC algorithm for the arrangement indexed by l is an increasing function of q and corresponds to $N_{2q}^{l} - 1$, where N_{2q}^{l} is the number of different virtual sensors of the virtual array associated with the couple (q, l). This maximal number has been computed in the paper for $1 \leq q \leq 4$ for a general array with no particular symmetries and whatever the value of q for a ULA. Another important result of the paper is that, while keeping, for $q > 1$, its asymptotic robustness to a strong background Gaussian noise whose spatial coherence is unknown, both the resolution of $2q$-MUSIC method, implemented from a finite number of snapshots, and its asymptotic robustness to modeling errors increase with q. This result jointly with the HO VA concept [9] allow to explain why, despite of their higher variance, $2q$-MUSIC methods with $q > 2$ may offer better performances than 2-MUSIC or 4-MUSIC methods when some resolution is required, i.e., in the presence of several sources, when the latter are poorly angularly separated or in the presence of modeling errors inherent in operational contexts. A consequence of this latter result is that, for given performances, $2q$-MUSIC method for $q > 1$ may allow to slacken some constraints about antennas' calibration or receivers' chains equalization.

REFERENCES

[1] P. O. Amblard, M. Gaeta, and J. L. Lacoume, "Statistics for complex variables and signals – Part I and II," *Signal Process.*, vol. 53, no. 1, pp. 1–25, Aug. 1996.

[2] G. Bienvenu and L. Kopp, "Optimality of high resolution array processing using the eigensystem approach," *IEEE Trans. Acoust. Speech, Signal Process.*, vol. 31, no. 5, pp. 1235–1247, Oct. 1983.

[3] Y. Bresler and A. Macovski, "Exact maximum likelihood parameter estimation of superimposed exponential signals in noise," *IEEE Trans. Acoust., Speech, Signal Process.*, vol. 34, no. 5, pp. 1081–1089, Oct. 1986.

[4] D. R. Brillinger, "A maximum likelihood approach to frequency wavenumber analysis," *IEEE Trans. Acoust., Speech, Signal Process.*, vol. 33, no. 4, pp. 1076–1085, Oct. 1985.

[5] J. P. Burg, "The relationship between maximum entropy spectra and maximum likelihood spectra," *Geophys.*, vol. 37, no. 2, pp. 375–376, Apr. 1972.

[6] J. Capon, "High resolution frequency-wavenumber spectrum analysis," *Proc. IEEE*, vol. 57, pp. 1408–1418, Feb. 1969.

[7] J. F. Cardoso, "Localisation et identification par la quadricovariance," *Traitement du Signal*, vol. 7, no. 5, 1990.

[8] J. F. Cardoso and E. Moulines, "Asymptotic performance analysis of direction finding algorithms based on fourth-order cumulants," *IEEE Trans. Signal Process.*, vol. 43, no. 1, pp. 214–224, Jan. 1995.

[9] P. Chevalier, L. Albera, A. Ferreol, and P. Comon, "On the virtual array concept for higher order array processing," *IEEE Trans. Signal Process.*, vol. 53, no. 4, pp. 1254–1271, Apr. 2005.

[10] P. Chevalier and A. Ferreol, "On the virtual array concept for the fourth-order direction finding problem," *IEEE Trans. Signal Process.*, vol. 47, no. 9, pp. 2592–2595, Sep. 1999.

[11] H. H. Chiang and C. L. Nikias, "The ESPRIT algorithm with high order statistics," in *Proc. Workshop Higher Order Statistics*, Vail, Jun. 1989, pp. 163–168.

[12] R. T. Compton Jr., *Adaptive Antennas – Concepts and Performance*. Englewood Cliffs, NJ: Prentice-Hall, 1988, p. 07632.

[13] A. V. Dandawate and G. B. Giannakis, "Asymptotic theory of mixed time averages and $\mathbb{K}$th-order cyclic-moment and cumulant statistics," *IEEE Trans. Inf. Theory*, vol. 41, no. 1, pp. 216–232, Jan. 1995.

[14] C. Demeure and P. Chevalier, "The smart antennas at thomson-CSF communications: Concepts, implementations, performances, applications," *Annales des Télécommunications*, vol. 53, no. 11-12, pp. 466–482, Nov.-Dec. 1998.

[15] M. C. Dogan and J. M. Mendel, "Applications of cumulants to array processing —Part I : Aperture extension and array calibration," *IEEE Trans. Signal Process.*, vol. 43, no. 5, pp. 1200–1216, May 1995.

[16] A. Ferreol and P. Chevalier, "On the behavior of current second and higher order blind source separation methods for cyclostationary sources," *IEEE Trans. Signal Process.*, vol. 48, no. 6, pp. 1712–1725, Jun. 2000.

[17] A. Ferreol, P. Chevalier, and L. Albera, "Higher order blind separation of non zero-mean cyclostationary sources," in *Proc. EUSIPCO 02*, Toulouse, France, Sep. 2002, pp. 103–106.

[18] ——, "Second order blind separation of first and second order cyclostationary sources—Application to AM, FSK, CPFSK and deterministic sources," *IEEE Trans. Signal Process.*, vol. 52, no. 4, pp. 845–861, Apr. 2004.

[19] A. Ferreol, P. Larzabal, and M. Viberg, "On the asymptotic performance analysis of subspace DOA estimation in the presence of modeling errors: Case of MUSIC," *IEEE Trans. Signal Process.*, vol. 54, no. 3, pp. 907–920, Mar. 2006.

[20] B. Friedlander, "A sensitivity analysis of the MUSIC algorithm," *IEEE Trans. Acoust. Speech, Signal Process.*, vol. 38, no. 10, pp. 1740–1751, Oct. 1990.

[21] E. Gönen and J. M. Mendel, "Applications of cumulants to array processing—Part VI : Polarization and direction of arrival estimation with minimally constrained arrays," *IEEE Trans. Signal Process.*, vol. 47, no. 9, pp. 2589–2592, Sep. 1999.

[22] P. Germain, A. Maguer, and L. Kopp, "Comparison of resolving power of array processing methods by using an analytical criterion," in *Proc. Int. Conf. Acoustics, Speech, Signal Processing (ICASSP)*, Glasgow, U.K., May 1989.

[23] M. Kaveh and A. J. Barabell, "The statistical performance of the MUSIC and the minimum norm algorithms in resolving plane waves in noise," *IEEE Trans. Acoust., Speech, Signal Process.*, vol. 34, no. 2, pp. 331–341, Apr. 1986.

[24] R. Kumaresan and D. W. Tufts, "Estimating the angles of arrival of multiple plane waves," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 19, no. 1, pp. 134–138, Jan. 1983.

[25] F. Li and R. J. Vaccaro, "Sensitivity analysis of DOA estimation algorithms to sensor errors," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 28, no. 3, pp. 708–717, Jul. 1992.

[26] P. M. Cullagh, "Tensor methods in statistics," in *Monographs on Statistics and Applied Probability*. London, U.K.: Chapman & Hall, 1987.

[27] A. Paulraj and T. Kailath, "Eigenstructure methods for direction of arrival estimation in the presence of unknown noise field," *IEEE Trans. Acoust., Speech, Signal Process.*, vol. 34, no. 1, pp. 13–20, Feb. 1986.

[28] B. Picinbono, "On circularity," *IEEE Trans. Signal Process.*, vol. 42, no. 12, pp. 3473–3482, Dec. 1994.

[29] B. Porat and B. Friedlander, "Analysis of the asymptotic relative efficiency of the MUSIC algorithm," *IEEE Trans. Acoust. Speech. Signal Process.*, vol. 36, no. 4, pp. 532–544, Apr. 1988.

[30] ——, "Direction finding algorithms based on higher order statistics," *IEEE Trans. Signal Process.*, vol. 39, no. 9, pp. 2016–2024, Sep. 1991.

[31] R. Roy, A. Paulraj, and T. Kailath, "ESPRIT—A subspace rotation approach to estimation of parameters of cisoids in noise," *IEEE Trans. Acoust., Speech, Signal Process.*, vol. 34, no. 5, pp. 1340–1342, Oct. 1986.

[32] R. O. Schmidt, "Multiple emitter location and signal parameter estimation," *IEEE Trans. Antennas Propag.*, vol. 34, no. 3, pp. 276–280, Mar. 1986.

[33] S. Shamsunder and G. B. Giannakis, "Modeling of non-Gaussian array data using cumulants: DOA estimation of more sources with less sensors," *Signal Process.*, vol. 30, no. 3, pp. 279–297, Feb. 1993.

[34] P. Stoica and A. Nehorai, "MUSIC, maximum likelihood and Cramer Rao bound," *IEEE Trans. Acoust., Speech, Signal Process.*, vol. 37, no. 12, pp. 720–741, May 1989.

[35] ——, "MUSIC, maximum likelihood and Cramer Rao bound: Further results and comparisons," *IEEE Trans. Acoust., Speech, Signal Process.*, vol. 38, no. 12, pp. 2140–2150, Dec. 1990.

[36] A. L. Swindlehurst and T. Kailath, "A performance analysis of subspaced-based methods in the presence of model errors, Part I: The MUSIC algorithm," *IEEE Trans. Signal Process.*, vol. 40, no. 7, pp. 1758–1773, Jul. 1992.

[37] M. Viberg and B. Ottersten, "Sensor array processing based on subspace fitting," *IEEE Trans. Acoust., Speech, Signal Process.*, vol. 39, no. 5, pp. 1110–1121, May 1991.

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