On the Virtual Array Concept for Higher Order Array Processing

Pascal Chevalier, Laurent Albera, Anne Ferréol, and Pierre Comon, *Senior Member, IEEE*

***Abstract*—For about two decades, many fourth order (FO) array processing methods have been developed for both direction finding and blind identification of non-Gaussian signals. One of the main interests in using FO cumulants only instead of second-order (SO) ones in array processing applications relies on the increase of both the effective aperture and the number of sensors of the considered array, which eventually introduces the FO *Vitual Array* concept presented elsewhere and allows, in particular, a better resolution and the processing of more sources than sensors. To still increase the resolution and the number of sources to be processed from a given array of sensors, new families of blind identification, source separation, and direction finding methods, at an order $m = 2q$ $(q \geq 2)$ only, have been developed recently. In this context, the purpose of this paper is to provide some important insights into the mechanisms and, more particularly, to both the resolution and the maximal processing capacity, of numerous $2q$th order array processing methods, whose previous methods are part of, by extending the Virtual Array concept to an arbitrary even order for several arrangements of the data statistics and for arrays with space, angular and/or polarization diversity.**

***Index Terms*—Blind source identification, higher order, HO direction finding, identifiability, space, angular, and polarization diversities, $2q$-MUSIC, virtual array.**

I. Introduction

For about two decades, many fourth order (FO) array processing methods have been developed for both direction finding [4], [6], [9], [21], [23] and blind identification [1], [5], [10], [12], [14], [17] of non-Gaussian signals. One of the main interests in using FO cumulants only instead of second-order (SO) ones in array processing applications relies on the increase of both the effective aperture and the number of sensors of the considered array, which eventually introduces the FO Virtual Array (VA) concept presented in [7], [15], and [16], allowing, in particular, both the processing of more sources than sensors and an increase in the resolution power of array processing methods.

In order to still increase both the resolution power of array processing methods and the number of sources to be processed from a given array of sensors, new families of blind identification, source separation, and direction finding methods, exploiting the data statistics at an arbitrary even order $m = 2q$ $(q \geq 2)$ only, have been developed recently in [3] and [8], respectively. More precisely, [8] mainly extends the well-known high resolution direction finding method called MUSIC [24] to an arbitrary even order $2q$, giving rise to the so-called $2q$-MUSIC methods, whose interests, for $q \geq 2$, are also shown in [8]. In particular, for operational contexts characterized by a high source density, such as airborne surveillance over urban areas, the use of Higher Order (HO) MUSIC methods for direction finding allows us to reduce or even to minimize the number of sensors of the array and, thus, the number of reception chains, which finally drastically reduces the overall cost. Besides, it is shown in [8] that, despite of their higher variance and contrary to some generally accepted ideas, $2q$-MUSIC methods with $q > 2$ may offer better performances than 2-MUSIC or 4-MUSIC methods when some resolution is required, i.e., in the presence of several sources, when the latter are poorly angularly separated or in the presence of modeling errors inherent in operational contexts. In the same spirit, to process both over and underdetermined mixtures of statistically independent non-Gaussian sources, [3] mainly extends the recently proposed FO blind source identification method called Independent Component Analysis using Redundancies (ICAR) [1] in the quadricovariance matrix to an arbitrary even-order $2q$, giving rise to the so-called $2q$-Blind Identification of Overcomplete MixturEs of sources (BIOME) methods, whose interests for $q > 2$ are shown in [3]. Note that the $2q$-BIOME method gives rise, for $q = 3$, to the sixth order method called Blind Identification of mixtures using Redundancies in the daTa Hexacovariance matrix (BIRTH) presented recently in [2]. In particular, it is shown in [2] and [3] that $2q$-BIOME methods, for $q \geq 3$, outperform all the existing Blind Source Identification (BSI) methods that are actually available, in terms of processing power of underdetermined mixtures of arbitrary statistically independent non-Gaussian sources.

Contrary to papers [3] and [8], the present paper does not focus on particular HO array processing methods for particular applications but rather aims at providing some important insights into the mechanisms of numerous HO methods and, thus, some explanations about their interests, through the extension of the VA concept introduced in [7], [15], and [16] for the FO array processing problems, to an arbitrary even-order $m = 2q$ $(q \geq 2)$ and for several arrangements of the $2q$th order data statistics for arrays with space, angular, and/or polarization diversity. This HO VA concept allows us, in particular, to show off both the increasing resolution and the increasing processing capacity of $2q$th order array processing methods as q increases. It allows us to solve not only the identifiability problem of HO methods

Manuscript received August 1, 2003; revised May 4, 2004. The associate editor coordinating the review of this manuscript and approving it for publication was Prof. Abdelhak M. Zoubir.

P. Chevalier and A. Ferréol are with Thalès-Communications, EDS/SPM/SBP, 92704 Colombes Cédex, France (e-mail: pascal.chevalier@fr.thalesgroup.com; Anne.ferreol@fr.thalesgroup.com).

L. Albera is with the Laboratoire Traitement du Signal et de l'Image (LTSI), Rennes University (INSERM U642), 35042 Rennes Cédex, France (e-mail: laurent.albera@univ-rennes1.fr).

P. Comon is with the I3S, Algorithmes-Euclide-B, F-06903 Sophia-Antipolis Cedex, France (E-mail: comon@i3s.unice.fr).

Digital Object Identifier 10.1109/TSP.2005.843703

presented in [3] and [8] in terms of maximal number of sources that can be processed by these methods from an array of N sensors but, in addition, that of all the array processing methods exploiting the algebraic structure of the $2q$th $(q \geq 2)$ order data statistics matrix only for particular arrangements of the latter. As a consequence of this result, the HO VA concept shows off the impact of the $2q$th order data statistics arrangement on the $2q$th order array processing method performances and, thus, the existence of an optimal arrangement of these statistics. This result is completely unknown by most of the researchers. Finally, one may think that the HO VA concept will spawn much practical research in array processing and will also be considered as a powerful tool for performance evaluation of HO array processing methods.

After an introduction of some notations, hypotheses, and data statistics in Section II, the VA concept is extended to even HO statistics in Section III, where the questions of both the optimal arrangement of the latter and the resolution of the VA is addressed. Some properties of the HO VA for arrays with space, angular, and/or polarization diversity are then presented in Section IV, where explicit upper bounds, that are reached for most array geometries, on the maximal number of independent non-Gaussian sources that can be processed by a $2q$th order method exploiting particular arrangements of the $2q$th order data statistics, are computed for $2q \leq 8$. Note that the restriction to values of $2q$ lower than or equal to 8 is not very restrictive since it corresponds to order of statistics that have the highest probability to be used for future applications. The results of Sections III and IV are then illustrated in Section V through the presentation of HO VA examples for both the Uniform Linear Array (ULA) and the Uniform Circular Arrays (UCA). Some practical situations for which the HO VA concept leads to better performance than SO or FO ones are pointed out and illustrated in Section VI through a direction finding application. Finally, Section VII concludes this paper.

II. Hypotheses, Notations, and Statistics of the Data

A. Hypotheses and Notations

We consider an array of N narrowband (NB) sensors, and we call $\boldsymbol{x}(t)$ the vector of complex amplitudes of the signals at the output of these sensors. Each sensor is assumed to receive the contribution of P zero-mean stationary and stastistically independent NB sources corrupted by a noise. Under these assumptions, the observation vector can approximately be written as follows:

$$\boldsymbol{x}(t) \approx \sum_{i=1}^{P} m_i(t)\boldsymbol{a}(\theta_i, \varphi_i) + \boldsymbol{v}(t) \triangleq A\,\boldsymbol{m}(t) + \boldsymbol{v}(t) \quad (1)$$

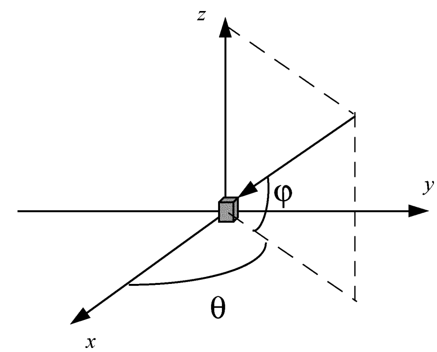

Fig. 1. Incoming signal in three dimensions.

where $\boldsymbol{v}(t)$ is the noise vector that is assumed zero-mean, $\boldsymbol{m}(t)$ is the vector whose components $m_i(t)$ are the complex amplitudes of the sources, θ_i and φ_i are the azimuth and the elevation angles of source i (Fig. 1), and A is the $(N \times P)$ matrix of the source steering vectors $\boldsymbol{a}(\theta_i, \varphi_i)$, which contains, in particular, the information about the direction of arrival of the sources. In particular, in the absence of coupling between sensors, component n of vector $\boldsymbol{a}(\theta_i, \varphi_i)$, which is denoted as $a_n(\theta_i, \varphi_i)$, can be written, in the general case of an array with space, angular, and polarization diversity, as (2), shown at the bottom of the page, [11] where λ is the wavelength, (x_n, y_n, z_n) are the coordinates of sensor n of the array, and $f_n(\theta_i, \varphi_i, p_i)$ is a complex number corresponding to the response of sensor n to a unit electric field coming from the direction (θ_i, φ_i) and having the state of polarization p_i (characterized by two angles in the wave plane) [11]. Let us recall that an array of sensors has space diversity if the sensors do not all have the same phase center. The array has angular and/or polarization diversity if the sensors do not have all the same radiating pattern and/or the same polarization, respectively.

B. Statistics of the Data

1) Presentation: The $2q$th $(q \geq 1)$ order array processing methods currently available exploit the information contained in the $(N^q \times N^q)$ $2q$th order circular covariance matrix, $C_{2q,x}$, whose entries are the $2q$th order circular cumulants of the data, $\mathrm{Cum}[x_{i_1}(t), \ldots, x_{i_q}(t), x_{i_{q+1}}(t)^*, \ldots, x_{i_{2q}}(t)^*]$ $(1 \leq i_j \leq N)(1 \leq j \leq 2q)$, where * corresponds to the complex conjugation. However, the latter entries can be arranged in the $C_{2q,x}$ matrix in different ways, and it is shown in the next section that the way these entries are arranged in the $C_{2q,x}$ matrix determines in particular the maximal processing power of the $2q$th order methods exploiting the algebraic structure of $C_{2q,x}$, such as the $2q$-MUSIC [8] or the $2q$-BIOME [3] methods. This result is new and seems to be completely unknown by most of the researchers.

In order to prove this important result in the next section, let us introduce an arbitrary integer l such that $(0 \leq l \leq q)$,

$$\begin{aligned} a_n(\theta_i, \varphi_i) &= a_n(\theta_i, \varphi_i, p_i) \\ &= f_n(\theta_i, \varphi_i, p_i) \exp\left\{ \frac{j2\pi\,[x_n \cos(\theta_i)\cos(\varphi_i) + y_n \sin(\theta_i)\cos(\varphi_i) + z_n \sin(\varphi_i)]}{\lambda} \right\} \end{aligned} \quad (2)$$

and let us arrange the $2q$-uplet $(i_1, \ldots, i_q, i_{q+1}, \ldots, i_{2q})$ of indices $i_j (1 \leq j \leq 2q)$ into two q-uplets indexed by l and defined by $(i_1, i_2, \ldots, i_l, i_{q+1}, \ldots, i_{2q-l})$ and $(i_{2q-l+1}, \ldots, i_{2q}, i_{l+1}, \ldots, i_q)$, respectively. As the indices $i_j (1 \leq j \leq 2q)$ vary from 1 to N, the two latter q-uplets take N^q values. Numbering, in a natural way, the N^q values of each of two latter q-uplets by the integers I_l and J_l, respectively, such that $1 \leq I_l, J_l \leq N^q$, we obtain

$$I_l \triangleq \sum_{j=1}^{l} N^{q-j}(i_j - 1) + \sum_{j=1}^{q-l} N^{q-l-j}(i_{q+j} - 1) + 1 \tag{3a}$$

$$J_l \triangleq \sum_{j=1}^{l} N^{q-j}(i_{2q-l+j} - 1) + \sum_{j=1}^{q-l} N^{q-l-j}(i_{l+j} - 1) + 1. \tag{3b}$$

Using the permutation invariance property of the cumulants, we deduce that $\mathrm{Cum}[x_{i_1}(t), \ldots, x_{i_q}(t), x_{i_{q+1}}(t)^*, \ldots, x_{i_{2q}}(t)^*] = \mathrm{Cum}[x_{i_1}(t), \ldots, x_{i_l}(t), x_{i_{q+1}}(t)^*, \ldots, x_{i_{2q-l}}(t)^*, x_{i_{2q-l+1}}(t)^*, \ldots, x_{i_{2q}}(t)^*, x_{i_{l+1}}(t), \ldots, x_{i_q}(t)]$ and assuming that the latter quantity is the element $[I_l, J_l]$ of the $C_{2q,x}$ matrix, thus noted $C_{2q,x}(l)$, it is easy to verify, from the Kronecker product definition, the hypotheses of Section II-A and under a Gaussian noise assumption that the $(N^q \times N^q)$ $C_{2q,x}(l)$ matrix can be written as

$$C_{2q,x}(l) \approx \sum_{i=1}^{P} c_{2q,m_i} \left[\boldsymbol{a}(\theta_i, \varphi_i)^{\otimes l} \otimes \boldsymbol{a}(\theta_i, \varphi_i)^{*\otimes(q-l)} \right] \times \left[\boldsymbol{a}(\theta_i, \varphi_i)^{\otimes l} \otimes \boldsymbol{a}(\theta_i, \varphi_i)^{*\otimes(q-l)} \right]^{\dagger} + \eta_2 V \delta(q-1) \tag{4}$$

where $c_{2q,m_i} \triangleq \mathrm{Cum}[m_{i_1}(t), \ldots, m_{i_q}(t), m_{i_{q+1}}(t)^*, \ldots, m_{i_{2q}}(t)^*]$, with $i_j = i (1 \leq j \leq 2q)$, is the $2q$th order circular autocumulant of $m_i(t)$, $\dagger$ corresponds to the conjugate transposition, η_2 is the mean power of the noise per sensor, V is the $(N \times N)$ spatial coherence matrix of the noise such that $\mathrm{Tr}[V] = N$, $\mathrm{Tr}[.]$ means Trace, $\delta(.)$ is the Kronecker symbol, $\otimes$ is the Kronecker product, and $\boldsymbol{a}^{\otimes l}$ is the $(N^l \times 1)$ vector defined by $\boldsymbol{a}^{\otimes l} \triangleq \boldsymbol{a} \otimes \boldsymbol{a} \otimes \ldots \otimes \boldsymbol{a}$ with a number of Kronecker product $\otimes$ equal to $l - 1$.

In particular, for $q = 1$ and $l = 1$, the $(N \times N)$ $C_{2q,x}(l)$ matrix corresponds to the well-known data covariance matrix (since the observations are zero-mean) defined by

$$R_x \triangleq C_{2,x}(1) = \mathrm{E}\left[\boldsymbol{x}(t)\boldsymbol{x}(t)^{\dagger}\right] \approx \sum_{i=1}^{P} c_{2,m_i} \boldsymbol{a}(\theta_i, \varphi_i)\boldsymbol{a}(\theta_i, \varphi_i)^{\dagger} + \eta_2 V. \tag{5}$$

For $q = 2$ and $l = 1$, the $(N^2 \times N^2)$ $C_{2q,x}(l)$ matrix corresponds to the classical expression of the data quadricovariance matrix used in [7] and [15] and in most of the papers dealing with FO array processing problems and is defined by

$$Q_x \triangleq C_{4,x}(1) \approx \sum_{i=1}^{P} c_{4,m_i} \left[\boldsymbol{a}(\theta_i, \varphi_i) \otimes \boldsymbol{a}(\theta_i, \varphi_i)^*\right] \times \left[\boldsymbol{a}(\theta_i, \varphi_i) \otimes \boldsymbol{a}(\theta_i, \varphi_i)^*\right]^{\dagger} \tag{6}$$

whereas for $q = 2$ and $l = 2$, the $(N^2 \times N^2)$ $C_{2q,x}(l)$ matrix corresponds to an alternative expression of the data quadricovariance matrix that is not often used and is defined by

$$\widetilde{Q}_x \triangleq C_{4,x}(2) \approx \sum_{i=1}^{P} c_{4,m_i} \left[\boldsymbol{a}(\theta_i, \varphi_i) \otimes \boldsymbol{a}(\theta_i, \varphi_i)\right] \times \left[\boldsymbol{a}(\theta_i, \varphi_i) \otimes \boldsymbol{a}(\theta_i, \varphi_i)\right]^{\dagger}. \tag{7}$$

2) Estimation: In situations of practical interest, the $2q$th order statistics of the data $\mathrm{Cum}[x_{i_1}(t), \ldots, x_{i_q}(t), x_{i_{q+1}}(t)^*, \ldots, x_{i_{2q}}(t)^*]$ are not known *a priori* and have to be estimated from L samples of data $\boldsymbol{x}(m) \triangleq \boldsymbol{x}(mT_e)$, $1 \leq m \leq L$, where T_e is the sample period.

For zero-mean stationary observations, using the ergodicity property, an empirical estimator of $\mathrm{Cum}[x_{i_1}(t), \ldots, x_{i_q}(t), x_{i_{q+1}}(t)^*, \ldots, x_{i_{2q}}(t)^*]$ that is asymptotically unbiased and consistent may be built from the well-known Leonov–Shiryaev formula [22], giving the expression of a nth order cumulant of $\boldsymbol{x}(t)$ as a function of its pth order moments $(1 \leq p \leq n)$, by replacing in the latter all the moments by their empirical estimate. More precisely, the Leonov–Shiryaev formula is given by

$$\mathrm{Cum}\left[x_{i_1}(t)^{\varepsilon 1}, x_{i_2}(t)^{\varepsilon 2}, \ldots, x_{i_n}(t)^{\varepsilon n}\right] = \sum_{p=1}^{n} (-1)^{p-1}(p-1)! \mathrm{E}\left[\prod_{j \in \mathrm{S}1} x_{i_j}(t)^{\varepsilon j}\right] \times \mathrm{E}\left[\prod_{j \in \mathrm{S}2} x_{i_j}(t)^{\varepsilon j}\right] \ldots \mathrm{E}\left[\prod_{j \in \mathrm{S}p} x_{i_j}(t)^{\varepsilon j}\right] \tag{8}$$

where $(\mathrm{S}1, \mathrm{S}2, \ldots, \mathrm{S}p)$ describes all the partitions in p sets of $(1, 2, \ldots, n)$, $\varepsilon_j = \pm 1$ $(1 \leq j \leq n)$ with the convention $x^1 = x$, and $x^{-1} = x^*$ and an empirical estimate of (8) is obtained by replacing in (8) in all the moments $E[x_{i_1}(t)^{\varepsilon 1} x_{i_2}(t)^{\varepsilon 2} \ldots x_{i_p}(t)^{\varepsilon p}]$ $(1 \leq p \leq n)$ by their empirical estimate, which is given by

$$\hat{\mathrm{E}}\left[x_{i_1}(t)^{\varepsilon 1} x_{i_2}(t)^{\varepsilon 2} \ldots x_{i_p}(t)^{\varepsilon p}\right](L) \triangleq \frac{1}{L} \sum_{m=1}^{L} x_{i_1}(m)^{\varepsilon 1} x_{i_2}(m)^{\varepsilon 2} \ldots x_{i_p}(m)^{\varepsilon p}. \tag{9}$$

Explicit expressions of (8) for $n = 2q$ with $1 \leq q \leq 3$ are given in Appendix A.

However, in radiocommunications contexts, most of the sources are no longer stationary but become cyclostationary (digital modulations). For zero-mean cyclostationary observations, the statistical matrix defined by (4) becomes time dependent, noted $C_{2q,x}(l)(t)$, and the theory developed in the paper can be extended without any difficulties by considering that $C_{2q,x}(l)$ is, in this case, the temporal mean, $\langle C_{2q,x}(l)(t)\rangle$, over an infinite interval duration, of the instantaneous statistics, $C_{2q,x}(l)(t)$. In these conditions, using a cyclo-ergodicity property, the matrix $C_{2q,x}(l)$ has to be estimated from the sampled data by a non empirical estimator such as that presented in [18] for $q = 2$. Note finally that this extension can also be applied to non zero-mean cyclostationary sources, such as some non linearly digitally modulated sources [20], provided that a non empirical statistic estimator, such as that presented in [20] for $q = 1$ and in [19] for $q = 2$, is used.

C. Related $2q$th order Array Processing Problems

A first family of $2q$th order array processing methods that are concerned with the theory developed in the next sections corresponds to the family of $2q$th order Blind Identification methods, which aim at blindly identifying the steering vectors of the sources $\boldsymbol{a}(\theta_i, \varphi_i)$ $(1 \le i \le P)$ from the exploitation of the algebraic structure of an estimate of the $C_{2q,x}(l)$ matrix for a particular choice of l. Such methods are described in [2] and [3]. A second family of methods concerned with the results of the paper corresponds to the $2q$th order subspace-based direction finding methods such as the $2q$-MUSIC method, presented in [8], which aims at estimating the angles of arrival of the sources (θ_i, φ_i) $(1 \le i \le P)$ from the exploitation of the algebraic structure of an estimate of the $C_{2q,x}(l)$ matrix for a particular choice of l.

III. Higher Order Virtual Array Concept

A. General Presentation

The VA concept has been introduced in [7], [15], and [16] for the classical FO array processing problem exploiting (6) only. In this section, we extend this concept to an arbitrary even order $m = 2q$ $(q \ge 2)$, for an arbitrary arrangement $C_{2q,x}(l)$ $(0 \le l \le q)$, of the data $2q$th order circular cumulants $\mathrm{Cum}[x_{i_1}(t), \ldots, x_{i_q}(t), x_{i_{q+1}}(t)^*, \ldots, x_{i_{2q}}(t)^*]$ $(1 \le i_j \le N)$ in the $C_{2q,x}$ matrix and for a general array with space, angular, and polarization diversities. This HO VA concept is presented in this section in the case of P statistically independent non-Gaussian sources.

Assuming no noise, we note that the matrices $C_{2q,x}(l)$ and R_x, which are defined by (4) and (5), respectively, have the same algebraic structure, where the auto-cumulant c_{2q,m_i} and the vector $[\boldsymbol{a}(\theta_i,\varphi_i)^{\otimes l} \otimes \boldsymbol{a}(\theta_i,\varphi_i)^{*\otimes(q-l)}]$ play, for $C_{2q,x}(l)$, the rule played for R_x by the power c_{2,m_i} and the steering vector $\boldsymbol{a}(\theta_i,\varphi_i)$, respectively. Thus, for the $2q$th order array processing methods exploiting (4), the $(N^q \times 1)$ vector $[\boldsymbol{a}(\theta_i,\varphi_i)^{\otimes l} \otimes \boldsymbol{a}(\theta_i,\varphi_i)^{*\otimes(q-l)}]$ can be considered as the *virtual steering vector* of the source i for the *true array* of N sensors with coordinates (x_n, y_n, z_n) and amplitude pattern $f_n(\theta,\varphi,p)$, $1 \le n \le N$. The N^q components of the vector $[\boldsymbol{a}(\theta_i,\varphi_i)^{\otimes l} \otimes \boldsymbol{a}(\theta_i,\varphi_i)^{*\otimes(q-l)}]$ correspond to the quantities $a_{k_1}(\theta_i,\varphi_i)a_{k_2}(\theta_i,\varphi_i)\ldots a_{k_l}(\theta_i,\varphi_i)a_{k_{l+1}}(\theta_i,\varphi_i)^* a_{k_{l+2}}(\theta_i,\varphi_i)^* \ldots a_{k_q}(\theta_i,\varphi_i)^*$ $(1 \le k_j \le N, 1 \le j \le q)$, where $a_{k_j}(\theta_i,\varphi_i)$ is the component k_j of vector $\boldsymbol{a}(\theta_i,\varphi_i)$. Using (2) in the latter components and numbering, in a natural way, the N^q values of the q-uplet $(k_1, k_2, \ldots, k_l, k_{l+1}, \ldots, k_q)$ by associating with the latter the integer K defined by

$$K \triangleq \sum_{j=1}^{q} N^{q-j}(k_j - 1) + 1 \tag{10}$$

we find that the component K of the vector $[\boldsymbol{a}(\theta_i,\varphi_i)^{\otimes l} \otimes \boldsymbol{a}(\theta_i,\varphi_i)^{*\otimes(q-l)}]$, noted $[\boldsymbol{a}(\theta_i,\varphi_i)^{\otimes l} \otimes \boldsymbol{a}(\theta_i,\varphi_i)^{*\otimes(q-l)}]_K$, takes the form (11), shown at the bottom of the page. Comparing (11) to (2), we deduce that the vector $[\boldsymbol{a}(\theta_i,\varphi_i)^{\otimes l} \otimes \boldsymbol{a}(\theta_i,\varphi_i)^{*\otimes(q-l)}]$ can also be considered as the *true steering vector* of the source i for the VA of N^q *Virtual Sensors* (VSs) with coordinates $(x^l_{k_1k_2\ldots k_q}, y^l_{k_1k_2\ldots k_q}, z^l_{k_1k_2\ldots k_q})$ and complex amplitude patterns $f^l_{k_1k_2\ldots k_q}(\theta,\varphi,p)$, $1 \le k_j \le N$ for $1 \le j \le q$, which are given by

$$\begin{aligned}\left(x^l_{k_1k_2\ldots k_q}, y^l_{k_1k_2\ldots k_q}, z^l_{k_1k_2\ldots k_q}\right) &= \left(\sum_{j=1}^{l} x_{k_j} - \sum_{u=1}^{q-l} x_{k_{l+u}}, \sum_{j=1}^{l} y_{k_j}\right. \\ &\quad \left. - \sum_{u=1}^{q-l} y_{k_{l+u}}, \sum_{j=1}^{l} z_{k_j} - \sum_{u=1}^{q-l} z_{k_{l+u}}\right)\end{aligned} \tag{12}$$

$$\begin{aligned}f^l_{k_1k_2\ldots k_q}&(\theta,\varphi,p) \\ &= \prod_{j=1}^{l}\prod_{u=1}^{q-l} f_{k_j}(\theta,\varphi,p) f_{k_{l+u}}(\theta,\varphi,p)^*\end{aligned} \tag{13}$$

$$\begin{aligned}\left[\boldsymbol{a}(\theta_i,\varphi_i)^{\otimes l} \otimes \boldsymbol{a}(\theta_i,\varphi_i)^{*\otimes(q-l)}\right]_K &= \left(\prod_{j=1}^{l}\prod_{u=1}^{q-l} f_{k_j}(\theta_i,\varphi_i,p_i) f_{k_{l+u}}(\theta_i,\varphi_i,p_i)^*\right) \\ \times \exp\Bigg\{ j2\pi \Bigg[&\left(\sum_{j=1}^{l} x_{k_j} - \sum_{u=1}^{q-l} x_{k_{l+u}}\right)\cos(\theta_i)\cos(\varphi_i) + \left(\sum_{j=1}^{l} y_{k_j} - \sum_{u=1}^{q-l} y_{k_{l+u}}\right)\sin(\theta_i)\cos(\varphi_i) \\ &+ \left(\sum_{j=1}^{l} z_{k_j} - \sum_{u=1}^{q-l} z_{k_{l+u}}\right)\sin(\varphi_i)\Bigg] \Big/ \lambda \Bigg\}\end{aligned} \tag{11}$$

which introduces in a very simple, direct, and short way the VA concept for the $2q$th order array processing problem for the arrangement $C_{2q,x}(l)$ and whatever the kind of diversity. Note that (13) shows that the complex amplitude response of a VS for given direction of arrival and polarization corresponds to a product of l complex amplitude responses of true sensors and $(q-l)$ conjugate ones for the considered direction of arrival and polarization.

Thus, as a summary, we can consider that the $2q$th order array processing problem of P statistically independent NB non-Gaussian sources from a given array of N sensors with coordinates (x_n, y_n, z_n) and complex amplitude patterns $f_n(\theta, \varphi, p)$, $1 \leq n \leq N$ is, for the arrangement $C_{2q,x}(l)$, similar to an SO array processing problem for which these P statistically independent NB sources impinge, with the *virtual powers* c_{2q,m_i} $(1 \leq i \leq P)$, on a VA of N^q VS having the coordinates $(x^l_{k_1k_2\ldots k_q}, y^l_{k_1k_2\ldots k_q}, z^l_{k_1k_2\ldots k_q})$ and the complex amplitude patterns $f^l_{k_1k_2\ldots k_q}(\theta, \varphi, p)$, $1 \leq k_j \leq N$ for $1 \leq j \leq q$, which are defined by (12) and (13) respectively. Thus, HO array processing may be used to replace sensors and hardware and, thus, to decrease the overall cost of a given system.

From this interpretation based on the HO VA concept, we naturally deduce that the 3-dB beamwidth of this VA controls the resolution power of $2q$th order array processing method for a finite observation duration and the considered arrangement, whereas the number of different sensors of this VA controls the maximal number of sources that can be processed by such methods for this arrangement. More precisely, as some of the N^q VS may coïncide, we note N^l_{2q} as the number of different VSs of the VA associated with the $2q$th order array processing problem for the arrangement $C_{2q,x}(l)$. Then, the maximum number of independent sources that can be processed by a $2q$th order BSI method exploiting the algebraic structure of $C_{2q,x}(l)$ is N^l_{2q}, whereas the $2q$th order direction finding methods exploiting the algebraic structure of $C_{2q,x}(l)$, such as the family of $2q$-MUSIC methods [8], are able to process up to $N^l_{2q} - 1$ non-Gaussian sources.

Another important result shown by (12) and (13) is that, for a given array of N sensors, the associated $2q$ $(q \geq 2)$th order VA depends on the parameter l and thus on the arrangement of the $2q$th order circular cumulants of the data in the $C_{2q,x}$ matrix. This new result not only shows off the importance of the chosen arrangement of the considered data $2q$th order cumulants on the processing capacity of the methods exploiting the algebraic structure of $C_{2q,x}$ but also raises the problem of the optimal arrangement of these cumulants for a given even order. This question is addressed in the next section.

Finally, note that (4) holds only for sources that are NB for the associated VA, i.e., for sources i such that the vector $[\boldsymbol{a}(\theta_i, \varphi_i)^{\otimes l} \otimes \boldsymbol{a}(\theta_i, \varphi_i)^{*\otimes(q-l)}]$ does not depend on the frequency parameter within the reception bandwidth, i.e., for source i in the reception bandwidth B such that

$$\frac{\pi B D_{q,l} \cos(\boldsymbol{k}_i, \mathbf{M}_1\mathbf{M}_{\max})}{c} \ll 1 \tag{14}$$

where c is the propagation velocity, $D_{q,l}$ is the aperture of the VA for the considered parameters q and l, $\boldsymbol{k}_i$ is wave vector for the source i, and $\mathbf{M}_1\mathbf{M}_{\max}$ is the vector whose norm is $D_{q,l}$ and whose direction is the line formed by the two most spaced VSs M_1 and $\mathrm{M}_{\max}$. As $D_{q,l}$ increases with q, the accepted reception bandwidth ensuring the NB assumption for the HO VA decreases with q. In particular, for HF or GSM links, the narrowband assumption for the HO VA is generally verified up to $q = 8$ or 10, i.e., up to a statistical order $m = 2q$ equal to 16, 18, or 20 from a classically used array of sensors for these applications [13].

B. Optimal Arrangement $C_{2q,x}(l)$

For a given value of q $(q \geq 2)$ and a given array of N sensors, we define in this paper the optimal arrangement $C_{2q,x}(l)$, which is denoted $C_{2q,x}(l_{\mathrm{opt}})$, as the one that maximizes the number of different VS N^l_{2q} of the associated VA, since the processing power of a $2q$th order method exploiting the algebraic structure of $C_{2q,x}(l)$ is directly related to the number of different VS of the associated VA.

To get more insight into $C_{2q,x}(l_{\mathrm{opt}})$, let us analyze (12) and (13). These expressions show that the q-uplets $(k_1, k_2, \ldots, k_l, k_{l+1}, \ldots, k_q)$ and $(k_{\sigma(1)}, k_{\sigma(2)}, \ldots, k_{\sigma(l)}, k_{\mu(l+1)}, \ldots, k_{\mu(q)})$, where $(\sigma(1), \sigma(2), \ldots, \sigma(l))$ and $(\mu(l+1), \mu(l+2), \ldots, \mu(q))$ are arbitrary permutations of $(1, 2, \ldots, l)$ and $(l+1, l+2, \ldots, q)$, respectively, give rise to the same VS (same coordinates and same radiation pattern) of the VA associated with $C_{2q,x}(l)$, which are defined by (12) and (13). The number of permutations of a given set of indices depends on the number of indices with different values in the set. For this reason, let us classify all the q-uplets $(k_1, k_2, \ldots, k_q)$ $(1 \leq k_j \leq N, 1 \leq j \leq q)$ in q families F_i $(1 \leq i \leq q)$ such that F_i corresponds to the set of q-uplets with i different elements k_j. For example, we have (15) and (16), shown at the bottom of the page. For the general case of an arbitrary array of N sensors, both the number of q-uplets of F_i and the number of different VSs of the VA associated with F_i for an arbitrary arrangement $C_{2q,x}(l)$ $(0 \leq l \leq q)$ are proportional to $N!/(N-i)!$. Indeed, among the i different elements k_j of F_i, once the value of one of them is chosen among N possibilities, there are still $N-1$ possibilities for the second one and then $N-2$ possibilities for the third one, and so on, and, finally, $N-i+1$ possibilities for the ith one, which finally corresponds to $N!/(N-i)!$ possible solutions for the different elements. Then, for each of the latter solutions,

$$\mathrm{F}_1 \triangleq \{(k_1, k_2, \ldots, k_q) \text{ such that } k_j = k \text{ for } 1 \leq j \leq q \text{ and } 1 \leq k \leq N\} \tag{15}$$

$$\mathrm{F}_q \triangleq \{(k_1, k_2, \ldots, k_q) \text{ such that } k_j \neq k_m \text{ for } 1 \leq j \neq m \leq q \text{ and } 1 \leq k_j, k_m \leq N\}. \tag{16}$$

the value of $q-i$ elements have to be chosen among the i considered different elements, finally giving rise to a number of q-uplets of F_i proportional to $N!/(N-i)!$. This quantity is equal to zero for $N<i$ but becomes a polynomial function of degree i with respect to variable N for $N \geq i$. Thus, as N becomes large, provided that $q \leq N$, the number of different VS of the VA for a given arrangement $C_{2q,x}(l)$, N_{2q}^{l} is mainly dominated by the number of different VSs associated with F_q for this arrangement $N_{2q}^{l}[\mathrm{F}_q]$. The number of q-uplets of F_q is exactly equal to $N!/(N-q)!$, whereas the number of different VSs associated with F_q for the arrangement $C_{2q,x}(l)$ is, for the general case of an arbitrary array of N sensors with no particular symmetries, equal to

$$N_{2q}^{l}[\mathrm{F}_q] = \frac{N!}{[(N-q)!(q-l)!l!]} \quad (q \leq N). \tag{17}$$

In fact, when all the k_j are different, the number of permutations $(\sigma(1), \sigma(2), \ldots, \sigma(l))$ and $(\mu(l+1), \mu(l+2), \ldots, \mu(q))$ of $(1, 2, \ldots, l)$ and $(l+1, l+2, \ldots, q)$ are equal to $l!$ and $(q-l)!$, respectively.

As a summary, in the general case of an arbitrary array of N sensors with no particular symmetries, for a given value of q $(q \geq 2)$ and for large values of N, the optimal arrangement $C_{2q,x}(l_{\mathrm{opt}})$ is such that l_{opt} maximizes $N_{2q}^{l}[\mathrm{F}_q]$ defined by (17) and, thus, minimizes the quantity $(q-l)!l!$ with respect to l $(0 \leq l \leq q)$. We deduce from this result that the arrangements $C_{2q,x}(l)$ and $C_{2q,x}(q-l)$ $(0 \leq l \leq q)$ give rise to the same number of VS (in fact, the first arrangement is the conjugate of the other, whatever the values of q and N). It is then sufficient to limit the analysis to $q/2 \leq l \leq q$ if q is even and to $(q+1)/2 \leq l \leq q$ if q is odd. We easily verify that

$$(q-l)!l! < (q-(l+1))!(l+1)! \quad \text{for } \frac{q}{2} \leq l \leq q-1, \quad \text{if } q \text{ is even} \tag{18}$$

$$(q-l)!l! < (q-(l+1))!(l+1)! \quad \text{for } \frac{(q+1)}{2} \leq l \leq q-1, \quad \text{if } q \text{ is odd} \tag{19}$$

which proves that $l_{\mathrm{opt}} = q/2$ if q is even and $l_{\mathrm{opt}} = (q+1)/2$ if q is odd. In other words, l_{opt} is, in all cases, the integer l that minimizes $|2l-q|$. It generates steering vectors $[\boldsymbol{a}(\theta_i, \varphi_i)^{\otimes l} \otimes \boldsymbol{a}(\theta_i, \varphi_i)^{*\otimes(q-l)}]$ for which the number of conjugate vectors is the least different from the number of nonconjugate vectors. In particular, for $q=2$, it corresponds to (6). We verify in Section IV for $q = 2$, 3, and 4 that this result, which is shown for large values of N, remains valid, whatever the value of N.

C. Virtual Array Resolution

To get more insights into the gain in resolution obtained with HO VA, let us compute the spatial correlation coefficient of two sources, with directions $\boldsymbol{\theta} = (\theta, \varphi)$ and $\boldsymbol{\theta}_0 = (\theta_0, \varphi_0)$, respectively, for the VA associated with statistical order $2q$ and arrangement indexed by l. This coefficient, which is noted $\alpha_{\theta,\theta_0}(2q,l)$ such that $0 \leq |\alpha_{\theta,\theta_0}(2q,l)| \leq 1$, is defined by the normalized inner product of the steering vectors $\boldsymbol{a}_{2q,l}(\theta,\varphi) \triangleq [\boldsymbol{a}(\theta,\varphi)^{\otimes l} \otimes \boldsymbol{a}(\theta,\varphi)^{*\otimes(q-l)}]$ and $\boldsymbol{a}_{2q,l}(\theta_0,\varphi_0) \triangleq [\boldsymbol{a}(\theta_0,\varphi_0)^{\otimes l} \otimes \boldsymbol{a}(\theta_0,\varphi_0)^{*\otimes(q-l)}]$ and can be written as

$$\alpha_{\theta,\theta_0}(2q,l) \triangleq \frac{\boldsymbol{a}_{2q,l}(\theta,\varphi)^{\dagger}\boldsymbol{a}_{2q,l}(\theta_0,\varphi_0)}{[\boldsymbol{a}_{2q,l}(\theta,\varphi)^{\dagger}\boldsymbol{a}_{2q,l}(\theta,\varphi)]^{\frac{1}{2}}[\boldsymbol{a}_{2q,l}(\theta_0,\varphi_0)^{\dagger}\boldsymbol{a}_{2q,l}(\theta_0,\varphi_0)]^{\frac{1}{2}}}. \tag{20}$$

For an array with space diversity only, this coefficient is proportional to the value, for the direction θ, of the complex amplitude pattern of the conventional beamforming in the direction θ_0 from the considered VA. It is shown in Appendix B that this coefficient (20) can be written as

$$\alpha_{\theta,\theta_0}(2q,l) = \frac{[\boldsymbol{a}(\theta,\varphi)^{\dagger}\boldsymbol{a}(\theta_0,\varphi_0)]^{l}\,[\boldsymbol{a}(\theta_0,\varphi_0)^{\dagger}\boldsymbol{a}(\theta,\varphi)]^{(q-l)}}{[\boldsymbol{a}(\theta,\varphi)^{\dagger}\boldsymbol{a}(\theta,\varphi)]^{\frac{q}{2}}\,[\boldsymbol{a}(\theta_0,\varphi_0)^{\dagger}\boldsymbol{a}(\theta_0,\varphi_0)]^{\frac{q}{2}}} \tag{21}$$

which implys that

$$|\alpha_{\theta,\theta_0}(2q,l)| = \left(\frac{|\boldsymbol{a}(\theta,\varphi)^{\dagger}\boldsymbol{a}(\theta_0,\varphi_0)|}{[\boldsymbol{a}(\theta,\varphi)^{\dagger}\boldsymbol{a}(\theta,\varphi)]^{\frac{1}{2}}[\boldsymbol{a}(\theta_0,\varphi_0)^{\dagger}\boldsymbol{a}(\theta_0,\varphi_0)]^{\frac{1}{2}}}\right)^{q} = |\alpha_{\theta,\theta_0}(2,1)|^{q}. \tag{22}$$

Expression (22) shows that despite the fact that $\alpha_{\theta,\theta_0}(2q,l)$ is a function of q and l, its modulus does not depend on l but only on q and on the normalized amplitude pattern $|\alpha_{\theta,\theta_0}(2,1)|$ of the considered array of N sensors for the pointing direction θ_0. Moreover, as $0 \leq |\alpha_{\theta,\theta_0}(2,1)| \leq 1$, we deduce from (22) that $|\alpha_{\theta,\theta_0}(2q,l)|$ is a decreasing function of q, which proves the increasing resolution of the HO VA as q increases. In particular, if we note $\theta_{3\,\mathrm{dB}}^{2q}$, the 3-dB beamwidth of the $2q$th order VA associated with a given array of N sensors, we find from (22) that $\theta_{3\,\mathrm{dB}}^{2q}$ can be easily deduced from the normalized amplitude pattern of the latter and is such that $|\alpha_{\theta,\theta_0}(2,1)| = 0.5^{1/q}$ for $\theta = \theta_{3\,\mathrm{dB}}^{2q}$, i.e., such that $|\alpha_{\theta,\theta_0}(2,1)| = 0.707, 0.794$, and 0.84 for $q = 2$, 3 and 4, respectively. As q increases, this generates $\theta_{3\,\mathrm{dB}}^{2q}$ values corresponding to a decreasing fraction of the 3-dB beamwidth $\theta_{3\,\mathrm{dB}}$ of the considered array of N sensors, and we will verify in Section V that $\theta_{3\,\mathrm{dB}}^{2q} = 0.84\theta_{3\,\mathrm{dB}}, 0.76\theta_{3\,\mathrm{dB}}$, and $0.71\theta_{3\,\mathrm{dB}}$ for $q = 2$, 3 and 4, respectively. Finally, (22) proves that rank-1 ambiguities (or grating lobes [11]) of the true and VA coincide, regardless of the values of q and l since the directions $\theta \neq \theta_0$ giving rise to $|\alpha_{\theta,\theta_0}(2q,l)| = 1$ are exactly the ones that give rise to $|\alpha_{\theta,\theta_0}(2,1)| = 1$. A consequence of this result is that a necessary and sufficient condition to obtain VA without any rank-1 ambiguities is that the considered array of N sensors have no rank-1 ambiguities.

IV. Properties of Higher Order Virtual Arrays

A. Case of an Array With Space, Angular, and Polarization Diversity

For an array with space, angular, and polarization diversities, the component n of vector $\boldsymbol{a}(\theta_i, \varphi_i)$ and the component K of $[\boldsymbol{a}(\theta_i, \varphi_i)^{\otimes l} \otimes \boldsymbol{a}(\theta_i, \varphi_i)^{*\otimes(q-l)}]$ are given by (2) and (11), respectively, which shows that the $2q$th $(q \geq 2)$ order VA associated with such an array is also an array with space, angular,

TABLE I
COORDINATES, COMPLEX RESPONSES AND MULTIPLICITY ORDER OF VS FOR SEVERAL VALUES OF l, FOR $q = 2$ AND FOR ARRAYS WITH SPACE, ANGULAR AND POLARIZATION DIVERSITIES

l	*VS coordinates*	*VS responses*	*VS multiplicities*
2	$2\boldsymbol{r}_{n1}$	f_{n1}^2	1
	$\boldsymbol{r}_{n1} + \boldsymbol{r}_{n2}$	$f_{n1} f_{n2}$	2
1	$\boldsymbol{r}_0$	$\lvert f_{n1}\rvert^2$	1
	$\boldsymbol{r}_{n1} - \boldsymbol{r}_{n2}$	$f_{n1} f_{n2}^*$	1

and polarization diversities, regardless of the arrangement of the $2q$th order circular cumulants of the data in the $C_{2q,x}$ matrix.

Ideally, it would have been interesting to obtain a general expression of the number of different VSs N_{2q}^l of the $2q$th order VA for the arrangement $C_{2q,x}(l)$ and for arbitrary array geometries. However, it does not seem possible to obtain such a result easily since the computation of the number of VSs that degenerate in a same one is very specific of the choice of q and l and of the array geometry. For this reason, we limit our subsequent analysis, for arbitrary array geometries, to some values of q $(2 \leq q \leq 4)$, which extends the results of [7] up to the eighth order for arbitrary arrangements of the data cumulants, despite the tedious character of the computations. Moreover, this analysis is not so much restrictive since the considered $2q$th order data statistics $(2 \leq q \leq 4)$ correspond in fact to the statistics that have the most probability to be used for future applications. Note that the general analysis for arbitrary values of q and l is possible for ULA and is presented in Section V.

To simplify the analysis, for each sensor n, $1 \leq n \leq N$, we note f_n as its complex response $f_n(\theta, \varphi, p)$, $\boldsymbol{r}_n \triangleq (x_n, y_n, z_n)$ as the triplet of its coordinates, $\boldsymbol{r}_0 \triangleq (0,0,0)$, and we define $\lambda \boldsymbol{r}_n \triangleq (\lambda x_n, \lambda y_n, \lambda z_n)$ and $\boldsymbol{r}_n + \boldsymbol{r}_m \triangleq (x_n + x_m, y_n + y_m, z_n + z_m)$. Moreover, for a given value of q, we define the order of multiplicity m of a given VS of the considered $2q$th order VA by the number of q-uplets $(k_1, k_2, \ldots, k_q)$, giving rise to this VS. When the order of multiplicity of a given VS is greater than 1, this VS can be considered to be weighted in amplitude by a factor corresponding to the order of multiplicity, and the associated VA then becomes an *amplitude tappered* array.

The coordinates, the complex responses, and the order of multiplicity of the VS of HO VA, which are deduced from (12) and (13), are presented in Tables I–III for $q = 2, 3$ and 4, respectively, and for several values of the parameter l. In these tables, the integers n_i take all the values between 1 and N $(1 \leq n_i \leq N$ for $1 \leq i \leq 4)$ but under the constraint that $n_i \neq n_j$ if $i \neq j$ for a given line of the tables. A VS is completely characterized by a line of a table for given values of the n_i.

The results of Tables I–III show that for arrays with sensors having different responses, the VA associated with the parameters (q, l) is *amplitude tappered* for $(q, l) = (2, 2)$, (3, 3), (3, 2), (4, 4), (4, 3) and (4, 2), whereas it is not for $(q, l) = (2, 1)$. In this latter case, the order of multiplicity of each VS is 1. Then, the number of different VS of the associated VA may be maximum for $q = 2$ and equal to N^2. It is, in particular, the case if the responses of all the VSs are different. However, for arbitrary values of q and l, the maximum number of VSs of the associated VA, noted $N_{\max}[2q, l]$, is generally strictly lower than

TABLE II
COORDINATES, COMPLEX RESPONSES AND MULTIPLICITY ORDER OF VS FOR SEVERAL VALUES OF l, FOR $q = 3$ AND FOR ARRAYS WITH SPACE, ANGULAR AND POLARIZATION DIVERSITIES

l	*VS coordinates*	*VS responses*	*VS multiplicities*
3	$3\boldsymbol{r}_{n1}$	f_{n1}^3	1
	$\boldsymbol{r}_{n1} + 2\boldsymbol{r}_{n2}$	$f_{n1} f_{n2}^2$	3
	$\boldsymbol{r}_{n1} + \boldsymbol{r}_{n2} + \boldsymbol{r}_{n3}$	$f_{n1} f_{n2} f_{n3}$	6
2	$\boldsymbol{r}_{n1}$	$f_{n1} \lvert f_{n1}\rvert^2$	1
	$\boldsymbol{r}_{n1}$	$f_{n1} \lvert f_{n2}\rvert^2$	2
	$2\boldsymbol{r}_{n2} - \boldsymbol{r}_{n1}$	$f_{n1}^* f_{n2}^2$	1
	$\boldsymbol{r}_{n1} + \boldsymbol{r}_{n2} - \boldsymbol{r}_{n3}$	$f_{n1} f_{n2} f_{n3}^*$	2

TABLE III
COORDINATES, COMPLEX RESPONSES AND MULTIPLICITY ORDER OF VS FOR SEVERAL VALUES OF l, FOR $q = 4$ AND FOR ARRAYS WITH SPACE, ANGULAR AND POLARIZATION DIVERSITIES

l	*VS coordinates*	*VS responses*	*VS multiplicities*
4	$4\boldsymbol{r}_{n1}$	f_{n1}^4	1
	$\boldsymbol{r}_{n1} + 3\boldsymbol{r}_{n2}$	$f_{n1} f_{n2}^3$	4
	$2\boldsymbol{r}_{n1} + 2\boldsymbol{r}_{n2}$	$f_{n1}^2 f_{n2}^2$	6
	$2\boldsymbol{r}_{n1} + \boldsymbol{r}_{n2} + \boldsymbol{r}_{n3}$	$f_{n1}^2 f_{n2} f_{n3}$	12
	$\boldsymbol{r}_{n1} + \boldsymbol{r}_{n2} + \boldsymbol{r}_{n3} + \boldsymbol{r}_{n4}$	$f_{n1} f_{n2} f_{n3} f_{n4}$	24
3	$2\boldsymbol{r}_{n1}$	$f_{n1}^2 \lvert f_{n1}\rvert^2$	1
	$\boldsymbol{r}_{n1} + \boldsymbol{r}_{n2}$	$f_{n1} f_{n2} \lvert f_{n2}\rvert^2$	3
	$3\boldsymbol{r}_{n2} - \boldsymbol{r}_{n1}$	$f_{n1}^* f_{n2}^3$	1
	$2\boldsymbol{r}_{n1}$	$f_{n1}^2 \lvert f_{n2}\rvert^2$	3
	$2\boldsymbol{r}_{n1} + \boldsymbol{r}_{n2} - \boldsymbol{r}_{n3}$	$f_{n1}^2 f_{n2} f_{n3}^*$	3
	$\boldsymbol{r}_{n2} + \boldsymbol{r}_{n3}$	$f_{n2} f_{n3} \lvert f_{n1}\rvert^2$	6
	$\boldsymbol{r}_{n1} + \boldsymbol{r}_{n2} + \boldsymbol{r}_{n3} - \boldsymbol{r}_{n4}$	$f_{n1} f_{n2} f_{n3} f_{n4}^*$	6
2	$\boldsymbol{r}_0$	$\lvert f_{n1}\rvert^4$	1
	$\boldsymbol{r}_{n1} - \boldsymbol{r}_{n2}$	$f_{n1} f_{n2}^* \lvert f_{n2}\rvert^2$	2
	$\boldsymbol{r}_{n2} - \boldsymbol{r}_{n1}$	$f_{n1}^* f_{n2} \lvert f_{n2}\rvert^2$	2
	$2\boldsymbol{r}_{n1} - 2\boldsymbol{r}_{n2}$	$f_{n1}^2 f_{n2}^{*2}$	1
	$\boldsymbol{r}_0$	$\lvert f_{n1}\rvert^2 \lvert f_{n2}\rvert^2$	4
	$2\boldsymbol{r}_{n1} - \boldsymbol{r}_{n2} - \boldsymbol{r}_{n3}$	$f_{n1}^2 f_{n2}^* f_{n3}^*$	2
	$\boldsymbol{r}_{n2} - \boldsymbol{r}_{n3}$	$f_{n2} f_{n3}^* \lvert f_{n1}\rvert^2$	4
	$-2\boldsymbol{r}_{n1} + \boldsymbol{r}_{n2} + \boldsymbol{r}_{n3}$	$f_{n2} f_{n3} f_{n1}^{*2}$	2
	$\boldsymbol{r}_{n1} + \boldsymbol{r}_{n2} - \boldsymbol{r}_{n3} - \boldsymbol{r}_{n4}$	$f_{n1} f_{n2} f_{n3}^* f_{n4}^*$	4

TABLE IV
$N_{\max}[2q, l]$ AS A FUNCTION OF N FOR SEVERAL VALUES OF q AND l AND FOR ARRAYS WITH SPACE, ANGULAR AND POLARIZATION DIVERSITIES

$m = 2q$	l	$N_{max}[2q, l]$
4 ($q = 2$)	2	$N(N+1)/2$
	1	N^2
6 ($q = 3$)	3	$N!/[6(N-3)!] + N(N-1) + N$
	2	$N!/[2(N-3)!] + 2N(N-1) + N$
8 ($q = 4$)	4	$N!/[24(N-4)!] + N!/[2(N-3)!] + 1.5N(N-1) + N$
	3	$N!/[6(N-4)!] + 1.5N!/(N-3)! + 3N(N-1) + N$
	2	$N!/[4(N-4)!] + 2N!/(N-3)! + 3.5N(N-1) + N$

N^q due to the amplitude tappering of the VA. Table IV shows precisely, for arrays with different sensors, the expression of $N_{\max}[2q, l]$, computed from the results from Tables I–III, as a function of N for $2 \le q \le 4$ and several values of l. Note that $N_{\max}[4, 1]$ has already been obtained in [7]. Note also that $N_{\max}[2q, l]$ corresponds to N_{2q}^l in most cases of sensors having different responses. We verify in Table IV that for a given value of q $(2 \le q \le 4)$ and the considered values of l, whatever the value of N, small or large, $N_{\max}[2q, l]$ is a decreasing function of l, which confirms the optimality of the arrangement $C_{2q,x}(l)$ for the integer l that minimizes $|2l - q|$, as discussed in Section III-B. Note, in addition, for a given value of N, and for optimal arrangements, the increasing values of $N_{\max}[2q, l]$ as q increases.

In order to quantify the results of Table IV, Table V summarizes the maximal number of different VS $N_{\max}[2q, l]$ of the associated VA for several values of N, q, and l. As N and q increase, note the increasing value of the loss in the processing power associated with the use of a suboptimal arrangement instead of the optimal one. For a given value of N, note the increasing value of $N_{\max}[2q, l]$ as q increases for optimal arrangements of the cumulants, whereas note the possible decreasing value of $N_{\max}[2q, l]$ as q increases when the arrangement moves from optimality to suboptimality (for example, $N_{\max}[6, 2] > N_{\max}[8, 4]$ for $2 \le N \le 5$).

B. Case of an Array With Angular and Polarization Diversity Only

For an array with angular and polarization diversities only, all the sensors of the array have the same phase center $\boldsymbol{r} \triangleq (x, y, z)$ but have different complex responses $f_n(\theta, \varphi, p)$, $1 \le n \le N$. Such an array is usually referred to as an array with colocalized sensors having different responses in angle and polarization. For such an array, (12) shows that for given values of q and l, the VSs of the associated VA have all the same coordinates given by $(2l - q)\boldsymbol{r}$, whereas their complex response is given by (13). This shows that the $2q$th $(q \ge 2)$ order VA associated with an array with angular and polarization diversities only is also an array with angular and polarization diversities only, whatever the arrangement of the $2q$th order circular cumulants of the data in the $C_{2q,x}$ matrix.

The complex responses of the colocalized VS of the $2q$th order VA for the arrangements $C_{2q,x}(l)$ are all presented in Tables I–III for $2 \le q \le 4$. In particular, all the upper bounds $N_{\max}[2q, l]$ presented in Tables I–III for an array with space, angular, and polarization diversities remain valid for an array of colocalized sensors with angular and polarization diversities only. This shows that for sensors having different complex responses, the geometry of the array does not generally play an important role in the maximal power capacity of the $2q$th order array processing methods exploiting the algebraic structure of $C_{2q,x}$ in terms of number of sources to be processed.

C. Case of an Array With Space Diversity Only

Let us consider in this section the particular case of an array with space diversity only. In this case, all the sensors of the array are identical, and the complex amplitude patterns of the latter $f_n(\theta, \varphi, p)$, $1 \le n \le N$ may be chosen to be equal to one. Under these assumptions, we deduce from (13) that for a given value of q $(q \ge 2)$, $f_{k_1 k_2 \ldots k_q}^l(\theta, \varphi, p) = 1$ whatever the q-uplet $(k_1, k_2, \ldots, k_q)$ and whatever the arrangement index l. This shows that the $2q$th $(q \ge 2)$ order VA associated with an array with space diversity only is also an array with space diversity only, whatever the arrangement of the $2q$th order circular cumulants of the data in the $C_{2q,x}$ matrix.

For such an array, the $2q$th order VA are presented hereafter for $(2 \le q \le 4)$, which extends the results of [7] up to the eighth order for arbitrary arrangements of the data cumulants. More precisely, for arrays with space diversity only, the coordinates and the order of multiplicity of the VSs of the HO VA, deduced from Tables I–III, are presented in Tables VI–VIII for $q = 2$, 3, and 4, respectively, and for several values of the parameter l. Again, in these tables, the integers n_i take all the values between 1 and N $(1 \le n_i \le N$ for $1 \le i \le 4)$ but under the constraint that $n_i \ne n_j$ if $i \ne j$ for a given line of the tables. A VS is completely characterized by a line of a table for given values of the n_i.

The results of Tables VI–VIII show that for arrays with identical sensors, the VA associated with the parameters (q, l) is always *amplitude tappered*, whatever the values of q and l, which implys in particular that $N_{2q}^l \le N_{\max}[2q, l] < N^q$. Table IX shows precisely, for arrays with identical sensors, the expression of $N_{\max}[2q, l]$ computed from results of Tables VI–VIII as a function of N for $2 \le q \le 4$ and several values of l. Note that $N_{\max}[4, 1]$ has already been obtained in [7]. Note also that $N_{\max}[2q, l]$ corresponds to N_{2q}^l in most cases of array geometries with no particular symmetries. We verify in Table IX that for a given value of q $(2 \le q \le 4)$ and the considered values of l, whatever the value of N, small or large, $N_{\max}[2q, l]$ is a decreasing function of l, which confirms the optimality of the arrangement $C_{2q,x}(l)$ for the integer l that minimizes $|2l - q|$, as discussed in Section III-B. Note, also, for a given value of N and for optimal arrangements, the increasing values of $N_{\max}[2q, l]$ as q increases. A comparison of Tables IV and IX shows that whatever the values of q and l, $N_{\max}[2q, l]$ can only remain constant or increase when an array with space diversity only is

TABLE V
$N_{\max}[2q, l]$ FOR SEVERAL VALUES OF N, q, AND l AND FOR ARRAYS WITH SPACE, ANGULAR, AND POLARIZATION DIVERSITIES

$m = 2q$	l	$N=2$	$N=3$	$N=4$	$N=5$	$N=6$	$N=7$	$N=8$
4 ($q = 2$)	2	3	6	10	15	21	28	36
	1	4	9	16	25	36	49	64
6 ($q = 3$)	3	4	10	20	35	56	84	120
	2	6	18	40	75	126	196	288
8 ($q = 4$)	4	5	15	35	70	126	210	330
	3	8	30	80	175	336	588	960
	2	9	36	100	225	441	784	1296

TABLE VI
COORDINATES AND MULTIPLICITY ORDER OF VSs FOR SEVERAL VALUES OF l FOR $q = 2$ AND FOR ARRAYS WITH SPACE DIVERSITY ONLY

l	*VS coordinates*	*VS multiplicities*
2	$2\mathbf{r}_{n1}$	1
	$\mathbf{r}_{n1} + \mathbf{r}_{n2}$	2
1	$\mathbf{r}_0$	N
	$\mathbf{r}_{n1} - \mathbf{r}_{n2}$	1

TABLE VII
COORDINATES AND MULTIPLICITY ORDER OF VSs FOR SEVERAL VALUES OF l, FOR $q = 3$ AND FOR ARRAYS WITH SPACE DIVERSITY ONLY

l	*VS coordinates*	*VS multiplicities*
3	$3\mathbf{r}_{n1}$	1
	$\mathbf{r}_{n1} + 2\mathbf{r}_{n2}$	3
	$\mathbf{r}_{n1} + \mathbf{r}_{n2} + \mathbf{r}_{n3}$	6
2	$\mathbf{r}_{n1}$	$2N-1$
	$2\mathbf{r}_{n1} - \mathbf{r}_{n2}$	1
	$\mathbf{r}_{n1} + \mathbf{r}_{n2} - \mathbf{r}_{n3}$	2

TABLE VIII
COORDINATES AND MULTIPLICITY ORDER OF VSs FOR SEVERAL VALUES OF l, FOR $q = 4$, AND FOR ARRAYS WITH SPACE DIVERSITY ONLY

l	*VS coordinates*	*VS multiplicities*
4	$4\mathbf{r}_{n1}$	1
	$\mathbf{r}_{n1} + 3\mathbf{r}_{n2}$	4
	$2\mathbf{r}_{n1} + 2\mathbf{r}_{n2}$	6
	$2\mathbf{r}_{n1} + \mathbf{r}_{n2} + \mathbf{r}_{n3}$	12
	$\mathbf{r}_{n1} + \mathbf{r}_{n2} + \mathbf{r}_{n3} + \mathbf{r}_{n4}$	24
3	$2\mathbf{r}_{n1}$	$3N-2$
	$\mathbf{r}_{n1} + \mathbf{r}_{n2}$	$6N-6$
	$3\mathbf{r}_{n1} - \mathbf{r}_{n2}$	1
	$2\mathbf{r}_{n1} + \mathbf{r}_{n2} - \mathbf{r}_{n3}$	3
	$\mathbf{r}_{n1} + \mathbf{r}_{n2} + \mathbf{r}_{n3} - \mathbf{r}_{n4}$	6
2	$\mathbf{r}_0$	$N(2N-1)$
	$\mathbf{r}_{n1} - \mathbf{r}_{n2}$	$4(N-1)$
	$2\mathbf{r}_{n1} - 2\mathbf{r}_{n2}$	1
	$2\mathbf{r}_{n1} - \mathbf{r}_{n2} - \mathbf{r}_{n3}$	2
	$-2\mathbf{r}_{n1} + \mathbf{r}_{n2} + \mathbf{r}_{n3}$	2
	$\mathbf{r}_{n1} + \mathbf{r}_{n2} - \mathbf{r}_{n3} - \mathbf{r}_{n4}$	4

replaced by an array with space, angular, and polarization diversities.

In order to quantify the results of Table IX, Table X summarizes the maximal number of different VSs $N_{\max}[2q, l]$ of the associated VA for several values of N, q, and l. Other results can be found in Table XII for odd and higher values of N. Again, the value of the loss in the processing power associated with the use of a suboptimal arrangement also increases as N and q increase. For a given value of N, we verify the increasing value of $N_{\max}[2q, l]$ as q increases for optimal arrangements of the cumulants.

V. VA EXAMPLES

In this section, the $2q$th order VA associated with particular arrays of sensors is described in order to illustrate the results obtained so far.

TABLE IX
$N_{\max}[2q, l]$ AS A FUNCTION OF N FOR SEVERAL VALUES OF q AND l AND FOR ARRAYS WITH SPACE DIVERSITY ONLY

m = 2q	***l***	$N_{max}[2q, l]$
4 (q = 2)	2	$N(N+1)/2$
	1	$N^2 - N + 1$
6 (q = 3)	3	$N!/[6(N-3)!] + N(N-1) + N$
	2	$N!/[2(N-3)!] + N(N-1) + N$
8 (q = 4)	4	$N!/[24(N-4)!] + N!/[2(N-3)!] + 1.5N(N-1) + N$
	3	$N!/[6(N-4)!] + N!/(N-3)! + 1.5N(N-1) + N$
	2	$N!/[4(N-4)!] + N!/(N-3)! + 2N(N-1) + 1$

A. Linear Array of N Identical Sensors

For a linear array, it is always possible to choose a coordinate system in which the sensor n has the coordinates $(x_n, 0, 0)$, $1 \leq n \leq N$. As a consequence, the VSs of the $2q$th order VA for the arrangement $C_{2q,x}(l)$ are, from (12), at coordinates

$$\left(x^l_{k_1k_2\ldots k_q}, y^l_{k_1k_2\ldots k_q}, z^l_{k_1k_2\ldots k_q}\right) = \left(\sum_{j=1}^{l} x_{k_j} - \sum_{u=1}^{q-l} x_{k_{l+u}}, 0, 0\right) \tag{23}$$

for $1 \leq k_j \leq N$ and $1 \leq j \leq q$. This shows that the $2q$th order VA is also a linear array whatever the arrangement $C_{2q,x}(l)$.

For a ULA, it is always possible to choose a coordinate system such that $x_n = n\, d$, where d is the interelement spacing, and the VA is the linear array composed of the sensors whose first coordinate is given by

$$x^l_{k_1k_2\ldots k_q} = \left(\sum_{j=1}^{l} k_j - \sum_{u=1}^{q-l} k_{l+u}\right) d \tag{24}$$

for $1 \leq k_j \leq N$ and $1 \leq j \leq q$. This shows that the $2q$th order VA is also a ULA with the same interelement spacing, whatever the arrangement $C_{2q,x}(l)$. Moreover, for given values of q, l, and N, the minimum and maximum values of (24), which are noted $x^l_{q,\min}$ and $x^l_{q,\max}$, respectively, are given by

$$x^l_{q,\min} = [l - (q-l)N]\, d = [l(1+N) - qN]\, d \tag{25}$$
$$x^l_{q,\max} = [lN - (q-l)]\, d = [l(1+N) - q]\, d \tag{26}$$

and the number of different VSs N^l_{2q} of the associated VA is easily deduced from (25) and (26) and is given by

$$N^l_{2q} = \frac{\left(x^l_{q,\max} - x^l_{q,\min}\right)}{d} + 1 = qN - (q-1) = q(N-1) + 1 \tag{27}$$

This is independent of l and means that for given values of q and N, the number of VSs is independent of the chosen arrangement $C_{2q,x}(l)$. In other words, in terms of processing power, for a given value of q and due to the symmetries of the array, all the arrangements $C_{2q,x}(l)$ are equivalent for a ULA. Besides, we deduce from (24) that

$$x^l_{k_1k_2\ldots k_q} = \left(\sum_{j=1}^{l} k_j - \sum_{u=2}^{q-l} k_{l+u}\right) d - k_{l+1} d \tag{28}$$
$$x^{l+1}_{k_1k_2k_2\ldots k_q} = \left(\sum_{j=1}^{l} k_j - \sum_{u=2}^{q-l} k_{l+u}\right) d + k_{l+1} d \tag{29}$$

which is enough to understand that for given values of q and N, the $2q$th order VA associated with $C_{2q,x}(l)$ is just a translation of $-(N+1)d$ of the VA associated with $C_{2q,x}(l+1)$. Indeed, when k_{l+1} varies from 1 to N, the quantity $k_{l+1}d$ varies from d to Nd and describes the N sensors of the ULA. In the same time, the quantity $-k_{l+1}d$ varys from $-d$ to $-Nd$ and describes the initial ULA translated of $-(N+1)d$. We then deduce from (28) and (29) that the coordinates $x^l_{k_1k_2\ldots k_q}$ and $x^{l+1}_{k_1k_2k_2\ldots k_q}$ are built in the same manner as two initial ULAs such that the first one is in translation with respect to the other, which proves that for a ULA, the $2q$th order VA (i.e., both the number of different VSs and the order of multiplicity of these VSs) is independent of the arrangement $C_{2q,x}(l)$.

Table XI summarizes, for a ULA, the number of different VSs N^l_{2q} given by (27) of the associated VA for several values of q and N. It is verified in [8] that the $2q$-MUSIC algorithm is able to process up to $N^l_{2q} - 1 = q(N-1)$ statistically independent non-Gaussian sources from an ULA of N sensors.

Comparing (27), which is quantified in Table XI, to $N_{\max}[2q, l]$, which is computed in Table IX and quantified in Table X, for $2 \leq q \leq 4$ and the associated values of l, we deduce that

$$N^l_{2q} = N_{\max}[2q, l] = q + 1, \quad \text{for} \quad N = 2 \tag{30}$$

since all the arrays with two sensors are ULA arrays, whereas $N^l_{2q} < N_{\max}[2q, l]$ for $N > 2$. Finally, to complete these results, we compute below for the ULA the order of multiplicity $m(i)$ of the associated VS i for $2 \leq q \leq 4$, and we illustrate some VA pattern related to a ULA. After tedious algebraic manipulations, indexing the VSs such that their first coordinate increases with their index, we obtain the following results.

1) Fourth Order VA ($q = 2$): For $q = 2$, the order of multiplicity $m(i)$ of the VS i is given by

$$m(i) = N - |N - \mathrm{i}|, \qquad 1 \leq i \leq 2N - 1. \tag{31}$$

This result has already been obtained in [7] for $l = 1$. These results are illustrated in Fig. 2, which shows the FO VA of a ULA of five sensors for which $d = \lambda/2$, together with the order of multiplicity of the VSs, with the x and y axes normalized by the wavelength λ.

TABLE X
$N_{\max}[2q, l]$ FOR SEVERAL VALUES OF N, q AND l AND FOR ARRAYS WITH SPACE DIVERSITY ONLY

m = 2q	**l**	**N = 2**	**N = 3**	**N = 4**	**N = 5**	**N = 6**	**N = 7**	**N = 8**
4 (q = 2)	**2**	3	6	10	15	21	28	36
	1	3	7	13	21	31	43	57
6 (q = 3)	**3**	4	10	20	35	56	84	120
	2	4	12	28	55	96	154	232
8 (q = 4)	**4**	5	15	35	70	126	210	330
	3	5	18	50	115	231	420	708
	2	5	19	55	131	271	505	869

TABLE XI
N_{2q}^{l} FOR SEVERAL VALUES OF q AND N FOR A ULA

m = 2q	**N = 2**	**N = 3**	**N = 4**	**N = 5**	**N = 6**	**N = 7**	**N = 8**
4 (q = 2)	3	5	7	9	11	13	15
6 (q = 3)	4	7	10	13	16	19	22
8 (q = 4)	5	9	13	17	21	25	29

2) Sixth Order VA ($q = 3$): For $q = 3$, the order of multiplicity $m(i)$ of the VS i is given by

$$m(i) = \frac{i(i+1)}{2}, \quad 1 \le i \le N \tag{32a}$$

$$m(i) = \frac{N(N+1)}{2} + (i-N)(2N-1-i) \qquad 1+N \le i \le L+N \tag{32b}$$

$$m(i) = \frac{N(N+1)}{2} + (i-N)(2N-1-i) \qquad 2N-1-L \le i \le 2N-2 \tag{32c}$$

$$m(i) = \frac{(3N-i-1)(3N-i)}{2} \qquad 2N-1 \le i \le 3N-2 \tag{32d}$$

where $L = (N-1)/2$ if N is odd and $L = N/2 - 1$ if N is even. These results are illustrated in Fig. 3, which shows the sixth order VA of a ULA of five sensors for which $d = \lambda/2$, together with the order of multiplicity of the VS, with the x and y axes normalized by the wavelength λ.

3) Eighth Order VA ($q = 4$): For $q = 4$, the order of multiplicity $m(i)$ of the VS i is given by

$$m(i) = \frac{\sum_{j=1}^{i} j(j+1)}{2} \quad 1 \le i \le N \tag{33a}$$

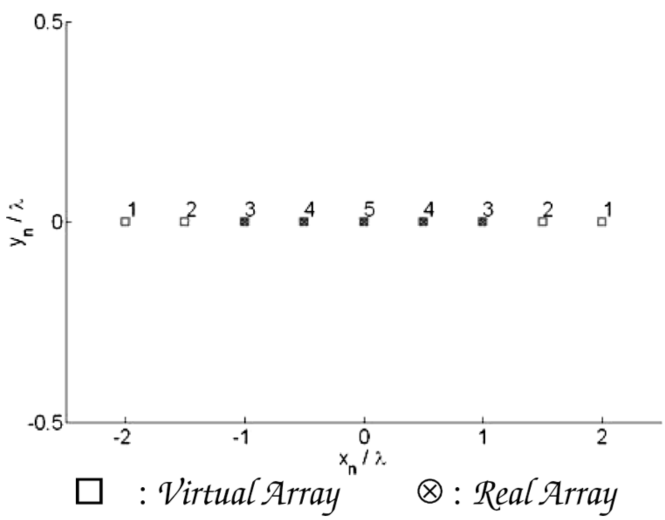

Fig. 2. Fourth order VA of a ULA of five sensors with the order of multiplicities of the VS.

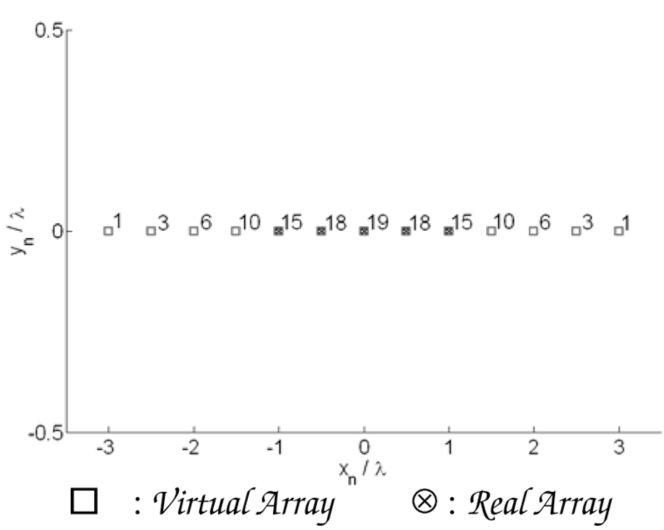

Fig. 3. Sixth order VA of a ULA of five sensors with the order of multiplicities of the VS.

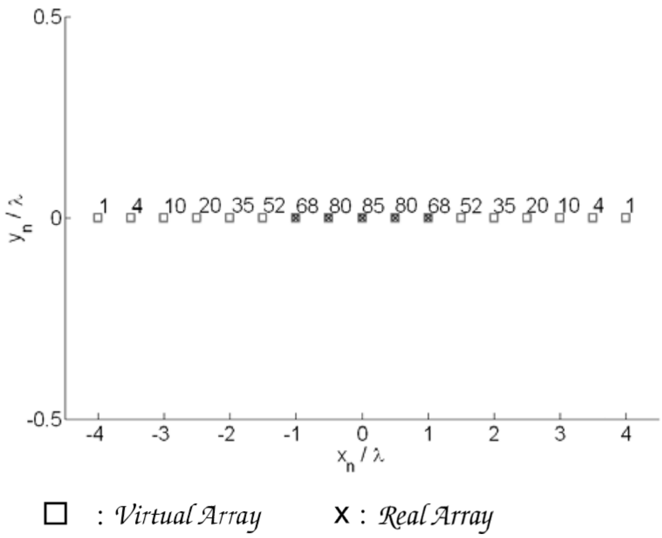

Fig. 4. Eighth order VA of a ULA of five sensors with the order of multiplicities of the VS.

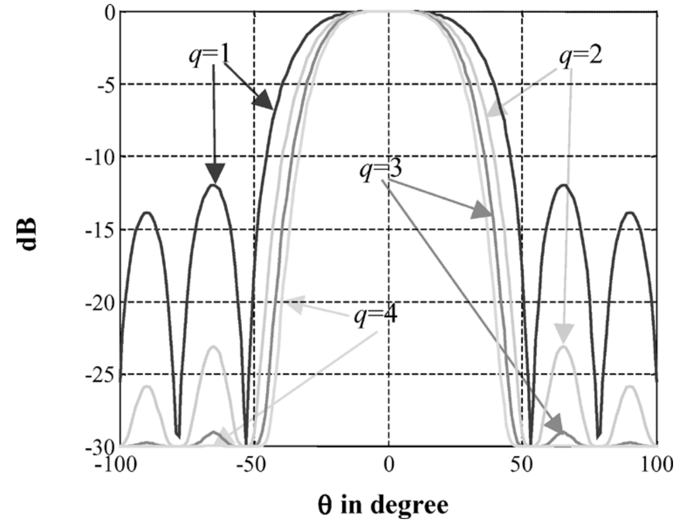

Fig. 5. VA pattern for $q = 1, 2, 3$, and 4, ULA with five sensors, $d = \lambda/2$, pointing direction: 0°.

$$m(i) = \frac{(i-N)N(N+1)}{2} + \sum_{j=1}^{i-N} j(N-j-1) + \frac{\sum_{j=i-N+1}^{N} j(j+1)}{2} \quad N+1 \le i \le 2N-1 \tag{33b}$$

$$m(i) = \frac{(3N-2-i)N(N+1)}{2} + \sum_{j=1}^{3N-2-i} j(N-j-1) + \frac{\sum_{j=3N-1-i}^{N} j(j+1)}{2} \quad 2N-1 \le i \le 3N-3 \tag{33c}$$

$$m(i) = \sum_{j=1}^{4N-2-i} \frac{j(j+1)}{2} \quad 3N-2 \le i \le 4N-3. \tag{33d}$$

These results are illustrated in Fig. 4, which shows the eighth order VA of a ULA of five sensors for which $d = \lambda/2$, together with the order of multiplicity of the VSs, with the x and y axes normalized by the wavelength λ.

4) VA Patterns: To complete these results and to illustrate the results of Section III-C related to the increasing resolution of HO VA as q increases, Fig. 5 shows the array pattern (the normalized inner product of associated steering vectors) of an HO VA associated with a ULA of five sensors equispaced half a wavelenght apart for $q = 1, 2, 3$, and 4 and for a pointing direction equal to 0°. Note the decreasing 3 dB beamwidth and sidelobe level of the array pattern as q increases in proportions given in Section III-C.

B. Circular Array of N Identical Sensors

For a UCA of N sensors, it is always possible to choose a coordinate system in which the sensor n has the coordinates $(R\cos\phi_n, R\sin\phi_n, 0)$ $1 \le n \le N$, where R is the radius of the array, and where $\phi_n \triangleq (n-1)2\pi/N$. We now analyze the associated $2q$th order VA for $2 \le q \le 4$ and for all the possible arrangements $C_{2q,x}(l)$.

1) Fourth order VA ($q = 2$):

a) $l = 2$: For $q = 2$ and $l = 2$, the coordinates of the associated VSs are $(R_{n1,n2}\cos\phi_{n1,n2}, R_{n1,n2}\sin\phi_{n1,n2}, 0)$, $1 \le n_1, n_2 \le N$, where

$$R_{n1,n2} = 2R\cos\left[\frac{(n_1 - n_2)\pi}{N}\right] \tag{34a}$$

$$\phi_{n1,n2} = \frac{(n_1 + n_2 - 2)\pi}{N}. \tag{34b}$$

It is then easy to show that these VSs lie on $1 + (N-1)/2$ different circles if N is odd, or $1 + N/2$ different circles if N is even. Moreover, for odd values of N, N different VSs lie uniformly spaced on each circle of the VA. We deduce that the VA of a UCA of N odd identical sensors has

$$N_4^2 = N\left[1 + \frac{(N-1)}{2}\right] = \frac{N(N+1)}{2} \tag{35}$$

different VSs, which corresponds with the associated upper-bound given in Table IX. The order of multiplicity of these sensors is given in Table VI. The previous results are illustrated in Table XII and Fig. 6. The latter shows the VA of a UCA of five sensors for which $R = 0.8\lambda$, together with the order of multiplicity of the VSs for $q = 2$ and $l = 2$. Table XII reports both the number of different sensors N_{2q}^l of the VA associated with a UCA of N sensors and the upper-bound $N_{\max}[2q, l]$ computed in Table IX for several values of q and l and for odd values of N.

b) $l = 1$: For $q = 2$ and $l = 1$, the coordinates of the associated VSs are $(R_{n1,n2}\cos\phi_{n1,n2}, R_{n1,n2}\sin\phi_{n1,n2}, 0)$, $1 \le n_1, n_2 \le N$, where

$$R_{n1,n2} = 2R\sin\left[\frac{(n_1 - n_2)\pi}{N}\right] \tag{36a}$$

$$\phi_{n1,n2} = \frac{\left(n_1 + n_2 - 2 + \frac{N}{2}\right)\pi}{N}. \tag{36b}$$

It is then easy to show that the VSs that are not at coordinates (0, 0, 0) lie on $(N-1)/2$ different circles if N is odd or $N/2$ different circles if N is even. Moreover, for odd values of N, $2N$

TABLE XII
$N_{\max}[2q, l]$ AND N_{2q}^{l} ASSOCIATED WITH A UCA FOR SEVERAL VALUES OF N, q, AND l AND FOR ARRAYS WITH SPACE DIVERSITY ONLY

		$N=3$		$N=5$		$N=7$		$N=9$		$N=11$	
$m=2q$	l	N_{max}	N_{2q}^{l}	N_{max}	N_{2q}^{l}	N_{max}	N_{2q}^{l}	N_{max}	N_{2q}^{l}	N_{max}	N_{2q}^{l}
4 ($q=2$)	2	6	6	15	15	28	28	45	45	66	66
	1	7	7	21	21	43	43	73	73	111	111
6 ($q=3$)	3	10	10	35	35	84	84	165	163	286	286
	2	12	12	55	55	154	154	333	306	616	616
8 ($q=4$)	4	15	15	70	70	210	210	495	477	1001	1001
	3	18	18	115	115	420	420	1125	918	2486	2486
	2	19	19	131	131	505	505	1405	1135	3191	3191

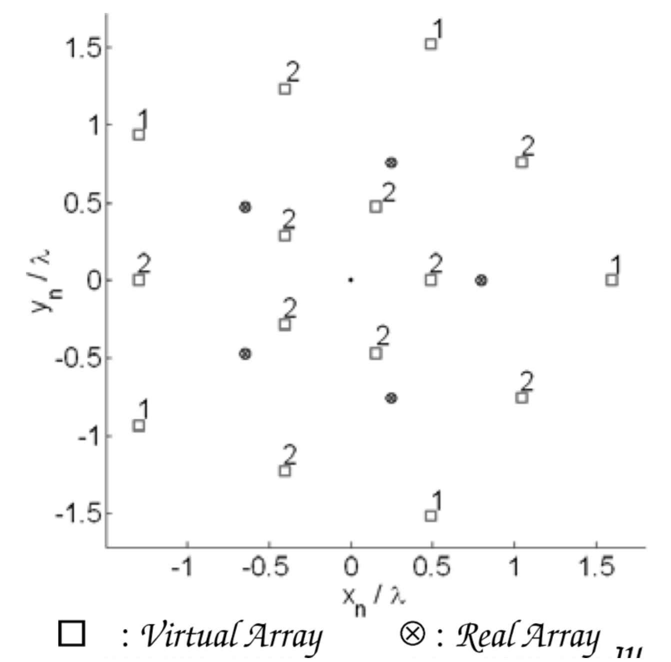

Fig. 6. Fourth order VA of a UCA of five sensors with the order of multiplicities of the VS for $(q, l) = (2, 2)$.

different VSs lie uniformly spaced on each circle of the VA. We deduce from this result that the VA of a UCA of N odd identical sensors has

$$N_4^2 = \frac{2N(N-1)}{2} + 1 = N^2 - N + 1 \tag{37}$$

different VSs, which corresponds to the associated upper bound given in Table IX. This result has already been obtained in [7]. The order of multiplicity of these sensors is given in Table VI. The previous results are illustrated in Fig. 7 and Table XII. In Fig. 7, the VA of a UCA of five sensors, for which $R = 0.8\lambda$, is shown together with the order of multiplicity of the VSs for $q = 2$ and $l = 1$.

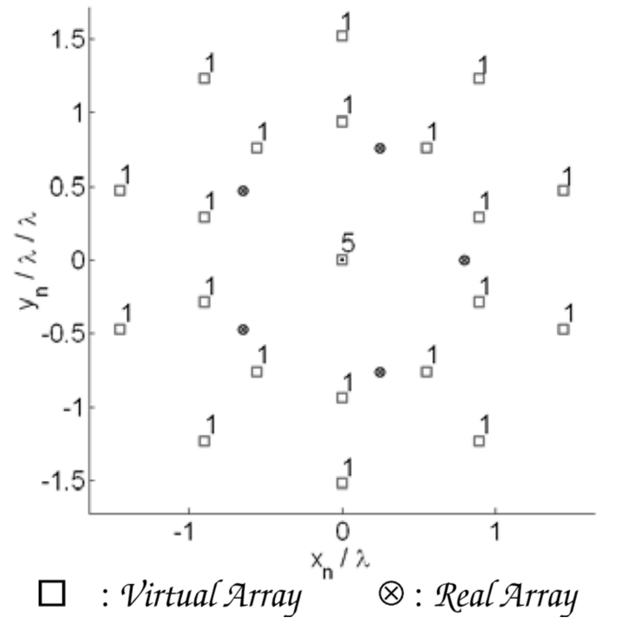

Fig. 7. Fourth order VA of a UCA of five sensors with the order of multiplicities of the VS for $(q, l) = (2, 1)$.

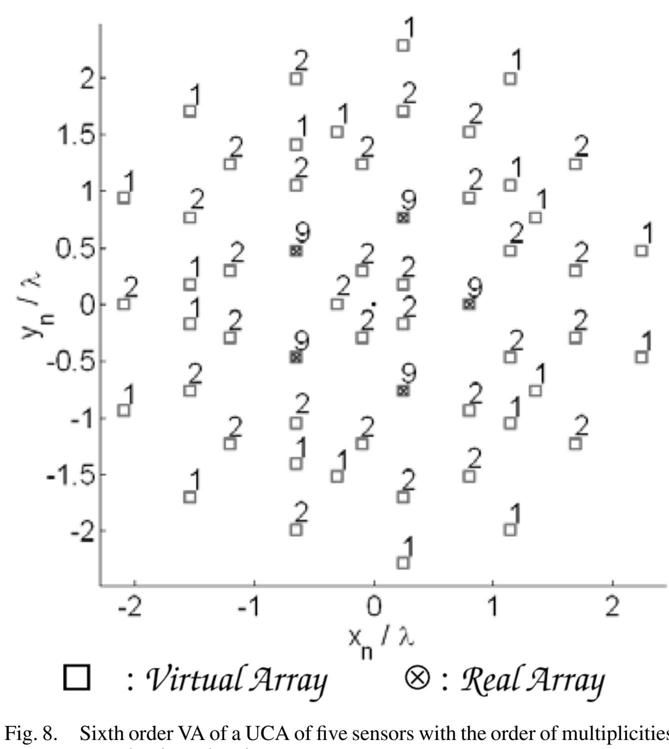

Fig. 8. Sixth order VA of a UCA of five sensors with the order of multiplicities of the VS for $(q, l) = (3, 2)$.

2) 2qth Order VA ($q > 2$): For $q > 2$, the analytical computation of the VA is more difficult. However, the simulations show that for given values of q and l, the number of different VSs N_{2q}^{l} of the VA corresponds to the upper bound $N_{\max}[2q, l]$ when N is a prime number. In this case, it is verified in [8] that the $2q$-MUSIC method is able to process up to $N_{2q}^{l} - 1 = N_{\max}[2q, l] - 1$ statistically independent non-Gaussian sources from a UCA of N sensors. Otherwise, N_{2q}^{l} remains smaller than $N_{\max}[2q, l]$. This result is illustrated in Table XII and Figs. 8 and 9. Figs. 8 and 9 show the VA of a UCA of five sensors for which

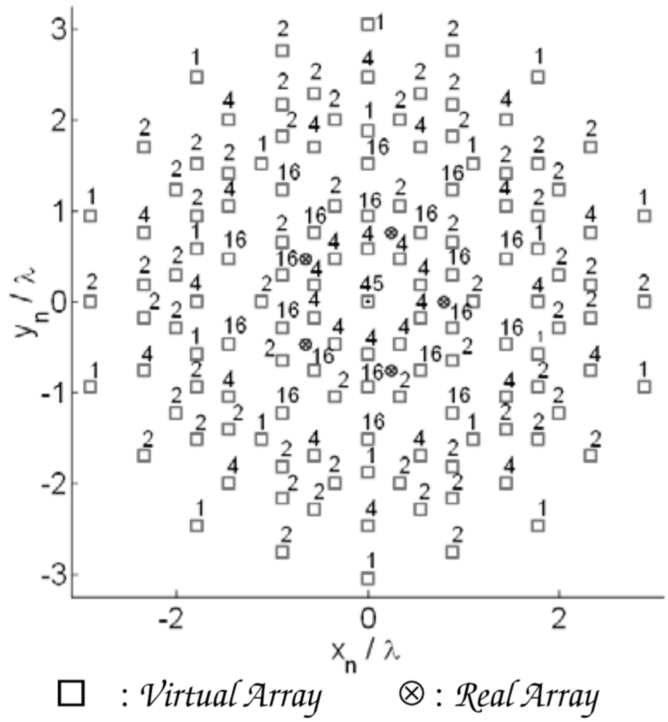

Fig. 9. Eighth order VA of a UCA of five sensors with the order of multiplicities of the VS for $(q, l) = (4, 2)$.

$R = 0.8\lambda$ together with the order of multiplicity of the VSs for $(q, l) = (3, 2)$ and $(q, l) = (4, 2)$, respectively.

VI. Illustration of the HO VA Interest Through a $2q$th Order Direction Finding Application

The strong potential of the HO VA concept is illustrated in this section through a $2q$th order direction finding application.

A. $2q$-MUSIC Method

Among the existing SO direction finding methods, the so-called High Resolution (HR) methods, developed from the beginning of the 1980s, are currently the most powerful in multisource contexts since they are characterized, in the absence of modeling errors, by an asymptotic resolution that becomes infinite, whatever the source signal-to-noise ratio (SNR). Among these HR methods, subspace-based methods such as the MUSIC (or 2-MUSIC) method [24] are the most popular. However, a first drawback of SO subspace-based methods such as the MUSIC method is that they are not able to process more than $N-1$ sources from an array of N sensors. A second drawback of these methods is that their performance may be strongly affected in the presence of modeling errors or when several poorly angularly separated sources have to be separated from a limited number of snapshots.

Mainly to overcome these limitations, FO direction finding methods [4], [6], [9], [21], [23] have been developed these two last decades, among which the extension of the MUSIC method to FO [23], called 4-MUSIC, is the most popular. FO direction finding methods allow in particular both an increase in the resolution power and the processing of more sources than sensors. In particular, it has been shown in [7] and Section IV of this paper that from an array of N sensors, the 4-MUSIC method may process up to $N(N-1)$ sources when the sensors are identical and up to $(N+1)(N-1)$ sources for different sensors.

In order to still increase both the resolution power of HR direction finding methods and the number of sources to be processed from a given array of sensors, the MUSIC method has been extended recently in [8] to an arbitrary even-order $2q (q \geq 1)$, giving rise to the so-called $2q$-MUSIC methods. For a given arrangement of the $2q$th order data statistics $C_{2q,x}(l)$ and after a source number estimation $\hat{P}$, the $2q$-MUSIC method [8] consists of finding the $\hat{P}$ couples (θ_i, φ_i) minimizing the estimated pseudo-spectrum defined by (38), shown at the bottom of the page, where $\hat{\Pi}_{2q-\mathrm{Music}(l)} \triangleq (I_{N^q} - \hat{E}_x \hat{E}_x^{\dagger})$, where I_{N^q} is the $(N^q \times N^q)$ identity matrix, and $\hat{E}_x$ is the $(N^q \times \hat{P})$ matrix of the $\hat{P}$ orthonormalized eigenvectors of the estimated statistical matrix $\hat{C}_{2q,x}(l)$ associated with the $\hat{P}$ strongest eigenvalues. Using the HO VA concept developed in the previous sections and to within the background noise and the source's SNR, the estimated pseudo-spectrum $\hat{C}_{2q-\mathrm{Music}(l)}(\theta, \varphi)$ can also be considered as the estimated pseudo-spectrum of the 2-MUSIC method implemented from the $2q$th order VA associated with the considered array of N sensors for the arrangement $\hat{C}_{2q,x}(l)$.

B. $2q$-MUSIC Performances

The performance of $2q$-MUSIC methods for $1 \leq q \leq 3$ and for arbitrary arrangements $C_{2q,x}(l)$ are analyzed in detail in [8] for both overdetermined $(P \leq N)$ and underdetermined $(P > N)$ mixtures of sources, both with and without modeling errors. In this context, the purpose of this section is not to present this performance analysis again but rather to illustrate the potential of the HO VA concept through the performance evaluation of $2q$-MUSIC methods on a simple example. To do so, we introduce a performance criterion in Section VI-B1 and describe the example in Section VI-B2. We assume that the sources have a zero elevation angle φ.

1) Performance Criterion: For each of the P considered sources and for a given direction finding method, two criterions are used in the following to quantify the quality of the associated direction-of-arrival estimation. For a given source, the first criterion is a probability of aberrant results generated by a given method for this source, and the second one is an averaged root mean square error (RMSE), computed from the nonaberrant results, which are generated by a given method for this source.

More precisely, for given values of q and l, a given number of snapshots L and a particular realization of the L observation vectors $\boldsymbol{x}(n)$ $(1 \leq n \leq L)$, the estimation $\hat{\theta}_p$ of the direction of

$$\hat{C}_{2q-\mathrm{Music}(l)}(\theta, \varphi) \triangleq \frac{\left[\boldsymbol{a}(\theta_i, \varphi)^{\otimes l} \otimes \boldsymbol{a}(\theta_i, \varphi)^{*\otimes(q-l)}\right]^{\dagger} \hat{\Pi}_{2q-\mathrm{Music}(l)} \left[\boldsymbol{a}(\theta_i, \varphi)^{\otimes l} \otimes \boldsymbol{a}(\theta_i, \varphi)^{*\otimes(q-l)}\right]}{\left[\boldsymbol{a}(\theta_i, \varphi)^{\otimes l} \otimes \boldsymbol{a}(\theta_i, \varphi)^{*\otimes(q-l)}\right]^{\dagger} \left[\boldsymbol{a}(\theta_i, \varphi)^{\otimes l} \otimes \boldsymbol{a}(\theta_i, \varphi)^{*\otimes(q-l)}\right]} \tag{38}$$

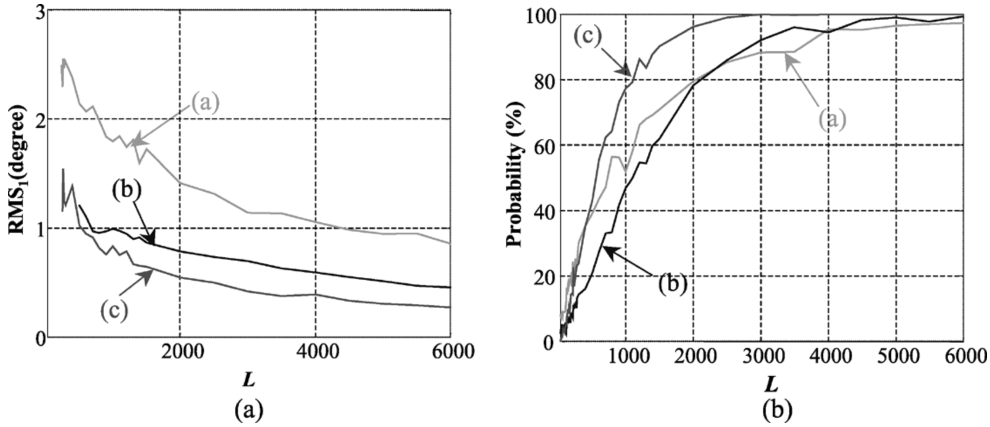

Fig. 10. RMS error of the source 1 and $p(\eta_1 \leq \eta)$ as a function of L. (a) 2-MUSIC. (b) 4-MUSIC. (c) 6-MUSIC. $P = 2, N = 3$, ULA, SNR= 5 dB, $\theta_1 = 90°$, $\theta_2 = 82,7°$. No modeling errors.

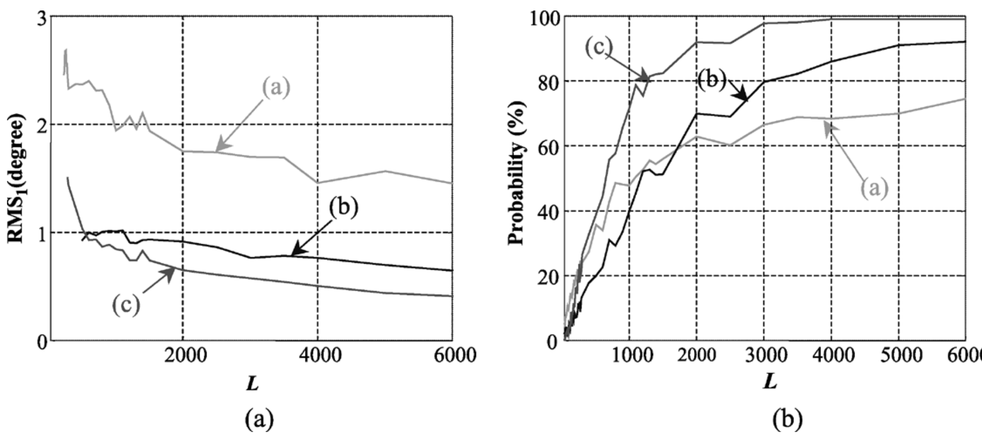

Fig. 11. RMS error of the source 1 and $p(\eta_1 \leq \eta)$ as a function of L. (a) 2-MUSIC. (b) 4-MUSIC. (c) 6-MUSIC. $P = 2, N = 3$, ULA, $SNR = 5$ dB, $\theta_1 = 90°$, $\theta_2 = 82,7°$. With modeling errors.

arrival of the source p $(1 \leq p \leq P)$ from $2q$-MUSIC is defined by

$$\hat{\theta}_p \triangleq \underset{\zeta_i}{\text{Arg}}\left(\underset{i}{\text{Min}}\,|\zeta_i - \theta_p|\right) \tag{39}$$

where the quantities $\zeta_i(1 \leq i \leq \hat{P})$ correspond to the $\hat{P}$ minima of the pseudo-spectrum $\hat{C}_{2q-\text{Music}(l)}(\theta)$ defined by (38) for $\varphi = 0$. To each estimate $\hat{\theta}_p$ $(1 \leq p \leq P)$, we associate the corresponding value of the pseudo-spectrum, which is defined by $\eta_p = \hat{C}_{2q-\text{Music}(l)}(\hat{\theta}_p)$. In this context, the estimate $\hat{\theta}_p$ is considered to be aberrant if $\eta_p > \eta$, where η is a threshold to be defined. In the following, $\eta = 0.1$.

Let us now consider M realizations of the L observation vectors $\boldsymbol{x}(n)$ $(1 \leq n \leq L)$. For a given method, the probability of abberant results for a given source p $\mathrm{p}(\eta_p > \eta)$ is defined by the ratio between the number of realizations for which $\hat{\theta}_p$ is aberrant, and the number M of realizations. From the nonaberrant realizations for the source p, we then define the averaged RMS error for the source p RMSE_p by the quantity

$$\text{RMSE}_p \triangleq \sqrt{\frac{1}{M_p}\sum_{m=1}^{M_p}|\hat{\theta}_{pm} - \theta_p|^2} \tag{40}$$

where M_p is the number of nonaberrant realizations for the source p, and $\hat{\theta}_{pm}$ is the estimate of θ_p for the nonaberrant realization m.

2) Performance Illustration: To illustrate the performance of $2q$-MUSIC methods, we assume that two statistically independent quadrature phase shift keying (QPSK) sources with a raise cosine pulse shape are received by a ULA of $N = 3$ omnidirectional sensors spaced half a wavelenght apart. The two QPSK sources have the same symbol duration $T = T_e$, where T_e is the sample period, the same roll-off $\mu = 0.3$, the same input SNR is equal to 5 dB, and the direction of arrival is equal to $\theta_1 = 90°$ and $\theta_2 = 82.7°$, respectively. Note that the normalized autocumulant of the QPSK symbols is equal to -1 at the FO and $+4$ at the sixth order.

Under these assumptions, Figs. 10 and 11 show the variations, as a function of the number of snapshots L, of the RMS error for the source 1 RMSE_1 and the associated probability of nonabberant results $\mathrm{p}(\eta_1 \leq \eta)$ (we obtain similar results for the source 2) estimated from $M = 300$ realizations at the output of both 2-MUSIC, 4-MUSIC, and 6-MUSIC methods for optimal arrangements of the considered statistics, without and with modeling errors, respectively. In the latter case, the steering vector $\boldsymbol{a}_p$ of the source p becomes an unknown function $\tilde{\boldsymbol{a}}(\theta_p) = \boldsymbol{a}(\theta_p) + \boldsymbol{e}(\theta_p)$ of θ_p, where $\boldsymbol{e}(\theta_p)$ is a modeling error vector that is assumed to be zero-mean, Gaussian, and circular with independent components such that $\mathrm{E}[\boldsymbol{e}(\theta_p)\,\boldsymbol{e}(\theta_p)^\dagger] = \sigma_{\mathrm{e}}^2\mathbf{I}_N$. Note that for omnidirectional sensors and small errors, σ_{e}^2 is the sum of the phase and amplitude error variances per reception chain. For the simulations, σ_{e} is chosen to be equal to 0.0174, which corresponds, for example, to a phase error with a standard deviation of 1° with no amplitude error.

Both in terms of probability of nonaberrant results and estimation precision, Figs. 10 and 11 show, for poorly angularly separated sources, the best behavior of the 6-MUSIC method with respect to 2-MUSIC and 4-MUSIC as soon as L becomes greater than 400 snapshots without modeling errors and 500 snapshots with modeling errors. For such values of L, the resolution gain and the better robustness to modeling errors obtained with 6-MUSIC with respect to 2-MUSIC and 4-MUSIC, due to the narrower 3 dB-beamwidth and the greater number of VSs of the associated sixth order VA, respectively, is higher than the loss due to a higher variance in the statistics estimates. A similar analysis can be done for 4-MUSIC with respect to 2-MUSIC as soon as L becomes greater than 2000 without modeling errors and 1700 snapshots with modeling errors.

Thus, the previous results show that despite their higher variance and contrary to some generally accepted ideas, $2q$-MUSIC methods with $q > 2$ may offer better performances than 2-MUSIC or 4-MUSIC methods when some resolution is required, i.e., in the presence of several sources, when the latter are poorly angularly separated or in the presence of modeling errors inherent in operational contexts, which definitely shows off the great interest of HO VA.

VII. Conclusion

In this paper, the VA concept, which was initially introduced in [7], [15], and [16] for the FO array processing problem and for a particular arrangement of the FO data statistics has been extended to an arbitrary even-order $m = 2q$ $(q \geq 2)$ for several arrangements of the $2q$th order data statistics and for general arrays with space, angular, and polarization diversities. This HO VA concept allows us to provide some important insights into the mechanisms of numerous HO methods and, thus, some explanations about their interests and performance. It allows us, in particular, not only to show off both the increasing resolution and the increasing processing capacity of the $2q$th order array processing methods as q increases but also to solve the identifiability problem of all the HO methods exploiting the algebraic structure of the $2q$th $(q \geq 2)$ order data statistics matrix only for particular arrangements of the latter. The maximal number of sources that can be processed by such methods reached for most of sensors responses and array geometries has been computed for $2 \leq q \leq 4$ and for several arrangements of the data statistics in the $C_{2q,x}$ matrix. For a given number of sensors, the array geometry together with the number of sensors with different complex responses in the array have been shown to be crucial parameters in the processing capacity of these HO methods. Another important result of the paper, which is completely unknown by most of the researchers, is that the way the $2q$th order data statistics are arranged generally controls the geometry and the number of VSs of the VA and, thus, the number of sources that can be processed by a $2q$th order method exploiting the algebraic structure of $C_{2q,x}$. This gives rise to the problem of the optimal arrangement of the data statistics, which has also been solved in the paper. In the particular case of a ULA of N identical sensors, it has been shown that all the considered arrangements of the data statistics are equivalent and give rise to VA with $N_{2q}^{l} = q(N-1)+1$ VSs, whereas when N is a prime number, the UCA of N identical sensors seems to generate VA with $N_{2q}^{l} = N_{\max}[2q, l]$ VSs, whatever the values of q and l. On the other hand, the HO VA concept allows us to explain why, despite their higher variance, HO array processing methods may offer better performances than SO or FO ones when some resolution is required, i.e., in the presence of several sources, when the latter are poorly angularly separated or in the presence of modeling errors inherent in operational contexts. Finally, one may think that the HO VA concept will spawn much practical research in array processing and will also be considered to be a powerful tool for performance evaluation of HO array processing methods.

Appendix A

We present in this Appendix explicit expressions of the Leonov–Shiryaev formula (8) for $q = 1$, 2 and 3, assuming zero-mean complex random vector $\boldsymbol{x}$.

$$\mathrm{Cum}\,[x_{i_1}, x_{i_2}] = \mathrm{E}\,[x_{i_1}\, x_{i_2}] \tag{A.1}$$

$$\begin{aligned}\mathrm{Cum}\,[x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}] &= \mathrm{E}\,[x_{i_1}\, x_{i_2}\, x_{i_3}\, x_{i_4}] \\ &\quad - \mathrm{E}\,[x_{i_1}\, x_{i_2}]\,\mathrm{E}\,[x_{i_3}\, x_{i_4}] \\ &\quad - \mathrm{E}\,[x_{i_1}\, x_{i_3}]\,\mathrm{E}\,[x_{i_2}\, x_{i_4}] \\ &\quad - \mathrm{E}\,[x_{i_1}\, x_{i_4}]\,\mathrm{E}\,[x_{i_2}\, x_{i_3}]\end{aligned} \tag{A.2}$$

$$\begin{aligned}\mathrm{Cum}\,[x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, x_{i_5}, x_{i_6}] &= \mathrm{E}\,[x_{i_1}\, x_{i_2}\, x_{i_3}\, x_{i_4}\, x_{i_5}\, x_{i_6}] \\ &\quad - \mathrm{E}\,[x_{i_1}\, x_{i_2}]\,\mathrm{E}\,[x_{i_3}\, x_{i_4}\, x_{i_5}\, x_{i_6}] \\ &\quad - \mathrm{E}\,[x_{i_1}\, x_{i_3}]\,\mathrm{E}\,[x_{i_2}\, x_{i_4}\, x_{i_5}\, x_{i_6}] \\ &\quad - \mathrm{E}\,[x_{i_1}\, x_{i_4}]\,\mathrm{E}\,[x_{i_2}\, x_{i_3}\, x_{i_5}\, x_{i_6}] \\ &\quad - \mathrm{E}\,[x_{i_1}\, x_{i_5}]\,\mathrm{E}\,[x_{i_2}\, x_{i_3}\, x_{i_4}\, x_{i_6}] \\ &\quad - \mathrm{E}\,[x_{i_1}\, x_{i_6}]\,\mathrm{E}\,[x_{i_2}\, x_{i_3}\, x_{i_4}\, x_{i_5}] \\ &\quad - \mathrm{E}\,[x_{i_2}\, x_{i_3}]\,\mathrm{E}\,[x_{i_1}\, x_{i_4}\, x_{i_5}\, x_{i_6}] \\ &\quad - \mathrm{E}\,[x_{i_2}\, x_{i_4}]\,\mathrm{E}\,[x_{i_1}\, x_{i_3}\, x_{i_5}\, x_{i_6}] \\ &\quad - \mathrm{E}\,[x_{i_2}\, x_{i_5}]\,\mathrm{E}\,[x_{i_1}\, x_{i_3}\, x_{i_4}\, x_{i_6}] \\ &\quad - \mathrm{E}\,[x_{i_2}\, x_{i_6}]\,\mathrm{E}\,[x_{i_1}\, x_{i_3}\, x_{i_4}\, x_{i_5}] \\ &\quad - \mathrm{E}\,[x_{i_3}\, x_{i_4}]\,\mathrm{E}\,[x_{i_1}\, x_{i_2}\, x_{i_5}\, x_{i_6}] \\ &\quad - \mathrm{E}\,[x_{i_3}\, x_{i_5}]\,\mathrm{E}\,[x_{i_1}\, x_{i_2}\, x_{i_4}\, x_{i_6}] \\ &\quad - \mathrm{E}\,[x_{i_3}\, x_{i_6}]\,\mathrm{E}\,[x_{i_1}\, x_{i_2}\, x_{i_4}\, x_{i_5}] \\ &\quad - \mathrm{E}\,[x_{i_4}\, x_{i_5}]\,\mathrm{E}\,[x_{i_1}\, x_{i_2}\, x_{i_3}\, x_{i_6}] \\ &\quad - \mathrm{E}\,[x_{i_4}\, x_{i_6}]\,\mathrm{E}\,[x_{i_1}\, x_{i_2}\, x_{i_3}\, x_{i_5}] \\ &\quad - \mathrm{E}\,[x_{i_5}\, x_{i_6}]\,\mathrm{E}\,[x_{i_1}\, x_{i_2}\, x_{i_3}\, x_{i_4}] \\ &\quad - \mathrm{E}\,[x_{i_1}\, x_{i_2}\, x_{i_3}]\,\mathrm{E}\,[x_{i_4}\, x_{i_5}\, x_{i_6}] \\ &\quad - \mathrm{E}\,[x_{i_1}\, x_{i_2}\, x_{i_4}]\,\mathrm{E}\,[x_{i_3}\, x_{i_5}\, x_{i_6}] \\ &\quad - \mathrm{E}\,[x_{i_1}\, x_{i_2}\, x_{i_5}]\,\mathrm{E}\,[x_{i_3}\, x_{i_4}\, x_{i_6}] \\ &\quad - \mathrm{E}\,[x_{i_1}\, x_{i_2}\, x_{i_6}]\,\mathrm{E}\,[x_{i_3}\, x_{i_4}\, x_{i_5}] \\ &\quad - \mathrm{E}\,[x_{i_1}\, x_{i_3}\, x_{i_4}]\,\mathrm{E}\,[x_{i_2}\, x_{i_5}\, x_{i_6}] \\ &\quad - \mathrm{E}\,[x_{i_1}\, x_{i_3}\, x_{i_5}]\,\mathrm{E}\,[x_{i_2}\, x_{i_4}\, x_{i_6}] \\ &\quad - \mathrm{E}\,[x_{i_1}\, x_{i_3}\, x_{i_6}]\,\mathrm{E}\,[x_{i_2}\, x_{i_4}\, x_{i_5}] \\ &\quad - \mathrm{E}\,[x_{i_1}\, x_{i_4}\, x_{i_5}]\,\mathrm{E}\,[x_{i_2}\, x_{i_3}\, x_{i_6}]\end{aligned}$$

$$\begin{aligned}
&- \mathrm{E}\left[x_{i_1}\, x_{i_4}\, x_{i_6}\right] \mathrm{E}\left[x_{i_2}\, x_{i_3}\, x_{i_5}\right] \\
&- \mathrm{E}\left[x_{i_1}\, x_{i_5}\, x_{i_6}\right] \mathrm{E}\left[x_{i_2}\, x_{i_3}\, x_{i_4}\right] \\
&+ 2\mathrm{E}\left[x_{i_1}\, x_{i_2}\right] \mathrm{E}\left[x_{i_3}\, x_{i_4}\right] \mathrm{E}\left[x_{i_5}\, x_{i_6}\right] \\
&+ 2\mathrm{E}\left[x_{i_1}\, x_{i_2}\right] \mathrm{E}\left[x_{i_3}\, x_{i_5}\right] \mathrm{E}\left[x_{i_4}\, x_{i_6}\right] \\
&+ 2\mathrm{E}\left[x_{i_1}\, x_{i_2}\right] \mathrm{E}\left[x_{i_3}\, x_{i_6}\right] \mathrm{E}\left[x_{i_4}\, x_{i_5}\right] \\
&+ 2\mathrm{E}\left[x_{i_1}\, x_{i_3}\right] \mathrm{E}\left[x_{i_2}\, x_{i_4}\right] \mathrm{E}\left[x_{i_5}\, x_{i_6}\right] \\
&+ 2\mathrm{E}\left[x_{i_1}\, x_{i_3}\right] \mathrm{E}\left[x_{i_2}\, x_{i_5}\right] \mathrm{E}\left[x_{i_4}\, x_{i_6}\right] \\
&+ 2\mathrm{E}\left[x_{i_1}\, x_{i_3}\right] \mathrm{E}\left[x_{i_2}\, x_{i_6}\right] \mathrm{E}\left[x_{i_4}\, x_{i_5}\right] \\
&+ 2\mathrm{E}\left[x_{i_1}\, x_{i_4}\right] \mathrm{E}\left[x_{i_2}\, x_{i_3}\right] \mathrm{E}\left[x_{i_5}\, x_{i_6}\right] \\
&+ 2\mathrm{E}\left[x_{i_1}\, x_{i_4}\right] \mathrm{E}\left[x_{i_2}\, x_{i_5}\right] \mathrm{E}\left[x_{i_3}\, x_{i_6}\right] \\
&+ 2\mathrm{E}\left[x_{i_1}\, x_{i_4}\right] \mathrm{E}\left[x_{i_2}\, x_{i_6}\right] \mathrm{E}\left[x_{i_3}\, x_{i_5}\right] \\
&+ 2\mathrm{E}\left[x_{i_1}\, x_{i_5}\right] \mathrm{E}\left[x_{i_2}\, x_{i_3}\right] \mathrm{E}\left[x_{i_4}\, x_{i_6}\right] \\
&+ 2\mathrm{E}\left[x_{i_1}\, x_{i_5}\right] \mathrm{E}\left[x_{i_2}\, x_{i_4}\right] \mathrm{E}\left[x_{i_3}\, x_{i_6}\right] \\
&+ 2\mathrm{E}\left[x_{i_1}\, x_{i_5}\right] \mathrm{E}\left[x_{i_2}\, x_{i_6}\right] \mathrm{E}\left[x_{i_3}\, x_{i_4}\right] \\
&+ 2\mathrm{E}\left[x_{i_1}\, x_{i_6}\right] \mathrm{E}\left[x_{i_2}\, x_{i_3}\right] \mathrm{E}\left[x_{i_4}\, x_{i_5}\right] \\
&+ 2\mathrm{E}\left[x_{i_1}\, x_{i_6}\right] \mathrm{E}\left[x_{i_2}\, x_{i_4}\right] \mathrm{E}\left[x_{i_3}\, x_{i_5}\right] \\
&+ 2\mathrm{E}\left[x_{i_1}\, x_{i_6}\right] \mathrm{E}\left[x_{i_2}\, x_{i_5}\right] \mathrm{E}\left[x_{i_3}\, x_{i_4}\right].
\end{aligned} \tag{A.3}$$

Appendix B

We show in this Appendix that the spatial correlation coefficient defined by (20) can be written as (21). To this aim, the property (B.1), given for arbitrary $(N \times 1)$ complex vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d}$, by

$$[\boldsymbol{a} \otimes \boldsymbol{b}^*]^\dagger [\boldsymbol{c} \otimes \boldsymbol{d}^*] = (\boldsymbol{a}^\dagger \boldsymbol{c})(\boldsymbol{d}^\dagger \boldsymbol{b}). \tag{B.1}$$

can easily be verified. Applying recurrently the property (B.1), we obtain

$$\left[\boldsymbol{a}^{\otimes l} \otimes \boldsymbol{a}^{*\otimes(q-l)}\right]^\dagger \left[\boldsymbol{b}^{\otimes l} \otimes \boldsymbol{b}^{*\otimes(q-l)}\right] = (\boldsymbol{a}^\dagger \boldsymbol{b})^l (\boldsymbol{b}^\dagger \boldsymbol{a})^{(q-l)}. \tag{B.2}$$

Then, applying (B.2) to (20), we obtain (21).

References

[1] L. Albera, A. Ferreol, P. Chevalier, and P. Comon, "ICAR, un algorithme d'ICA à convergence rapide robuste au bruit," in *Proc. GRETSI*, Paris, France, Sep. 2003.

[2] L. Albera, A. Ferreol, P. Comon, and P. Chevalier, "Sixth order blind identification of underdetermined mixtures (BIRTH) of sources," in *Proc. ICA*, Nara, Japan, Apr. 2003, pp. 909–914.

[3] ——, "Blind Identification of Overcomplete MixturEs of sources (BIOME)," *Linear Algebra and Its Applications J.*, vol. 391, pp. 3–30, Nov. 2004.

[4] J. F. Cardoso, "Localization et Identification par la quadricovariance," *Traitement du Signal*, vol. 7, no. 5, Jun. 1990.

[5] ——, "Super-symetric decomposition of the fourth-order cumulant tensor—blind identification of more sources than sensors," in *Proc. ICASSP*, Toronto, ON, Canada, May 1991, pp. 3109–3112.

[6] J. F. Cardoso and E. Moulines, "Asymptotic performance analysis of direction finding algorithms based on fourth order cumulants," *IEEE Trans. Signal Process.*, vol. 43, no. 1, pp. 214–224, Jan. 1995.

[7] P. Chevalier and A. Ferreol, "On the virtual array concept for the fourth order direction finding problem," *IEEE Trans. Signal Process.*, vol. 47, no. 9, pp. 2592–2595, Sep. 1999.

[8] P. Chevalier, A. Ferreol, and L. Albera, "High resolution direction finding from higher order statistics: The $2q$-MUSIC algorithm," *IEEE Trans. Signal Process.*, submitted for publication.

[9] H. H. Chiang and C. L. Nikias, "The ESPRIT algorithm with high order statistics," in *Proc. Workshop Higher Order Statistics*, Vail, CO, Jun. 1989, pp. 163–168.

[10] P. Comon, "Blind channel identification and extraction of more sources than sensors," in *Proc. SPIE Conf.*, San Diego, CA, Jul. 1998, pp. 2–13.

[11] R. T. Compton Jr., *Adaptive Antennas—Concepts and Performance*. Englewood Cliffs, NJ: Prentice-Hall, 1988.

[12] L. De Lathauwer, B. De Moor, and J. Vandewalle, "ICA techniques for more sources than sensors," in *Proc. Workshop Higher Order Statistics*, Caesara, Israel, Jun. 1999.

[13] C. Demeure and P. Chevalier, "The smart antennas at Thomson-CSF Communications: Concepts, implementations, performances, applications," *Annales Télécommun.*, vol. 53, no. 11–12, pp. 466–482, Nov.–Dec. 1998.

[14] M. C. Dogan and J. M. Mendel, "Cumulant-based blind optimum beamforming," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 30, no. 3, pp. 722–741, Jul. 1994.

[15] ——, "Applications of cumulants to array processing—part I: aperture extension and array calibration," *IEEE Trans. Signal Process.*, vol. 43, no. 5, pp. 1200–1216, May 1995.

[16] M. C. Dogan and J. M. Mendel, "Method and Apparatus for Signal Analysis Employing a Virtual Cross Correlation Computer," Patent no. 5 459 668, Oct. 1995.

[17] A. Ferreol, L. Albera, and P. Chevalier, "Fourth Order Blind Identification of Underdetermined Mixtures of sources (FOBIUM)," in *Proc. ICASSP*, Hong Kong, Apr. 2003, pp. 41–44.

[18] A. Ferreol and P. Chevalier, "On the behavior of current second and higher order blind source separation methods for cyclostationary sources," *IEEE Trans. Signal Process.*, vol. 48, no. 6, pp. 1712–1725, June 2000. Errata: vol. 50, no. 4, p 990, Apr. 2002.

[19] A. Ferreol, P. Chevalier, and L. Albera, "Higher order blind separation of non zero-mean cyclostationary sources," in *Proc. EUSIPCO*, Toulouse, France, Sep. 2002, pp. 103–106.

[20] ——, "Second order blind separation of first and second order cyclostationary sources—application to AM, FSK, CPFSK and deterministic sources," *IEEE Trans. Signal Process.*, vol. 52, no. 4, pp. 845–861, Apr. 2004.

[21] E. Gönen and J. M. Mendel, "Applications of cumulants to array processing—part VI: polarization and direction of arrival estimation with minimally constrained arrays," *IEEE Trans. Signal Process.*, vol. 47, no. 9, pp. 2589–2592, Sep. 1999.

[22] P. Mc Cullagh, "Tensor methods in statistics," in *Monographs on Statistics and Applied Probability*. London, U.K.: Chapman and Hall, 1987.

[23] B. Porat and B. Friedlander, "Direction finding algorithms based on higher order statistics," *IEEE Trans. Signal Process.*, vol. 39, no. 9, pp. 2016–2024, Sep. 1991.

[24] R. O. Schmidt, "Multiple emitter location and signal parameter estimation," *IEEE Trans. Antennas. Propag.*, vol. AP-34, no. 3, pp. 276–280, Mar. 1986.

Pascal Chevalier received the M.Sc. degree from Ecole Nationale Supérieure des Techniques Avancées (ENSTA), Paris, France, and the Ph.D. degree from South-Paris University in 1985 and 1991, respectively.

Since 1991, he has been with Thalés-Communications, Colonbes, France, where he has shared industrial activities (studies, experimentations, expertise, management), teaching activities both in French engineering schools (Supelec, ENST, ENSTA), and French Universities (Cergy-Pontoise), and research activities. Since 2000, he has also been acting as Technical Manager and Architect of the array processing subsystem as part of a national program of military satellite telecommunications. He has bee a Thalés Expert since 2003.His present research interests are in array processing techniques, either blind or informed, second order or higher order, spatial- or spatio-temporal, time-invariant or time-varying, especially for cyclostationary signals, linear or nonlinear, and particularly widely linear for noncircular signals, for applications such as TDMA and CDMA radiocommunications networks, satellite telecommunications, spectrum monitoring, and HF/VUHF passive listening. He is author or co-author of about 100 papers, including journals, conferences, patents, and chapters of books.

Dr. Chevalier was a member of the THOMSON-CSF Technical and Scientifical Council from 1993 to 1998. He co-received the "2003 Science and Defense" Award from the french Ministry of Defence for its work as a whole about array processing for military radiocommunications. He is presently a EURASIP member and an emeritus member of the Societé des Electriciens et des Electroniciens (SEE).

Laurent Albera was born in Massy, France, in 1976. He received the DESS degree in mathematics. In 2001, he received the DEA degree in authomatic and signal processing from the University of Science (Paris XI), Orsay, France, and the Ph.D. degree in science from the University of Nice, Sophia-Antipolis, France.

He is now Associate Professor with the University of Rennes I, Rennes, France, and is affiliated with the Laboratoire Traitement du Signal et de l'Image (LTSI). His research interests include high order statistics, multidimensional algebra, blind deconvolution and equalization, digital communications, statistical signal and array processing, and numerical analysis. More exactly, since 2000, he has been involved with blind source separation (BSS) and independent component analysis (ICA) by processing both the cyclostationary source case and the underdetermined mixture identification problem.

Anne Ferréol was born in 1964 in Lyon, France. She received the M.Sc. degree from ICPI-Lyon, Lyon, France, and the Mastère degree from Ecole Nationale Supérieure des Télécommunications (ENST), Paris, France, in 1988 and 1989, respectively. She is currently pursuing the Ph.D. degree with the Ecole Normale Supérieure de Cachan, France, in collaboration with both SATIE Laboratory and THALES Communications.

Since 1989, she has been with Thomson-CSF-Communications, Gennevilliers, France, in the array processing department.

Ms. Ferréol co-received the 2003 "Science and Defense" Award from the french Ministry of Defence for its work as a whole about array processing for military radiocomunications.

Pierre Comon (M'87–SM'95) graduated in 1982, and received the Doctorate degree in 1985, both from the University of Grenoble, Grenoble, France. He later received the Habilitation to Lead Researches in 1995 from the University of Nice, Sophia Antipolis, France.

For nearly 13 years, he has been in industry, first with Crouzet-Sextant, Valence, France, between 1982 and 1985, and then with Thomson Marconi, Sophia-Antipolis, France, between 1988 and 1997. He spent 1987 with the ISL Laboratory, Stanford University, Stanford, CA. He joined the Eurecom Institute, Sophia Antipolis, in 1997 and left during the Fall of 1998. He was an Associate Research Director with CNRS, Sophia-Antipolis, from 1994 to 1998. He has been Research Director at Laboratory I3S, CNRS, since 1998. His research interests include high order statistics, blind deconvolution and equalization, digital communications, and statistical signal and array processing.

Dr. Comon was Associate Editor of the IEEE TRANSACTIONS ON SIGNAL PROCESSING from 1995 to 1998 and a member of the French National Committee of Scientific Research from 1995 to 2000. He was the coordinator of the European Basic Research Working Group ATHOS from 1992 to 1995. Between 1992 and 1998, he was a member of the Technical and Scientific Council of the Thomson Group. Between July 2001 and January 2004, he acted as the launching Associate Editor of the IEEE TRANSACTIONS ON CIRCUITS AND SYSTEMS I in the area of Blind Techniques. He was IEEE Distinguished Lecturer from 2002 to 2003.