ON THE VIRTUAL ARRAY CONCEPT FOR HIGHER ORDER ARRAY PROCESSING

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Revised version : April 2004

EDICS SP 1-NLHO, 1-DETC

ABSTRACT

For about two decades, many Fourth-Order (FO) array processing methods have been developed for both direction finding and blind identification of non Gaussian signals. One of the main interests in using FO cumulants only instead of Second Order (SO) ones in array processing applications relies on the increase of both the effective aperture and the number of sensors of the considered array, which eventually introduces the FO *Vitual Array* concept presented in [15-16] and [7] and which allows in particular a better resolution and the processing of more sources than sensors. To still increase the resolution and the number of sources to be processed from a given array of sensors, new families of blind identification, source separation and direction finding methods, at an order $m = 2q$ ($q \geq 2$) only, have been developed recently in [3] and [8] respectively. In this context, the purpose of this paper is to provide some important insights into the mechanisms, and more particularly to both the resolution and the maximal processing capacity, of numerous $2q$-th order array processing methods, whose methods [3] and [8] are part of, by extending the Virtual Array concept to an arbitrary even order, for several arrangements of the data statistics and for arrays with space, angular and/or polarization diversity.

Keywords : *Higher order, Virtual array, Blind source identification, HO direction finding, 2q-MUSIC, Identifiability, Space, angular and polarization diversities*

I. INTRODUCTION

For about two decades, many FO array processing methods have been developed for both direction finding [4] [6] [9] [21] [23] and blind identification [1] [5] [10] [12] [14] [17] of non Gaussian signals. One of the main interests in using FO cumulants only instead of SO ones in array processing applications relies on the increase of both the effective aperture and the number of sensors of the considered array, which eventually introduces the FO Virtual Array (VA) concept presented in [15-16] and [7], allowing in particular both the processing of more sources than sensors and an increase in the resolution power of array processing methods.

In order to still increase both the resolution power of array processing methods and the number of sources to be processed from a given array of sensors, new families of blind identification, source separation and direction finding methods, exploiting the data statistics at an arbitrary even order $m = 2q$ ($q \geq 2$) only, have been developed recently in [3] and [8] respectively. More precisely, the reference [8] mainly extends the well-known high resolution direction finding method called MUSIC [24] to an arbitrary even order $2q$, giving rise to the so-called $2q$-MUSIC methods whose interests, for $q \geq 2$, are also shown in [8]. In particular, for operational contexts characterized by a high sources density, such as airborne surveillance over urban areas, the use of Higher Order (HO) MUSIC methods for direction finding allows to reduce or even to minimize the number of sensors of the array and thus the number of reception chains, which finally drastically reduces the overall cost. Besides, it is shown in [8] that, despite of their higher variance and contrary to some generally accepted ideas, $2q$-MUSIC methods with $q > 2$ may offer better performances than 2-MUSIC or 4-MUSIC methods when some resolution is required, i.e. in the presence of several sources, when the latter are poorly angularly separated or in the presence of modelling errors inherent in operational contexts. In the same spirit, to process both over and underdetermined mixtures of statistically independent non Gaussian sources, the reference [3] mainly extends the recently proposed FO blind source identification method called ICAR [1] (Independent Component Analysis using Redundancies in the quadricovariance matrix) to an arbitrary even order $2q$, giving rise to the so-called $2q$-BIOME methods (Blind Identification of Overcomplete MixturEs of sources), whose interests for $q > 2$ are shown in [3]. Note that the $2q$-BIOME method gives rise, for $q = 3$, to the sixth order method called BIRTH (Blind Identification of mixtures using Redundancies in the daTa Hexacovariance matrix) presented recently in [2]. In particular, it is shown in [2] and [3] that $2q$-BIOME methods, for $q \geq 3$, outperform all the existing Blind Source Identification (BSI) methods actually available, in terms of processing power of underdetermined mixtures of arbitrary statistically independent non Gaussian sources.

Contrary to papers [8] and [3], the present paper does not focus on particular HO array processing methods for particular applications but rather aims at providing some important insights into the mechanisms of numerous HO methods and thus some explanations about their interests, through the extension of the VA concept, introduced in [15-16] and [7] for the FO array processing problems, to an arbitrary even order $m = 2q$ ($q \geq 2$) and for several arrangements of the $2q$-th order data statistics, for arrays with space, angular and/or polarisation diversity. This HO VA concept allows in particular to show off both the increasing resolution and the increasing processing capacity of $2q$-th order array processing methods as q increases. It allows to solve not only the identifiability problem of HO methods presented in [8] and [3], in terms of maximal number of sources which can be processed by these methods from an array of N sensors, but also that of all the array processing methods exploiting the algebraic structure of the $2q$-th ($q \geq 2$) order data statistics matrix only, for particular arrangements of the latter. As a consequence of this result, the HO VA concept shows off the impact of the $2q$-th order data statistics arrangement on the $2q$-th order array processing method performances and thus the existence of an optimal arrangement of these statistics, result completely unknown by most of the researchers. Finally, one may think that the HO VA concept will spawn much practical research in array processing and will also be considered as a powerful tool for performance evaluation of HO array processing methods.

After an introduction of some notations, hypotheses and data statistics in section II, the VA concept is extended to even HO statistics in Section III where the questions of both the optimal arrangement of the latter and the resolution of VA is addressed. Some properties of the HO VA for arrays with space, angular and/or polarisation diversity are then presented in section IV where explicit upper-bounds, reached for most array geometries, on the maximal number of independent non Gaussian sources that can be processed by a $2q$-th order method exploiting particular arrangements of the $2q$-th order data statistics, are computed for $2q \leq 8$. Note that the restriction to values of $2q$ lower than or equal to 8 is not so much restrictive since it corresponds to order of statistics which have the highest probability to be used for future applications. The results of sections III and IV are then illustrated in section V through the presentation of HO VA examples for both Uniform Linear Array (ULA) and Uniform Circular Arrays (UCA). Some practical situations for which the HO VA concept leads to better performance than SO or FO ones are pointed out and illustrated in section VI through a direction finding application. Finally section VII concludes this paper.

II. HYPOTHESES, NOTATIONS AND STATISTICS OF THE DATA

A. Hypotheses and Notations

We consider an array of N narrow-band (NB) sensors and we call $\boldsymbol{x}(t)$ the vector of complex amplitudes of the signals at the output of these sensors. Each sensor is assumed to receive the contribution of P zero-mean stationary and stastistically independent NB sources corrupted by a noise. Under these assumptions, the observation vector can approximately be written as follows

$$\boldsymbol{x}(t) \approx \sum_{i=1}^{P} m_i(t)\, \boldsymbol{a}(\theta_i, \varphi_i) + \boldsymbol{v}(t) \triangleq A\, \boldsymbol{m}(t) + \boldsymbol{v}(t) \tag{1}$$

where $\boldsymbol{v}(t)$ is the noise vector, assumed zero-mean, $\boldsymbol{m}(t)$ is the vector whose components $m_i(t)$ are the complex amplitudes of the sources, θ_i and φ_i are the azimuth and the elevation angles of source i (Figure 1), A is the (NxP) matrix of the source steering vectors $\boldsymbol{a}(\theta_i, \varphi_i)$, which contains in particular the information about the direction of arrival of the sources. In particular, in the absence of coupling between sensors, component n of vector $\boldsymbol{a}(\theta_i, \varphi_i)$, denoted $a_n(\theta_i, \varphi_i)$, can be written, in the general case of an array with space, angular and polarization diversity, as [11]

$$a_n(\theta_i, \varphi_i) = a_n(\theta_i, \varphi_i, p_i) = f_n(\theta_i, \varphi_i, p_i) \exp\{j2\pi[x_n \cos(\theta_i) \cos(\varphi_i) + y_n \sin(\theta_i) \cos(\varphi_i) + z_n \sin(\varphi_i)] / \lambda\} \tag{2}$$

where λ is the wavelength, (x_n, y_n, z_n) are the coordinates of sensor n of the array, $f_n(\theta_i, \varphi_i, p_i)$ is a complex number corresponding to the response of sensor n to a unit electric field coming from the direction (θ_i, φ_i) and having the state of polarization p_i (characterized by two angles in the wave plane) [11]. Let us recall that an array of sensors has space diversity if the sensors have not all the same phase center. The array has angular and/or polarization diversity if the sensors have not all the same radiating pattern and/or the same polarization, respectively.

Figure 1

B. Statistics of the data

B1. *Presentation*

The $2q$-th ($q \geq 1$) order array processing methods currently available exploit the information contained in the (N^qxN^q) $2q$–th order circular covariance matrix, $C_{2q,x}$, whose entries are the $2q$–th order circular cumulants of the data, Cum[$x_{i_1}(t)$,…, $x_{i_q}(t)$, $x_{i_{q+1}}(t)^*$, …, $x_{i_{2q}}(t)^*$] ($1 \leq i_j \leq N$) ($1 \leq j \leq 2q$), where * corresponds to the complex conjugation. However, the latter entries can be arranged in the $C_{2q,x}$ matrix in different ways and it is shown in the next section that the way these entries are arranged in the $C_{2q,x}$ matrix determines in particular the maximal processing power of the $2q$-th order methods exploiting the algebraic structure of $C_{2q,x}$, such as the $2q$-MUSIC [8] or the $2q$-BIOME [3] methods. This result is new and seems to be completely unknown by most of the researchers.

In order to prove this important result in the next section, let us introduce an arbitrary integer l such that ($0 \leq l \leq q$) and let us arrange the $2q$–uplet, (i_1,.., i_q, i_{q+1},.., i_{2q}), of indices i_j ($1 \leq j \leq 2q$) into two q-uplets indexed by l and defined by (i_1, i_2,…, i_l, i_{q+1},…, i_{2q-l}) and (i_{2q-l+1},…, i_{2q}, i_{l+1},…, i_q) respectively. As the indices i_j ($1 \leq j \leq 2q$) varys from 1 to N, the two latter q-uplets take N^q values. Numbering, in a natural way, the N^q values of each of two latter q-uplets by the integers I_l and J_l respectively, such that $1 \leq I_l$, J_l $\leq N^q$, we obtain

$$I_l \triangleq \sum_{j=1}^{l} N^{q-j}\,(i_j - 1) + \sum_{j=1}^{q-l} N^{q-l-j}\,(i_{q+j} - 1) + 1 \tag{3a}$$

$$J_l \triangleq \sum_{j=1}^{l} N^{q-j}\,(i_{2q-l+j} - 1) + \sum_{j=1}^{q-l} N^{q-l-j}\,(i_{l+j} - 1) + 1 \tag{3b}$$

Using the permutation invariance property of the cumulants, we deduce that Cum[$x_{i_1}(t)$,…, $x_{i_q}(t)$, $x_{i_{q+1}}(t)^*$, …, $x_{i_{2q}}(t)^*$] = Cum[$x_{i_1}(t)$,…, $x_{i_l}(t)$, $x_{i_{q+1}}(t)^*$, …, $x_{i_{2q-l}}(t)^*$, $x_{i_{2q-l+1}}(t)^*$,.., $x_{i_{2q}}(t)^*$, $x_{i_{l+1}}(t)$,.., $x_{i_q}(t)$] and assuming that the latter quantity is the element [I_l , J_l] of the $C_{2q,x}$ matrix, thus noted $C_{2q,x}(l)$, it is easy to verify, from the Kronecker product definition, the hypotheses of section A and under a Gaussian noise assumption, that the (N^qxN^q) $C_{2q,x}(l)$ matrix can be written as

$$C_{2q,x}(l) \approx \sum_{i=1}^{P} c_{2q,m_i} \, [\boldsymbol{a}(\theta_i, \varphi_i)^{\otimes l} \otimes \boldsymbol{a}(\theta_i, \varphi_i)^{*\otimes(q-l)}] \, [\boldsymbol{a}(\theta_i, \varphi_i)^{\otimes l} \otimes \boldsymbol{a}(\theta_i, \varphi_i)^{*\otimes(q-l)}]^{\dagger} + \eta_2 \, V \, \delta(q-1) \tag{4}$$

where $c_{2q,m_i} \overset{\Delta}{=} \mathrm{Cum}[m_{i_1}(t),\ldots, m_{i_q}(t), m_{i_{q+1}}(t)^*, \ldots, m_{i_{2q}}(t)^*]$, with $i_j = i$ $(1 \le j \le 2q)$, is the $2q$-th order circular autocumulant of $m_i(t)$, † corresponds to the conjugate transposition, η_2 is the mean power of the noise per sensor, V is the $(N \times N)$ spatial coherence matrix of the noise such that $\mathrm{Tr}[V] = N$, Tr[.] means Trace, $\delta(.)$ is the Kronecker symbol, $\otimes$ is the Kronecker product and $\boldsymbol{a}^{\otimes l}$ is the $(N^l \mathrm{x} 1)$ vector defined by $\boldsymbol{a}^{\otimes l} \overset{\Delta}{=} \boldsymbol{a} \otimes \boldsymbol{a} \otimes \ldots . \otimes \boldsymbol{a}$ with a number of Kronecker product $\otimes$ equal to $l - 1$.

In particular, for $q = 1$ and $l = 1$, the $(N \times N)$ $C_{2q,x}(l)$ matrix corresponds to the well-known data covariance matrix (since the observations are zero-mean) defined by

$$R_x \overset{\Delta}{=} C_{2,x}(1) = \mathrm{E}[\boldsymbol{x}(t)\, \boldsymbol{x}(t)^{\dagger}] \approx \sum_{i=1}^{P} c_{2,m_i} \, \boldsymbol{a}(\theta_i, \varphi_i)\, \boldsymbol{a}(\theta_i, \varphi_i)^{\dagger} + \eta_2 \, V \tag{5}$$

For $q = 2$ and $l = 1$, the $(N^2 \mathrm{x} N^2)$ $C_{2q,x}(l)$ matrix corresponds to the classical expression of the data quadricovariance matrix, used in [15] and [7] and in most of the papers dealing with FO array processing problems, and defined by

$$Q_x \overset{\Delta}{=} C_{4,x}(1) \approx \sum_{i=1}^{P} c_{4,m_i} \, [\boldsymbol{a}(\theta_i, \varphi_i) \otimes \boldsymbol{a}(\theta_i, \varphi_i)^*] \, [\boldsymbol{a}(\theta_i, \varphi_i) \otimes \boldsymbol{a}(\theta_i, \varphi_i)^*]^{\dagger} \tag{6}$$

whereas for $q = 2$ and $l = 2$, the $(N^2 \mathrm{x} N^2)$ $C_{2q,x}(l)$ matrix corresponds to an alternative expression of the data quadricovariance matrix, not often used and defined by

$$\tilde{Q}_x \overset{\Delta}{=} C_{4,x}(2) \approx \sum_{i=1}^{P} c_{4,m_i} \, [\boldsymbol{a}(\theta_i, \varphi_i) \otimes \boldsymbol{a}(\theta_i, \varphi_i)] \, [\boldsymbol{a}(\theta_i, \varphi_i) \otimes \boldsymbol{a}(\theta_i, \varphi_i)]^{\dagger} \tag{7}$$

B2. *Estimation*

In situations of practical interests, the $2q$-th order statistics of the data, $\mathrm{Cum}[x_{i_1}(t),\ldots, x_{i_q}(t), x_{i_{q+1}}(t)^*, \ldots, x_{i_{2q}}(t)^*]$, are not known a priori and have to be estimated from L samples of data, $\boldsymbol{x}(l) \overset{\Delta}{=} \boldsymbol{x}(lT_e)$, $1 \le l \le L$, where T_e is the sample period.

For zero-mean stationary observations, using the ergodicity property, an empirical estimator of $\mathrm{Cum}[x_{i_1}(t),\ldots, x_{i_q}(t), x_{i_{q+1}}(t)^*, \ldots, x_{i_{2q}}(t)^*]$, asymptotically unbiased and consistent, may be built from the well-known Leonov-Shiryaev formula [22], giving the expression of a n-th order cumulant of $\boldsymbol{x}(t)$ as a function of its p-th order moments ($1 \leq p \leq n$), by replacing in the latter all the moments by their empirical estimate. More precisely, the Leonov-Shiryaev formula is given by

$$\mathrm{Cum}[x_{i_1}(t)^{\varepsilon 1}, x_{i_2}(t)^{\varepsilon 2},\ldots, x_{i_n}(t)^{\varepsilon n}] = \sum_{p=1}^{n} (-1)^{p-1} (p-1)!\, \mathrm{E}[\prod_{j \in \mathrm{S1}} x_{i_j}(t)^{\varepsilon j}]\, \mathrm{E}[\prod_{j \in \mathrm{S2}} x_{i_j}(t)^{\varepsilon j}] \,..\, \mathrm{E}[\prod_{j \in \mathrm{S}p} x_{i_j}(t)^{\varepsilon j}] \quad (8)$$

where (S1, S2,…., Sp) describes all the partitions in p sets of $(1, 2,\ldots, n)$, $\varepsilon_j = \pm 1$ ($1 \leq j \leq n$) with the convention $x^1 = x$ and $x^{-1} = x^*$ and an empirical estimate of (8) is obtained by replacing in (8) all the moments $\mathrm{E}[x_{i_1}(t)^{\varepsilon 1} x_{i_2}(t)^{\varepsilon 2} \ldots x_{i_p}(t)^{\varepsilon p}]$ ($1 \leq p \leq n$) by their empirical estimate given by

$$\hat{\mathrm{E}}[x_{i_1}(t)^{\varepsilon 1} x_{i_2}(t)^{\varepsilon 2} \ldots x_{i_p}(t)^{\varepsilon p}](L) \triangleq \frac{1}{L} \sum_{l=1}^{L} x_{i_1}(l)^{\varepsilon 1} x_{i_2}(l)^{\varepsilon 2} \ldots x_{i_p}(l)^{\varepsilon p} \quad (9)$$

Explicit expressions of (8) for $n = 2q$ with $1 \leq q \leq 3$ are given in Appendix A.

However, in radiocommunications contexts, most of the sources are no longer stationary but become cyclostationary (digital modulations). For zero-mean cyclostationary observations, the statistical matrix defined by (4) becomes time dependent, noted $C_{2q,x}(l)(t)$, and the theory developed in the paper can be extended without any difficulties by considering that $C_{2q,x}(l)$ is, in this case, the temporal mean, $< C_{2q,x}(l)(t) >$, over an infinite interval duration, of the instantaneous statistics, $C_{2q,x}(l)(t)$. In these conditions, using a cyclo-ergodicity property, the matrix $C_{2q,x}(l)$ has to be estimated from the sampled data by a non empirical estimator such as that presented in [18] for $q = 2$. Note finally that this extension can also be applied to non zero-mean cyclostationary sources, such as some non linearly digitally modulated sources [20], provided that a non empirical statistic estimator, such as that presented in [20] for $q = 1$ and in [19] for $q = 2$, is used.

C. Related 2q-th order array processing problems

A first family of 2q-th order array processing methods which are concerned by the theory developed in the next sections corresponds to the family of 2q-th order Blind Identification methods, which aim at blindly identifying the steering vectors of the sources, $\boldsymbol{a}(\theta_i, \varphi_i)$ ($1 \leq i \leq P$), from the exploitation of the algebraic structure of an estimate of the $C_{2q,x}(l)$ matrix for a particular choice of l. Such methods are described in [2-3]. A second family of methods concerned by the results of the paper corresponds to the $2q$-th order subspace-based direction finding methods such as the $2q$-MUSIC method, presented in [8], which aims at estimating the angles of arrival of the sources, (θ_i, φ_i) $(1 \leq i \leq P)$, from the exploitation of the algebraic structure of an estimate of the $C_{2q,x}(l)$ matrix for a particular choice of l.

III. HIGHER ORDER VIRTUAL ARRAY CONCEPT

A. General presentation

The VA concept has been introduced in [15-16] and [7] for the classical FO array processing problem exploiting expression (6) only. In this section, we extend this concept to an arbitrary even order $m = 2q$ $(q \geq 2)$, for an arbitrary arrangement, $C_{2q,x}(l)$ $(0 \leq l \leq q)$, of the data $2q$-th order circular cumulants, $\text{Cum}[x_{i_1}(t),\ldots, x_{i_q}(t), x_{i_{q+1}}(t)^*, \ldots, x_{i_{2q}}(t)^*]$ $(1 \leq i_j \leq N)$, in the $C_{2q,x}$ matrix and for a general array with space, angular and polarization diversities. This HO VA concept is presented in this section in the case of P statistically independent non Gaussian sources.

Assuming no noise, we note that the matrices $C_{2q,x}(l)$ and R_x, defined by (4) and (5) respectively, have the same algebraic structure, where the auto-cumulant c_{2q,m_i} and the vector $[\boldsymbol{a}(\theta_i, \varphi_i)^{\otimes l} \otimes \boldsymbol{a}(\theta_i, \varphi_i)^{*\otimes(q-l)}]$ play, for $C_{2q,x}(l)$, the rule played for R_x by the power c_{2,m_i} and the steering vector $\boldsymbol{a}(\theta_i, \varphi_i)$ respectively. Thus, for the $2q$-th order array processing methods exploiting expression (4), the $(N^q \times 1)$ vector $[\boldsymbol{a}(\theta_i, \varphi_i)^{\otimes l} \otimes \boldsymbol{a}(\theta_i, \varphi_i)^{*\otimes(q-l)}]$ can be considered as the *virtual steering vector* of the source i for the *true array* of N sensors with coordinates (x_n, y_n, z_n) and amplitude pattern $f_n(\theta, \varphi, p)$, $1 \leq n \leq N$. The N^q components of the vector $[\boldsymbol{a}(\theta_i, \varphi_i)^{\otimes l} \otimes \boldsymbol{a}(\theta_i, \varphi_i)^{*\otimes(q-l)}]$ correspond to the quantities $a_{k_1}(\theta_i, \varphi_i)\, a_{k_2}(\theta_i, \varphi_i) \ldots a_{k_l}(\theta_i, \varphi_i)\, a_{k_{l+1}}(\theta_i, \varphi_i)^* a_{k_{l+2}}(\theta_i, \varphi_i)^* \ldots a_{k_q}(\theta_i, \varphi_i)^*$ $(1 \leq k_j \leq N,\ 1 \leq j \leq q)$, where $a_{k_j}(\theta_i, \varphi_i)$ is the component k_j of vector $\boldsymbol{a}(\theta_i, \varphi_i)$. Using (2) in the latter components and numbering, in a natural way, the N^q values of the q-uplet $(k_1, k_2, \ldots, k_l, k_{l+1}, \ldots, k_q)$ by associating to the latter the integer K defined by

$$K \triangleq \sum_{j=1}^{q} N^{q-j}\,(k_j - 1) + 1 \qquad (10)$$

we find that the component K of the vector $[\boldsymbol{a}(\theta_i, \varphi_i)^{\otimes l} \otimes \boldsymbol{a}(\theta_i, \varphi_i)^{*\otimes(q-l)}]$, noted $[\boldsymbol{a}(\theta_i, \varphi_i)^{\otimes l} \otimes \boldsymbol{a}(\theta_i, \varphi_i)^{*\otimes(q-l)}]_K$, takes the form

$$[\boldsymbol{a}(\theta_i, \varphi_i)^{\otimes l} \otimes \boldsymbol{a}(\theta_i, \varphi_i)^{*\otimes(q-l)}]_K = \Big(\prod_{j=1}^{l} \prod_{u=1}^{q-l} f_{k_j}(\theta_i, \varphi_i, p_i) f_{k_{l+u}}(\theta_i, \varphi_i, p_i)^*\Big)$$

$$\exp\Big\{j2\pi\Big[(\sum_{j=1}^{l} x_{k_j} - \sum_{u=1}^{q-l} x_{k_{l+u}}) \cos(\theta_i)\cos(\varphi_i) + (\sum_{j=1}^{l} y_{k_j} - \sum_{u=1}^{q-l} y_{k_{l+u}}) \sin(\theta_i)\cos(\varphi_i)$$

$$+ (\sum_{j=1}^{l} z_{k_j} - \sum_{u=1}^{q-l} z_{k_{l+u}}) \sin(\varphi_i)\Big] / \lambda\Big\} \qquad (11)$$

Comparing (11) to (2), we deduce that the vector $[\boldsymbol{a}(\theta_i, \varphi_i)^{\otimes l} \otimes \boldsymbol{a}(\theta_i, \varphi_i)^{*\otimes(q-l)}]$ can also be considered as the *true steering vector* of the source i for the VA of N^q *Virtual Sensors* (VS) with coordinates, $(x^l_{k_1k_2..k_q}, y^l_{k_1k_2..k_q}, z^l_{k_1k_2..k_q})$, and complex amplitude patterns, $f^l_{k_1k_2..k_q}(\theta, \varphi, p)$, $1 \leq k_j \leq N$ for $1 \leq j \leq q$, given by

$$(x^l_{k_1k_2..k_q}, y^l_{k_1k_2..k_q}, z^l_{k_1k_2..k_q}) = (\sum_{j=1}^{l} x_{k_j} - \sum_{u=1}^{q-l} x_{k_{l+u}}, \sum_{j=1}^{l} y_{k_j} - \sum_{u=1}^{q-l} y_{k_{l+u}}, \sum_{j=1}^{l} z_{k_j} - \sum_{u=1}^{q-l} z_{k_{l+u}}) \qquad (12)$$

$$f^l_{k_1k_2..k_q}(\theta, \varphi, p) = \prod_{j=1}^{l} \prod_{u=1}^{q-l} f_{k_j}(\theta, \varphi, p)\, f_{k_{l+u}}(\theta, \varphi, p)^* \qquad (13)$$

which introduces in a very simple, direct and short way the VA concept for the $2q$-th order array processing problem, for the arrangement $C_{2q,x}(l)$ and whatever the kind of diversity. Note that expression (13) shows that the complex amplitude response of a VS for given direction of arrival and polarization corresponds to a product of l complex amplitude responses of true sensors and $(q-l)$ conjugate ones for the considered direction of arrival and polarization.

Thus, as a summary, we can consider that the $2q$-th order array processing problem of P statistically independent NB non Gaussian sources from a given array of N sensors with coordinates (x_n, y_n, z_n) and complex amplitude patterns $f_n(\theta, \varphi, p)$, $1 \leq n \leq N$, is, for the arrangement $C_{2q,x}(l)$, similar to a SO array processing problem for which these P statistically independent NB sources impinge, with the *virtual powers* c_{2q,m_i} $(1 \leq i \leq P)$, on a VA of N^q VS having the coordinates $(x^l_{k_1k_2..k_q}, y^l_{k_1k_2..k_q}, z^l_{k_1k_2..k_q})$ and the complex amplitude patterns $f^l_{k_1k_2..k_q}(\theta, \varphi, p)$, $1 \leq k_j \leq N$ for $1 \leq j \leq q$, defined by (12) and (13) respectively. Thus HO array processing may be used to replace sensors and hardware and thus to decrease overall cost of a given system.

From this interpretation based on the HO VA concept, we naturally deduce that the 3dB beamwidth of this VA controls the resolution power of $2q$-th order array processing method for a finite observation duration and the considered arrangement whereas the number of different sensors of this VA controls the maximal number of sources which can be processed by such methods for this arrangement. More precisely, as some of the N^q VS may coïncide, we note N_{2q}^{l} the number of different VS of the VA associated with the $2q$-th order array processing problem for the arrangement $C_{2q,x}(l)$. Then, the maximum number of independent sources that can be processed by a $2q$-th order BSI method exploiting the algebraic structure of $C_{2q,x}(l)$ is N_{2q}^{l} whereas $2q$-th order direction finding methods exploiting the algebraic structure of $C_{2q,x}(l)$, such as the family of $2q$-MUSIC methods [8], are able to process up to $N_{2q}^{l} - 1$ non Gaussian sources.

Another important result shown by expressions (12) and (13) is that, for a given array of N sensors, the associated $2q$-th ($q \geq 2$) order VA depends on the parameter l and thus on the arrangement of the $2q$-th order circular cumulants of the data in the $C_{2q,x}$ matrix. This new result not only shows off the importance of the chosen arrangement of the considered data $2q$-th order cumulants on the processing capacity of the methods exploiting the algebraic structure of $C_{2q,x}$, but also raises the problem of the optimal arrangement of these cumulants for a given even order. This question is addressed in the next section.

Finally, note that expression (4) holds only for sources which are NB for the associated VA, i.e for sources i such that the vector $[\boldsymbol{a}(\theta_i, \varphi_i)^{\otimes l} \otimes \boldsymbol{a}(\theta_i, \varphi_i)^{*\otimes(q-l)}]$ does not depend on the frequency parameter within the reception bandwidth, i.e. for source i in the reception bandwidth B such that

$$\pi B D_{q,l} \cos(\boldsymbol{k}_i, \mathbf{M}_1\mathbf{M}_{\max}) / c << 1 \qquad (14)$$

where c is the propagation velocity, $D_{q,l}$ is the aperture of the VA for the considered parameters q and l, $\boldsymbol{k}_i$ is wave vector for the source i and $\mathbf{M}_1\mathbf{M}_{\max}$ is the vector whose norm is $D_{q,l}$ and whose direction is the line formed by the two most spaced VS M_1 and $M_{\max}$. As $D_{q,l}$ increases with q, the accepted reception bandwidth ensuring the NB assumption for the HO VA decreases with q. In particular, for HF or GSM links, the narrow-band assumption for the HO VA is generally verified up to q = 8 or 10, i.e up to a statistical order $m = 2q$ equal to 16, 18 or 20, from classically used array of sensors for these applications [13].

B. Optimal arrangement $C_{2q,x}(l)$

For a given value of q ($q \geq 2$) and a given array of N sensors, we define in this paper the optimal arrangement $C_{2q,x}(l)$, denoted $C_{2q,x}(l_{opt})$, as the one that maximizes the number of different VS, N_{2q}^{l}, of the associated VA, since the processing power of a $2q$-th order method exploiting the algebraic structure of $C_{2q,x}(l)$ is directly related to the number of different VS of the associated VA.

To get more insight into $C_{2q,x}(l_{opt})$, let us analyse the expressions (12) and (13). These expressions show that the q-uplets (k_1, k_2, …., k_l, k_{l+1},….., k_q) and ($k_{\sigma(1)}$, $k_{\sigma(2)}$, …., $k_{\sigma(l)}$, $k_{\mu(l+1)}$,….., $k_{\mu(q)}$), where ($\sigma(1)$, $\sigma(2)$, …. , $\sigma(l)$) and ($\mu(l+1)$, $\mu(l+2)$, …. , $\mu(q)$) are arbitrary permutations of (1, 2, …. , l) and (l+1, l+2, …., q) respectively, give rise to the same VS (same coordinates and same radiation pattern) of the VA associated with $C_{2q,x}(l)$, defined by (12) and (13). The number of permutations of a given set of indices depends on the number of indices with different values in the set. For this reason, let us classify all the q-uplets (k_1, k_2,….., k_q) ($1 \leq k_j \leq N$, $1 \leq j \leq q$) in q families F_i ($1 \leq i \leq q$) such that F_i corresponds to the set of q-uplets with i different elements k_j. For example

$$F_1 \overset{\Delta}{=} \{ (k_1, k_2,\ldots, k_q) \,/\, k_j = k \text{ for } 1 \leq j \leq q \text{ and } 1 \leq k \leq N \} \tag{15}$$

$$F_q \overset{\Delta}{=} \{ (k_1, k_2,\ldots, k_q) \,/\, k_j \neq k_m \text{ for } 1 \leq j \neq m \leq q \text{ and } 1 \leq k_j, k_m \leq N \} \tag{16}$$

For the general case of an arbitrary array of N sensors, both the number of q-uplets of F_i and the number of different VS of the VA associated with F_i for an arbitrary arrangement $C_{2q,x}(l)$ ($0 \leq l \leq q$) are proportional to $N\,!/\,(N-i)\,!$. Indeed among the i different elements k_j of F_i, once the value of one of them is chosen among N possibilities, there are still $N-1$ possibilities for the second one and then $N-2$ possibilities for the third one and so on and finally $N-i+1$ possibilities for the i-th one, which finally corresponds to $N\,!/\,(N-i)\,!$ possible solutions for the different elements. Then for each of the latter solutions, the value of $q-i$ elements have to be chosen among the i considered different elements, giving rise finally to a number of q-uplets of F_i proportional to $N\,!/\,(N-i)\,!$. This quantity is equal to zero for $N < i$, but becomes a polynomial function of degree i with respect to variable N for $N \geq i$. Thus, as N becomes large, provided that $q \leq N$, the number of different VS of the VA for a given arrangement $C_{2q,x}(l)$, N_{2q}^{l}, is mainly dominated by the number of different VS associated with F_q for this arrangement, $N_{2q}^{l}[F_q]$. The number of q-uplets of F_q is exactely equal to $N!/ (N-q)!$ whereas the number of different VS associated with F_q for the arrangement $C_{2q,x}(l)$ is, for the general case of an arbitrary array of N sensors with no particular symmetries, equal to

$$N_{2q}^{l}[F_q] \quad = \quad N!/ [(N-q)!\, (q-l)!\, l!] \qquad (q \leq N) \tag{17}$$

In fact, when all the k_j are different, the number of permutations ($\sigma(1)$, $\sigma(2)$, …. , $\sigma(l)$) and ($\mu(l+1)$, $\mu(l+2)$, …. , $\mu(q)$) of (1, 2, …. , l) and ($l+1$, $l+2$, …., q) are equal to $l!$ and $(q-l)!$, respectively.

As a summary, in the general case of an arbitrary array of N sensors with no particular symmetries, for a given value of q ($q \geq 2$) and for large values of N, the optimal arrangement $C_{2q,x}(l_{opt})$ is such that l_{opt} maximizes $N_{2q}^{l}[F_q]$ defined by (17) and thus minimizes the quantity $(q-l)!\, l!$ with respect to l ($0 \leq l \leq q$). We deduce from this result that the arrangements $C_{2q,x}(l)$ and $C_{2q,x}(q-l)$ ($0 \leq l \leq q$) give rise to the same number of VS (in fact the first arrangement is the conjugate of the other whatever the values of q and N). It is then sufficient to limit the analysis to $q/2 \leq l \leq q$ if q is even and to $(q+1)/2 \leq l \leq q$ if q is odd. We easily verify that

$$(q-l)!\, l! \ < \ (q-(l+1))!\ (l+1)! \qquad \text{for} \quad q/2 \leq l \leq q-1 \qquad \text{if } q \text{ is even} \tag{18}$$

$$(q-l)!\, l! \ < \ (q-(l+1))!\ (l+1)! \qquad \text{for} \quad (q+1)/2 \leq l \leq q-1 \qquad \text{if } q \text{ is odd} \tag{19}$$

which proves that $l_{opt} = q/2$ if q is even and $l_{opt} = (q+1)/2$ if q is odd. In other words, l_{opt} is, in all cases, the integer l which minimizes $|2l - q|$. It generates steering vectors $[\boldsymbol{a}(\theta_i, \varphi_i)^{\otimes l} \otimes \boldsymbol{a}(\theta_i, \varphi_i)^{*\otimes(q-l)}]$ for which the number of conjugate vectors is the least different from the number of non conjugate vectors. In particular, for $q = 2$, it corresponds to expression (6). We verify in section IV for $q = 2$, 3 and 4 that this result, shown for large values of N, remains valid whatever the value of N.

C. Virtual Array resolution

To get more insights into the gain in resolution obtained with HO VA, let us compute the spatial correlation coefficient of two sources, with directions $\boldsymbol{\theta} = (\theta, \varphi)$ and $\boldsymbol{\theta}_0 = (\theta_0, \varphi_0)$ respectively, for the VA associated with statistical order $2q$ and arrangement indexed by l. This coefficient, noted $\alpha_{\theta,\theta_0}(2q, l)$ and such that $0 \leq | \alpha_{\theta,\theta_0}(2q, l) | \leq 1$, is defined by the normalized inner product of the steering vectors $\boldsymbol{a}_{2q,l}(\theta, \varphi) \triangleq [\boldsymbol{a}(\theta, \varphi)^{\otimes l} \otimes \boldsymbol{a}(\theta, \varphi)^{*\otimes(q-l)}]$ and $\boldsymbol{a}_{2q,l}(\theta_0, \varphi_0) \triangleq [\boldsymbol{a}(\theta_0, \varphi_0)^{\otimes l} \otimes \boldsymbol{a}(\theta_0, \varphi_0)^{*\otimes(q-l)}]$ and can be written as

$$\alpha_{\theta,\theta_0}(2q, l) \triangleq \frac{\boldsymbol{a}_{2q,l}(\theta, \varphi)^{\dagger}\, \boldsymbol{a}_{2q,l}(\theta_0, \varphi_0)}{[\boldsymbol{a}_{2q,l}(\theta, \varphi)^{\dagger}\, \boldsymbol{a}_{2q,l}(\theta, \varphi)]^{1/2}\, [\boldsymbol{a}_{2q,l}(\theta_0, \varphi_0)^{\dagger}\, \boldsymbol{a}_{2q,l}(\theta_0, \varphi_0)]^{1/2}} \quad (20)$$

For an array with space diversity only, this coefficient is proportional to the value, for the direction θ, of the complex amplitude pattern of the conventional beamforming in the direction θ_0 from the considered VA. It is shown in Appendix B that this coefficient (20) can be written as

$$\alpha_{\theta,\theta_0}(2q, l) = \frac{[\boldsymbol{a}(\theta, \varphi)^{\dagger}\, \boldsymbol{a}(\theta_0, \varphi_0)]^{l}\, [\boldsymbol{a}(\theta_0, \varphi_0)^{\dagger}\, \boldsymbol{a}(\theta, \varphi)]^{(q-l)}}{[\boldsymbol{a}(\theta, \varphi)^{\dagger}\, \boldsymbol{a}(\theta, \varphi)]^{q/2}\, [\boldsymbol{a}(\theta_0, \varphi_0)^{\dagger}\, \boldsymbol{a}(\theta_0, \varphi_0)]^{q/2}} \quad (21)$$

which implys that

$$\begin{aligned} |\, \alpha_{\theta,\theta_0}(2q, l)\,| &= \left(\frac{|\, \boldsymbol{a}(\theta, \varphi)^{\dagger}\, \boldsymbol{a}(\theta_0, \varphi_0)\,|}{[\boldsymbol{a}(\theta, \varphi)^{\dagger}\, \boldsymbol{a}(\theta, \varphi)]^{1/2}\, [\boldsymbol{a}(\theta_0, \varphi_0)^{\dagger}\, \boldsymbol{a}(\theta_0, \varphi_0)]^{1/2}} \right)^{q} \\ &= |\, \alpha_{\theta,\theta_0}(2, 1)\,|^{q} \end{aligned} \quad (22)$$

Expression (22) shows that despite of the fact that $\alpha_{\theta,\theta_0}(2q, l)$ is a function of q and l, its modulus does not depend on l but only depends on q and on the normalized amplitude pattern, $|\alpha_{\theta,\theta_0}(2, 1)|$, of the considered array of N sensors for the pointing direction θ_0. Moreover, as $0 \leq |\alpha_{\theta,\theta_0}(2, 1)| \leq 1$, we deduce from (22) that $|\alpha_{\theta,\theta_0}(2q, l)|$ is a decreasing function of q, which proves the increasing resolution of the HO VA as q increases. In particular, if we note, θ_{3dB}^{2q}, the 3dB beamwidth of the $2q$-th order VA associated with a given array of N sensors, we find from (22) that θ_{3dB}^{2q} can be easily deduced from the normalized amplitude pattern of the latter and is such that $|\alpha_{\theta,\theta_0}(2, 1)| = 0.5^{\,1/q}$ for $\theta = \theta_{3dB}^{2q}$, i.e such that $|\alpha_{\theta,\theta_0}(2, 1)| = 0.707$, 0.794 and 0.84 for $q = 2$, 3 and 4 respectively. As q increases, this generates θ_{3dB}^{2q} values corresponding to a decreasing fraction of the 3dB beamwidth, θ_{3dB}, of the considered array of N sensors and we will verify in section V that $\theta_{3dB}^{2q} = 0.84\ \theta_{3dB}$, $0.76\ \theta_{3dB}$ and $0.71\ \theta_{3dB}$ for $q = 2$, 3 and 4 respectively. Finally, expression (22) proves that rank-1 ambiguities (or grating lobes [11]) of the true and VA coïncide whatever the values of q and l, since the directions $\theta \neq \theta_0$ giving rise to $|\alpha_{\theta,\theta_0}(2q, l)| = 1$ are exactly the ones which give rise to $|\alpha_{\theta,\theta_0}(2, 1)| = 1$. A consequence of this result is that a necessary and sufficient condition to obtain

VA without any rank-1 ambiguities is that the considered array of N sensors have no rank-1 ambiguities.

IV. PROPERTIES OF HIGHER ORDER VIRTUAL ARRAYS

A. Case of an array with space, angular and polarization diversity

For an array with space, angular and polarization diversities, the component n of vector $\boldsymbol{a}(\theta_i, \varphi_i)$ and the component K of $[\boldsymbol{a}(\theta_i, \varphi_i)^{\otimes l} \otimes \boldsymbol{a}(\theta_i, \varphi_i)^{*\otimes(q-l)}]$ are given by (2) and (11) respectively, which shows that the $2q$-th ($q \geq 2$) order VA associated with such an array is also an array with space, angular and polarization diversities, whatever the arrangement of the $2q$-th order circular cumulants of the data in the $C_{2q,x}$ matrix.

Ideally, it would have been interesting to obtain a general expression of the number of different VS, N_{2q}^{l}, of the $2q$-th order VA for the arrangement $C_{2q,x}(l)$ and for arbitrary array geometries. However, it does not seem possible to obtain such a result easily since the computation of the number of VS which degenerate in a same one is very specific of the choice of q and l and of the array geometry. For this reason, we limit our subsequent analysis, for arbitrary array geometries, to some values of q ($2 \leq q \leq 4$), which extends the results of [7] up to the eighth order for arbitrary arrangements of the data cumulants, despite of the tedious character of the computations. Moreover, this analysis is not so much restrictive since the considered $2q$-th order data statistics ($2 \leq q \leq 4$) correspond in fact to the statistics which have the most probability to be used for future applications. Note that the general analysis for arbitrary values of q and l is possible for ULA and is presented in section V.

To simplify the analysis, for each sensor n, $1 \leq n \leq N$, we note f_n its complex response $f_n(\theta, \varphi, p)$, $\boldsymbol{r}_n \triangleq (x_n, y_n, z_n)$ the triplet of its coordinates, $\boldsymbol{r}_0 \triangleq (0, 0, 0)$ and we define $\lambda\, \boldsymbol{r}_n \triangleq (\lambda\, x_n, \lambda\, y_n, \lambda\, z_n)$ and $\boldsymbol{r}_n + \boldsymbol{r}_m \triangleq (x_n + x_m, y_n + y_m, z_n + z_m)$. Moreover for a given value of q, we define the order of multiplicity, m, of a given VS of the considered $2q$-th order VA by the number of q-uplets $(k_1, k_2, \ldots, k_q)$ giving rise to this VS. When the order of multiplicity of a given VS is greater than 1, this VS can be considered as weighted in amplitude by a factor corresponding to the order of multiplicity and the associated VA then becomes an *amplitude tappered* array.

The coordinates, the complex responses and the order of multiplicity of the VS of HO VA, deduced from (12) and (13), are presented in tables 1, 2 and 3 for q = 2, 3 and 4 respectively and for several values of the parameter l. In these tables the integers n_i take all the values between 1 and N ($1 \leq n_i \leq N$ for $1 \leq i \leq 4$) but under the constraint that $n_i \neq n_j$ if $i \neq j$ for a given line of the tables. A VS is completely characterized by a line of a table for given values of the n_i.

The results of tables 1 to 3 show that for arrays with sensors having different responses, the VA associated with the parameters (q, l) is *amplitude tappered* for (q, l) = (2, 2), (3, 3), (3, 2), (4, 4), (4, 3) and (4, 2), whereas it is not for (q, l) = (2, 1). In this latter case the order of multiplicity of each VS is 1 and then, the number of different VS of the associated VA may be maximum for q = 2 and equal to N^2. It is in particular the case if the responses of all the VS are different. However, for arbitrary values of q and l, the maximum number of VS of the associated VA, noted $N_{max}[2q, l]$, is generally strictly lower than N^q due to the amplitude tappering of the VA. Table 4 shows precisely, for arrays with different sensors, the expression of $N_{max}[2q, l]$, computed from results of tables 1 to 3, as a function of N for $2 \leq q \leq 4$ and several values of l. Note that $N_{max}[4, 1]$ has already been obtained in [7]. Note also that $N_{max}[2q, l]$ corresponds to N_{2q}^{l} in most cases of sensors having different responses. We verify on table 4 that for a given value of q ($2 \leq q \leq 4$) and the considered values of l, whatever the value of N, small or large, $N_{max}[2q, l]$ is a decreasing function of l, which confirms the optimality of the arrangement $C_{2q,x}(l)$ for the integer l which minimizes $|2l - q|$, as discussed in section III.B. Note also, for a given value of N and for optimal arrangements, the increasing values of $N_{max}[2q, l]$ as q increases.

In order to quantify the results of table 4, table 5 summarizes the maximal number of different VS, $N_{max}[2q, l]$, of the associated VA for several values of N, q and l. As N and q increase, note the increasing value of the loss in the processing power associated with the use of a sub-optimal arrangement instead of the optimal one. For a given value of N, note the increasing value of $N_{max}[2q, l]$ as q increases for optimal arrangements of the cumulants whereas note the possible decreasing value of $N_{max}[2q, l]$ as q increases when the arrangement moves from optimality to suboptimality (for example $N_{max}[6, 2] > N_{max}[8, 4]$ for $2 \leq N \leq 5$).

Table 1

Table 2

Table 3

Table 4

Table 5

B. Case of an array with angular and polarization diversity only

For an array with angular and polarization diversities only, all the sensors of the array have the same phase center, $\boldsymbol{r} \triangleq (x, y, z)$, but have different complex responses, $f_n(\theta, \varphi, p)$, $1 \leq n \leq N$. Such an array is usually referred to an array with colocalized sensors, having different responses in angle and polarization. For such an array, the expression (12) shows that for given values of q and l the VS of the associated VA have all the same coordinates given by $(2l - q)\ \boldsymbol{r}$ while their complex response is given by (13). This shows that the $2q$-th ($q \geq 2$) order VA associated with an array with angular and polarization diversities only is also an array with angular and polarization diversities only, whatever the arrangement of the $2q$-th order circular cumulants of the data in the $C_{2q,x}$ matrix.

The complex responses of the colocalized VS of the $2q$-th order VA for the arrangements $C_{2q,x}(l)$ are all presented in tables 1 to 3 for $2 \leq q \leq 4$. In particular, all the upper-bounds, $N_{max}[2q, l]$, presented in tables 1 to 3 for an array with space, angular and polarization diversities remain valid for an array of colocalized sensors with angular and polarization diversities only. This shows that for sensors having different complex responses, the geometry of the array does not generally play an important role in the maximal power capacity of the $2q$-th order array processing methods exploiting the algebraic structure of $C_{2q,x}$, in terms of number of sources to be processed.

C. Case of an array with space diversity only

Let us consider in this section the particular case of an array with space diversity only. In this case, all the sensors of the array are identical and the complex amplitude patterns of the latter, $f_n(\theta, \varphi, p)$, $1 \leq n \leq N$, may be chosen to be equal to one. Under these assumptions, we deduce from (13) that, for a given value of q ($q \geq 2$), $f^{l}_{k_1 k_2 .. k_q}(\theta, \varphi, p) = 1$ whatever the q-uplet $(k_1, k_2, \ldots, k_q)$ and whatever the arrangement index l. This shows that the $2q$-th ($q \geq 2$) order VA associated with an array with space diversity only is also an array with space diversity only whatever the arrangement of the $2q$-th order circular cumulants of the data in the $C_{2q,x}$ matrix.

For such an array, the $2q$-th order VA are presented hereafter for ($2 \leq q \leq 4$), which extends the results of [7] up to the eighth order for arbitrary arrangements of the data cumulants. More precisely, for arrays with space diversity only, the coordinates and the order of multiplicity of the VS of HO VA, deduced from tables 1 to 3, are presented in tables 6, 7 and 8 for $q = 2$, 3 and 4 respectively and for several values of the parameter l. Again, in these tables the integers n_i takes all the values between 1 and N ($1 \leq n_i \leq N$ for $1 \leq i \leq 4$) but under the constraint that $n_i \neq n_j$ if $i \neq j$ for a given line of the tables. A VS is completely characterized by a line of a table for given values of the n_i.

The results of tables 6 to 8 show that for arrays with identical sensors, the VA associated with the parameters (q, l) is always *amplitude tappered* whatever the values of q and l, which implys in particular that $N_{2q}^{l} \leq N_{max}[2q, l] < N^q$. Table 9 shows precisely, for arrays with identical sensors, the expression of $N_{max}[2q, l]$, computed from results of tables 6 to 8, as a function of N for $2 \leq q \leq 4$ and several values of l. Note that $N_{max}[4, 1]$ has already been obtained in [7]. Note also that $N_{max}[2q, l]$ corresponds to N_{2q}^{l} in most cases of array geometries with no particular symmetries. We verify on table 9 that for a given value of q ($2 \leq q \leq 4$) and the considered values of l, whatever the value of N, small or large, $N_{max}[2q, l]$ is a decreasing function of l, which confirms the optimality of the arrangement $C_{2q,x}(l)$ for the integer l which minimizes $|2l - q|$, as discussed in section III.B. Note also, for a given value of N and for optimal arrangements, the increasing values of $N_{max}[2q, l]$ as q increases. Comparison of tables 4 and 9 shows that whatever the values of q and l, $N_{max}[2q, l]$ can only remain constant or increase when an array with space diversity only is replaced by an array with space, angular and polarization diversities.

In order to quantify the results of table 9, table 10 summarizes the maximal number of different VS, $N_{max}[2q, l]$, of the associated VA for several values of N, q and l. Other results can be found in table 12 for odd and higher values of N. Again, the value of the loss in the processing power associated with the use of a sub-optimal arrangement also increases as N and q increase. For a given value of N, we verify the increasing value of $N_{max}[2q, l]$ as q increases for optimal arrangements of the cumulants.

Table 6

Table 7

Table 8

Table 9

Table 10

V. VIRTUAL ARRAY EXAMPLES

In this section, the $2q$-th order VA associated with particular arrays of sensors is described, in order to illustrate the results obtained so far.

A. Linear array of *N* identical sensors

For a linear array, it is always possible to choose a coordinate system in which the sensor n has the coordinates $(x_n, 0, 0)$, $1 \leq n \leq N$. As a consequence, the VS of the $2q$-th order VA for the arrangement $C_{2q,x}(l)$ are, from (12), at coordinates

$$(x^l_{k_1 k_2 .. k_q}, y^l_{k_1 k_2 .. k_q}, z^l_{k_1 k_2 .. k_q}) = \left(\sum_{j=1}^{l} x_{k_j} - \sum_{u=1}^{q-l} x_{k_{l+u}}, 0, 0\right) \tag{23}$$

for $1 \leq k_j \leq N$ and $1 \leq j \leq q$. This shows that the $2q$-th order VA is also a linear array whatever the arrangement $C_{2q,x}(l)$.

For a ULA, it is always possible to choose a coordinate system such that $x_n = n\, d$, where d is the interelement spacing, and the VA is the linear array composed of the sensors whose first coordinate is given by

$$x^l_{k_1 k_2 .. k_q} = \left(\sum_{j=1}^{l} k_j - \sum_{u=1}^{q-l} k_{l+u}\right) d \tag{24}$$

for $1 \leq k_j \leq N$ and $1 \leq j \leq q$. This shows that the $2q$-th order VA is also a ULA with the same interelement spacing, whatever the arrangement $C_{2q,x}(l)$. Moreover, for given values of q, l and N, the minimum and maximum values of (24), noted $x^l_{q,\min}$ and $x^l_{q,\max}$ respectively, are given by

$$x^l_{q,\min} = [\, l - (q-l)\, N \,]\, d = [\, l\,(1+N) - qN \,]\, d \tag{25}$$

$$x^l_{q,\max} = [\, l\, N - (q-l) \,]\, d = [\, l\,(1+N) - q \,]\, d \tag{26}$$

and the number of different VS, N^l_{2q}, of the associated VA is easily deduced from (25) and (26) and is given by

$$N^l_{2q} = (x^l_{q,\max} - x^l_{q,\min}) / d + 1 = qN - (q-1) = q(N-1) + 1 \tag{27}$$

This is independent of l, and means that, for given values of q and N, the number of VS is independent of the chosen arrangement $C_{2q,x}(l)$. In other words, in terms of processing power, for a given value of q and due to the symmetries of the array, all the arrangements $C_{2q,x}(l)$ are equivalent for a ULA. Besides, we deduce from (24) that

$$x^{l}_{k_1 k_2 .. k_q} = \left(\sum_{j=1}^{l} k_j - \sum_{u=2}^{q-l} k_{l+u}\right) d - k_{l+1}\, d \qquad (28)$$

$$x^{l+1}_{k_1 k_2 k_2 .. k_q} = \left(\sum_{j=1}^{l} k_j - \sum_{u=2}^{q-l} k_{l+u}\right) d + k_{l+1}\, d \qquad (29)$$

which is enough to understand that, for given values of q and N, the $2q$-th order VA associated with $C_{2q,x}(l)$ is just a translation of $-(N+1)d$ of the VA associated with $C_{2q,x}(l+1)$. Indeed, when k_{l+1} varies from 1 to N, the quantity $k_{l+1}\, d$ varies from d to Nd and describes the N sensors of the ULA. In the same time, the quantity $-k_{l+1}d$ varys from $-d$ to $-Nd$ and describes the initial ULA translated of $-(N+1)d$. We then deduce from (28) and (29) that the coordinates $x^{l}_{k_1 k_2 .. k_q}$ and $x^{l+1}_{k_1 k_2 k_2 .. k_q}$ are built in the same manner from two initial ULA's such that the first one is in translation with respect to the other, which proves that for a ULA, the $2q$-th order VA (i.e both the number of different VS and the order of multiplicity of these VS) is independent of the arrangement $C_{2q,x}(l)$.

Table 11 summarizes, for a ULA, the number of different VS, N^{l}_{2q} given by (27), of the associated VA for several values of q and N. It is verified in [8] that the $2q$-MUSIC algorithm is able to process up to $N^{l}_{2q} - 1 = q(N-1)$ statistically independent non Gaussian sources from an ULA of N sensors.

Table 11

Comparing (27), quantified in table 11, to $N_{max}[2q, l]$, computed in table 9 and quantified in table 10, for $2 \leq q \leq 4$ and the associated values of l, we deduce that

$$N^{l}_{2q} = N_{max}[2q, l] = q + 1 \qquad \text{for} \qquad N = 2 \qquad (30)$$

since all the arrays with 2 sensors are ULA arrays, whereas $N^{l}_{2q} < N_{max}[2q, l]$ for $N > 2$. Finally, to complete these results, we compute below for the ULA the order of multiplicity $m(i)$ of the associated VS i for $2 \leq q \leq 4$ and we illustrate some VA pattern related to a ULA. After tedious algebraic manipulations, indexing the VS such that their first coordinate increases with their index, we obtain the following results :

A1. *4-th order VA (q = 2)*

For $q = 2$, the order of multiplicity, $m(i)$ of the VS i is given by

$$m(i) = N - | N - i | \qquad 1 \leq i \leq 2N - 1 \qquad (31)$$

This result has already been obtained in [7] for $l = 1$. These results are illustrated in figure 2 which shows the FO VA of a ULA of 5 sensors for which $d = \lambda/2$, together with the order of multiplicity of the VS, with the x and y axes normalized by the wavelength λ.

Figure 2

A2. *6-th order VA (q = 3)*

For $q = 3$, the order of multiplicity, $m(i)$ of the VS i is given by

$$m(i) = i\,(i + 1)/2 \qquad 1 \leq i \leq N \qquad (32a)$$

$$m(i) = N(N + 1)/2 + (i - N)(2N - 1 - i) \qquad 1 + N \leq i \leq L + N \qquad (32b)$$

$$m(i) = N(N + 1)/2 + (i - N)(2N - 1 - i) \qquad 2N - 1 - L \leq i \leq 2N - 2 \qquad (32c)$$

$$m(i) = (3N - i - 1)\,(3N - i) / 2 \qquad 2N - 1 \leq i \leq 3N - 2 \qquad (32d)$$

where $L = (N - 1)/2$ if N is odd and $L = N/2 - 1$ if N is even. These results are illustrated in figure 3, which shows the 6th-order VA of a ULA of 5 sensors for which $d = \lambda/2$, together with the order of multiplicity of the VS, with the x and y axes normalized by the wavelength λ.

Figure 3

A3. *8-th order VA (q = 4)*

For $q = 4$, the order of multiplicity, $m(i)$ of the VS i is given by

$$m(i) = \sum_{j=1}^{i} j\,(j + 1)/2 \qquad 1 \leq i \leq N \qquad (33a)$$

$$m(i) = (i - N)N(N + 1)/2 + \sum_{j=1}^{i-N} j\,(N - j - 1) + \sum_{j=i-N+1}^{N} j\,(j + 1)/2$$

$$N+1 \leq i \leq 2N-1 \quad (33b)$$

$$m(i) = (3N-2-i)N(N+1)/2 + \sum_{j=1}^{3N-2-i} j\,(N-j-1) + \sum_{j=3N-1-i}^{N} j\,(j+1)/2$$

$$2N-1 \leq i \leq 3N-3 \quad (33c)$$

$$m(i) = \sum_{j=1}^{4N-2-i} j\,(j+1)/2 \qquad 3N-2 \leq i \leq 4N-3 \quad (33d)$$

These results are illustrated in figure 4, which shows the 8th-order VA of a ULA of 5 sensors for which $d = \lambda/2$, together with the order of multiplicity of the VS, with the x and y axes normalized by the wavelength λ.

A4. *VA patterns*

To complete these results and to illustrate the results of section III.C related to the increasing resolution of HO VA as q increases, Figure 5 shows the array pattern (the normalized inner product of associated steering vectors) of HO VA associated to a ULA of 5 sensors equispaced half a wavelenght apart for q = 1, 2, 3 and 4 and for a pointing direction equal to 0°. Note the decreasing 3dB beamwidth and sidelobes level of the array pattern as q increases in proportions given in section III.C.

Figure 4

Figure 5

B. Circular array of *N* identical sensors

For a UCA of N sensors, it is always possible to choose a coordinate system in which the sensor n has the coordinates ($R\cos\phi_n$, $R\sin\phi_n$, 0) $1 \leq n \leq N$, where R is the radius of the array and where $\phi_n \triangleq (n-1)\, 2\pi / N$. We now analyse the associated 2q-th order VA for $2 \leq q \leq 4$ and for all the possible arrangements $C_{2q,x}(l)$.

B1. *4-th order VA (q = 2)*

a) $l = 2$

For $q = 2$ and $l = 2$, the coordinates of the associated VS are ($R_{n1,n2} \cos\phi_{n1,n2}$, $R_{n1,n2} \sin\phi_{n1,n2}$, 0), $1 \leq n_1, n_2 \leq N$, where

$$R_{n1,n2} = 2R\cos[(n_1 - n_2)\pi/N] \quad (34a)$$

$$\phi_{n1,n2} = (n_1 + n_2 - 2)\,\pi/N \quad (34b)$$

It is then easy to show that these VS lie on 1 + (N − 1)/2 different circles if N is odd, or 1+ N/2 different circles if N is even. Moreover, for odd values of N, N different VS lie on each circle of the VA, uniformly spaced. We deduce that the VA of a UCA of N odd identical sensors has

$$N_4^2 = N[1 + (N - 1)/2] = N(N + 1)/2 \quad (35)$$

different VS, which corresponds to the associated upper-bound given in table 9. The order of multiplicity of these sensors is given in table 6. The previous results are illustrated in table 12 and figure 6. The latter shows the VA of a UCA of 5 sensors for which $R = 0.8\,\lambda$, together with the order of multiplicity of the VS, for $q = 2$ and $l = 2$. Table 12 reports both the number of different sensors, N_{2q}^l, of the VA associated to a UCA of N sensors, and the upper-bound, $N_{max}[2q, l]$, computed in table 9, for several values of q and l and for odd values of N.

Figure 6

b) $l = 1$

For $q = 2$ and $l = 1$, the coordinates of the associated VS are ($R_{n1,n2}\cos\phi_{n1,n2}$, $R_{n1,n2}\sin\phi_{n1,n2}$, 0), $1 \le n_1, n_2 \le N$, where

$$R_{n1,n2} = 2R\sin[(n_1 - n_2)\pi/N] \quad (36a)$$

$$\phi_{n1,n2} = (n_1 + n_2 - 2 + N/2)\,\pi/N \quad (36b)$$

It is then easy to show that the VS that are not at coordinates (0, 0, 0) lie on (N − 1)/2 different circles if N is odd, or N/2 different circles if N is even. Moreover, for odd values of N, 2N different VS lie on each circle of the VA, uniformly spaced. We deduce from this result that the VA of a UCA of N odd identical sensors has

$$N_4^2 = 2N(N - 1)/2 + 1 = N^2 - N + 1 \quad (37)$$

different VS, which corresponds to the associated upper-bound given in table 9, result already obtained in [7]. The order of multiplicity of these sensors is given in table 6. The previous results are illustrated in figure 7 and table 12. In figure 7, the VA of a UCA of 5 sensors, for which $R = 0.8\ \lambda$, is shown together with the order of multiplicity of the VS, for $q = 2$ and $l = 1$.

Figure 7

B2. *2q-th order VA (q > 2)*

For $q > 2$, the analytical computation of the VA is more difficult. However, the simulations show that, for given values of q and l, the number of different VS, N_{2q}^{l}, of the VA corresponds to the upper-bound, $N_{max}[2q, l]$, when N is a prime number. In this case, it is verified in [8] that the 2q-MUSIC method is able to process up to $N_{2q}^{l} - 1 = N_{max}[2q, l] - 1$ statistically independent non Gaussian sources from a UCA of N sensors. Otherwise, N_{2q}^{l} remains smaller than $N_{max}[2q, l]$. This result is illustrated in Table 12, Figure 8 and Figure 9. Figures 8 and 9 show the VA of a UCA of 5 sensors for which $R = 0.8\ \lambda$, together with the order of multiplicity of the VS, for $(q, l) = (3, 2)$ and $(q, l) = (4, 2)$ respectively.

Figure 8

Figure 9

Table 12

VI. ILLUSTRATION OF THE HO VIRTUAL ARRAY INTEREST THROUGH A 2q-TH ORDER DIRECTION FINDING APPLICATION

The strong potential of the HO VA concept is illustrated in this section through a 2q-th order direction finding application.

A. 2*q*-MUSIC method

Among the existing SO direction finding methods, the so-called High Resolution (HR) methods, developed from the beginning of the eighties, are currently the most powerful in multi-sources contexts since they are characterized, in the absence of modelling errors, by an asymptotic resolution which becomes infinite whatever the source Signal to Noise Ratio (SNR). Among these HR methods, subspace-based methods such as the MUSIC (or 2-MUSIC) method [24] are the most popular. However, a first drawback of SO subspace-based methods such as the MUSIC method is that they are not able to process more than $N - 1$ sources from an array of N sensors. A second drawback of these methods is that their performance may be strongly affected in the presence of modelling errors or when several poorly angularly separated sources with a low SNR have to be separated from a limited number of snapshots.

Mainly to overcome these limitations, FO direction finding methods [4] [6] [9] [21] [23] have been developed these two last decades, among which the extension of the MUSIC method to FO [23], called 4-MUSIC, is the most popular. FO direction finding methods allow in particular both an increase in the resolution power and the processing of more sources than sensors. In particular, it has been shown in [7] and section IV of this paper that, from an array of N sensors, the 4-MUSIC method may process up to $N(N-1)$ sources when the sensors are identical and up to $(N+1)(N-1)$ sources for different sensors.

In order to still increase both the resolution power of HR direction finding methods and the number of sources to be processed from a given array of sensors, the MUSIC method has been extended recently in [8] to an arbitrary even order $2q$ ($q \geq 1$) giving rise to the so-called $2q$-MUSIC methods. For a given arrangement of the $2q$-th order data statistics, $C_{2q,x}(l)$, and after a source number estimation, $\hat{P}$, the $2q$-MUSIC method [8] consists to find the $\hat{P}$ couples (θ_i, φ_i) minimizing the estimated pseudo-spectrum defined by

$$\hat{C}_{2q\text{-}Music(l)}(\theta, \varphi) \triangleq \frac{[\boldsymbol{a}(\theta_i, \varphi_i)^{\otimes l} \otimes \boldsymbol{a}(\theta_i, \varphi_i)^{*\otimes(q-l)}]^\dagger \, \hat{\Pi}_{2q\text{-}Music(l)} \, [\boldsymbol{a}(\theta_i, \varphi_i)^{\otimes l} \otimes \boldsymbol{a}(\theta_i, \varphi_i)^{*\otimes(q-l)}]}{[\boldsymbol{a}(\theta_i, \varphi_i)^{\otimes l} \otimes \boldsymbol{a}(\theta_i, \varphi_i)^{*\otimes(q-l)}]^\dagger \, [\boldsymbol{a}(\theta_i, \varphi_i)^{\otimes l} \otimes \boldsymbol{a}(\theta_i, \varphi_i)^{*\otimes(q-l)}]} \quad (38)$$

where $\hat{\Pi}_{2q\text{-}Music(l)} \triangleq (\mathrm{I}_{N^q} - \hat{E}_x \hat{E}_x^\dagger)$, with I_{N^q} the $(N^q \times N^q)$ identity matrix and $\hat{E}_x$ the $(N^q \times \hat{P})$ matrix of the $\hat{P}$ orthonormalized eigenvectors of the estimated statistical matrix, $\hat{C}_{2q,x}(l)$, associated with the $\hat{P}$ strongest eigenvalues. Using the HO VA concept developed in the previous sections and to within the background noise and the sources SNR, the estimated pseudo-spectrum $\hat{C}_{2q\text{-}Music(l)}(\theta, \varphi)$ can also be considered as the estimated pseudo-spectrum of the 2-MUSIC method implemented from the $2q$-th order VA associated with the considered array of N sensors for the arrangement $\hat{C}_{2q,x}(l)$.

B. 2*q*-MUSIC performances

The performance of $2q$-MUSIC methods for $1 \leq q \leq 3$ and for arbitrary arrangements, $C_{2q,x}(l)$, are analysed in detail in [8] for both overdetermined ($P \leq N$) and underdetermined ($P > N$) mixtures of sources, $2q$-th order correlated or not, both with and without modelling errors. In this context, the purpose of this section is not to present again this performance analysis but is rather to illustrate the potential of the HO VA concept through the performance evaluation of $2q$-MUSIC methods on a simple example. To do so, we introduce a performance criterion in section B1 and describe the example is section B2. We assume that the sources have a zero elevation angle φ.

B1. *Performance criterion*

For each of the P considered sources and for a given direction finding method, two criterions are used in the following to quantify the quality of the associated direction of arrival estimation. For a given source, the first criterion is a probability of aberrant results generated by a given method for this source and the second one is an averaged Root Mean Square Error (RMSE), computed from the non aberrant results, generated by a given method for this source.

More precisely, for given values of q and l, a given number of snapshots, L, and a particular realization of the L observation vectors $\boldsymbol{x}(l)$ $(1 \leq l \leq L)$, the estimation, $\hat{\theta}_p$, of the direction of arrival of the source p $(1 \leq p \leq P)$ from $2q$-MUSIC is defined by

$$\hat{\theta}_p \overset{\Delta}{=} \underset{\zeta_i}{\mathrm{Arg}} \left(\underset{i}{\mathrm{Min}} \, |\zeta_i - \theta_p| \right) \tag{39}$$

where the quantities ζ_i $(1 \leq i \leq \hat{P})$ correspond to the $\hat{P}$ minima of the pseudo-spectrum $\hat{C}_{2q\text{-}Music(l)}(\theta)$ defined by (38) for $\varphi = 0$. To each estimate $\hat{\theta}_p$ $(1 \leq p \leq P)$, we associate the corresponding value of the pseudo-spectrum, defined by $\eta_p = \hat{C}_{2q\text{-}Music(l)}(\hat{\theta}_p)$. In this context, the estimate $\hat{\theta}_p$ is considered to be aberrant if $\eta_p > \eta$, where η is a treshold to be defined. In the following $\eta = 0.1$.

Let us now consider M realizations of the L observation vectors $\boldsymbol{x}(l)$ $(1 \leq l \leq L)$. For a given method, the probability of abberant results for a given source p, $p(\eta_p > \eta)$, is defined by the ratio between the number of realizations for which $\hat{\theta}_p$ is aberrant and the number of realizations M. From the non aberrant realizations for the source p, we then define the averaged RMS error for the source p, RMSE_p, by the quantity

$$\mathrm{RMSE}_p \overset{\Delta}{=} \sqrt{\frac{1}{M_p} \sum_{m=1}^{M_p} \left| \hat{\theta}_{pm} - \theta_p \right|^2} \tag{40}$$

where M_p is the number of non aberrant realizations for the source p and $\hat{\theta}_{pm}$ is the estimate of θ_p for the non aberrant realization m.

B2. *Performance illustration*

To illustrate the performance of $2q$-MUSIC methods, we assume that 2 statistically independent QPSK sources with a raise cosine pulse shape are received by a ULA of N = 3 omnidirectional sensors spaced half a wavelenght apart. The 2 QPSK sources have the same symbol duration $T = T_e$, where T_e is the sample period, the same roll-off $\mu = 0.3$, the same input SNR equal to 5 dB and a direction of arrival equal to $\theta_1 = 90°$ and $\theta_2 = 82.7°$ respectively. Note that the normalized autocumulant of the QPSK symbols is equal to –1 at the FO and +4 at the Sixth Order.

Under these assumptions, Figures 10 and 11 show the variations, as a function of the number of snapshots L, of the RMS error for the source 1, RMSE_1, and the associated probability of non abberant results, $p(\eta_1 \leq \eta)$, (we obtain similar results for the source 2), estimated from M = 300 realizations, at the output of both 2-MUSIC, 4-MUSIC and 6-MUSIC methods for optimal arrangements of the considered statistics, without and with modelling errors respectively. In the latter case, the steering vector $\boldsymbol{a}_p$ of the source p becomes an unknown function, $\tilde{\boldsymbol{a}}(\theta_p) = \boldsymbol{a}(\theta_p) + \boldsymbol{e}(\theta_p)$, of θ_p, where $\boldsymbol{e}(\theta_p)$ is a modelling error vector assumed zero-mean, Gaussian, circular with independent components such that $E[\mathbf{e}_p \, \mathbf{e}_p^{\dagger}] = \sigma_e^2 \, \mathbf{I}_N$. Note that for omnidirectional sensors and small errors, σ_e^2 is the sum of the phase and amplitude error variances per reception chain. For the simulations, σ_e is chosen to be equal to 0.0174, which corresponds for example to a phase error with a standard deviation of 1° without any amplitude error.

Both in terms of probability of non aberrant results and estimation precision, figures 10 and 11 show, for poorly angularly separated sources, the best behavior of the 6-MUSIC method with respect to 2-MUSIC and 4-MUSIC as soon as L becomes greater than 400 snapshots without modelling errors and 500 snapshots with modelling errors. For such values of L, the resolution gain and the better robustness to modelling errors obtained with 6-MUSIC with respect to 2-MUSIC and 4-MUSIC, due to the narrower 3dB-beamwidth and the greater number of VS of the associated 6-th order VA respectively, is higher than the loss due to a higher variance in the statistics estimates. A similar analysis can be done for 4-MUSIC with respect to 2-MUSIC as soon as L becomes greater than 2000 without modelling errors and 1700 snapshots with modelling errors.

Thus, the previous results show that, despite of their higher variance and contrary to some generally accepted ideas, $2q$-MUSIC methods with $q > 2$ may offer better performances than 2-MUSIC or 4-MUSIC methods when some resolution is required, i.e. in the presence of several sources, when the latter are poorly angularly separated or in the presence of modelling errors inherent in operational contexts, which definitely shows off the great interest of HO VA.

Figure 10

Figure 11

VII. CONCLUSION

In this paper, the VA concept, initially introduced in [15-16] and [7] for the FO array processing problem and for a particular arrangement of the FO data statistics, has been extended to an arbitrary even order $m = 2q$ ($q \geq 2$), for several arrangements of the $2q$-th order data statistics and for general arrays with space, angular and polarization diversities. This HO VA concept allows to provide some important insights into the mechanisms of numerous HO methods and thus some explanations about their interests and performance. It allows in particular not only to show off both the increasing resolution and the increasing processing capacity of $2q$-th order array processing methods as q increases but also to solve the identifiability problem of all the HO methods exploiting the algebraic structure of the $2q$-th ($q \geq 2$) order data statistics matrix only, for particular arrangements of the latter. The maximal number of sources that can be processed by such methods, reached for most of sensors responses and array geometries, has been computed for $2 \leq q \leq 4$ and for several arrangements of the data statistics in the $C_{2q,x}$ matrix. For a given number of sensors, the array geometry together with the number of sensors with different amplitude patterns in the array have been shown to be crucial parameters in the processing capacity of these HO methods. Another important result of the paper, completely unknown by most of the researchers, is that the way the $2q$-th order data statistics are arranged generally controls the geometry and the number of VS of the VA and thus the number of sources that can be processed by a $2q$-th order method exploiting the algebraic structure of $C_{2q,x}$. This gives rise to the problem of the optimal arrangement of the data statistics, which has also been solved in the paper. In the particular case of a ULA of N identical sensors, it has been shown that all the considered arrangements of the data statistics are equivalent and give rise to VA with $N_{2q}^{l} = q(N - 1) + 1$ VS, while when N is a prime number, the UCA of N identical sensors seems to generate VA with $N_{2q}^{l} = N_{max}[2q, l]$ VS whatever the values of q and l. On the other hand, the HO VA concept allows to explain why, despite of their higher variance, HO array processing methods may offer better performances than SO or FO ones when some resolution is required, i.e. in the presence of several sources, when the latter are poorly angularly separated or in the presence of modelling errors inherent in operational contexts. Finally, one may think that the HO VA concept will spawn much practical research in array processing and will also be considered as a powerful tool for performance evaluation of HO array processing methods.

APPENDIX A

We present in this Appendix explicit expressions of the Leonov-Shiryaev formula (8) for q = 1, 2 and 3, assuming zero-mean complex random vector $\boldsymbol{x}$.

$$\mathrm{Cum}[x_{i_1}, x_{i_2}] = \mathrm{E}[x_{i_1} x_{i_2}] \tag{A.1}$$

$$\mathrm{Cum}[x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}] = \mathrm{E}[x_{i_1} x_{i_2} x_{i_3} x_{i_4}] - \mathrm{E}[x_{i_1} x_{i_2}]\,\mathrm{E}[x_{i_3} x_{i_4}] - \mathrm{E}[x_{i_1} x_{i_3}]\,\mathrm{E}[x_{i_2} x_{i_4}] - \mathrm{E}[x_{i_1} x_{i_4}]\,\mathrm{E}[x_{i_2} x_{i_3}] \tag{A.2}$$

$$\begin{aligned}
\mathrm{Cum}[x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, x_{i_5}, x_{i_6}] = {} & \mathrm{E}[x_{i_1} x_{i_2} x_{i_3} x_{i_4} x_{i_5} x_{i_6}] - \mathrm{E}[x_{i_1} x_{i_2}]\,\mathrm{E}[x_{i_3} x_{i_4} x_{i_5} x_{i_6}] - \\
& \mathrm{E}[x_{i_1} x_{i_3}]\,\mathrm{E}[x_{i_2} x_{i_4} x_{i_5} x_{i_6}] - \mathrm{E}[x_{i_1} x_{i_4}]\,\mathrm{E}[x_{i_2} x_{i_3} x_{i_5} x_{i_6}] - \mathrm{E}[x_{i_1} x_{i_5}]\,\mathrm{E}[x_{i_2} x_{i_3} x_{i_4} x_{i_6}] - \\
& \mathrm{E}[x_{i_1} x_{i_6}]\,\mathrm{E}[x_{i_2} x_{i_3} x_{i_4} x_{i_5}] - \mathrm{E}[x_{i_2} x_{i_3}]\,\mathrm{E}[x_{i_1} x_{i_4} x_{i_5} x_{i_6}] - \mathrm{E}[x_{i_2} x_{i_4}]\,\mathrm{E}[x_{i_1} x_{i_3} x_{i_5} x_{i_6}] - \\
& \mathrm{E}[x_{i_2} x_{i_5}]\,\mathrm{E}[x_{i_1} x_{i_3} x_{i_4} x_{i_6}] - \mathrm{E}[x_{i_2} x_{i_6}]\,\mathrm{E}[x_{i_1} x_{i_3} x_{i_4} x_{i_5}] - \mathrm{E}[x_{i_3} x_{i_4}]\,\mathrm{E}[x_{i_1} x_{i_2} x_{i_5} x_{i_6}] - \\
& \mathrm{E}[x_{i_3} x_{i_5}]\,\mathrm{E}[x_{i_1} x_{i_2} x_{i_4} x_{i_6}] - \mathrm{E}[x_{i_3} x_{i_6}]\,\mathrm{E}[x_{i_1} x_{i_2} x_{i_4} x_{i_5}] - \mathrm{E}[x_{i_4} x_{i_5}]\,\mathrm{E}[x_{i_1} x_{i_2} x_{i_3} x_{i_6}] - \\
& \mathrm{E}[x_{i_4} x_{i_6}]\,\mathrm{E}[x_{i_1} x_{i_2} x_{i_3} x_{i_5}] - \mathrm{E}[x_{i_5} x_{i_6}]\,\mathrm{E}[x_{i_1} x_{i_2} x_{i_3} x_{i_4}] - \\
& \mathrm{E}[x_{i_1} x_{i_2} x_{i_3}]\,\mathrm{E}[x_{i_4} x_{i_5} x_{i_6}] - \mathrm{E}[x_{i_1} x_{i_2} x_{i_4}]\,\mathrm{E}[x_{i_3} x_{i_5} x_{i_6}] - \mathrm{E}[x_{i_1} x_{i_2} x_{i_5}]\,\mathrm{E}[x_{i_3} x_{i_4} x_{i_6}] - \\
& \mathrm{E}[x_{i_1} x_{i_2} x_{i_6}]\,\mathrm{E}[x_{i_3} x_{i_4} x_{i_5}] - \mathrm{E}[x_{i_1} x_{i_3} x_{i_4}]\,\mathrm{E}[x_{i_2} x_{i_5} x_{i_6}] - \mathrm{E}[x_{i_1} x_{i_3} x_{i_5}]\,\mathrm{E}[x_{i_2} x_{i_4} x_{i_6}] - \\
& \mathrm{E}[x_{i_1} x_{i_3} x_{i_6}]\,\mathrm{E}[x_{i_2} x_{i_4} x_{i_5}] - \mathrm{E}[x_{i_1} x_{i_4} x_{i_5}]\,\mathrm{E}[x_{i_2} x_{i_3} x_{i_6}] - \mathrm{E}[x_{i_1} x_{i_4} x_{i_6}]\,\mathrm{E}[x_{i_2} x_{i_3} x_{i_5}] - \\
& \mathrm{E}[x_{i_1} x_{i_5} x_{i_6}]\,\mathrm{E}[x_{i_2} x_{i_3} x_{i_4}] + \\
& 2\mathrm{E}[x_{i_1} x_{i_2}]\mathrm{E}[x_{i_3} x_{i_4}]\mathrm{E}[x_{i_5} x_{i_6}] + 2\mathrm{E}[x_{i_1} x_{i_2}]\mathrm{E}[x_{i_3} x_{i_5}]\mathrm{E}[x_{i_4} x_{i_6}] + 2\mathrm{E}[x_{i_1} x_{i_2}]\mathrm{E}[x_{i_3} x_{i_6}]\mathrm{E}[x_{i_4} x_{i_5}] + \\
& 2\mathrm{E}[x_{i_1} x_{i_3}]\mathrm{E}[x_{i_2} x_{i_4}]\mathrm{E}[x_{i_5} x_{i_6}] + 2\mathrm{E}[x_{i_1} x_{i_3}]\mathrm{E}[x_{i_2} x_{i_5}]\mathrm{E}[x_{i_4} x_{i_6}] + 2\mathrm{E}[x_{i_1} x_{i_3}]\mathrm{E}[x_{i_2} x_{i_6}]\mathrm{E}[x_{i_4} x_{i_5}] + \\
& 2\mathrm{E}[x_{i_1} x_{i_4}]\mathrm{E}[x_{i_2} x_{i_3}]\mathrm{E}[x_{i_5} x_{i_6}] + 2\mathrm{E}[x_{i_1} x_{i_4}]\mathrm{E}[x_{i_2} x_{i_5}]\mathrm{E}[x_{i_3} x_{i_6}] + 2\mathrm{E}[x_{i_1} x_{i_4}]\mathrm{E}[x_{i_2} x_{i_6}]\mathrm{E}[x_{i_3} x_{i_5}] + \\
& 2\mathrm{E}[x_{i_1} x_{i_5}]\mathrm{E}[x_{i_2} x_{i_3}]\mathrm{E}[x_{i_4} x_{i_6}] + 2\mathrm{E}[x_{i_1} x_{i_5}]\mathrm{E}[x_{i_2} x_{i_4}]\mathrm{E}[x_{i_3} x_{i_6}] + 2\mathrm{E}[x_{i_1} x_{i_5}]\mathrm{E}[x_{i_2} x_{i_6}]\mathrm{E}[x_{i_3} x_{i_4}] + \\
& 2\mathrm{E}[x_{i_1} x_{i_6}]\mathrm{E}[x_{i_2} x_{i_3}]\mathrm{E}[x_{i_4} x_{i_5}] + 2\mathrm{E}[x_{i_1} x_{i_6}]\mathrm{E}[x_{i_2} x_{i_4}]\mathrm{E}[x_{i_3} x_{i_5}] + 2\mathrm{E}[x_{i_1} x_{i_6}]\mathrm{E}[x_{i_2} x_{i_5}]\mathrm{E}[x_{i_3} x_{i_4}]
\end{aligned} \tag{A.3}$$

APPENDIX B

We show in this appendix that the spatial correlation coefficient defined by (20) can be written as (21). To this aim, it can easily be verified the property (B.1) given, for arbitraries (N x 1) complex vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$, by

$$[\, \boldsymbol{a} \otimes \boldsymbol{b}^{*}]^{\dagger} \, [\, \boldsymbol{c} \otimes \boldsymbol{d}^{*}] \quad = \quad (\boldsymbol{a}^{\dagger}\boldsymbol{c}) \, (\boldsymbol{d}^{\dagger}\boldsymbol{b}) \tag{B.1}$$

Applying recurrently the property (B.1), we obtain

$$[\boldsymbol{a}^{\otimes l} \ \otimes \ \boldsymbol{a}^{*\otimes(q-l)}]^{\dagger} \, [\boldsymbol{b}^{\otimes l} \ \otimes \ \boldsymbol{b}^{*\otimes(q-l)}] \quad = \quad (\boldsymbol{a}^{\dagger}\boldsymbol{b})^{l} \, (\boldsymbol{b}^{\dagger}\boldsymbol{a})^{(q-l)} \tag{B.2}$$

Then, applying (B.2) to (20), we obtain (21).

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l	VS coordinates	VS responses	VS multiplicities
2	$2\boldsymbol{r}_{n1}$	f_{n1}^{2}	1
	$\boldsymbol{r}_{n1} + \boldsymbol{r}_{n2}$	$f_{n1} f_{n2}$	2
1	$\boldsymbol{r}_{0}$	$\lvert f_{n1} \rvert^{2}$	1
	$\boldsymbol{r}_{n1} - \boldsymbol{r}_{n2}$	$f_{n1} f_{n2}^{*}$	1

Table 1 - *Coordinates, Complex responses and Multiplicity order of VS for several values of l, for q = 2 and for arrays with space, angular and polarization diversities*

l	VS coordinates	VS responses	VS multiplicities
3	$3\boldsymbol{r}_{n1}$	f_{n1}^{3}	1
	$\boldsymbol{r}_{n1} + 2\boldsymbol{r}_{n2}$	$f_{n1} f_{n2}^{2}$	3
	$\boldsymbol{r}_{n1} + \boldsymbol{r}_{n2} + \boldsymbol{r}_{n3}$	$f_{n1} f_{n2} f_{n3}$	6
2	$\boldsymbol{r}_{n1}$	$f_{n1} \lvert f_{n1} \rvert^{2}$	1
	$\boldsymbol{r}_{n1}$	$f_{n1} \lvert f_{n2} \rvert^{2}$	2
	$2\boldsymbol{r}_{n2} - \boldsymbol{r}_{n1}$	$f_{n1}^{*} f_{n2}^{2}$	1
	$\boldsymbol{r}_{n1} + \boldsymbol{r}_{n2} - \boldsymbol{r}_{n3}$	$f_{n1} f_{n2} f_{n3}^{*}$	2

Table 2 - *Coordinates, Complex responses and Multiplicity order of VS for several values of l, for q = 3 and for arrays with space, angular and polarization diversities*

l	*VS coordinates*	*VS responses*	*VS multiplicities*
4	$4\mathbf{r}_{n1}$	$f_{n1}^{\;4}$	1
	$\mathbf{r}_{n1} + 3\mathbf{r}_{n2}$	$f_{n1}\, f_{n2}^{\;3}$	4
	$2\mathbf{r}_{n1} + 2\mathbf{r}_{n2}$	$f_{n1}^{\;2}\, f_{n2}^{\;2}$	6
	$2\mathbf{r}_{n1} + \mathbf{r}_{n2} + \mathbf{r}_{n3}$	$f_{n1}^{\;2}\, f_{n2}\, f_{n3}$	12
	$\mathbf{r}_{n1} + \mathbf{r}_{n2} + \mathbf{r}_{n3} + \mathbf{r}_{n4}$	$f_{n1}\, f_{n2}\, f_{n3}\, f_{n4}$	24
3	$2\mathbf{r}_{n1}$	$f_{n1}^{\;2}\lvert f_{n1}\rvert^2$	1
	$\mathbf{r}_{n1} + \mathbf{r}_{n2}$	$f_{n1}\, f_{n2}\lvert f_{n2}\rvert^2$	3
	$3\mathbf{r}_{n2} - \mathbf{r}_{n1}$	$f_{n1}^{\;*}\, f_{n2}^{\;3}$	1
	$2\mathbf{r}_{n1}$	$f_{n1}^{\;2}\lvert f_{n2}\rvert^2$	3
	$2\mathbf{r}_{n1} + \mathbf{r}_{n2} - \mathbf{r}_{n3}$	$f_{n1}^{\;2}\, f_{n2}\, f_{n3}^{\;*}$	3
	$\mathbf{r}_{n2} + \mathbf{r}_{n3}$	$f_{n2}\, f_{n3}\lvert f_{n1}\rvert^2$	6
	$\mathbf{r}_{n1} + \mathbf{r}_{n2} + \mathbf{r}_{n3} - \mathbf{r}_{n4}$	$f_{n1}\, f_{n2}\, f_{n3}\, f_{n4}^{\;*}$	6
2	$\mathbf{r}_0$	$\lvert f_{n1}\rvert^4$	1
	$\mathbf{r}_{n1} - \mathbf{r}_{n2}$	$f_{n1}\, f_{n2}^{\;*}\lvert f_{n2}\rvert^2$	2
	$\mathbf{r}_{n2} - \mathbf{r}_{n1}$	$f_{n1}^{\;*}\, f_{n2}\lvert f_{n2}\rvert^2$	2
	$2\mathbf{r}_{n1} - 2\mathbf{r}_{n2}$	$f_{n1}^{\;2}\, f_{n2}^{\;*2}$	1
	$\mathbf{r}_0$	$\lvert f_{n1}\rvert^2\,\lvert f_{n2}\rvert^2$	4
	$2\mathbf{r}_{n1} - \mathbf{r}_{n2} - \mathbf{r}_{n3}$	$f_{n1}^{\;2}\, f_{n2}^{\;*}\, f_{n3}^{\;*}$	2
	$\mathbf{r}_{n2} - \mathbf{r}_{n3}$	$f_{n2}\, f_{n3}^{\;*}\lvert f_{n1}\rvert^2$	4
	$-2\mathbf{r}_{n1} + \mathbf{r}_{n2} + \mathbf{r}_{n3}$	$f_{n2}\, f_{n3}\, f_{n1}^{\;*2}$	2
	$\mathbf{r}_{n1} + \mathbf{r}_{n2} - \mathbf{r}_{n3} - \mathbf{r}_{n4}$	$f_{n1}\, f_{n2}\, f_{n3}^{\;*}\, f_{n4}^{\;*}$	4

Table 3 - *Coordinates, Complex responses and Multiplicity order of VS for several values of l, for $q = 4$ and for arrays with space, angular and polarization diversities*

$m = 2q$	l	$N_{max}[2q, l]$
4 ($q = 2$)	2	$N(N+1)/2$
	1	N^2
6 ($q = 3$)	3	$N!/[6(N-3)!] + N(N-1) + N$
	2	$N!/[2(N-3)!] + 2N(N-1) + N$
8 ($q = 4$)	4	$N!/[24(N-4)!] + N!/[2(N-3)!] + 1.5N(N-1) + N$
	3	$N!/[6(N-4)!] + 1.5N!/(N-3)! + 3N(N-1) + N$
	2	$N!/[4(N-4)!] + 2N!/(N-3)! + 3.5N(N-1) + N$

Table 4 - $N_{max}[2q, l]$ *as a function of* N *for several values of* q *and* l *and for arrays with space, angular and polarization diversities*

$m = 2q$	l	$N=2$	$N=3$	$N=4$	$N=5$	$N=6$	$N=7$	$N=8$
4 ($q = 2$)	2	3	6	10	15	21	28	36
	1	4	9	16	25	36	49	64
6 ($q = 3$)	3	4	10	20	35	56	84	120
	2	6	18	40	75	126	196	288
8 ($q = 4$)	4	5	15	35	70	126	210	330
	3	8	30	80	175	336	588	960
	2	9	36	100	225	441	784	1296

Table 5 - $N_{max}[2q, l]$ *for several values of* N, q *and* l *and for arrays with space, angular and polarization diversities*

l	VS coordinates	VS multiplicities
2	$2\mathbf{r}_{n1}$	1
	$\mathbf{r}_{n1} + \mathbf{r}_{n2}$	2
1	$\mathbf{r}_0$	N
	$\mathbf{r}_{n1} - \mathbf{r}_{n2}$	1

Table 6 - *Coordinates and Multiplicity order of VS for several values of* l *, for* $q = 2$ *and for arrays with space diversity only*

l	VS coordinates	VS multiplicities
3	$3\mathbf{r}_{n1}$	1
	$\mathbf{r}_{n1} + 2\mathbf{r}_{n2}$	3
	$\mathbf{r}_{n1} + \mathbf{r}_{n2} + \mathbf{r}_{n3}$	6
2	$\mathbf{r}_{n1}$	$2N - 1$
	$2\mathbf{r}_{n1} - \mathbf{r}_{n2}$	1
	$\mathbf{r}_{n1} + \mathbf{r}_{n2} - \mathbf{r}_{n3}$	2

Table 7 - *Coordinates and Multiplicity order of VS for several values of* l *, for* $q = 3$ *and for arrays with space diversity only*

l	VS coordinates	VS multiplicities
4	$4\mathbf{r}_{n1}$	1
	$\mathbf{r}_{n1} + 3\mathbf{r}_{n2}$	4
	$2\mathbf{r}_{n1} + 2\mathbf{r}_{n2}$	6
	$2\mathbf{r}_{n1} + \mathbf{r}_{n2} + \mathbf{r}_{n3}$	12
	$\mathbf{r}_{n1} + \mathbf{r}_{n2} + \mathbf{r}_{n3} + \mathbf{r}_{n4}$	24
3	$2\mathbf{r}_{n1}$	$3N - 2$
	$\mathbf{r}_{n1} + \mathbf{r}_{n2}$	$6N - 6$
	$3\mathbf{r}_{n1} - \mathbf{r}_{n2}$	1
	$2\mathbf{r}_{n1} + \mathbf{r}_{n2} - \mathbf{r}_{n3}$	3
	$\mathbf{r}_{n1} + \mathbf{r}_{n2} + \mathbf{r}_{n3} - \mathbf{r}_{n4}$	6
2	$\mathbf{r}_0$	$N(2N - 1)$
	$\mathbf{r}_{n1} - \mathbf{r}_{n2}$	$4(N - 1)$
	$2\mathbf{r}_{n1} - 2\mathbf{r}_{n2}$	1
	$2\mathbf{r}_{n1} - \mathbf{r}_{n2} - \mathbf{r}_{n3}$	2
	$-2\mathbf{r}_{n1} + \mathbf{r}_{n2} + \mathbf{r}_{n3}$	2
	$\mathbf{r}_{n1} + \mathbf{r}_{n2} - \mathbf{r}_{n3} - \mathbf{r}_{n4}$	4

Table 8 - *Coordinates and Multiplicity order of VS for several values of l, for $q = 4$ and for arrays with space diversity only*

$m = 2q$	l	$N_{max}[2q, l]$
4 ($q = 2$)	2	$N(N+1)/2$
	1	$N^2 - N + 1$
6 ($q = 3$)	3	$N!/[6(N-3)!] + N(N-1) + N$
	2	$N!/[2(N-3)!] + N(N-1) + N$
8 ($q = 4$)	4	$N!/[24(N-4)!] + N!/[2(N-3)!] + 1.5N(N-1) + N$
	3	$N!/[6(N-4)!] + N!/(N-3)! + 1.5N(N-1) + N$
	2	$N!/[4(N-4)!] + N!/(N-3)! + 2N(N-1) + 1$

Table 9 - *$N_{max}[2q, l]$ as a function of N for several values of q and l and for arrays with space diversity only*

$m = 2q$	l	$N = 2$	$N = 3$	$N = 4$	$N = 5$	$N = 6$	$N = 7$	$N = 8$
4 ($q = 2$)	2	3	6	10	15	21	28	36
	1	3	7	13	21	31	43	57
6 ($q = 3$)	3	4	10	20	35	56	84	120
	2	4	12	28	55	96	154	232
8 ($q = 4$)	4	5	15	35	70	126	210	330
	3	5	18	50	115	231	420	708
	2	5	19	55	131	271	505	869

Table 10 - *$N_{max}[2q, l]$ for several values of N, q and l and for arrays with space diversity only*

m = 2q	N = 2	N = 3	N = 4	N = 5	N = 6	N = 7	N = 8
4 (q = 2)	3	5	7	9	11	13	15
6 (q = 3)	4	7	10	13	16	19	22
8 (q = 4)	5	9	13	17	21	25	29

Table 11 - *N_{2q}^{l} for several values of q and N for a ULA*

		N = 3		N = 5		N = 7		N = 9		N = 11	
m = 2q	l	N_{max}	N_{2q}^{l}	N_{max}	N_{2q}^{l}	N_{max}	N_{2q}^{l}	N_{max}	N_{2q}^{l}	N_{max}	N_{2q}^{l}
4 (q = 2)	2	6	6	15	15	28	28	45	45	66	66
	1	7	7	21	21	43	43	73	73	111	111
6 (q = 3)	3	10	10	35	35	84	84	165	163	286	286
	2	12	12	55	55	154	154	333	306	616	616
8 (q = 4)	4	15	15	70	70	210	210	495	477	1001	1001
	3	18	18	115	115	420	420	1125	918	2486	2486
	2	19	19	131	131	505	505	1405	1135	3191	3191

Table 12 - *$N_{max}[2q, l]$ and N_{2q}^{l} associated to a UCA for several values of N, q and l and for arrays with space diversity only*

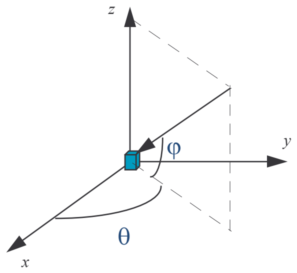
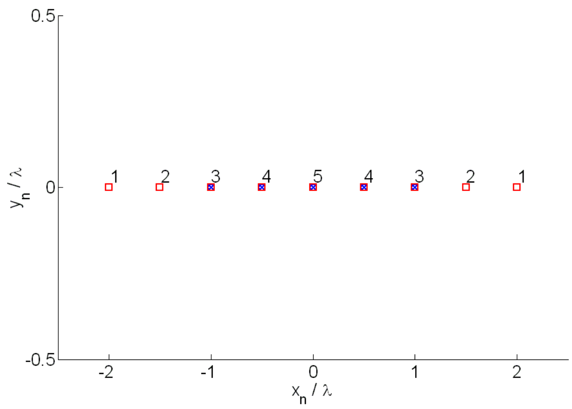
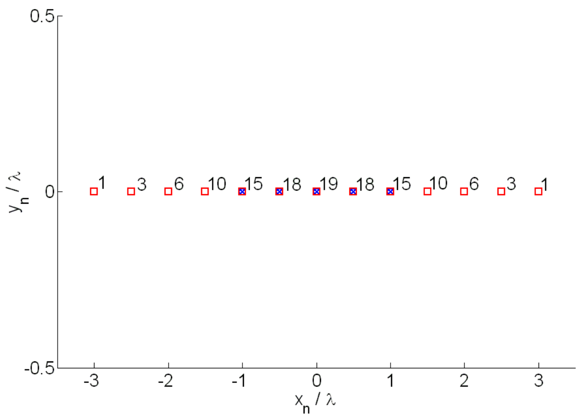


Figure 1 - *An incoming signal in three dimensions*



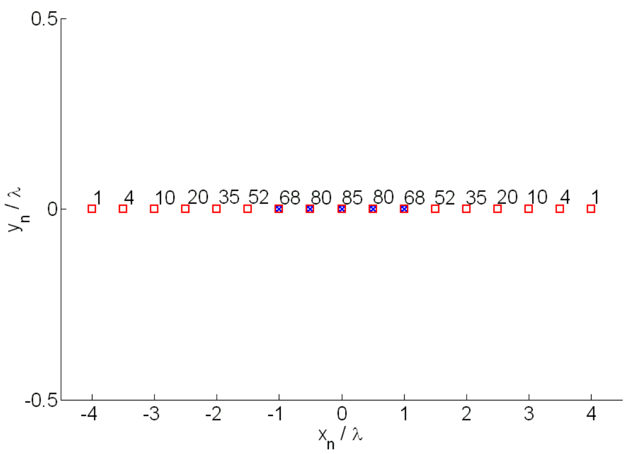
□ : *Virtual Array* x : *Real Array*

Figure 2 - *4^{th}-order VA of a ULA of 5 sensors with the order of multiplicities of the VS*



□ : *Virtual Array* x : *Real Array*

Figure 3 – *6^{th}-order VA of a ULA of 5 sensors with the order of multiplicities of the VS*



□ : *Virtual Array* x : *Real Array*

Figure 4 – *8th-order VA of a ULA of 5 sensors with the order of multiplicities of the VS*

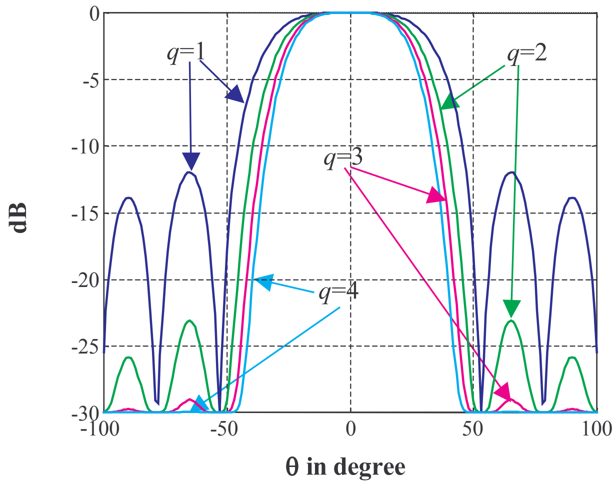
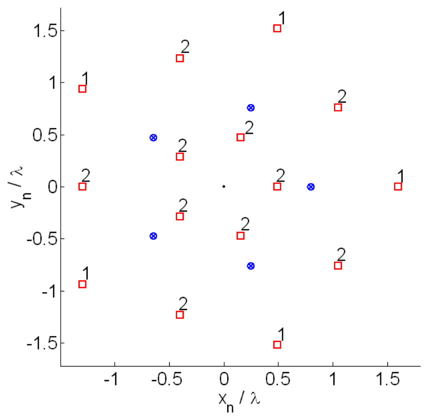
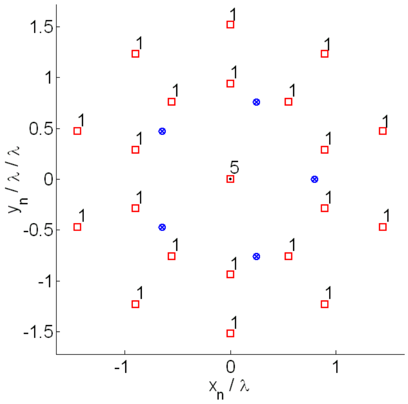


Figure 5 – *VA pattern for q = 1, 2, 3, 4, ULA with 5 sensors, d = λ /2, Pointing direction : 0°*



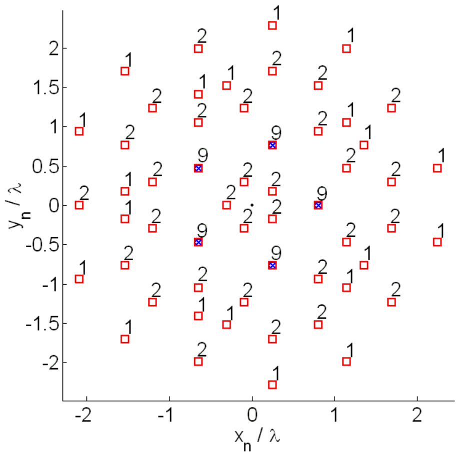
□ : *Virtual Array* ⊗ : *Real Array*

Figure 6 – *4th-order VA of a UCA of 5 sensors with the order of multiplicities of the VS for (q, l) = (2, 2)*



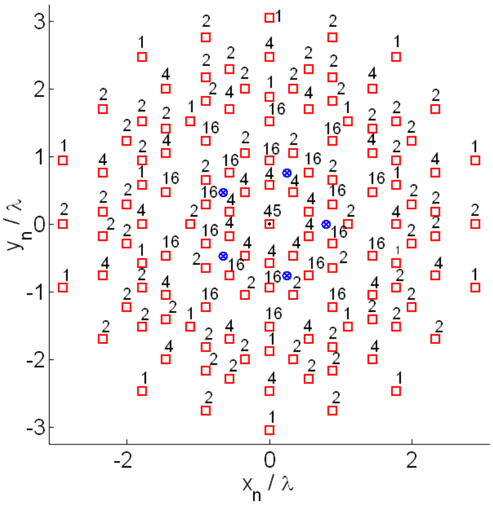
□ : *Virtual Array* ⊗ : *Real Array*

Figure 7 – *4th-order VA of a UCA of 5 sensors with the order of multiplicities of the VS for (q, l) = (2, 1)*



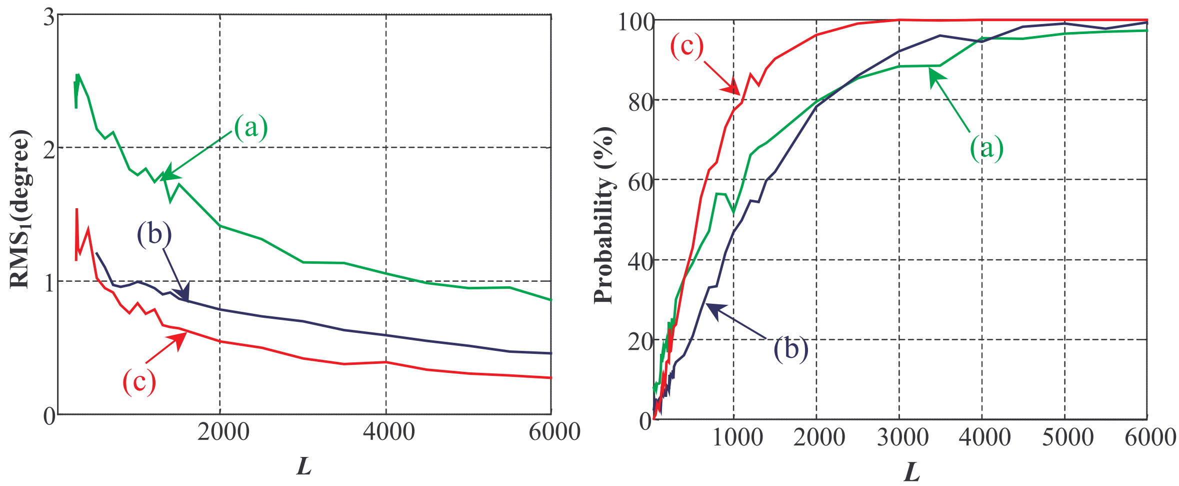
□ : *Virtual Array* ⊗ : *Real Array*

Figure 8 – *6th-order VA of a UCA of 5 sensors with the order of multiplicities of the VS for (q, l) = (3, 2)*



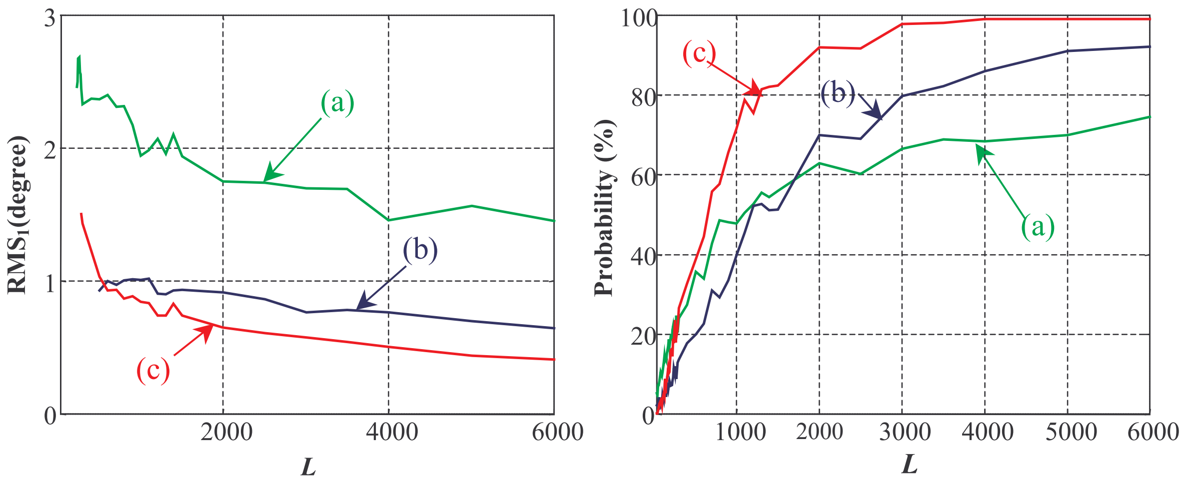
□ : *Virtual Array* ⊗ : *Real Array*

Figure 9 – *8^{th}-order VA of a UCA of 5 sensors with the order of multiplicities of the VS for (q, l) = (4, 2)*



(a) (b)

Figure 10 – *RMS error of the source 1 and $p(\eta_1 \leq \eta)$ as a function of L, (a) 2-MUSIC, (b) 4-MUSIC, (c) 6-MUSIC, P = 2, N = 3, ULA, SNR = 5 dB, θ_1 =90°, θ_2 =82,7°, no modelling errors*



(a) (b)

Figure 11 – *RMS error of the source 1 and $p(\eta_1 \leq \eta)$ as a function of L, (a) 2-MUSIC, (b) 4-MUSIC, (c) 6-MUSIC, P = 2, N = 3, ULA, SNR = 5 dB, θ_1 =90°, θ_2 =82,7°, with modelling errors*

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BIOGRAPHIES

Pascal Chevalier

Pascal Chevalier received the M.Sc degree from Ecole Nationale Supérieure des Techniques Avancées (ENSTA) and the Ph.D. degree from South-Paris University, France, in 1985 and 1991 respectively.

Since 1991 he has shared industrial activities (studies, experimentations, expertises, management), teaching activities both in French engineer schools (Supelec, ENST, ENSTA) and French Universities (Cergy-Pontoise) and research activities. Since 2000, he has also been acting as Technical Manager and Architect of the array processing sub-system as part of a national program of military satellite telecommunications. He is currently a Thalès Expert since 2003.

His present research interests are in array processing techniques, either blind or informed, second order or higher order, spatial-or spatio-temporal, Time-Invariant or Time-Varying especially for cyclostationary signals, linear or non linear and particularly widely linear for non circular signals, for applications such as TDMA and CDMA radiocommunications networks, satellite telecommunications, spectrum monitoring and HF/VUHF passive listening.

Dr Chevalier has been a member of the THOMSON-CSF Technical and Scientifical Council between 1995 and 1998. He co-received the 2003 "Science and Defense" Award from the french Ministry of Defence for its work as a whole about array processing for military radiocommunications. He is author or co-author of about 100 papers (Journal, Conferences, Patents and Chapters of books).

Dr. Chevalier is presently an EURASIP member and a senior member of the Societé des Electriciens et des Electroniciens (SEE).

Laurent Albera

Laurent Albera was born in Massy, France, in 1976. He received from University of Science of Orsay (Paris XI), France, the M.S. degree both in Mathematical Engineering in 2000 and in Signal Processing Engineering in 2001. Currently, he's preparing for the PH.D. degree from University of Science of Nice Sophia-Antipolis, France, in collaboration with both I3S Laboratory and THALES Communications. Since 2000, he has currently worked in

Blind Source Separation and Independent Component Analysis (ICA). His research interests especially include both the cyclostationary source case and the underdetermined mixture problem.

Anne Férréol

Anne Férréol was born in 1964 in Lyon, France. She received the M. Sc degree from ICPI-Lyon and the Mastère degree from Ecole Nationale Supérieure des Télécommunications (ENST) in 1988 and 1989 respectively.

Since 1989 she has been working at Thomson-CSF-Communications, in the array processing department. She is currently preparing for the PH.D. degree from Ecole Normale Supérieure de Cachan, France, in collaboration with both SATIE Laboratory and THALES Communications.

She co-received the 2003 “Science and Defense” Award from the french Ministry of Defence for its work as a whole about array processing for military radiocommunications.

Her current interests concern direction finding and blind source separation.

Pierre Comon

Pierre Comon (M'87 - SM'95) graduated in 1982, and received the Doctorate degree in 1985, both from the University of Grenoble, France. He later received the Habilitation to Lead Researches in 1995, from the University of Nice, France.

He has been for nearly 13 years in industry, first with Crouzet-Sextant, Valence, France, between 1982 and 1985, and then with Thomson Marconi, Sophia-Antipolis, France, between 1988 and 1997. He spent 1987 with the ISL laboratory, Stanford University, CA. He joined in 1997 the Eurecom Institute, Sophia-Antipolis, France, and left during the fall of 1998. He was an Associate Research Director with CNRS from 1994 to 1998. He is now Research Director with CNRS since 1998 at laboratory I3S, Sophia-Antipolis, France. His research interests include High-Order statistics, Blind deconvolution and Equalization, Digital communications, and Statistical Signal and Array Processing.

Dr. Comon was Associate Editor of the IEEE Transactions on Signal Processing from 1995 to 1998, and a member of the French National Committee of Scientific Research from

1995 to 2000. He was the coordinator of the European Basic Research Working Group ATHOS, from 1992 to 1995. Between 1992 and 1998, he was a member of the Technical and Scientific Council of the Thomson Group. Between July 2001 and January 2004 he acted as launching Associate Editor with the IEEE Transactions on Circuits and Systems I, in the area of Blind Techniques. He has also been IEEE Distinguished Lecturer over the period 2002-2003.