ICAR: A Tool for Blind Source Separation Using Fourth-Order Statistics Only

Laurent Albera, Anne Ferréol, Pascal Chevalier, and Pierre Comon, *Senior Member, IEEE*

Abstract—**The problem of blind separation of overdetermined mixtures of sources, that is, with fewer sources than (or as many sources as) sensors, is addressed in this paper. A new method, called Independent Component Analysis using Redundancies in the quadricovariance (ICAR), is proposed in order to process complex data. This method, without any whitening operation, only exploits some redundancies of a particular quadricovariance matrix of the data. Computer simulations demonstrate that ICAR offers in general good results and even outperforms classical methods in several situations: ICAR i) succeeds in separating sources with low signal-to-noise ratios, ii) does not require sources with different second-order or/and first-order spectral densities, iii) is asymptotically not affected by the presence of a Gaussian noise with unknown spatial correlation, iv) is not sensitive to an over estimation of the number of sources.**

Index Terms—**Blind source separation, fourth-order statistics, independent component analysis, overdetermined mixtures.**

I. INTRODUCTION

INDEPENDENT component analysis (ICA) plays an important role in various application areas, including radiocommunications, radar, sonar, seismology, radio astronomy, data analysis, speech [4], and medical diagnosis, [20]. In digital radiocommunications contexts, for instance, if some sources are received by an array of sensors, and if the channel delay spread associated with the different sensors is significantly smaller than the symbol durations for each source, a *static* mixture of complex sources is observed on the sensors. On the other hand, in electrocardiography (ECG), it is possible to record the electrical activity of a fetal heart from ECG recordings measured on the mother's skin. These *cutaneous* recordings can also be considered, in a first approximation, as instantaneous linear mixtures of potential signals generated by underlying bioelectric phenomena [20], hence, again, the *static* model considered.

The goal of blind source separation (BSS) is to restore transmitted sources from the sole observation of sensor data. In some applications, however, sources are not sought, and it is sufficient to identify the (static) mixture. Direction-of-arrival (DOA) estimation problems belong to this class [37] since the column vectors of the mixture contain all the information necessary to determine the location of transmitters. The column vectors of the mixture are the so-called source steering vectors. It is thus legitimate to distinguish between *blind identification* of source mixtures and *blind extraction* of sources; we will go back to this in Section II.

Some algorithms utilize second-order (SO) statistics as the classical principal component analysis (PCA) in factor analysis. In contrast, ICA attempts to restore the independence of outputs using higher order statistics. The consequence is that the indeterminacy is reduced so that ICA allows blind identification of the static mixture, and transmitted sources can eventually be extracted. More precisely, the ICA concept relies on the core assumption that i) sources should be independent in some way. Additionally, when a contrast functional is sought to be maximized, ii) the mixture has to be *overdetermined*, which means that there should be at most as many sources as sensors [40]. In fact, there must exist a linear source separator [15] in the latter framework.

On the other hand, the more general case where there may be more sources than sensors is often referred to as blind identification of *underdetermined* mixtures and is not considered in this paper but is addressed elsewhere; see [3], [7], [17], [19], [22], [25], [35], and references therein.

Since the first paper related to higher order (HO) BSS, published in 1985 [30], many concepts and algorithms have come out. For instance, the ICA concept was proposed a few years later, as well as the maximization of a fourth-order (FO) contrast criterion (subsequently referred to as COM2) [15]. At the same time, a matrix approach was developed in [8] and gave rise to the joint diagonalization (JAD) [9]. A few years later, Hyvarinen *et al.* developed the FastICA method: first for signals with values in the real field [31] and later for complex signals [6], using the fixed-point algorithm to maximize an FO contrast. This algorithm is of deflation type, as is that of Delfosse *et al.* [21], and must extract one source at a time, although some versions of FastICA extract all sources simultaneously. In addition, Comon proposed a simple solution [16], named COM1 in this paper, to the maximization of another FO contrast function previously published in [18], [33], and [36]. Another algorithm of interest is SO blind identification (SOBI), based only on SO statistics, developed independently by several authors in the 1990s and addressed in depth later in [5].

Each of these methods suffers from limitations. To start with, the SOBI algorithm is unable to restore components that have similar spectral densities. Moreover, the JADE method is very sensitive to an overestimation of the number of sources, as shown in the simulation section of this paper and in [2]. Note that in electronic warfare contexts, the number of sources needs to be estimated and may be overestimated, especially for low

signal-to-noise ratios [38], [41]. On the other hand, although the previous methods [6], [8], [15], [16], [31] perform under some reasonable assumptions, they may be strongly affected by a Gaussian noise with unknown spatial correlation, as shown in the simulation section of this paper. Such a noise appears, for instance, in some radiocommunications applications. It is, in particular, the case for ionospheric radiocommunications in the high-frequency (HF) band where the external noise, which is composed of multiple emitting sources (industrial noise, atmospheric noise…) is much stronger than the thermal noise generated by the receivers. In order to deal with the correlated noise problem, Ferréol *et al.* [26] and Abed-Meraim *et al.* [1] have proposed a new family of BSS methods, respectively, exploiting the potential cyclostationarity of the received sources. In fact, the latter family of algorithms uses cyclic statistics of the data. Note that a cyclic covariance matrix associated with a stationary noise is null for nonzero cyclic frequencies. Consequently, these cyclic methods allow the optimal separation of independent cyclostationary sources, even in the presence of a stationary noise with unknown spatial correlation. However, the use of cyclic methods is more complex because of the estimation of cyclic frequencies and time delays. To overcome this drawback, Ferréol *et al.* have recently introduced the FO blind identification of underdetermined mixtures of sources (FOBIUM) algorithm [25], which, without an SO whitening step, performs BSS, even in the presence of a Gaussian noise with unknown spatial correlation. Nevertheless, since FOBIUM is an extension of the SOBI method to FO statistics, it requires sources with different FO spectral densities. FOBIUM also allows us to address the underdetermined case, but this is out of the scope of the present paper.

In order to overcome the limitations of the previous algorithms, the method called Independent Component Analysis using Redundancies in the quadricovariance (ICAR) shortly presented in [2] is proposed in this paper and addresses the case of complex mixture and sources in the presence of additive (possibly spatially correlated) Gaussian noise. Only based on FO statistics, ICAR skips the SO whitening step, in contrast with classical methods [5], [6], [8], [15], [16], [31] and, consequently, is asymptotically not affected by the presence of a Gaussian noise with unknown spatial correlation. Actually, ICAR exploits redundancies in a particular FO statistical matrix of the data, called *quadricovariance*. The latter algorithm assumes sources to have nonzero FO marginal cumulants with the same sign, which is an assumption that is verified in most radiocommunications contexts. Indeed, the kurtosis of most of radiocommunications signals is negative. For example, M-phase shift keying (PSK) constellations have a kurtosis equal to -2 for $M = 2$ and to -1 for $M \geq 4$. Continuous-phase modulations (CPMs), among which we find the GMSK modulation (GSM standard), are such that their kurtosis is smaller than or equal to -1, due to their constant modulus. Furthermore, the performance of ICAR is also analyzed in this paper, in different practical situations through computer simulations, and compared to those of classical algorithms, namely SOBI, COM1, COM2, JADE, FastICA, and FOBIUM. It appears that ICAR exhibits good results in most cases, even when classical methods fail.

The paper is organized as follows. Section II introduces the BSS problem and assumptions needed in ICAR. Section III defines the SO and FO statistics considered in the paper, and Section IV describes the ICAR concept in detail. Computer results are reported in Section V. Section VI concludes the paper.

II. ASSUMPTIONS AND PROBLEM FORMULATION

A noisy mixture of P statistically independent narrowband (NB) sources $s_p(k)$ is assumed to be received by an array of N sensors. In accordance with the usual practice [34], only complex envelopes of NB signals are considered. The vector of complex envelopes of the signals at the sensor outputs $\boldsymbol{x}(k)$ is thus given by

$$\boldsymbol{x}(k) = \boldsymbol{A}\,\boldsymbol{s}(k) + \boldsymbol{\nu}(k) \qquad (1)$$

where $\boldsymbol{A}$, $\boldsymbol{s}(k)$, and $\boldsymbol{\nu}(k)$ are the $N \times P$ constant mixing matrix, the $P \times 1$ source with components $s_p(k)$, and $N \times 1$ noise random vectors, respectively. In addition, for any fixed index k, $\boldsymbol{s}(k)$ and $\boldsymbol{\nu}(k)$ are statistically independent. We further assume the following hypotheses:

A1) Vector $\boldsymbol{s}(k)$ is stationary and ergodic[1] with components *a priori* in the complex field and mutually uncorrelated at order 4.

A2) Noise vector $\boldsymbol{\nu}(k)$ is stationary, ergodic, and Gaussian with components *a priori* in the complex field as well.

A3) FO marginal source cumulants, called *kurtosis* (if normalized) and defined in Section III-B, are not null, and all have the same sign.

A4) The mixture matrix $\boldsymbol{A}$ does not contains no null entry.

A5) $\boldsymbol{A}$ is a full column rank matrix.

Note that sources with null kurtosis are tolerated but cannot be seen and processed by ICAR. Such sources will be considered to be noise. Moreover, the second part of A3) will be discussed in Section IV-C1. Assumption A4) is not a strong assumption, in particular, in digital radiocommunications contexts, since it is more than just reasonable to assume the array of sensors in good repair. On the other hand, if the nth sensor is defective, the nth row of $\boldsymbol{A}$ will be null. It is then necessary to erase the contribution of this sensor and to assume that we have $N - 1$ sensor outputs instead of N. As far as the masking phenomenon is concerned, it is more rare and may produce at most one null component in each column of $\boldsymbol{A}$ for arrays with space diversity. Forthcoming works will consist of studying the ICAR robustness with respect to this pathological phenomenon. As far as A5) is concerned, it implies necessarily that $P \leq N$. Under the previous assumptions, the problems addressed in this paper are both the blind identification and the blind extraction of the sources using solely the FO statistics of the data. The goal of blind mixture identification (BMI) is to blindly identify the mixing matrix $\boldsymbol{A}$ to within a trivial matrix $\boldsymbol{T}$; recall that a trivial matrix is of the form $\boldsymbol{\Lambda}\boldsymbol{\Pi}$, where $\boldsymbol{\Lambda}$ is invertible diagonal, and $\boldsymbol{\Pi}$ is a permutation. On the other hand, the goal of blind source extraction (BSE) or BSS is to blindly find a matrix $\boldsymbol{W}$, yielding a $P \times 1$ output vector $\boldsymbol{y}(k) = \boldsymbol{W}^H \boldsymbol{x}(k)$ corresponding to the best estimate $\hat{\boldsymbol{s}}(k)$ of the vector $\boldsymbol{s}(k)$ up to a multiplicative trivial matrix. Superscript $(^H)$ denotes the complex conjugate transpose of a matrix.

---

[1]The *cyclostationary* and *cycloergodic* case is addressed in Section III-D.

III. SO AND FO DATA STATISTICS

A. SO Statistics

The SO statistics considered in the paper are given by

$$C_{i_1,\boldsymbol{x}}^{i_2}(k) = Cum\{x_{i_1}(k), x_{i_2}(k)^*\}. \tag{2}$$

Function (2) is well known as the SO *cumulant* of $\boldsymbol{x}(k)$. Consequently, the SO *marginal cumulant* of source $s_p(k)$ is defined by

$$C_{p,\boldsymbol{s}}^{p}(k) = Cum\{s_p(k), s_p(k)^*\}. \tag{3}$$

B. FO Statistics

The FO statistics considered in the paper are given by

$$C_{i_1,i_2,\boldsymbol{x}}^{i_3,i_4}(k) = Cum\{x_{i_1}(k), x_{i_2}(k), x_{i_3}(k)^*, x_{i_4}(k)^*\} \tag{4}$$

where two terms $x_i(k)$ are not conjugate, and two terms are conjugate. Function (4) is well known as the FO *cumulant* of $\boldsymbol{x}(k)$. Consequently, the FO *marginal cumulant* of source $s_p(k)$ is defined by

$$C_{p,p,\boldsymbol{s}}^{p,p}(k) = Cum\{s_p(k), s_p(k), s_p(k)^*, s_p(k)^*\}. \tag{5}$$

Likewise, the *kurtosis* of source $s_p(k)$ is given by

$$\kappa_{p,p,\boldsymbol{s}}^{p,p}(k) = \frac{C_{p,p,\boldsymbol{s}}^{p,p}(k)}{(\gamma_p(0))^2} \tag{6}$$

where $\gamma_p(0)$ is the variance of source $s_p(k)$. Note that in the presence of stationary sources, SO (2) and FO (4) statistics do not depend on time k; therefore, they can be denoted by $C_{i_1,\boldsymbol{x}}^{i_2}$ and $C_{i_1,i_2,\boldsymbol{x}}^{i_3,i_4}$, respectively.

C. Matrix Arrangement

1) SO and FO Statistical Matrices: SO and FO statistics computed according to (2) and (4) may be arranged in two Hermitian statistical matrices $\boldsymbol{R_x} = \mathcal{C}_{2,\boldsymbol{x}}$ and $\boldsymbol{Q_x} = \mathcal{C}_{4,\boldsymbol{x}}$ of size $N \times N$ and $N^2 \times N^2$, respectively. These matrices are called the *covariance* and the *quadricovariance* of $\boldsymbol{x}(k)$, respectively. We limit ourselves to arrangements of SO and FO statistics that give different results in terms of the maximum number of processed sources at the output of the BSS methods. The impact of the chosen way to arrange statistics in a matrix is analyzed in [12]. It is shown in [12], through extensions of the Virtual Array concept initially introduced in [23] and [14] for the FO data statistics, that there exists an optimal arrangement of the FO cumulants in a quadricovariance matrix with respect to the maximal number of statistically independent sources to be processed by a method exploiting the algebraic structure of this quadricovariance. As far as SO statistics are concerned, there is a unique nonredundant way to store them in a matrix $\boldsymbol{R_x}$ under constraints of hermicity. Consider, indeed, the following arrangement:

$$R_{\boldsymbol{x}}(i_1, i_2) = C_{i_1,\boldsymbol{x}}^{i_2} \tag{7}$$

where $R_{\boldsymbol{x}}(i_1, i_2)$ is the (i_1, i_2)th entry of matrix $\boldsymbol{R_x}$; the other possible arrangement $R_{\boldsymbol{x}}'(i_2, i_1) = C_{i_1,\boldsymbol{x}}^{i_2}$ just leads to $\boldsymbol{R_x^H}$ and,

hence, to the same result in terms of the maximum number of processed sources. On the other hand, there are two distinct nonredundant ways associated with FO statistics under constraints of hermicity, which can be indexed by the integer ℓ ($\ell \in \{0, 1\}$). Each way yields a statistical matrix $\mathcal{C}_{4,\boldsymbol{x}}^{\ell}$ such that its (I_1^{ℓ}, I_2^{ℓ})th entry ($1 \le I_1^{\ell}, I_2^{\ell} \le N^2$) is given by

$$\mathcal{C}_{4,\boldsymbol{x}}^{\ell}\left(I_1^{\ell}, I_2^{\ell}\right) = C_{i_1,i_2,\boldsymbol{x}}^{i_3,i_4} \tag{8}$$

where for any ℓ belonging to $\{0, 1\}$ and for all i_1, i_2, i_3, i_4 ($1 \le i_1, i_2, i_3, i_4 \le N$)

$$I_1^{\ell} = \begin{cases} i_2 + N(i_1 - 1), & \text{if } \ell = 0 \\ i_4 + N(i_1 - 1), & \text{if } \ell = 1 \end{cases} \tag{9}$$

and

$$I_2^{\ell} = \begin{cases} i_4 + N(i_3 - 1), & \text{if } \ell = 0 \\ i_2 + N(i_3 - 1), & \text{if } \ell = 1 \end{cases}. \tag{10}$$

Note that the optimal arrangement is shown in [12] to correspond to $\ell = 1$, and for this reason, we consider this arrangement in the following sections. Therefore, matrices $\mathcal{C}_{4,\boldsymbol{s}}^{1}, \mathcal{C}_{4,\boldsymbol{x}}^{1}$ will be denoted by $\boldsymbol{Q_s}$ and $\boldsymbol{Q_x}$, respectively.

Remark 1: Another perhaps more intuitive (especially for readers familiar with Matlab) way to present the construction of $\boldsymbol{Q_x}$ is the following: First, construct a four-dimensional (4-D) tensor $\boldsymbol{T}$, whose elements are given by

$$\boldsymbol{T}(i_4, i_1, i_2, i_3) = C_{i_1,i_2,\boldsymbol{x}}^{i_3,i_4}.$$

The matrix $\boldsymbol{Q_x}$ is then given by a simple Matlab reshape operation as follows:

$$\boldsymbol{Q_x} = reshape(\boldsymbol{T}, N^2, N^2).$$

2) Multilinearity Property: The SO and FO statistical matrices of the data $\boldsymbol{R_x}$ and $\boldsymbol{Q_x}$ have a special structure, due to the multilinearity property under change of coordinate systems, which is enjoyed by all moments and cumulants. Since sources and noise are independent, this property can be expressed, for SO statistical matrices and according to (7), by

$$\boldsymbol{R_x} = \boldsymbol{A}\boldsymbol{R_s}\boldsymbol{A}^H + \boldsymbol{R_\nu}. \tag{11}$$

Similarly, according to (8)–(10), and since noise is Gaussian and independent of sources, the FO cumulant matrix can be expressed as follows, using the multilinearity property associated with $\ell = 1$:

$$\boldsymbol{Q_x} = [\boldsymbol{A} \otimes \boldsymbol{A}^*]\boldsymbol{Q_s}[\boldsymbol{A} \otimes \boldsymbol{A}^*]^H. \tag{12}$$

The $P \times P$ matrix $\boldsymbol{R_s}$ and the $P^2 \times P^2$ matrix $\boldsymbol{Q_s}$ are the SO and FO statistical matrices of $\boldsymbol{s}(k)$, respectively. $\boldsymbol{R_\nu}$ denotes the $N \times N$ SO statistical matrix of $\boldsymbol{\nu}(k)$.

D. Statistical Estimation

In practical situations, SO and FO statistics have to be estimated from components of $\boldsymbol{x}(k)$. If components are stationary and ergodic, sample statistics may be used to estimate (2) and (4). Nevertheless, if sources are cyclostationary, cycloergodic,

and potentially nonzero-mean, SO and FO continuous time average statistics have to be used instead of (2) and (4), such as

$$C_{i_1, \boldsymbol{x}}^{i_2} = \left\langle C_{i_1, \boldsymbol{x}}^{i_2}(k) \right\rangle_c \tag{13}$$

and

$$C_{i_1, i_2, \boldsymbol{x}}^{i_3, i_4} = \left\langle C_{i_1, i_2, \boldsymbol{x}}^{i_3, i_4}(k) \right\rangle_c \tag{14}$$

where $\langle \cdot \rangle_c$ is the continuous time average operation defined by

$$\forall f, \; f : t \longmapsto f(t), \quad \langle f(t) \rangle_c = \lim_{T \to +\infty} \frac{1}{T} \int_{-T/2}^{T/2} f(t) dt. \tag{15}$$

These continuous-time temporal mean statistics need some knowledge on cyclic frequencies of the received signal and are thus computed using, for instance, the unbiased and consistent estimators described in [27], [29], and [28]. Moreover, by ordering these continuous-time temporal mean statistics in matrices $\boldsymbol{R_x}$ and $\boldsymbol{Q_x}$ by means of (7)–(10), respectively, expressions (11) and (12) remain valid.

IV. ICAR METHOD

In this section, we present a new method of BSS named ICAR, which exploits the algebraic structure of an alternative expression of matrix $\boldsymbol{Q_x}$.

A. Matrix Notation

Define a columnwise Kronecker product denoted $\varnothing$ and referred to as the Khatri–Rao product [24], [39]. For any $N \times P$ rectangular matrices $\boldsymbol{G}$ and $\boldsymbol{H}$, the columns of the $N^2 \times P$ matrix $\boldsymbol{G} \varnothing \boldsymbol{H}$ are defined as $\boldsymbol{g}_j \otimes \boldsymbol{h}_j$, where $\otimes$ denotes the usual Kronecker product, if $\boldsymbol{g}_j$ and $\boldsymbol{h}_j$ denote the columns of $\boldsymbol{G}$ and $\boldsymbol{H}$, respectively.

B. Core Equation

The ICAR method exploits several redundancies present in the quadricovariance matrix of the data $\boldsymbol{Q_x}$. Although most BSS algorithms, such as JADE, exploit expression (12), the ICAR method uses an alternative form, which is described by

$$\boldsymbol{Q_x} = [\boldsymbol{A} \varnothing \boldsymbol{A}^*] \, \boldsymbol{\zeta_s} \, [\boldsymbol{A} \varnothing \boldsymbol{A}^*]^H \tag{16}$$

where the $P \times P$ diagonal matrix $\boldsymbol{\zeta_s} = \mathrm{Diag} \left[C_{1,1,\boldsymbol{s}}^{1,1}, C_{2,2,\boldsymbol{s}}^{2,2}, \right.$ $\left. \ldots, C_{P,P,\boldsymbol{s}}^{P,P} \right]$ (i.e., $\forall (p_1, p_2), \; 1 \leq p_1, p_2 \leq P, \; \boldsymbol{\zeta_s}(p_1, p_2) = C_{p_1, p_1, \boldsymbol{s}}^{p_1, p_1}$ if $p_1 = p_2$, 0 otherwise) is full rank, in contrast to $\boldsymbol{Q_s}$ in (12), and where the $N^2 \times P$ matrix $\boldsymbol{A} \varnothing \boldsymbol{A}^*$ is defined by

$$\boldsymbol{A} \varnothing \boldsymbol{A}^* = [\, \boldsymbol{a}_1 \otimes \boldsymbol{a}_1^* \;\; \boldsymbol{a}_2 \otimes \boldsymbol{a}_2^* \;\; \cdots \;\; \boldsymbol{a}_P \otimes \boldsymbol{a}_P^* \,] \tag{17}$$

and can be written as

$$\boldsymbol{A} \varnothing \boldsymbol{A}^* = \left[\, [\boldsymbol{A}^* \boldsymbol{\Phi}_1]^T \;\; [\boldsymbol{A}^* \boldsymbol{\Phi}_2]^T \;\; \cdots \;\; [\boldsymbol{A}^* \boldsymbol{\Phi}_N]^T \,\right]^T \tag{18}$$

with

$$\boldsymbol{\Phi}_n = \mathrm{Diag}[\, A(n,1), \, A(n,2), \, \ldots, \, A(n,P) \,]. \tag{19}$$

In other words, the nonzero elements of the $P \times P$ diagonal matrix $\boldsymbol{\Phi}_n$ are the components of the nth row of matrix $\boldsymbol{A}$. In addition, note that (16) can be easily derived from (12). Indeed,

the latter equation straightly implies, in view of the structure of the diagonal noninversible matrix $\boldsymbol{Q_s}$, that the only column vectors of matrix $\boldsymbol{A} \otimes \boldsymbol{A}^*$, which generate matrix $\boldsymbol{Q_x}$, are $\boldsymbol{a}_p \otimes \boldsymbol{a}_p^*$ $(1 \leq p \leq P)$; hence, we have the result (16).

C. ICAR Concept

The algorithm proposed proceeds in three stages. First, a unitary matrix $\boldsymbol{V}$ is estimated in the Least Square (LS) sense and allows the estimation of $\boldsymbol{A} \varnothing \boldsymbol{A}^*$ from $\boldsymbol{Q_x}$ (16). In a second stage, several algorithms may be thought of in order to compute an estimate of $\boldsymbol{A}$ from $\boldsymbol{A} \varnothing \boldsymbol{A}^*$. Finally, an estimation of sources $\boldsymbol{s}(k)$ is computed using the estimate of $\boldsymbol{A}$.

1) Identification of $\boldsymbol{A} \varnothing \boldsymbol{A}^*$: Matrix $\boldsymbol{A} \varnothing \boldsymbol{A}^*$ is an unobservable square root of $\boldsymbol{Q_x}$ to within a diagonal matrix, as shown by (16). In this context, the idea is to build an observable square root $\boldsymbol{Q_x}^{1/2}$, of $\boldsymbol{Q_x}$, differing from $[\boldsymbol{A} \varnothing \boldsymbol{A}^*] \boldsymbol{\zeta_s}^{1/2}$ only by a unitary matrix $\boldsymbol{V}$ and then to identify the latter from the exploitation of the algebraic structure of $\boldsymbol{Q_x}$. Therefore, consider the following proposition.

Proposition 1: If $\boldsymbol{A}$ is of full column rank and contains no null entries, then the $N^2 \times P$ matrix $\boldsymbol{A} \varnothing \boldsymbol{A}^*$ is full column rank.

The proof is given in Appendix A. Therefore, Proposition 1 and assumption A3) together prove that matrix $\boldsymbol{Q_x}$, which is given by (16), is of rank P. Moreover, assumption A3) and (16) imply that $\boldsymbol{Q_x}$ is positive if the FO marginal source cumulants are positive, which we assume in the following. Thus, a square root of $\boldsymbol{Q_x}$, which is denoted $\boldsymbol{Q_x}^{1/2}$ and defined such that $\boldsymbol{Q_x} = \boldsymbol{Q_x}^{1/2}[\boldsymbol{Q_x}^{1/2}]^H$, may be computed. If the FO marginal source cumulants are negative, matrix $-\boldsymbol{Q_x}$ can be considered instead for computing the square root. In the case where there are terms with a different sign, our derivation can be reformulated in terms of an unknown $\boldsymbol{J}$-unitary matrix[2] $\boldsymbol{V}$ instead of unitary. Then, we deduce from (16) that matrix $[\boldsymbol{A} \varnothing \boldsymbol{A}^*] \boldsymbol{\zeta_s}^{1/2}$ is a natural square root of $\boldsymbol{Q_x}$. Yet, another possibility is to compute this square root via the eigenvalue decomposition (EVD) of $\boldsymbol{Q_x}$, which is given by

$$\boldsymbol{Q_x} = \boldsymbol{E_s} \, \boldsymbol{L_s} \, \boldsymbol{E_s}^H \tag{20}$$

where $\boldsymbol{L_s}$ is the real-valued diagonal matrix of the nonzero eigenvalues of $\boldsymbol{Q_x}$. Since matrix $\boldsymbol{Q_x}$ is of rank P, $\boldsymbol{L_s}$ is of size $P \times P$. Besides, $\boldsymbol{E_s}$ is the $N^2 \times P$ matrix of the associated orthonormalized eigenvectors. Consequently, a square root of $\boldsymbol{Q_x}$ can be computed as

$$\boldsymbol{Q_x}^{1/2} = \boldsymbol{E_s} \, \boldsymbol{L_s}^{1/2} \tag{21}$$

where $\boldsymbol{L_s}^{1/2}$ denotes a square root of $\boldsymbol{L_s}$.

Proposition 2: For a full-rank matrix $\boldsymbol{A} \varnothing \boldsymbol{A}^*$, source kurtoses are not null and have all the same sign if and only if the diagonal elements of $\boldsymbol{L_s}$ are not null and have also the same sign, corresponding to that of the FO marginal source cumulants.

The proof is given in Appendix B. In addition, (34) can be rewritten as

$$\boldsymbol{Q_x}^{1/2} = \boldsymbol{E_s} \, \boldsymbol{L_s}^{1/2} = [\boldsymbol{A} \varnothing \boldsymbol{A}^*] \boldsymbol{\zeta_s}^{1/2} \boldsymbol{V}^H \tag{22}$$

---

[2]A $\boldsymbol{J}$-unitary matrix $\boldsymbol{V}$ is such that $\boldsymbol{V} \boldsymbol{J} \boldsymbol{V}^H = \boldsymbol{J}$, where $\boldsymbol{J}$ is a sign diagonal matrix.

showing the link between $Q_x^{1/2}$ and $A \oslash A^*$. Plugging (18) into (22), matrix $Q_x^{1/2}$ can be eventually rewritten as

$$Q_x^{1/2} = \left[[A^* \Phi_1 \zeta_s^{1/2} V^H]^T \cdots [A^* \Phi_N \zeta_s^{1/2} V^H]^T \right]^T = \left[\Gamma_1^T \cdots \Gamma_N^T \right]^T \tag{23}$$

where the N matrix blocks Γ_n of size $N \times P$ are given by

$$\forall n, \ 1 \le n \le N, \quad \Gamma_n = A^* \Phi_n \zeta_s^{1/2} V^H. \tag{24}$$

Proposition 3: For any n ($1 \le n \le N$), matrix Γ_n is full column rank.

The proof is given in Appendix C. Using Proposition 3, the pseudo-inverse $\Gamma_n^\sharp$ of the $N \times P$ matrix Γ_n is defined by

$$\forall n, \ 1 \le n \le N, \quad \Gamma_n^\sharp = (\Gamma_n^H \Gamma_n)^{-1} \Gamma_n^H. \tag{25}$$

Then, consider the $N(N-1)$ matrices Θ_{n_1,n_2} in

$$\forall (n_1, n_2), \ 1 \le n_1 \ne n_2 \le N, \quad \Theta_{n_1,n_2} = \Gamma_{n_1}^\sharp \Gamma_{n_2} \tag{26}$$

which can be rewritten, from (24) and (25), as

$$\begin{aligned}\Theta_{n_1,n_2} &= V \zeta_s^{-1/2} \Phi_{n_1}^{-1} \Phi_{n_2} \zeta_s^{1/2} V^H \\ &= V \Phi_{n_1}^{-1} \zeta_s^{-1/2} \zeta_s^{1/2} \Phi_{n_2} V^H = V \Phi_{n_1}^{-1} \Phi_{n_2} V^H \end{aligned} \tag{27}$$

where $\zeta_s^{1/2}$, Φ_{n_1}, Φ_{n_2}, and $D_{n_1,n_2} = \Phi_{n_1}^{-1} \Phi_{n_2}$ are $P \times P$ diagonal full-rank matrices [the full-rank character of matrices Φ_{n_1}, Φ_{n_2}, and D_{n_1,n_2} is due to assumption A4)]. It appears from (27) that matrix V jointly diagonalizes the $N(N-1)$ matrices Θ_{n_1,n_2}.

Proposition 4: If A is of full column rank and contains no null entries, then for all pairs (p_1, p_2), $1 \le p_1 \ne p_2 \le P$, at least one pair (n_1, n_2), $1 \le n_1 \ne n_2 \le N$ exists such that $D_{n_1,n_2}(p_1, p_1) \ne D_{n_1,n_2}(p_2, p_2)$.

The proof is given in Appendix D. Under Proposition 4, [5] allows the assertion that if V_{sol} jointly diagonalizes matrices Θ_{n_1,n_2}, then V_{sol} and V are related through $V_{\text{sol}} = VT$, where T is a trivial unitary matrix. Therefore, matrix V_{sol} allows, in accordance with (22), the recovery of $A \oslash A^*$ up to a multiplicative trivial matrix:

$$Q_x^{1/2} V_{\text{sol}} = [A \oslash A^*] \zeta_s^{1/2} T. \tag{28}$$

2) Identification of Mixture A: Three algorithms are proposed in this section to identify A from the estimate $Q_x^{1/2} V_{\text{sol}}$ of $A \oslash A^*$. These algorithms optimize differently the compromise between performance and complexity.

Note that (28) can be rewritten from (18) in the form of N matrix blocks $\Sigma_n = A^* \Phi_n \zeta_s^{1/2} T$ of size $N \times P$ as

$$Q_x^{1/2} V_{\text{sol}} = \left[\Sigma_1^T \ \Sigma_2^T \cdots \Sigma_N^T \right]^T. \tag{29}$$

Therefore, a first approach to estimate A up to a trivial matrix, called ICAR1 in the sequel, consists of merely keeping the matrix block Σ_1^* made up of the N first rows of $Q_x^{1/2} V_{\text{sol}}$ such that

$$\Sigma_1 = A^* \Phi_1 \zeta_s^{1/2} T \tag{30}$$

where Φ_1 and $\zeta_s^{1/2}$ are diagonal matrices, and where T is a unitary trivial matrix.

It is also possible to take into account all the matrix blocks Σ_n^* and to compute their average. This yields a second algorithm, named ICAR2, of higher complexity.

A third algorithm, called ICAR3, is now described, and yields a more accurate solution to the BMI problem: Since matrix $A \oslash A^*$, which is given by (17), has been identified from the previous section by $Q_x^{1/2} V_{\text{sol}}$ to within a trivial matrix, ICAR3 consists first of mapping each $N^2 \times 1$ column vector b_p of $Q_x^{1/2} V_{\text{sol}}$ into an $N \times N$ matrix B_p (the nth column of B_p is made up from the N consecutive entries of vector b_p, between $[N(n-1)+1]$ and Nn) and, second, of diagonalizing each matrix B_p^*.

Proposition 5: For any matrix B_p ($1 \le p \le P$) built from $Q_x^{1/2} V_{\text{sol}}$, there exists a unique column vector a_q ($1 \le q \le P$) of A such that the eigenvector of B_p^* associated with the largest eigenvalue corresponds, up to a scale factor, to a_q.

The proof is given in Appendix E. In addition, the indeterminacy of the norms of columns of A is related to matrices Φ_p, $\zeta_s^{1/2}$, a unitary diagonal matrix (whose product by a permutation matrix gives T), and the way to identify A from matrix $Q_x^{1/2} V_{\text{sol}}$. As far as the permutation indeterminacy is concerned, it is related to matrix T.

3) Extraction of the P Independent Components: Finally, to estimate the signal vector $s(k)$ for any value k, it is sufficient, under A5), to apply a linear filter built from the identified matrix A; such a filter may be the Spatial Matched Filter (SMF) given by [11] $W = R_x^{-1} A$, which is optimal in the presence of decorrelated signals. In practical situations, since matrix A is estimated up to a trivial matrix according to Section IV-C2, neither the order of sources $s(k)$ nor their amplitude can be identified.

D. Implementation of the ICAR Methods

The different steps of the ICAR method are summarized hereafter when K samples of the observations $x(k)$ ($1 \le k \le K$) are available.

```
Step 1 Compute an estimate of FO statis-
tics C^{i3,i4}_{i1,i2,x} from the K samples x(k) and
store them, using the (ℓ = 1)-arrangement,
into matrix Q̂_x, which is an estimate of
Q_x.
Step 2 Compute the EVD of the Hermitian
matrix Q̂_x, and estimate P̂ (the number of
sources) from this EVD. Restrict Q̂_x to
the P̂ principal components: Q̂_x = Ês L̂s Ês^H,
where L̂s is the diagonal matrix of the P̂
eigenvalues of largest modulus, and Ês is
the matrix of the associated eigenvectors.
Step 3 Estimate the sign ε of the diagonal
elements of L̂s.
```

Step 4 Compute a square root matrix $[\epsilon\widehat{Q}_x]^{1/2}$ of $\epsilon\widehat{Q}_x$: $[\epsilon\widehat{Q}_x]^{1/2} = \widehat{E}_s|\widehat{L}_s|^{1/2}$, where $|\cdot|$ denotes the absolute value operator.

Step 5 Compute from $[\epsilon\widehat{Q}_x]^{1/2}$ the N matrices $\widehat{\Gamma}_n$, construct matrices $\widehat{\Theta}_{n_1,n_2} = [\widehat{\Gamma}_{n_1}^{\sharp}\,\widehat{\Gamma}_{n_2}]$ for all (n_1,n_2), $1 \le n_1 \ne n_2 \le N$, and compute $\widehat{V}_{\mathrm{sol}}$, which is an estimate of V_{sol}, from the joint diagonalization of the $N(N-1)$ matrices $\widehat{\Theta}_{n_1,n_2}$; one possible joint diagonalization algorithm may be found in [9].

Step 6 Compute an estimate $\widehat{A}$ of the mixture A from the $N^2 \times P$ matrix $[[\epsilon\widehat{Q}_x]^{1/2}\widehat{V}_{\mathrm{sol}}]$ by either one of the following:

1) (ICAR1) taking the matrix block made up of the N first rows of $[[\epsilon\widehat{Q}_x]^{1/2}\widehat{V}_{\mathrm{sol}}]^*$;

2) (ICAR2) taking the average of the N matrix blocks, of size $N \times P$, made up of the successive rows of $[[\epsilon\widehat{Q}_x]^{1/2}\widehat{V}_{\mathrm{sol}}]^*$;

3) (ICAR3) taking each column vector $\widehat{b}_p$ of $[[\epsilon\widehat{Q}_x]^{1/2}\widehat{V}_{\mathrm{sol}}]$, remodeling them into $N \times N$ matrices $\widehat{B}_p$, and building the matrix whose pth column vector is the eigenvector of matrix $\widehat{B}_p^*$ associated with the largest eigenvalue.

Step 7 Estimate the signal vector $s(k)$ for any value k by applying to $x(k)$ a linear filter built from $\widehat{A}$, like, for example, the SMF defined by $\widehat{W} = \widehat{R}_x^{-1}\widehat{A}$.

V. COMPUTER RESULTS

In this section, a comparative performance analysis of seven BSS methods (SOBI, COM1, COM2, JADE, FastICA, FOBIUM, and ICAR) in various scenarios is presented. For this purpose, we consider a Uniform Linear Array (ULA) of $N = 4$ sensors, except for Fig. 2, where $N = 2$, equispaced half a wavelength apart [32]. $P = 2$ quadrature phase shift keying (QPSK) sources are linearly modulated with a pulse shape filter corresponding to 1/2-Nyquist filters with a rolloff equal to 0.3 [34]. In addition, the $P = 2$ sources have the same symbol period T and the same signal-to-noise ratio (SNR) equal to 15 dB, except for Fig. 4(a) and (b). The sources are assumed to be well angularly separated, except for Fig. 6, where the other cases are also considered. The source carrier residuals are such that $f_{c1}T_e = 0$, $f_{c2}T_e = 0.65$, except for Fig. 2, where $f_{c2}T_e = 0$. The sample period T_e corresponds to the symbol period T. As a result, the used SO and FO statistics are time invariant so that classical sample estimators may be employed. As far as the background noise is concerned, it is temporally and spatially white, except for Section V-A2. Eventually, the simulation results are averaged over 200 realizations. Note that we resample the sources and the noise between these 200 experiments. On the other hand, the mixing matrix does not change, except for Fig. 6, where its influence on the BSS methods performance is pointed out.

Moreover, the criterion used in this paper, in order to evaluate the performance of BSS algorithms, is the well-known signal-to-interference plus noise matio maximum (SINRM)
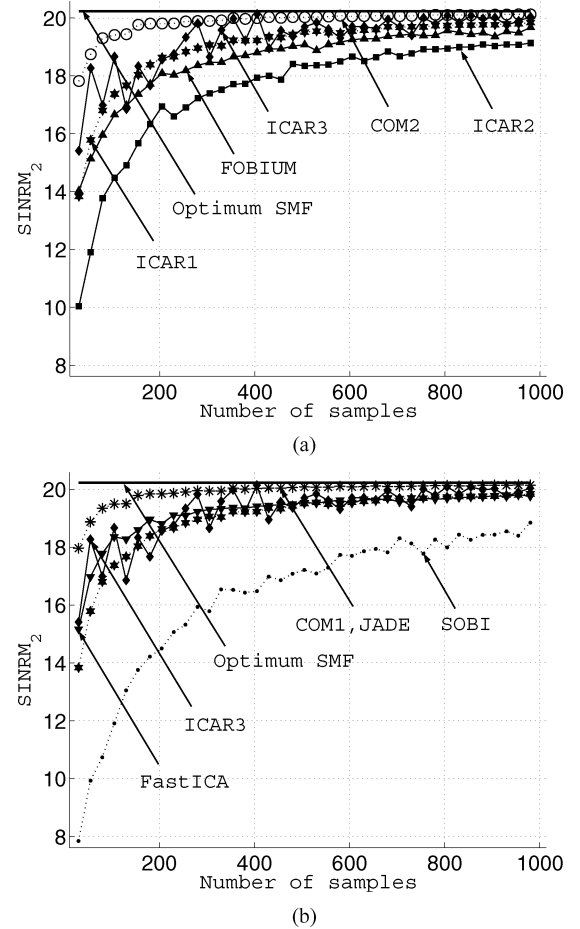

Fig. 1. Behavior of BSS methods in the presence of a white noise. (a) Case I. (b) Case II.

criterion defined in [11, Sec. 3]. In other words, for each source $s_p(k)$ $(1 \le p \le P)$, the SINR for the source p at the output of a spatial filter w_i is defined by

$$\mathrm{SINR}_p[w_i] = \gamma_p(0)\frac{|w_i^H a_p|^2}{w_i^H R_{\nu_p} w_i} \tag{31}$$

where $\gamma_p(0)$ is the variance of the pth source. Moreover, R_{ν_p} is the total noise covariance matrix for source p, corresponding to matrix R_x in the absence of source p. In these conditions, the restitution quality of source p at the output of separator W, whose columns are the w_i, can be evaluated by the maximum value of $\mathrm{SINR}_p[w_i]$ when i varies from 1 to P and may be denoted SINRM_p.

1) White Noise Case: The performance of ICAR at the output of the considered source separator is first illustrated in the presence of a spatially and temporally white Gaussian noise and compared with some well-known BSS algorithms. Fig. 1(a) and (b) show the variations of SINRM_2 (source 2 performance) at the output of the previous methods as a function of the number of samples. Fig. 1(a) and (b) show the good performance of the ICAR algorithm, especially ICAR3 (the third method given in Section IV-C2), facing the well-known SOBI, COM1, COM2, JADE, FastICA, and FOBIUM methods. As for the SOBI method, it requires about 450 snapshots to obtain good results, due to a mild difference between the spectral
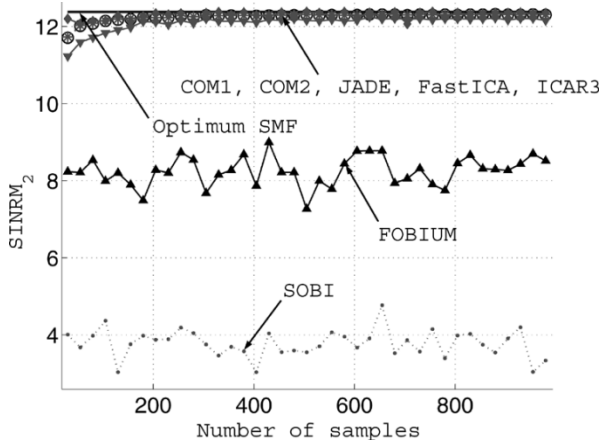
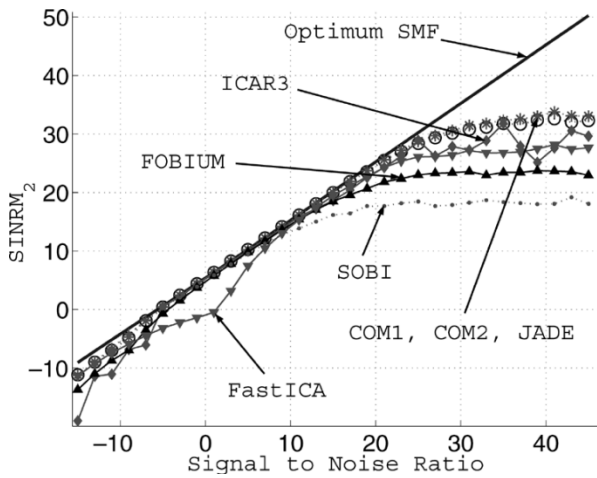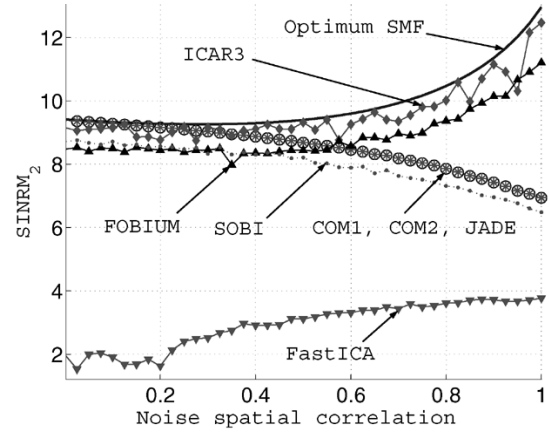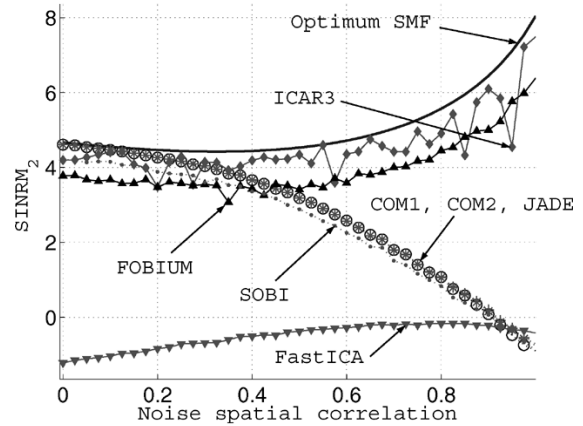Fig. 2. Behavior of BSS methods for sources with identical trispectra.

Fig. 3. Behavior of BSS methods for different SNRs.

Fig. 4. Behavior of BSS methods for a colored noise. (a) Case I: an SNR of 5 dB. (b) Case II: an SNR of 0 dB.

densities of the sources. Note that similar results have been obtained for the other source. In addition, since the best results between the three ICAR methods are obtained, in particular, for ICAR3, we report in the following sections the comparison results only for this third method. Contrary to the other figures, Fig. 2 shows performance results when the two QPSKs are chosen *in baseband*, i.e., taking $f_{c1} T_e = f_{c2} T_e = 0$, which implies that the two source signals have identical trispectra. Consequently, the SOBI and FOBIUM algorithms are unable to separate them correctly. However, we note that the FOBIUM method seems to be more robust than SOBI with respect to a spectrum difference of the sources. Moreover, other simulations have shown that the FOBIUM results are better as quotient N/P increases, even if they remain suboptimal.

Fig. 3 shows, for a number of 400 samples, the variations of SINRM_2 at the output of the previous methods as a function of the input SNR, which are identical for the two sources. All the BSS methods have approximately the same behavior. First, when the SNR is very small, they do not succeed perfectly in extracting the third source. On the contrary, for SNRs between -4 and 20 dB, the source separation is optimal. Finally, although the variations of SINRM_2 for SNRs greater than 20 dB are somewhat surprising, this result has already been observed by Monzingo and Miller in [32] for optimal separators

when mixture A is known. Note that similar results have been obtained for the other source.

2) Colored Noise Case: Then, the ICAR3 method is compared with the other algorithms in the presence of a Gaussian noise with unknown spatial correlation. Fig. 4(a) and (b) shows the variations of SINRM_2 at the output of the previous methods as a function of the noise spatial correlation factor ρ. The SNR of the two sources is taken to be equal to 5 dB [see Fig. 4(a)] and then 0 dB [see Fig. 4(b)]. In addition, 400 samples are used to extract the two sources. Note that the Gaussian noise model employed in this simulation is the sum of an internal noise $\boldsymbol{\nu_{in}}(k)$ and an external noise $\boldsymbol{\nu_{out}}(k)$ of covariance matrices $R_{\boldsymbol{\nu}}^{in}$ and $R_{\boldsymbol{\nu}}^{out}$, respectively, such that

$$R_{\boldsymbol{\nu}}^{in}(r,q) \overset{\text{def}}{=} \frac{\sigma^2 \delta(r-q)}{2} \quad R_{\boldsymbol{\nu}}^{out}(r,q) \overset{\text{def}}{=} \frac{\sigma^2 \rho^{|r-q|}}{2} \quad (32)$$

where σ^2, ρ are the total noise variance per sensor and the noise spatial correlation factor, respectively. Note that $R_{\boldsymbol{\nu}}(r,q) \overset{\text{def}}{=} R_{\boldsymbol{\nu}}^{in}(r,q) + R_{\boldsymbol{\nu}}^{out}(r,q)$ is the (r,q)th component of the total noise covariance matrix. It appears in Fig. 4(a) that FOBIUM and ICAR3 are insensitive to a Gaussian noise with unknown spatial correlation, whereas ICAR3 seems to be a bit more robust than FOBIUM. On the other hand, the well-known COM1, COM2, JADE, and SOBI methods are strongly affected as soon as the
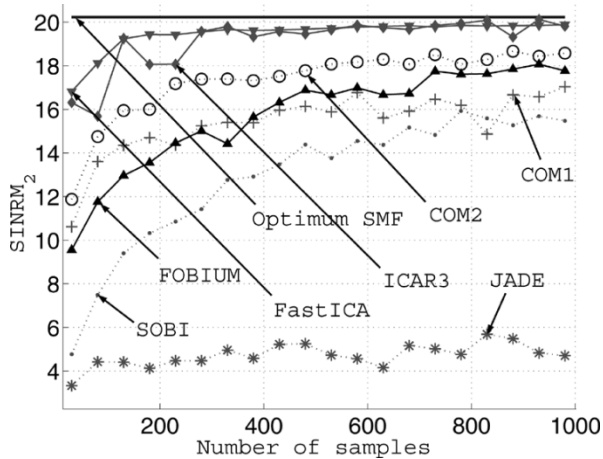
Fig. 5. Behavior of BSS methods for an overestimated number of sources.

Fig. 6. Behavior of BSS methods for different source spatial correlations.

noise spatial correlation increases beyond 0.5. In fact, the classical BSS methods require a prior spatial whitening based on SO moments. This stage theoretically needs the perfect knowledge of the noise covariance. If this is not the case, a whitening of the observed data is performed instead, which is biased. ICAR does not suffer from this drawback since it uses only FO cumulants, which are (asymptotically) insensitive to Gaussian noise, regardless of its space/time color. Note the poor performance of FastICA due to the presence of weak sources. Besides, similar results have been observed for source 1. As far as Fig. 4(b) is concerned, it confirms the fact that the performance differences between ICAR3 and the classical BSS methods increases as the source SNR decreases.

3) Overestimation of the Number of Sources: On the other hand, in operational contexts, the number of sources may be overestimated. It is then interesting to compare the ICAR method with other algorithms in such situations. To this aim, we assume that the estimated number of sources is equal to $\widehat{P} = 3$. Fig. 5 shows the variations of $SINRM_2$ (source 2 performance) at the output of the previous methods as a function of the number of samples, whereas the input SNR of the two sources is assumed to be equal to 15 dB. Similar results have been observed for source 1. More particularly, it appears that the FastICA and ICAR3 methods are robust with respect to an overestimation of the number of sources, whereas, in this simulation configuration, the JADE algorithm loses 15 dB, for less than 1000 samples, with respect to the case where $\widehat{P} = 2$. As for the other methods, such as the FOBIUM algorithm, they are also affected by this overestimation, but less than the JADE algorithm since they lose, on average, 3 dB. The explanation of this surprising phenomenon is not easy and is beyond the scope of this paper. However, a similar behavior had been observed in [10] and [13] when comparing JADE and COM algorithms. The lack of robustness of JADE stems from the fact that only a subset of cross-cumulants are minimized, which means that some cross-cumulants are implicitly maximized along with marginal ones.

4) Mixing Matrix Influence: Finally, the performance of the seven BSS methods (SOBI, COM1, COM2, JADE, FastICA,
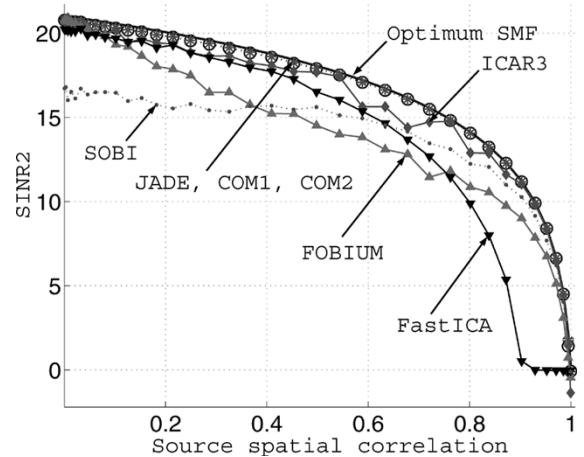
FOBIUM, and ICAR3) are compared for different mixing matrices. Indeed, Fig. 6 shows the variations of $SINRM_2$ at the output of the previous methods as a function of the source spatial correlation $c_{1,2}$, which is defined as the normalized modulus of the scalar product between the two steering vectors, i.e., the two column vectors of matrix $\boldsymbol{A}$:

$$c_{1,2} = \frac{1}{N} \left| \boldsymbol{a}_1^H \boldsymbol{a}_2 \right|. \tag{33}$$

The input SNR of the two sources is assumed to be equal to 15 dB. Similar results have been observed for source 1. More particularly, it appears that ICAR3 presents results that are generally close to the optimum SMF, except for some isolated values. In addition, FastICA seems to be more sensitive to sources that are not angularly separated enough. On the other hand, this simulation section allows the evaluation of the robutness of the previous methods with respect to assumption A5), which is a basic but needed assumption in BSS, as shown in Fig. 6.

VI. CONCLUSION

The ICAR algorithm, exploiting the information contained in the data statistics at fourth order only, has been proposed in this paper. This algorithm allows the processing of overdetermined (including square) mixtures of sources, provided the latter have marginal FO cumulants with the same sign, which is generally the case in radio communications contexts. Three conclusions can be drawn: First, in the presence of a Gaussian noise that is spatially and temporally white, the proposed method yields satisfactory results. Second, contrary to most BSS algorithms, the ICAR method is not sensitive to a Gaussian colored noise, whose spatial coherence is unknown. Last, the ICAR algorithm is robust with respect to an overestimation of the number of sources, which is not the case for some methods such as JADE. Forthcoming works include the search for a contrast criterion associated with ICAR in order to accurately analyze its performance.

APPENDIX A
PROOF OF PROPOSITION 1

The proof of Proposition 1 follows immediately from (18), (19), and assumption A4), i.e., matrix A contains no null entry. In fact, suppose that $A \oslash A^*$ is not full column rank. Then, there exists some $P \times 1$ vector $\beta \neq 0$ such that $[A \oslash A^*]\beta = 0$, which, due to the structure of $A \oslash A^*$ (18) implies that for all $1 \leq n \leq N$, $A^* \Phi_n \beta = 0$. Therefore, it implies that A cannot be full column rank [since matrices Φ_n are $P \times P$ diagonal with nonzero entries, due to (19) and assumption A4)], which contradicts the fact that A is of full column rank A5).

APPENDIX B
PROOF OF PROPOSITION 2

The proof is straightforward. In fact, two square roots of a matrix (here Q_x) are always equal to within a unitary matrix, which yields

$$[A \oslash A^*]\zeta_s^{1/2} = E_s L_s^{1/2} V \qquad (34)$$

for some $P \times P$ unitary matrix V. Equation (34) shows that the right-hand side is the SVD of the left-hand side, hence, the Proposition 2 result, since $E_s^H[A \oslash A^*]\zeta_s[A \oslash A^*]^H E_s = L_s$ is a real positive matrix.

APPENDIX C
PROOF OF PROPOSITION 3

A^* is a full column rank matrix according to A5). The diagonal matrices $\zeta_s^{1/2}$ and Φ_n (note that the diagonal elements of the latter are components of A), are invertible according to A3) and A4), respectively (in other words, because source kurtoses are not null and because matrix A contains no null entry). As far as the square matrix V is concerned, it is invertible because of its unitary structure. Therefore, matrices Γ_n are the product of a full column rank matrix A^* and an invertible matrix $F_n = \Phi_n \zeta_s^{1/2} V^H$. The fact that this particular product is of full column rank remains to be proven. In fact, suppose that $A^* F_n$ is not of full column rank. Then, there exists some $P \times 1$ vector $\beta \neq 0$ such that $A^* F_n \beta = 0$. Therefore, it implies that A^* cannot be full column rank (since matrices F_n are $P \times P$ invertible), which contradicts the first sentence of this section.

APPENDIX D
PROOF OF PROPOSITION 4

If assumptions A4) and A5) are equivalent to assuming that A with no null entries and of full column rank, respectively, then Proposition 4 may be rewritten as

$$A4) + A5) \Rightarrow \{ \forall (p_1, p_2), 1 \leq p_1 \neq p_2 \leq P, \exists (n_1, n_2)$$
$$1 \leq n_1 \neq n_2 \leq N \text{ s.t. } D_{n_1, n_2}(p_1, p_1) \neq D_{n_1, n_2}(p_2, p_2) \} \quad (35)$$

where s.t. means *such that*. To prove assertion (35), assume the contrary:

$$A4) + A5) \Rightarrow \{ \exists (p_1, p_2), 1 \leq p_1 \neq p_2 \leq P, \forall (n_1, n_2)$$
$$1 \leq n_1 \neq n_2 \leq N \text{ s.t. } D_{n_1, n_2}(p_1, p_1) = D_{n_1, n_2}(p_2, p_2) \}. \quad (36)$$

This implies, since $D_{n_1, n_2} = \Phi_{n_1}^{-1} \Phi_{n_2}$ are $P \times P$ diagonal full rank matrices, that $\exists (p_1, p_2), 1 \leq p_1 \neq p_2 \leq P, \forall (n_1, n_2)$, $1 \leq n_1 \neq n_2 \leq N$ s.t.

$$\frac{\Phi_{n_2}(p_1, p_1)}{\Phi_{n_1}(p_1, p_1)} = \frac{\Phi_{n_2}(p_2, p_2)}{\Phi_{n_1}(p_2, p_2)} \qquad (37)$$

which is equivalent, according to (19), to

$$\frac{A(n_2, p_1)}{A(n_1, p_1)} = \frac{A(n_2, p_2)}{A(n_1, p_2)}. \qquad (38)$$

This means that

$$\exists (p_1, p_2), 1 \leq p_1 \neq p_2 \leq P \text{ s.t. } a_{p_1} \propto a_{p_2}. \qquad (39)$$

In other words, assuming (36) under A4) and A5) implies that at least two column vectors of A are collinear, which contradicts A5). Consequently, assertion (35) and, hence, Proposition 4 are true.

APPENDIX E
PROOF OF PROPOSITION 5

Each column b_p of $Q_x^{1/2} V_{\text{sol}}$ is defined, according to (28), by

$$\forall p, 1 \leq p \leq P, \quad b_p = \lambda_{\xi(p)} a_{\xi(p)} \otimes a_{\xi(p)}^* \qquad (40)$$

where $\xi(\cdot)$ is a bijective function of $\{1, 2, \ldots, P\}$ into itself (i.e., a permutation), and where $|\lambda_p| = [|C_{p,p,s}^{p,p}|]^{1/2}$, $|\cdot|$ denotes the complex modulus operator. Therefore, we transform the P vectors b_p of size $N^2 \times 1$ into $N \times N$ matrices B_p, where the (n_1, n_2)th component of B_p corresponds to the $[N(n_2 - 1) + n_1]$th component of b_p such that

$$B_p = \lambda_{\xi(p)} [a_{\xi(p)} a_{\xi(p)}^H]^*. \qquad (41)$$

Note that B_p is a rank one matrix. Consequently, a simple diagonalization of each matrix B_p^* indeed allows the extraction in a unique way up to a scale and permutation factor, of each column vector of A.

REFERENCES

[1] K. Abed-Meraim, Y. Xiang, J. H. Manton, and Y. Hua, "Blind source separation using second-order cyclostationary statistics," *IEEE Trans. Signal Process.*, vol. 49, no. 4, pp. 694–701, Apr. 2001.

[2] L. Albera, A. Ferreol, P. Chevalier, and P. Comon, "ICAR: Independent component analysis using redundancies," in *Proc IEEE Int. Symp. Circuits Syst.*, Vancouver, BC, Canada, May 23–26, 2004.

[3] L. Albera, A. Ferreol, P. Comon, and P. Chevalier, "Sixth order blind identification of underdetermined mixtures (BIRTH) of sources," in *Proc. Fourth Int. Symp. Independent Component Analysis Blind Signal Separation*, Nara, Japan, Apr. 1–4, 2003, pp. 909–914.

[4] F. Asano, S. Ikeda, M. Ogawa, H. Asoh, and N. Kitawaki, "Combined approach of array processing and independent component analysis for blind separation of acoustic signals," *IEEE Trans. Speech Audio Process.*, vol. 11, no. 3, pp. 204–215, May 2003.

[5] A. Belouchrani, K. Abed-Meraim, J.-F. Cardoso, and E. Moulines, "A blind source separation technique using second-order statistics," *IEEE Trans. Signal Process.*, vol. 45, no. 2, pp. 434–444, Feb. 1997.

[6] E. Bingham and H. Hyvarinen, "A fast fixed-point algorithm for independent component analysis of complex valued signals," *Int. J. Neural Syst.*, vol. 10, no. 1, pp. 1–8, 2000.

[7] J.-F. Cardoso, "Super-symetric decomposition of the fourth-order cumulant tensor. Blind identification of more sources than sensors," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Process.*, Toronto, ON, Canada, May 1991, pp. 3109–3112.

[8] J.-F. Cardoso and A. Souloumiac, "Blind beamforming for non-Gaussian signals," *Proc. Inst. Elect. Eng. F*, vol. 140, no. 6, pp. 362–370, Dec. 1993.

[9] ——, "Jacobi angles for simultaneous diagonalization," *SIAM J. Matrix Anal. Appl.*, vol. 17, no. 1, pp. 161–164, 1996.

[10] P. Chevalier, "Performances des séparateurs aveugles de sources aux ordres supérieurs," in *Proc. GRETSI Colloque sur le Traitement du Signal et des Images*, Juan-Les-Pins, France, Sep. 1995.

[11] ——, "Optimal separation of independent narrow-band sources: Concept and performances," in *Signal Process.*, 1999, vol. 73, pp. 27–47.

[12] P. Chevalier, L. Albera, A. Ferreol, and P. Comon, "On the virtual array concept for higher order array processing," *IEEE Trans. Signal Process.*, vol. 53, no. 4, pp. 1254–1271, Apr. 2005.

[13] P. Chevalier, V. Capdevielle, and P. Comon, "Performance of HO blind source separation methods: Experimental results on ionospheric HF links," in *First Int. Symp. Independent Component Analysis Blind Signal Separation*, Aussois, France, Jan. 1999, pp. 443–448.

[14] P. Chevalier and A. Ferreol, "On the virtual array concept for the fourth-order direction finding problem," *IEEE Trans. Signal Process.*, vol. 47, no. 9, pp. 2592–2595, Sep. 1999.

[15] P. Comon, "Independent component analysis, A new concept ?," *Signal Process.*, vol. 36, no. 3, pp. 287–314, Apr. 1994.

[16] ——, "From source separation to blind equalization, contrast-based approaches," in *Proc. Int. Conf. Image Signal Process.*, Agadir, Morocco, May 3–5, 2001, pp. 20–32.

[17] ——, "Blind identification and source separation in 2×3 under-determined mixtures," *IEEE Trans. Signal Process.*, vol. 52, no. 1, pp. 11–22, Jan. 2004.

[18] P. Comon and E. Moreau, "Improved contrast dedicated to blind separation in communications," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Process.*, Munich, Germany, Apr. 20–24, 1997, pp. 3453–3456.

[19] L. De Lathauwer, P. Comon, B. De Moor, and J. Vandewalle, "ICA algorithms for 3 sources and 2 sensors," in *Proc. IEEE Signal Process. Workshop Higher-Order Statist.*, Caesarea, Israel, Jun. 14–16, 1999, pp. 116–120.

[20] L. De Lathauwer, B. De Moor, and J. Vandewalle, "Fetal electrocardiogram extraction by blind source subspace separation," *IEEE Trans. Biomed. Eng., Special Section Adv. Statist. Signal Process. Biomed.*, vol. 47, no. 5, pp. 567–572, May 2000.

[21] N. Delfosse and P. Loubaton, "Adaptive blind separation of independent sources: A deflation approach," in *Signal Process.*, 1995, vol. 45, pp. 59–83.

[22] K. I. Diamantaras, "Blind separation of multiple binary sources using a single linear mixture," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Process.*, Istanbul, Turkey, Jun. 5–9, 2000, pp. 2889–2892.

[23] M. C. Dogan and J. M. Mendel, "Applications of cumulants to array processing — Part I: Aperture extension and array calibration," *IEEE Trans. Signal Process.*, vol. 43, no. 5, pp. 1200–1216, May 1995.

[24] J. Eriksson and V. Koivunen, "Identifiability, separability and uniqueness of linear ICA models," *IEEE Signal Process. Lett.*, vol. 11, no. 7, pp. 601–604, Jul. 2004.

[25] A. Ferreol, L. Albera, and P. Chevalier, "Fourth order blind identification of underdetermined mixtures of sources (FOBIUM)," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Process.*, Hong Kong, Apr. 6–10, 2003, pp. 41–44.

[26] A. Ferreol and P. Chevalier, "Higher order blind source separation using cyclostationarity property of the signals," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Process.*, Munich, Germany, Apr. 21–24, 1997, pp. 4061–4064.

[27] ——, "On the behavior of current second and higher order blind source separation methods for cyclostationary sources," *IEEE Trans. Signal Process.*, vol. 48, no. 6, pp. 1712–1725, Jun. 2000.

[28] A. Ferreol, P. Chevalier, and L. Albera, "Higher order blind separation of non zero-mean cyclostationary sources," in *Proc. Eur. Signal Process. Conf.*, vol. 5, Toulouse, France, Sep. 3–6, 2002, pp. 103–106.

[29] ——, "Second order blind separation of first and second order cyclostationary sources — Application to AM, FSK, CPFSK and deterministic sources," *IEEE Trans. Signal Process.*, vol. 52, no. 4, pp. 845–861, Apr. 2004.

[30] J. Herault, C. Jutten, and B. Ans, "Détection de grandeurs primitives dans un message composite par une architecture de calcul neuromimétique en apprentissage non supervisé," in *Proc. Dixième Colloque sur le Traitement du Signal et des Images*, Nice, France, Sep. 1985, pp. 1017–1022.

[31] A. Hyvarinen and E. Oja, "A fast fixed-point algorithm for independent component analysis," *Neural Comput.*, vol. 9, no. 7, pp. 1483–1492, 1997.

[32] R. A. Monzingo and T. W. Miller, *Introduction to Adaptive Arrays*. New York: Wiley, 1980.

[33] E. Moreau and O. Macchi, "New self-adaptive algorithms for source separation based on contrast functions," in *Proc. IEEE Signal Processing Workshop Higher Order Statistics*, Lake Tahoe, CA, Jun. 1993, pp. 215–219.

[34] J. G. Proakis, *Digital Communications*, Third ed. New York: McGraw-Hill, 1995.

[35] A. Taleb, "An algorithm for the blind identification of N independent signal with 2 sensors," in *Proc. Sixteenth Symp. Signal Process. Applicat.*, Kuala Lumpur, Malaysia, Aug. 13–16, 2001, pp. 5–8.

[36] N. Thirion and E. Moreau, "New criteria for blind signal separation," in *Proc. IEEE Workshop Statistical Signal Array Process.*, Philadelphia, PA, Aug. 2000, pp. 344–348.

[37] M. Viberg and B. Ottersten, "Sensor array processing based on subspace fitting," *IEEE Trans. Signal Process.*, vol. 39, no. 5, pp. 1110–1121, May 1991.

[38] M. Wax and T. Kailath, "Detection of signals by information theoretic criteria," *IEEE Trans. Acoust., Speech, Signal Process.*, vol. ASSP-33, no. 2, pp. 387–392, Apr. 1985.

[39] A. Yeredor, "Non-orthogonal joint diagonalization in the least-squares sense with application in blind source separation," *IEEE Trans. Signal Process.*, vol. 50, no. 7, pp. 1545–1553, Jul. 2002.

[40] L. Zhang, A. Cichocki, and S. Amari, "Natural gradient algorithm to blind separation of overdetermined mixture with additive noises," *IEEE Signal Process. Lett.*, vol. 6, no. 11, pp. 293–295, Nov. 1999.

[41] L.-C. Zhao, P. Krishnaiah, and Z.-D. Bai, "Remarks on certain criteria for detection of number of signals," *IEEE Trans. Acoust., Speech, Signal Process.*, vol. ASSP-35, no. 1, pp. 129–132, Feb. 1987.

Laurent Albera was born in Massy, France, in 1976. He received the DESS degree in mathematics. In 2001, he received the DEA degree in automatic and signal processing from the University of Science (Paris XI), Orsay, France, and the Ph.D. degree in science from the University of Nice, Sophia-Antipolis, France.

He is now Assistant Professor with the University of Rennes I, Rennes, France, and is affiliated with the Laboratoire Traitement du Signal et de l'Image (LTSI). His research interests include high order statistics, multidimensional algebra, blind deconvolution and equalization, digital communications, statistical signal and array processing, and numerical analysis. More exactly, since 2000, he has been involved with blind source separation (BSS) and independent component analysis (ICA) by processing both the cyclostationary source case and the underdetermined mixture identification problem.

Anne Ferréol was born in 1964 in Lyon, France. She received the M.Sc. degree from ICPI-Lyon, Lyon, France, and the Mastère degree from Ecole Nationale Supérieure des Télécommunications (ENST), Paris, France, in 1988 and 1989, respectively. She is currently pursuing the Ph.D. degree with the Ecole Normale Supérieure de Cachan, France, in collaboration with both SATIE Laboratory and THALES Communications.

Since 1989, she has been with Thomson-CSF-Communications, Gennevilliers, France, in the array processing department.

Ms. Ferréol co-received the 2003 "Science and Defense" Award from the French Ministry of Defence for its work as a whole about array processing for military radiocommunications.

Pascal Chevalier received the M.Sc. degree from Ecole Nationale Supérieure des Techniques Avancées (ENSTA), Paris, France, and the Ph.D. degree from South-Paris University in 1985 and 1991, respectively.

Since 1991, he has been with Thalés-Communications, Colonbes, France, where he has shared industrial activities (studies, experimentations, expertise, management), teaching activities both in French engineering schools (Supelec, ENST, ENSTA), and French Universities (Cergy-Pontoise), and research activities. Since 2000, he has also been acting as Technical Manager and Architect of the array processing subsystem as part of a national program of military satellite telecommunications. He has been a Thalés Expert since 2003. His present research interests are in array processing techniques, either blind or informed, second order or higher order, spatial- or spatio-temporal, time-invariant or time-varying, especially for cyclostationary signals, linear or nonlinear, and particularly widely linear for noncircular signals, for applications such as TDMA and CDMA radiocommunications networks, satellite telecommunications, spectrum monitoring, and HF/VUHF passive listening. He is author or co-author of about 100 papers, including journals, conferences, patents, and chapters of books.

Dr. Chevalier was a member of the THOMSON-CSF Technical and Scientific Council from 1993 to 1998. He co-received the "2003 Science and Defense" Award from the French Ministry of Defence for its work as a whole about array processing for military radiocommunications. He is presently a EURASIP member and an emeritus member of the Societé des Electriciens et des Electroniciens (SEE).

Pierre Comon (M'87–SM'95) graduated in 1982, and received the Doctorate degree in 1985, both from the University of Grenoble, Grenoble, France. He later received the Habilitation to Lead Researches in 1995 from the University of Nice, Sophia Antipolis, France.

For nearly 13 years, he has been in industry, first with Crouzet-Sextant, Valence, France, between 1982 and 1985, and then with Thomson Marconi, Sophia-Antipolis, France, between 1988 and 1997. He spent 1987 with the ISL Laboratory, Stanford University, Stanford, CA. He joined the Eurecom Institute, Sophia Antipolis, in 1997 and left durisng the Fall of 1998. He was an Associate Research Director with CNRS, Sophia-Antipolis, from 1994 to 1998. He has been Research Director at Laboratory I3S, CNRS, since 1998. His research interests include high order statistics, blind deconvolution and equalization, digital communications, and statistical signal and array processing.

Dr. Comon was Associate Editor of the IEEE TRANSACTIONS ON SIGNAL PROCESSING from 1995 to 1998 and a member of the French National Committee of Scientific Research from 1995 to 2000. He was the coordinator of the European Basic Research Working Group ATHOS from 1992 to 1995. Between 1992 and 1998, he was a member of the Technical and Scientific Council of the Thomson Group. Between July 2001 and January 2004, he acted as the launching Associate Editor of the IEEE TRANSACTIONS ON CIRCUITS AND SYSTEMS I in the area of Blind Techniques. He was IEEE Distinguished Lecturer from 2002 to 2003.