L2 PCGS - Outils Mathématiques 4

Corrigé du CC2

Exercice 1.

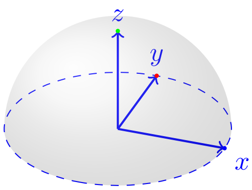
1) $D = \{(r,\theta) \mid 1 \leqslant r \leqslant 2 \text{ et } \theta \in [-\pi, \pi]\}$.

2) $Aire(D) = Aire(D(0,0),2)) - Aire(D(0,0),1)) = 4\pi - \pi = \boxed{3\pi}$

3) le passage en coordonnées polaires nous donne

$$\iint_D x^2\,dxdy = \int_{-\pi}^{\pi}\int_1^2 (r\cos\theta)^2 r dr d\theta = \int_{-\pi}^{\pi}\cos^2\theta d\theta \int_1^2 r^3 dr = \int_{-\pi}^{\pi}\frac{1+\cos(2\theta)}{2}d\theta\int_1^2 r^3 dr = \left[\frac{\theta}{2}+\frac{\sin(2\theta)}{4}\right]_{-\pi}^{\pi}\times\left[\frac{r^4}{4}\right]_1^2 = \boxed{\frac{15\pi}{4}}$$

Exercice 2.

1) 

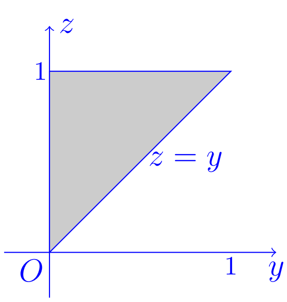
2) $Volume(\Omega) = \frac{1}{2}Volume(B(0,0,0),R)) = \frac{1}{2}\times\frac{4\pi R^3}{3} = \boxed{\frac{2\pi R^3}{3}}$

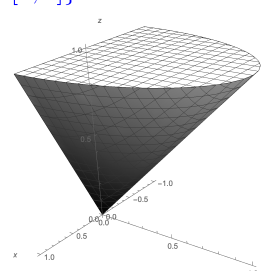
3) En coordonnées sphériques: $\Omega = \{(r,\theta,\phi) \mid \ r\in[0,R], \theta\in[-\pi,\pi] \text{ et } \phi\in[0,\frac{\pi}{2}]\}$, $z^2 = r^2\sin^2(\phi)$ et $dxdydz = r^2\cos(\phi)drd\theta d\phi$. Alors

$$\iiint_\Omega z^2 dxdydz = \int_{-\pi}^{\pi}d\theta\int_0^{\frac{\pi}{2}}\cos(\phi)\sin^2(\phi)d\phi\int_0^R r^4 dr = [\theta]_{-\pi}^{\pi}\times\left[\frac{\sin^3(\phi)}{3}\right]_0^{\frac{\pi}{2}}\times\left[\frac{r^5}{5}\right]_0^R = \boxed{\frac{2\pi R^5}{15}}.$$

Exercice 3.

1) Sur le plan yOz, $x=0$, d'où $y^2 \leqslant z^2$ et puisque $y \geqslant 0$ et $z\in[0,1]$ on aura $0\leqslant y\leqslant z$, ainsi l'intersection est l'ensemble $\{(0,y,z) \mid 0\leqslant y\leqslant z \ \text{ et } z\in[0,1]\}$.





3) En coordonnées cylindriques: $D = \{(r,\theta,z) \ \ 0\leqslant r\leqslant z, z\in[0,1] \text{ et } \theta\in[0,\pi]\}$, $x + y^2 = r\cos(\theta) + r^2\sin^2(\theta)$ et $dxdydz = rdrd\theta dz$ d'où

$$\iiint_D (x+y^2)dxdydz = \int_0^1\left(\int_0^z\left(\int_0^\pi (r^2\cos(\theta)+r^3\sin^2(\theta))d\theta\right)dr\right)dz$$

$$= \int_0^1\left(\int_0^z\left(\int_0^\pi r^2\cos(\theta)+r^3\left(\frac{1-\cos(2\theta)}{2}\right)d\theta\right)dr\right)dz = \int_0^1\left(\int_0^z\left[r^2\sin(\theta)+r^3\left(\frac{\theta}{2}-\frac{\sin(2\theta)}{4}\right)\right]_0^\pi dr\right)dz$$

$$= \frac{\pi}{2}\int_0^1\left(\int_0^z r^3 dr\right)dz = \frac{\pi}{2}\int_0^1\left[\frac{r^4}{4}\right]_0^z dz = \frac{\pi}{2}\int_0^1\frac{z^4}{4}dz = \frac{\pi}{2}\left[\frac{z^5}{20}\right]_0^1 = \boxed{\frac{\pi}{40}}$$

Remarque 0.1 *La fonction $f(x,y,z) = x$ est impaire et D est invariant par $(x,y,z)\to(-x,y,z)$, d'où $\iiint_D xdxdydz = 0$, ainsi $\iiint_D(x+y^2)dxdydz = \iiint_D xdxdydz + \iiint_D y^2dxdydz = \iiint_D y^2dxdydz$.*

Question bonus: En coordonnées cylindriques: $x = x, y = r\cos(\theta), z = r\sin(\theta)$, $dxdydz = rdrd\theta dx$, $\frac{1}{(4-x)^2\sqrt{y^2+z^2}} = \frac{1}{(4-x)^2 r}$ et $\Omega = \{(x, r\cos\theta, r\sin\theta) \mid 1\leqslant r\leqslant\sqrt{4-x}, \quad \theta\in[-\pi,\pi] \text{ et } 0\leqslant x\leqslant 3\}$ d'où

$$\iiint_\Omega \frac{1}{(4-x)^2\sqrt{y^2+z^2}}dxdydz = \int_{-\pi}^{\pi}d\theta\int_0^3\frac{1}{(4-x)^2}\left(\int_1^{\sqrt{4-x}}dr\right)dx = 2\pi\int_0^3\frac{\sqrt{4-x}-1}{(4-x)^2}dx = 2\pi\left[\frac{2}{\sqrt{4-x}}-\frac{1}{(4-x)}\right]_0^3 = \boxed{\frac{\pi}{2}}$$