Eigenvalues and subelliptic estimates for non-selfadjoint semiclassical operators with double characteristics

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Abstract: For a class of non-selfadjoint h–pseudodifferential operators with double characteristics, we give a precise description of the spectrum and establish accurate semiclassical resolvent estimates in a neighborhood of the origin. Specifically, assuming that the quadratic approximations of the principal symbol of the operator along the double characteristics enjoy a partial ellipticity property along a suitable subspace of the phase space, namely their singular space, we give a precise description of the spectrum of the operator in an $\mathcal{O}(h)$–neighborhood of the origin. Moreover, when all the singular spaces are reduced to zero, we establish accurate semiclassical resolvent estimates of subelliptic type, which depend directly on algebraic properties of the Hamilton maps associated to the quadratic approximations of the principal symbol.

Keywords and Phrases: non-selfadjoint operator, eigenvalue, resolvent estimate, subelliptic estimates, double characteristics, singular space, pseudodifferential calculus, Wick calculus, FBI transform, Grushin problem

Contents

1 Introduction

In this work, we are concerned with the analysis of spectral properties for general non-selfadjoint pseudodifferential operators with double characteristics. This study was initiated in [11], and our purpose here is to complement the results of [11] on two essential points, as we describe below. Assume that we are given a non-selfadjoint semiclassical pseudodifferential operator

$$P = P^w(x, hD_x; h),\ 0 < h \leq 1;$$

defined by the semiclassical Weyl quantization of the symbol $P(x, \xi; h)$,

$$P^w(x, hD_x; h)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^{2n}} e^{i(x-y)\cdot\xi} P\Big(\frac{x+y}{2}, h\xi; h\Big) u(y) dy d\xi,$$

with a semiclassical asymptotic expansion

$$P(x, \xi; h) \sim \sum_{j=0}^{+\infty} h^j p_j(x, \xi),$$

such that its principal symbol p_0 has a non-negative real part

$$\text{Re } p_0(X) \geq 0,\ X = (x, \xi) \in \mathbf{R}^{2n},$$

and such that we have a finite number of doubly characteristic points X_0 for the operator,

$$p_0(X_0) = \nabla p_0(X_0) = 0.$$

Our interest is in studying spectral properties and the resolvent growth of the operator P in a fixed neighborhood of the origin. In the previous work [11], we established an accurate semiclassical a priori estimate

$$h|| \, u \, ||_{L^2} \leq C_0|| \, (P - hz)u \, ||_{L^2}, \quad |z| \leq C, \tag{1.1}$$

valid in an $\mathcal{O}(h)$-neighborhood of the origin, when the quadratic approximations q of the principal symbol p_0 at the doubly characteristic points enjoy the partial ellipticity property

$$(x,\xi) \in S, \; q(x,\xi) = 0 \Rightarrow (x,\xi) = 0. \tag{1.2}$$

Here S is a suitable subspace of the phase space, namely the singular space associated to q [10], and the spectral parameter z in (1.1) avoids a discrete set depending on the values of the subprincipal symbol p_1 and the spectra of the quadratic approximations of the principal symbol p_0 at the doubly characteristic points. The a priori estimate (1.1) gives a first localization and bounds on the low lying eigenvalues of the operator P, i.e., when restricting the attention to an $\mathcal{O}(h)$-neighborhood of the origin in the complex spectral plane. In the first part of the present work, we shall push this analysis further and give a precise description of the spectrum of the operator P in an $\mathcal{O}(h)$-neighborhood of the origin, with complete semiclassical asymptotic expansions for the eigenvalues. That such a study is planned by the authors was mentioned in [11].

In the second part of this work, we shall be concerned with the behavior of the resolvent norm of P in a sufficiently small but fixed neighborhood of the origin. We shall actually show that this behavior is linked to subelliptic properties of the quadratic approximations of the principal symbol p_0 at the doubly characteristic points, and that the positive integers k_0 appearing in the resolvent estimates

$$h^{\frac{2k_0}{2k_0+1}}|z|^{\frac{1}{2k_0+1}}\|u\|_{L^2} \leq C_0\|Pu - zu\|_{L^2},$$

depend directly on the loss of derivatives associated to the subelliptic properties of these quadratic operators. We shall show how the positive integers k_0 are intrinsically associated to the structure of the doubly characteristic set, and how they are completely characterized by algebraic properties of the Hamilton maps associated to the quadratic approximations of the principal symbol.

As in [11], the starting point for this work has been the general study of the Kramers-Fokker-Planck type operators carried out by F. Hérau, J. Sjöstrand and C. Stolk in [8]. This study has been a major breakthrough in the understanding of the spectral properties of some general classes of pseudodifferential operators that are neither selfadjoint nor elliptic. We draw our inspiration considerably from this work and use many techniques developed in the analysis of [8]. By using some of these techniques, together with the recent improvements in the understanding of spectral and subelliptic properties of non-elliptic quadratic operators obtained in [10] and [20], here we are able to extend to a large class of non-selfadjoint semiclassical pseudodifferential operators with double characteristics the results proved in [8] for the case of operators of Kramers-Fokker-Planck type.

1.1 Miscellaneous facts about quadratic differential operators

Before giving the precise statement of the main results contained in this article, we shall recall miscellaneous facts and notation concerning quadratic differential operators. Associated to a complex-valued quadratic form

$$\begin{array}{rcl} q : \mathbf{R}_x^n \times \mathbf{R}_\xi^n & \rightarrow & \mathbf{C} \\ (x,\xi) & \mapsto & q(x,\xi), \end{array}$$

with $n \in \mathbf{N}^*$, is the Hamilton map $F \in M_{2n}(\mathbf{C})$ uniquely defined by the identity

$$q\big((x,\xi);(y,\eta)\big) = \sigma\big((x,\xi),F(y,\eta)\big),\ (x,\xi) \in \mathbf{R}^{2n}, (y,\eta) \in \mathbf{R}^{2n}, \tag{1.3}$$

where $q\big(\cdot;\cdot\big)$ stands for the polarized form associated to the quadratic form q and σ is the canonical symplectic form on $\mathbf{R}^{2n}$,

$$\sigma\big((x,\xi),(y,\eta)\big) = \xi.y - x.\eta,\ (x,\xi) \in \mathbf{R}^{2n}, (y,\eta) \in \mathbf{R}^{2n}. \tag{1.4}$$

It follows directly from the definition of the Hamilton map F that its real and imaginary parts, denoted respectively by $\textrm{Re } F$ and $\textrm{Im } F$,

$$\textrm{Re } F = \frac{1}{2}(F+\overline{F}),\ \textrm{Im } F = \frac{1}{2i}(F-\overline{F}),$$

with $\overline{F}$ being the complex conjugate of F, are the Hamilton maps associated to the quadratic forms $\textrm{Re } q$ and $\textrm{Im } q$, respectively; and that a Hamilton map is always skew-symmetric with respect to σ. This fact is just a consequence of the properties of the skew-symmetry of the symplectic form and the symmetry of the polarized form,

$$\forall X,Y \in \mathbf{R}^{2n},\ \sigma(X,FY) = q(X;Y) = q(Y;X) = \sigma(Y,FX) = -\sigma(FX,Y). \tag{1.5}$$

We defined in [10] the singular space S associated to the quadratic symbol q as the following intersection of kernels,

$$S = \Big(\bigcap_{j=0}^{2n-1} \mathrm{Ker}\big[\mathrm{Re}\ F(\mathrm{Im}\ F)^j\big]\Big) \bigcap \mathbf{R}^{2n}, \tag{1.6}$$

where F stands for the Hamilton map of q, and we proved in Theorem 1.2.2 in [10], that when a quadratic symbol q with a non-negative real part is elliptic on its singular space S,

$$(x,\xi) \in S,\ q(x,\xi) = 0 \Rightarrow (x,\xi) = 0, \tag{1.7}$$

then the spectrum of the quadratic operator $q^w(x,D_x)$ is only composed of eigenvalues of finite multiplicity and is given by

$$\sigma\big(q^w(x,D_x)\big) = \Big\{ \sum_{\substack{\lambda \in \sigma(F),\\ -i\lambda \in \mathbf{C}_+ \cup (\Sigma(q|_S) \setminus \{0\})}} \big(r_\lambda + 2k_\lambda\big)(-i\lambda) : k_\lambda \in \mathbf{N} \Big\}. \tag{1.8}$$

Here r_λ is the dimension of the space of generalized eigenvectors of F in $\mathbf{C}^{2n}$ belonging to the eigenvalue $\lambda \in \mathbf{C}$, and

$$\Sigma(q|_S) = \overline{q(S)} \text{ and } \mathbf{C}_+ = \{z \in \mathbf{C} : \mathrm{Re}\ z > 0\}.$$

It follows from (1.6) that the closure of the range of q along S, $\Sigma(q|_S)$, satisfies $\Sigma(q|_S) \subset i\mathbf{R}$.

Remark. Equivalently, one can describe the singular space as the subset in the phase space where all the Poisson brackets $H_{\mathrm{Im}\, q}^k \mathrm{Re}\ q$, $k \in \mathbf{N}$, are vanishing,

$$S = \{X \in \mathbf{R}^{2n} : H_{\mathrm{Im}\, q}^k \mathrm{Re}\ q(X) = 0, k \in \mathbf{N}\}.$$

The singular space is therefore exactly the set of points X_0 in the phase space where the real part of q under the flow generated by the Hamilton vector field associated to its imaginary part Im q,

$$t \mapsto \mathrm{Re}\ q(e^{tH_{\mathrm{Im}\, q}} X_0),$$

vanishes to an infinite order at $t = 0$. We refer to Section 2 in [10] to find all the arguments needed to establish this second equivalent description of the singular space.

We shall finish this subsection by recalling that quadratic operators with a zero singular space $S = \{0\}$, enjoy noticeable subelliptic properties. Specifically, when $q^w(x, D_x)$ stands for a quadratic operator whose Weyl symbol q has a non-negative real part $\text{Re } q \geq 0$, and a zero singular space $S = \{0\}$, it was established in [20] that it fulfills the subelliptic estimate

$$\big\|\big(\langle(x,\xi)\rangle^{2/(2k_0+1)}\big)^w u\big\|_{L^2} \leq C\big(\|q^w(x,D_x)u\|_{L^2} + \|u\|_{L^2}\big),\ u \in \mathcal{S}(\mathbf{R}^n), \tag{1.9}$$

with a loss of $2k_0/(2k_0+1)$ derivatives, where $\langle(x,\xi)\rangle = (1+|x|^2+|\xi|^2)^{1/2}$ and k_0 stands for the smallest integer $0 \leq k_0 \leq 2n-1$ such that

$$\Big(\bigcap_{j=0}^{k_0} \text{Ker}\big[\text{Re } F(\text{Im } F)^j\big]\Big) \bigcap \mathbf{R}^{2n} = \{0\}.$$

Such a non-negative integer k_0 is well-defined since $S = \{0\}$.

1.2 Statement of the main results

Let us now state the main results contained in this paper. Let $m \geq 1$ be a C^∞ order function on $\mathbf{R}^{2n}$ fulfilling

$$\exists C_0 \geq 1, N_0 > 0,\ m(X) \leq C_0\langle X - Y\rangle^{N_0} m(Y),\ X, Y \in \mathbf{R}^{2n}, \tag{1.10}$$

where $\langle X\rangle = (1+|X|^2)^{\frac{1}{2}}$, and let $S(m)$ be the symbol class

$$S(m) = \big\{a \in C^\infty(\mathbf{R}^{2n}, \mathbf{C}) : \forall \alpha \in \mathbf{N}^{2n}, \exists C_\alpha > 0, \forall X \in \mathbf{R}^{2n},\ |\partial_X^\alpha a(X)| \leq C_\alpha m(X)\big\}.$$

We shall assume in the following, as we may, that m belongs to its own symbol class $m \in S(m)$.

Considering a symbol $P(x, \xi; h)$ with a semiclassical asymptotic expansion in the symbol class $S(m)$,

$$P(x,\xi;h) \sim \sum_{j=0}^{+\infty} h^j p_j(x,\xi), \tag{1.11}$$

with some $p_j \in S(m)$, $j \in \mathbf{N}$, independent of the semiclassical parameter h, such that its principal symbol p_0 has a non-negative real part

$$\text{Re } p_0(X) \geq 0,\ X = (x,\xi) \in \mathbf{R}^{2n}, \tag{1.12}$$

we shall study the operator

$$P = P^w(x, hD_x; h),\ 0 < h \leq 1, \tag{1.13}$$

defined by the h-Weyl quantization of the symbol $P(x, \xi; h)$, that is, the Weyl quantization of the symbol $P(x, h\xi; h)$,

$$P^w(x, hD_x; h)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^{2n}} e^{i(x-y)\cdot\xi} P\Big(\frac{x+y}{2}, h\xi; h\Big) u(y)dyd\xi. \tag{1.14}$$

We shall make the important assumption that $\operatorname{Re} p_0$ is elliptic at infinity in the sense that for some $C > 1$, we have

$$\operatorname{Re} p_0(X) \geq \frac{m(X)}{C}, \quad |X| \geq C. \tag{1.15}$$

The ellipticity assumption (1.15) implies that, for $h > 0$ small enough and when equipped with the domain

$$\mathcal{D}(P) = H(m) := (m^w(x, hD))^{-1} \left(L^2(\mathbf{R}^n)\right),$$

the operator P becomes closed and densely defined on $L^2(\mathbf{R}^n)$. Furthermore, another basic consequence of (1.12) and (1.15) is that when $z \in \operatorname{neigh}(0, \mathbf{C})$, the analytic family of operators

$$P - z : H(m) \to L^2(\mathbf{R}^n),$$

is Fredholm of index 0, for all $h > 0$ small enough — see, e.g., [2]. An application of analytic Fredholm theory allows us then to conclude that the spectrum of P in a small but fixed neighborhood of $0 \in \mathbf{C}$ is discrete and consists of eigenvalues of finite algebraic multiplicity.

We shall assume that the characteristic set of the real part of the principal symbol p_0,

$$(\text{Re } p_0)^{-1}(0) \subset \mathbf{R}^{2n},$$

is finite, so that we may write it as

$$(\text{Re } p_0)^{-1}(0) = \{X_1, ..., X_N\}. \tag{1.16}$$

The sign assumption (1.12) implies in particular that we have

$$d\text{Re } p_0(X_j) = 0,$$

for all $1 \leq j \leq N$, and we shall actually assume that these points are all doubly characteristic for the full principal symbol p_0,

$$p_0(X_j) = dp_0(X_j) = 0,\ 1 \leq j \leq N, \tag{1.17}$$

so that we may write

$$p_0(X_j + Y) = q_j(Y) + \mathcal{O}(Y^3), \tag{1.18}$$

when $Y \to 0$. Here q_j is the quadratic form which begins the Taylor expansion of the principal symbol p_0 at X_j. Notice that the sign assumption (1.12) implies that the complex-valued quadratic forms q_j have non-negative real parts,

$$\mathrm{Re}\, q_j \geq 0, \tag{1.19}$$

when $1 \leq j \leq N$. We shall assume throughout the present work that when $1 \leq j \leq N$, the quadratic form q_j is elliptic along the associated singular space S_j introduced in (1.6), in the sense of (1.2).

The following result was established in [11], under the assumptions above: let $C > 1$ and assume that $z \in \mathbf{C}$ with $|z| \leq C$ is such that for all $1 \leq j \leq N$, we have $z - p_1(X_j) \notin \Omega_j$, where $\Omega_j \subset \mathbf{C}$ is a fixed neighborhood of the spectrum of the quadratic operator $q_j^w(x, D_x)$. Then for all $h > 0$ small enough, the following a priori estimate holds,

$$h||\, u\,|| \leq \mathcal{O}(1)||\,(P - hz)u\,||, \quad u \in \mathcal{S}(\mathbf{R}^n). \tag{1.20}$$

Here $||\cdot||$ is the L^2–norm on $\mathbf{R}^n$. In view of the observations made above, we see that the estimate (1.20) extends to all of $\mathcal{D}(P) = H(m)$, since the Schwartz space $\mathcal{S}(\mathbf{R}^n)$ is dense in the latter. The operator $P - hz : H(m) \to L^2(\mathbf{R}^n)$ is therefore injective with closed range, and thus invertible, thanks to the Fredholm property. We conclude that when $z \in \mathbf{C}$ is as above, then hz is not an eigenvalue of P and the resolvent estimate

$$(P - hz)^{-1} = \mathcal{O}\left(\frac{1}{h}\right) : L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n) \tag{1.21}$$

holds true.

The following is the first main result of this work.

Theorem 1.1 *Let us make the assumptions* (1.12)*,* (1.15)*,* (1.16)*, and* (1.17)*. Assume furthermore that the quadratic form q_j introduced in* (1.18) *is elliptic along the* *singular space S_j, when $1 \leq j \leq N$. Let $C > 0$. Then there exists $h_0 > 0$ such that for all $0 < h \leq h_0$, the spectrum of the operator P in the open disc in the complex plane $D(0, Ch)$ is given by the eigenvalues of the form,*

$$z_{j,k} \sim h\left(\lambda_{j,k} + p_1(X_j) + h^{1/N_{j,k}}\lambda_{j,k,1} + h^{2/N_{j,k}}\lambda_{j,k,2} + \ldots\right), \; 1 \leq j \leq N. \tag{1.22}$$

Here $\lambda_{j,k}$ are the eigenvalues in $D(0, C)$ of $q_j^w(x, D_x)$ given in (1.8), repeated according to their algebraic multiplicity, and $N_{j,k}$ is the dimension of the corresponding generalized eigenspace. (Possibly after changing $C > 0$, we may assume that $|\lambda_{j,k} + p_1(X_j)| \neq C$ for all k, $1 \leq j \leq N$.)

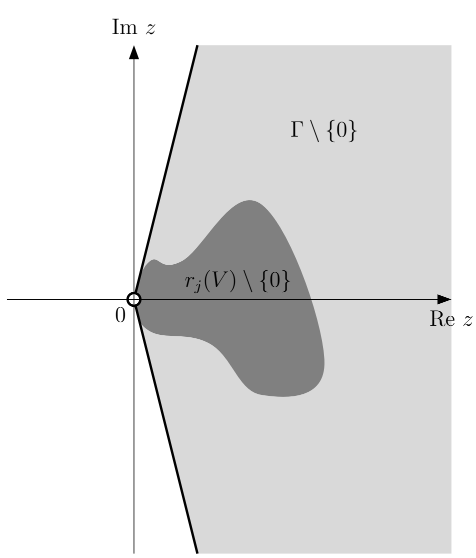
We now come to state the second main result of this work. In doing so, let us introduce the symbols

$$r_j(Y) = p_0(X_j + Y) - q_j(Y), \; 1 \leq j \leq N. \tag{1.23}$$

We shall assume that there exists a closed angular sector Γ with vertex at 0 and a neighborhood V of the origin in $\mathbf{R}^{2n}$ such that for all $1 \leq j \leq N$,

$$r_j(V) \setminus \{0\} \subset \Gamma \setminus \{0\} \subset \{z \in \mathbf{C} : \mathrm{Re}\, z > 0\}. \tag{1.24}$$

Figure 1: The range of r_j.



By denoting F_j the Hamilton maps and S_j the singular spaces associated to the quadratic forms q_j, we shall also assume that all the singular spaces are reduced to zero,

$$S_j = \{0\}, \tag{1.25}$$

when $1 \leq j \leq N$. According to the definition of the singular space (1.6), one can therefore consider the smallest integers, $0 \leq k_j \leq 2n-1$, such that

$$\Big(\bigcap_{l=0}^{k_j} \text{Ker}\big[\text{Re } F_j(\text{Im } F_j)^l\big]\Big) \bigcap \mathbf{R}^{2n} = \{0\}. \tag{1.26}$$

Defining the integer

$$k_0 = \max_{j=1,...,N} k_j, \tag{1.27}$$

in $\{0,...,2n-1\}$, we shall establish the following result:

Theorem 1.2 *Consider a symbol $P(x,\xi;h)$ with a semiclassical expansion in the class $S(m)$ fulfilling the assumptions* (1.12), (1.15), (1.16), (1.17) *and* (1.24). *When all the quadratic forms q_j, $1 \leq j \leq N$, defined in* (1.18) *have zero singular spaces $S_j = \{0\}$, then for any constant $C_0 > 0$ sufficiently small, there exist positive constants $0 < h_0 \leq 1$, $C \geq 1$ and $c_0 > 0$ such that for all $0 < h \leq h_0$, $u \in \mathcal{S}(\mathbf{R}^n)$ and $z \in \Omega_h$,*

$$h^{\frac{2k_0}{2k_0+1}}|z|^{\frac{1}{2k_0+1}}\|u\|_{L^2} \leq c_0\|Pu - zu\|_{L^2}, \tag{1.28}$$

where $P = P^w(x,hD_x;h)$, k_0 is the integer defined in (1.27) *and Ω_h denotes the set*

$$\Omega_h = \Big\{z \in \mathbf{C} : \text{Re } z \leq \frac{1}{C}h^{\frac{2k_0}{2k_0+1}}|z|^{\frac{1}{2k_0+1}},\ Ch \leq |z| \leq C_0\Big\}. \tag{1.29}$$

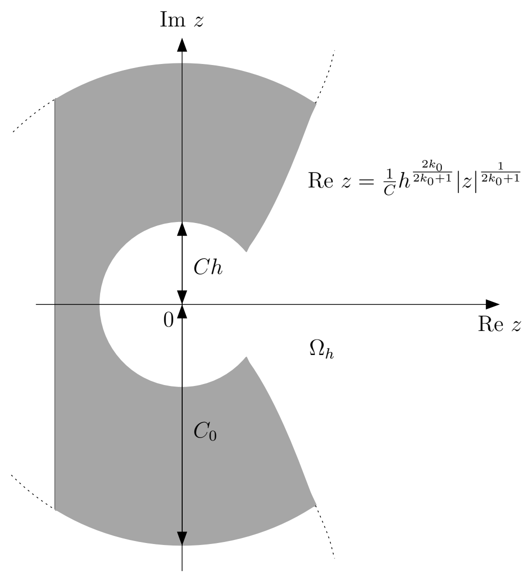
The set Ω_h defined in (1.29) is represented on Figure 2. We may also notice that when $z \in \Omega_h$, then Theorem 1.2 implies that z is in the resolvent set of P, and the resolvent estimate

$$(P-z)^{-1} = \mathcal{O}\left(h^{-\frac{2k_0}{2k_0+1}}\,|z|^{-\frac{1}{2k_0+1}}\right) : L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$$

holds.

Notice that the quantity $h^{\frac{2k_0}{2k_0+1}}|z|^{\frac{1}{2k_0+1}}$, which appears in the estimate (1.28), when $Ch \leq |z| \leq C_0$, increases when the spectral parameter z moves away from the origin at a rate, which depends on the maximal loss of derivatives $2k_0/(2k_0+1)$

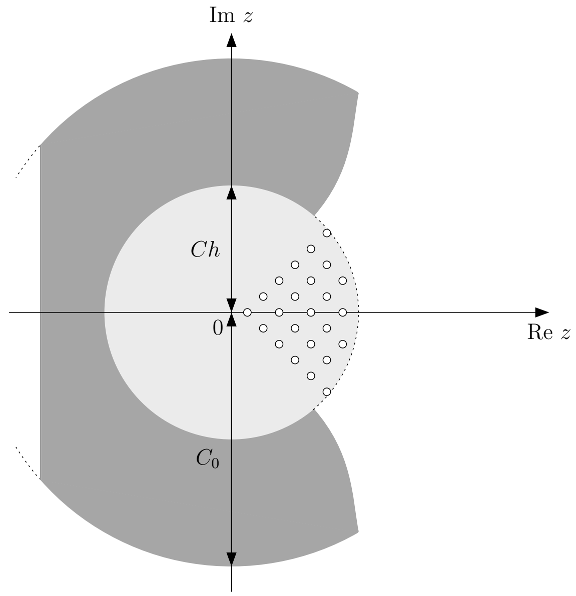
Figure 2: Set Ω_h.



appearing in the subelliptic estimates (1.9), fulfilled by the quadratic approximations of the principal symbol at the doubly characteristic points. When the spectral parameter is of the order of magnitude of h, we recover the semiclassical hypoelliptic a priori estimate (1.20), proved in [11], with a loss of the full power of the semiclassical parameter. Theorem 1.2 and Theorem 1 in [11], together with the description of the spectrum of P, given in Theorem 1.1, give therefore an almost complete picture of the spectral properties and the growth of the resolvent norm of a non-selfadjoint semiclassical pseudodifferential operator with double characteristics fulfilling the assumptions of Theorems 1.2 near the doubly characteristic set. These results underline the basic rôle played by the singular space in the analysis of the general structure of double characteristics.

Coming back to Theorem 1.2, we would like to stress the fact that the non-negative integer k_0 defined in (1.27), $0 \leq k_0 \leq 2n-1$, measuring the maximal loss of derivatives $2k_0/(2k_0+1)$ appearing in the subelliptic estimates (1.9) fulfilled by the quadratic approximations of the principal symbol at doubly characteristic points and the rate of growth of the resolvent norm when the spectral parameter z moves away from the origin in the estimate (1.28); can actually take any value in the set $\{0, ..., 2n-1\}$, when $n \geq 1$. Explicit local models for the quadratic approximations

Figure 3: The estimate $h^{\frac{2k_0}{2k_0+1}}|z|^{\frac{1}{2k_0+1}}\|u\|_{L^2} \leq c\|Pu-zu\|_{L^2}$ is fulfilled when z belongs to the dark grey region of the figure; whereas the estimate $h\|u\|_{L^2} \leq \|Pu - zu\|_{L^2}$ is fulfilled in the light grey one.



of the principal symbol at doubly characteristic points for which the integer k_0 can take any value in the set $\{0, ..., 2n - 1\}$ are given for example by the following symbols:

- Case $k_0 = 0$: According to the definition of the Hamilton map, this is the case of any quadratic symbol q with a positive definite real part Re $q > 0$.

- Case $k_0 = 1$: Consider a Fokker-Plank operator with a nondegenerate quadratic potential tensorized with a harmonic oscillator in other symplectic variables

$$\xi_2^2 + x_2^2 + i(x_2\xi_1 - x_1\xi_2) + \sum_{j=3}^{n}(\xi_j^2 + x_j^2).$$

- Case $k_0 = 2p$, with $1 \leq p \leq n - 1$: Consider

$$\xi_1^2 + x_1^2 + i(\xi_1^2 + 2x_2\xi_1 + \xi_2^2 + 2x_3\xi_2 + ... + \xi_p^2 + 2x_{p+1}\xi_p + \xi_{p+1}^2) + \sum_{j=p+2}^{n}(\xi_j^2 + x_j^2).$$

- Case $k_0 = 2p+1$, with $1 \leq p \leq n-1$: Consider

$$x_1^2 + i(\xi_1^2 + 2x_2\xi_1 + \xi_2^2 + 2x_3\xi_2 + ... + \xi_p^2 + 2x_{p+1}\xi_p + \xi_{p+1}^2) + \sum_{j=p+2}^{n} (\xi_j^2 + x_j^2).$$

We refer the reader to [20] for more details concerning those examples.

Remark. The basic rôle played by conditions of subelliptic type for the understanding of resolvent estimates for non-selfadjoint operators of principal type was first stressed in [2]. See also [18, 19] for specific cases. These results were recently improved by W. Bordeaux Montrieux in a model situation [1] and in the general case by J. Sjöstrand in [24].

In [8], the authors obtain a result analogous to Theorem 1.1 and a resolvent estimate similar to (1.28), in the case when $k_0 = 1$. These results are obtained using assumptions of subelliptic type for the principal symbol of the operator, both locally near the doubly characteristic points, and at infinity. Our analysis does not consider such a general situation where the ellipticity may fail both locally and at infinity. The purpose of the present work, as well as of [11], is to understand deeper the phenomena occurring near the doubly characteristic set, and therefore we simplify parts of the analysis of [8] by requiring a property of ellipticity at infinity (1.15) for the real part of the principal symbol p_0, whereas we weaken the assumptions of subelliptic type at the doubly characteristic points. The assumption of subelliptic type for the principal symbol p_0 of the operator near a doubly characteristic point, say here $X_0 = 0$,

$$\exists \varepsilon_0 > 0, \ \text{Re } p_0(X) + \varepsilon_0 H_{\text{Im} p_0}^2 \text{Re } p_0(X) \sim |X|^2,$$

made in [8], implies (See Section 4 in [11]) that the singular space S associated to the quadratic approximation q of the principal symbol p_0 at $X_0 = 0$ is reduced to $\{0\}$. More specifically, the singular space S is equal to zero after the intersection of exactly two kernels,

$$S = \text{Ker}(\text{Re } F) \cap \text{Ker}\big[\text{Re } F(\text{Im } F)\big] \bigcap \mathbf{R}^{2n} = \{0\}.$$

This explains why the integer k_0 is equal to 1 in the case studied in [8].

In the proof of Theorem 1.1, we rely upon the techniques developed in [8], [10], [11], and similarly to [8], the proof proceeds by solving a globally well-posed Grushin problem for the operator P in a suitable microlocally weighted L^2–space, introduced in [11]. The main technical tool in the first part of the paper is therefore a systematic use of the FBI–Bargmann transformation as well as of the associated weighted spaces of holomorphic functions.

The proof of Theorem 1.2 uses elements of the Wick calculus, whose main features are recalled in the appendix (Section A). This proof also depends crucially on the construction of weight functions performed in [20] (Proposition 2.0.1) for the quadratic approximations of the principal symbol at the doubly characteristic points. The method used in this proof, by starting with weights built for quadratic symbols in order to deal with the general doubly characteristic case, largely accounts for the assumption (1.24). We shall need this assumption in our proof of Theorem 1.2. Nevertheless, this hypothesis may be relevant only technically.

The plan of the paper is as follows. In Section 2, we study quadratic differential operators with quadratic symbols q, elliptic along the associated singular spaces, and derive some Gaussian decay estimates for the generalized eigenfunctions, thereby completing the corresponding discussion in [10]. This study is instrumental in Section 3, devoted to the construction of a globally well-posed Grushin proof for the operator P and to the proof of Theorem 1.1. Theorem 1.2 is established in Section 4. As alluded to above, the proof makes use of some elements of the Wick calculus, and the relevant facts concerning those techniques are reviewed in the appendix.

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2 Gaussian decay of eigenfunctions in the quadratic case

In this section we shall be concerned with a quadratic form q on $\mathbf{R}^{2n}$ such that $\operatorname{Re} q \geq 0$ and with q being elliptic along the associated singular space S, introduced in (1.6). It follows then from [10] (Section 1.4.1) that the singular space $S \subset \mathbf{R}^{2n}$ is symplectic. We have the following decomposition,

$$\mathbf{R}^{2n} = S^{\sigma\perp} \oplus S, \tag{2.1}$$

where $S^{\sigma\perp}$ is the orthogonal space of S with respect to the symplectic form σ in $\mathbf{R}^{2n}$, and let us recall from [10] (Section 2) that we have linear symplectic coordinates

(x', ξ') in $S^{\sigma\perp}$ and (x'', ξ'') in S, respectively, such that if

$$X = (x, \xi) = (X'; X'') = (x', \xi'; x'', \xi'') \in \mathbf{R}^{2n} = \mathbf{R}^{2n'} \times \mathbf{R}^{2n''}, \tag{2.2}$$

then

$$q(x, \xi) = q_1(x', \xi') + iq_2(x'', \xi''), \quad q_1 = q|_{S^{\sigma\perp}}, \quad iq_2 = q|_S. \tag{2.3}$$

We know furthermore from [10] (Proposition 2.0.1) that the symplectic coordinates may be chosen such that the elliptic quadratic form q_2 satisfies

$$q_2(x'', \xi'') = \varepsilon_0 \sum_{j=1}^{n''} \frac{\lambda_j}{2} \left(x_j''^2 + \xi_j''^2 \right), \quad \lambda_j > 0, \quad \varepsilon_0 \in \{\pm 1\}, \tag{2.4}$$

while q_1 enjoys the following averaging property: for each $T > 0$, the quadratic form

$$\langle \operatorname{Re} q_1 \rangle_T (x', \xi') = \frac{1}{T} \int_0^T \operatorname{Re} q_1 \left(\exp\left(tH_{\operatorname{Im} q_1} \right)(x', \xi') \right) \, dt \tag{2.5}$$

is positive definite in (x', ξ'). In what follows, in order to fix the ideas, we take $\varepsilon_0 = 1$ in (2.4).

Following [11] (Section 2), let us introduce the quadratic weight function,

$$G_0(X) = -\int J\left(-\frac{t}{T}\right) \operatorname{Re} q \left(\exp\left(tH_{\operatorname{Im} q} \right)(X) \right) \, dt, \quad T > 0, \tag{2.6}$$

where J is a compactly supported piecewise affine function satisfying

$$J'(t) = \delta(t) - 1_{[-1,0]}(t),$$

and $1_{[-1,0]}$ the characteristic function of the set $[0, 1]$. It follows that

$$H_{\operatorname{Im} q} G_0 = \langle \operatorname{Re} q \rangle_{T, \operatorname{Im} q} - \operatorname{Re} q, \tag{2.7}$$

where

$$\langle \operatorname{Re} q \rangle_{T, \operatorname{Im} q}(X) = \frac{1}{T} \int_0^T \operatorname{Re} q(\exp\left(tH_{\operatorname{Im} q} \right)(X)) \, dt.$$

From (2.3) and (2.4) we see that G_0 is a function of X' only, so that $G_0 = G_0(X')$, $X' = (x', \xi') \in \mathbf{R}^{2n'}$. Following [8] and [11], we shall therefore consider an IR-deformation of the real phase space $S^{\sigma\perp} = \mathbf{R}^{2n'}$, associated to the quadratic weight G_0, viewed as a function on $\mathbf{R}^{2n'}$. Let us set

$$\Lambda_\delta = \{ X' + i\delta H_{G_0}(X'); \, X' \in \mathbf{R}^{2n'} \} \subset \mathbf{C}^{2n'}, \quad 0 \leq \delta \leq 1. \tag{2.8}$$

We then know that for all $\delta > 0$ small enough, Λ_δ is a linear IR-manifold, and, as explained for instance in [9] (Section 4), there exists a linear canonical transformation

$$\kappa_\delta : \mathbf{R}^{2n'} \to \Lambda_\delta, \tag{2.9}$$

such that

$$\kappa_\delta(X') = X' + i\delta H_{G_0}(X') + \mathcal{O}(\delta^2 |X'|). \tag{2.10}$$

We introduce next the standard FBI-Bargmann transformation along $S^{\sigma\perp} \simeq \mathbf{R}^{2n'}$,

$$T'u(x') = \widetilde{C} h^{-3n'/4} \int e^{\frac{i}{h}\varphi(x',y')} u(y')\, dy', \quad x' \in \mathbf{C}^{n'}, \quad \widetilde{C} > 0, \tag{2.11}$$

where $\varphi(x', y') = \frac{i}{2}(x' - y')^2$. Associated to T' there is a complex linear canonical transformation

$$\kappa_{T'} : \mathbf{C}^{2n'} \ni (y', \eta') \mapsto (x', \xi') = (y' - i\eta', \eta') \in \mathbf{C}^{2n'}, \tag{2.12}$$

mapping the real phase space $\mathbf{R}^{2n'}$ onto the linear IR-manifold

$$\Lambda_{\Phi_0} = \left\{ \left(x', \frac{2}{i} \frac{\partial \Phi_0}{\partial x'}(x') \right) : x' \in \mathbf{C}^{n'} \right\}, \tag{2.13}$$

where

$$\Phi_0(x') = \frac{1}{2} \left(\operatorname{Im} x' \right)^2.$$

For a suitable choice of $\widetilde{C} > 0$ in (2.11), we know that the map T' takes $L^2(\mathbf{R}^{n'})$ unitarily onto $H_{\Phi_0,h}(\mathbf{C}^{n'})$. Here and in what follows, when $\Phi \in C^\infty(\mathbf{C}^{n'})$ is a suitable smooth strictly plurisubharmonic weight function close to Φ_0 in (2.13), we shall let $H_{\Phi,h}(\mathbf{C}^{n'})$ stand for the closed subspace of $L^2(\mathbf{C}^{n'}; e^{-\frac{2\Phi}{h}} L(dx'))$, consisting of functions that are entire holomorphic. The integration element $L(dx')$ stands here for the Lebesgue measure on $\mathbf{C}^{n'}$.

Following [11] (Section 3), we write next

$$\kappa_{T'}(\Lambda_\delta) = \Lambda_{\Phi_\delta} := \left\{ \left(x', \frac{2}{i} \frac{\partial \Phi_\delta}{\partial x'}(x') \right); x' \in \mathbf{C}^{n'} \right\}, \tag{2.14}$$

for $0 \leq \delta \leq \delta_0$ with $\delta_0 > 0$ small enough, where $\Phi_\delta(x')$ is a strictly plurisubharmonic quadratic form on $\mathbf{C}^{n'}$, given by

$$\Phi_\delta(x') = \text{v.c.}_{(y',\eta') \in \mathbf{C}^{n'} \times \mathbf{R}^{n'}} \left(-\operatorname{Im} \varphi(x', y') - (\operatorname{Im} y') \cdot \eta' + \delta G_0(\operatorname{Re} y', \eta') \right). \tag{2.15}$$

The unique critical point $(y'(x'), \eta(x'))$ giving the corresponding critical value in (2.15) satisfies

$$y'(x') = \operatorname{Re} x' + \mathcal{O}(\delta\,|x'|), \quad \eta'(x') = -\operatorname{Im} x' + \mathcal{O}(\delta\,|x'|), \tag{2.16}$$

and as in [10], [11], we obtain that

$$\Phi_\delta(x') = \Phi_0(x') + \delta G_0(\operatorname{Re} x', -\operatorname{Im} x') + \mathcal{O}(\delta^2\,|x'|^2). \tag{2.17}$$

Let us set $Q_1 = q_1^w(x', hD_{x'})$ and recall from [23] the exact Egorov property

$$T'Q_1 u = \widetilde{Q}_1 T' u,\ \ u \in \mathcal{S}(\mathbf{R}^{n'}), \tag{2.18}$$

where $\widetilde{Q}_1$ is a semiclassical quadratic differential operator on $\mathbf{C}^{n'}$ whose Weyl symbol $\widetilde{q}_1$ satisfies

$$\widetilde{q}_1 \circ \kappa_{T'} = q_1, \tag{2.19}$$

with $\kappa_{T'}$ given in (2.12).

Continuing to follow [23], let us also recall that when realizing $\widetilde{Q}_1$ as an unbounded operator on $H_{\Phi_0,h}(\mathbf{C}^{n'})$, we may first use the contour integral representation

$$\widetilde{Q}_1 u(x') = \frac{1}{(2\pi h)^{n'}} \iint_{\theta' = \frac{2}{i}\frac{\partial \Phi_0}{\partial x'}\left(\frac{x'+y'}{2}\right)} e^{\frac{i}{h}(x'-y')\cdot\theta'} \widetilde{q}_1\Big(\frac{x'+y'}{2}, \theta'\Big) u(y')\, dy'\, d\theta',$$

and then, using that the symbol $\widetilde{q}_1$ is holomorphic, by a contour deformation we obtain the following formula for $\widetilde{Q}_1$ as an unbounded operator on $H_{\Phi_0,h}(\mathbf{C}^{n'})$,

$$\widetilde{Q}_1 u(x') = \frac{1}{(2\pi h)^{n'}} \iint_{\theta' = \frac{2}{i}\frac{\partial \Phi_0}{\partial x'}\left(\frac{x'+y'}{2}\right) + it\overline{(x'-y')}} e^{\frac{i}{h}(x'-y')\cdot\theta'} \widetilde{q}_1\Big(\frac{x'+y'}{2}, \theta'\Big) u(y')\, dy'\, d\theta', \tag{2.20}$$

for any $t > 0$. Furthermore, the operator $\widetilde{Q}_1$ can also be viewed as an unbounded operator

$$\widetilde{Q}_1 : H_{\Phi_\delta,h}(\mathbf{C}^{n'}) \to H_{\Phi_\delta,h}(\mathbf{C}^{n'}), \tag{2.21}$$

defined for $0 < \delta \leq \delta_0$, with $\delta_0 > 0$ sufficiently small. Indeed, when defining the operator in (2.21), it suffices to make a contour deformation in (2.20) and set

$$\widetilde{Q}_1 u(x') = \frac{1}{(2\pi h)^{n'}} \iint_{\theta' = \frac{2}{i}\frac{\partial \Phi_\delta}{\partial x'}\left(\frac{x'+y'}{2}\right) + it\overline{(x'-y')}} e^{\frac{i}{h}(x'-y')\cdot\theta'} \widetilde{q}_1\Big(\frac{x'+y'}{2}, \theta\Big) u(y')\, dy'\, d\theta', \tag{2.22}$$

for any $t > 0$. We then know from the general theory [17], [22], that the operator in (2.21) is unitarily equivalent to the quadratic operator on $L^2(\mathbf{R}^{n'})$, whose Weyl symbol is given by the quadratic form

$$X' \mapsto q_1\left(\kappa_\delta(X')\right), \quad X' \in \mathbf{R}^{2n'}, \tag{2.23}$$

with κ_δ introduced in (2.9), (2.10). In particular, using (2.5), (2.7), and (2.10), we see as in [10] (p.827) that the real part of the quadratic form in (2.23) is positive definite, and from [10] (p.828) we also know that the spectrum of $\widetilde{Q}_1$ acting on $H_{\Phi_0,h}(\mathbf{C}^{n'})$ agrees with the spectrum of $\widetilde{Q}_1$ acting on $H_{\Phi_\delta,h}(\mathbf{C}^{n'})$, for all $0 < \delta \leq \delta_0$, $\delta_0 > 0$ small enough, including the multiplicities. For future reference, let us recall from [10] the explicit description of the spectrum of $\widetilde{Q}_1$, which is given by

$$\mathrm{Spec}(\widetilde{Q}_1) = \left\{ h \sum_{\substack{\lambda \in \sigma(F_1) \\ \mathrm{Im}\lambda > 0}} (r_\lambda + 2k_\lambda)\frac{\lambda}{i},\, k_\lambda \in \mathbf{N} \right\}. \tag{2.24}$$

Here, F_1 is the Hamilton map associated to the quadratic form q_1 and r_λ is the dimension of the generalized eigenspace of F_1 in $\mathbf{C}^{2n'}$ corresponding to the eigenvalue $\lambda \in \mathbf{C}$ of the Hamilton map F_1.

In the remainder of this section, we shall be concerned exclusively with the case of $(h = 1)$ quantization, and we shall then write $H_{\Phi_0}(\mathbf{C}^{n'}) = H_{\Phi_0,h=1}(\mathbf{C}^{n'})$, and similarly for $H_{\Phi_\delta}(\mathbf{C}^{n'})$. The following result is a slight generalization of the corresponding statement from [10].

Proposition 2.1 *There exists $\eta > 0$ and $\delta_0 > 0$ small enough, such that the generalized eigenvectors u of the operators*

$$\widetilde{Q}_1(x', D_{x'}) : H_{\Phi_0}(\mathbf{C}^{n'}) \to H_{\Phi_0}(\mathbf{C}^{n'})$$

and

$$\widetilde{Q}_1(x', D_{x'}) : H_{\Phi_\delta}(\mathbf{C}^{n'}) \to H_{\Phi_\delta}(\mathbf{C}^{n'}), \quad 0 < \delta \leq \delta_0,$$

agree and satisfy

$$u \in H_{\Phi_0 - \eta|x'|^2}(\mathbf{C}^{n'}). \tag{2.25}$$

Proof: The statement of the proposition was established in the work [10], in the case when u is an eigenvector of $\widetilde{Q}_1$. When treating the case of generalized eigenvectors, we may argue in a way similar to [10] (p.829-831), and consider the restriction of the heat semigroup, viewed as a bounded operator,

$$e^{-t\widetilde{Q}_1} : H_{\Phi_0}(\mathbf{C}^{n'}) \to H_{\Phi_t}(\mathbf{C}^{n'}), \quad 0 < t \leq t_0, \tag{2.26}$$

$t_0 > 0$ small enough, to a generalized eigenspace $E_{\lambda_0} \subset H_{\Phi_0}(\mathbf{C}^{n'})$ of $\widetilde{Q}_1$, associated to an eigenvalue λ_0. The space E_{λ_0} is finite-dimensional, and the restriction of $\widetilde{Q}_1 - \lambda_0$ to E_{λ_0} is nilpotent. It was shown in [10] (Lemma 3.1.2) that for each $t > 0$ small enough, there exists $\alpha = \alpha(t) > 0$ such that the quadratic form Φ_t satisfies

$$\Phi_t(x') \leq \Phi_0(x') - \alpha \left|x'\right|^2, \quad x' \in \mathbf{C}^{n'}.$$

Notice that the map $e^{-t\widetilde{Q}_1} : E_{\lambda_0} \to E_{\lambda_0}$ is bijective for any $t \geq 0$. Indeed, the generalized eigenspace E_{λ_0} is stable under the action of the operator $\widetilde{Q}_1$ and its restriction to this finite-dimensional space

$$\widetilde{Q}_1|_{E_{\lambda_0}} : E_{\lambda_0} \to E_{\lambda_0},$$

is a bounded operator. This implies that the restriction of the semigroup to the space $(e^{-t\widetilde{Q}_1})|_{E_{\lambda_0}}$ coincides with the exponential of the bounded operator $-t\widetilde{Q}_1|_{E_{\lambda_0}}$, which is always bijective. It follows therefore that the generalized eigenvectors $u \in H_{\Phi_0}(\mathbf{C}^{n'})$ of $\widetilde{Q}_1$ acting on $H_{\Phi_0}(\mathbf{C}^{n'})$, belong to $H_{\Phi_\delta}(\mathbf{C}^{n'})$, for $\delta > 0$ small enough, and satisfy (2.25). Considering the action of the heat semigroup on the corresponding generalized eigenspace of the operator $\widetilde{Q}_1$ acting on $H_{\Phi_\delta}(\mathbf{C}^{n'})$ and repeating the arguments following the statement of Lemma 3.1.2 in [10], we obtain the statement of the proposition. □

Having obtained the exponential decay properties of the generalized eigenvectors of $\widetilde{Q}_1$, we return to the full quadratic operator $Q = q^w(x, D_x)$ in (2.3), and introduce the corresponding quadratic differential operator $\widetilde{Q}$ on the FBI transform side, given by

$$TQu = \widetilde{Q}Tu, \quad u \in \mathcal{S}(\mathbf{R}^n).$$

Here the full FBI-Bargmann transformation T is given by

$$T = T' \otimes T'' : L^2(\mathbf{R}^n) = L^2(\mathbf{R}^{n'}) \otimes L^2(\mathbf{R}^{n''}) \to H_{\Phi_0}(\mathbf{C}^{n'}) \otimes H_{\Phi_0}(\mathbf{C}^{n''}) = H_{\Phi_0}(\mathbf{C}^n),$$

with the partial transform T'' along the singular space S being defined similarly to (2.11). Associated to T'' and to T, we have the linear canonical transformations $\kappa_{T''}$ and κ_T, with $\kappa_T = \kappa_{T'} \otimes \kappa_{T''}$, so that $\kappa_T(y, \eta) = (y - i\eta, \eta)$. The splitting of the coordinates (2.2) induces, by means of κ_T, the corresponding splitting of the coordinates in $\mathbf{C}^n$, so that we can write $x = (x', x'') \in \mathbf{C}^n = \mathbf{C}^{n'} \times \mathbf{C}^{n''}$. We have, in view of (2.3),

$$\widetilde{Q}(x, D_x) = \widetilde{Q}_1(x', D_{x'}) + i\widetilde{Q}_2(x'', D_{x''}), \tag{2.27}$$

where the symbol $\widetilde{q}_2$ of the quadratic operator $\widetilde{Q}_2(x'', D_{x''})$ is given by $\widetilde{q}_2 = q_2 \circ \kappa_{T''}^{-1}$. We shall be concerned with the generalized eigenfunctions of the operator $\widetilde{Q}(x, D_x)$ in (2.27) acting on the weighted space

$$H_{\Phi_\delta}(\mathbf{C}^n) = H_{\Phi_\delta}(\mathbf{C}^{n'}) \otimes H_{\Phi_0}(\mathbf{C}^{n''}),$$

with $\delta > 0$ small enough fixed. Here in the left hand side,

$$\Phi_\delta(x) = \Phi_\delta(x') + \Phi_0(x''),$$

and an application of (2.13) and (2.17) shows that

$$\Phi_\delta(x) = \Phi_0(x) + \delta G_0(\operatorname{Re} x', -\operatorname{Im} x') + \mathcal{O}(\delta^2 \left|x'\right|^2).$$

Let us recall from [10] (p.843) that the spectrum of $\widetilde{Q}(x, D_x)$ is given by

$$\sigma(\widetilde{Q}(x, D_x)) = \sigma(\widetilde{Q}_1(x', D_{x'})) + i\sigma(\widetilde{Q}_2(x'', D_{x''})),$$

with the spectrum of $\widetilde{Q}_1(x', D_{x'})$ given in (2.24), and furthermore, from (2.4), we know that the spectrum of $\widetilde{Q}_2(x'', D_{x''})$ consists of the eigenvalues of the form

$$\mu_{\alpha''} = \sum_{j=1}^{n''} \frac{\lambda_j}{2} \left(2\alpha_j'' + 1\right), \quad \alpha'' \in \mathbf{N}^{n''}.$$

The corresponding eigenfunctions are given by

$$\Phi_{\alpha''}(x'') = (T''\varphi_{\alpha''})(x''), \tag{2.28}$$

where

$$\varphi_{\alpha''}(y'') = H_{\alpha''}(y'')e^{-(y'')^2/2}$$

are the Hermite functions, with $H_{\alpha''}(y'')$ being the Hermite polynomials on $\mathbf{R}^{n''}$. It is clear that the eigenfunctions $\Phi_{\alpha''}(x'')$ form an orthonormal basis of $H_{\Phi_0}(\mathbf{C}^{n''})$, and a straightforward computation shows that the functions $\Phi_{\alpha''}(x'')$ are of the form

$$\Phi_{\alpha''}(x'') = p_{\alpha''}(x'')e^{-(x'')^2/4},$$

where $p_{\alpha''}(x'')$ is a holomorphic polynomial on $\mathbf{C}^{n''}$. In particular, we have

$$\Phi_{\alpha''} \in H_{\Phi_0 - \eta|x''|^2}(\mathbf{C}^{n''}), \tag{2.29}$$

for some fixed $\eta > 0$.

Let $u \in H_{\Phi_\delta}(\mathbf{C}^n)$, and let us write

$$u(x', x'') = \sum_{\alpha'' \in \mathbf{N}^{n''}} u_{\alpha''}(x') \Phi_{\alpha''}(x'').$$

Using that

$$\left(\widetilde{Q}(x, D_x) - \lambda\right) u = \sum_{\alpha'' \in \mathbf{N}^{n''}} \left[(\widetilde{Q}_1(x', D_{x'}) + i\mu_{\alpha''} - \lambda) u_{\alpha''}(x')\right] \Phi_{\alpha''}(x''), \tag{2.30}$$

we see that u is a generalized eigenvector of $\widetilde{Q}(x, D_x)$ corresponding to an eigenvalue $\lambda \in \mathbf{C}$, precisely when u is of the form

$$u(x', x'') = \sum_{\alpha''} u_{\alpha''}(x') \Phi_{\alpha''}(x''), \tag{2.31}$$

where the summation extends over all $\alpha'' \in \mathbf{N}^{n''}$ for which

$$\lambda - i\mu_{\alpha''} \in \sigma(\widetilde{Q}_1(x', D_{x'})),$$

and $u_{\alpha''}(x') \in H_{\Phi_\delta}(\mathbf{C}^{n'})$ is a generalized eigenvector of $\widetilde{Q}_1(x', D_{x'})$ associated to the eigenvalue $\lambda - i\mu_{\alpha''}$. Since, according to (2.24), $\sigma(\widetilde{Q}_1)$ is contained in a proper closed cone in $\mathbf{C}$ of the form $|\mathrm{Im}\, z| \leq C \mathrm{Re}\, z$, $C > 0$, it follows that the sum in (2.31) contains a fixed finite number of terms, when $|\lambda| = \mathcal{O}(1)$. Combining Proposition 2.1, (2.29), and (2.31), we obtain the following result, which summarizes the discussion pursued in this section.

Proposition 2.2 *There exists $\eta > 0$ such that for all $0 \leq \delta \leq \delta_0$, with $\delta_0 > 0$ small enough, the generalized eigenvectors u of the quadratic operator $\widetilde{Q}(x, D_x)$ acting on $H_{\Phi_\delta}(\mathbf{C}^n)$, satisfy*

$$u \in H_{\Phi_0 - \eta|x|^2}(\mathbf{C}^n).$$

Remark. The discussion in this section, together with the corresponding analysis in Section 3 in [10], can be considered as a natural generalization of Remark 11.7 in [8]. For future reference, let us also remark that from [8], [9], [21], we know that the generalized eigenfunctions u of the operator $\widetilde{Q}(x, D_x)$ are such that the inverse FBI transform $T^{-1}u \in L^2(\mathbf{R}^n)$ is of the form

$$T^{-1}u = p(x)e^{i\Phi(x)}, \tag{2.32}$$

where p is a polynomial on $\mathbf{R}^n$ and $\Phi(x)$ is a complex quadratic form, and according to Proposition 2.2), we have $\operatorname{Im}\Phi > 0$. Furthermore, the positive Lagrangian subspace $\{(x, \Phi'(x));\, x \in \mathbf{C}^n\}$ is the stable outgoing manifold for the Hamilton flow of the quadratic form

$$(x,\xi) \mapsto \frac{1}{i}e^{-i\theta}q(x,\xi), \quad (x,\xi) \in \mathbf{R}^{2n},$$

where $\theta > 0$ is sufficiently small but fixed.

3 Global Grushin problem

Throughout this section, we shall make the simplifying assumption that the integer N introduced in (1.16) satisfies $N = 1$, and that the corresponding doubly characteristic point is $X_1 = (0,0) \in \mathbf{R}^{2n}$. This assumption serves merely to simplify the notation in the proofs and does not cause any loss of generality. In particular, we write

$$p_0(X) = q(X) + \mathcal{O}(X^3),$$

where q is a quadratic form, to which Proposition 2.2 applies.

When proving Theorem 1.1, it will be convenient to work with symbols in the class $S(1)$, bounded together with all of their derivatives, similarly to what was done in [11]. Let us begin this section by describing therefore a reduction to the case when $m = 1$. When doing so, we notice that for all $h > 0$ sufficiently small, the operator

$$P + 1 : H(m) \to L^2(\mathbf{R}^n)$$

is bijective, and by an application of Beals's lemma, we know that $(P+1)^{-1} \in \operatorname{Op}_h(S((\frac{1}{m})))$, see [3], p.99-100. Let

$$\widetilde{P} = (P+1)^{-1}P \in \operatorname{Op}_h\left(S(1)\right),$$

with the leading symbol given by

$$\widetilde{p}_0 = \frac{p_0}{p_0 + 1}. \tag{3.1}$$

Furthermore, by holomorphic functional calculus [6], or by an explicit calculation using the Weyl calculus [3] (use formula (8.11) p.100), we see that the subprincipal symbol of $\widetilde{P}$ is given by

$$\widetilde{p}_1 = \frac{p_1}{(p_0+1)^2}. \tag{3.2}$$

It follows from (3.1) that the leading symbol $\widetilde{p}_0$ of the bounded h-pseudodifferential operator $\widetilde{P}$ satisfies $\operatorname{Re}\widetilde{p}_0 \geq 0$, and that $\operatorname{Re}\widetilde{p}_0$ is elliptic near infinity in the class $S(1)$. Furthermore, $\widetilde{p}_0$ vanishes precisely at the origin, with

$$\widetilde{p}_0(X) = q(X) + \mathcal{O}(X^3), \quad \widetilde{p}_1(0) = p_1(0).$$

In order to deduce the asymptotic description of the eigenvalues for the operator P from the corresponding description for the operator $\widetilde{P}$, we notice that the resolvents of P and $\widetilde{P}$ are related as follows, for $z \in \operatorname{neigh}(0, \mathbf{C})$,

$$\left(\widetilde{P} - z\right)^{-1} = (1-z)^{-1}\left(P - \frac{z}{1-z}\right)^{-1}(P+1).$$

Hence, $z \in \operatorname{neigh}(0, \mathbf{C})$ is an eigenvalue of $\widetilde{P}$ precisely when $z/(1-z)$ is an eigenvalue of P, and the multiplicities agree. In what follows, we shall therefore be concerned exclusively with the case when $m = 1$.

3.1 Grushin problem in the quadratic case

In this subsection, we shall describe a well-posed Grushin problem for the elliptic quadratic operator $\widetilde{Q}(x, D_x)$ defined in (2.27), acting on the weighted space $H_{\Phi_\delta}(\mathbf{C}^n)$, for $\delta > 0$ small enough but fixed. Let $\lambda_0 \in \mathbf{C}$ be an eigenvalue of $\widetilde{Q}(x, D_x)$, and let $E_{\lambda_0} \subset H_{\Phi_\delta}(\mathbf{C}^n)$ be the corresponding finite-dimensional generalized eigenspace. According to Proposition 2.2, we have

$$E_{\lambda_0} \subset H_{\Phi_0 - \eta|x|^2}(\mathbf{C}^n), \quad \eta > 0.$$

Let $e_1, \ldots, e_{N_0}$ be a basis for E_{λ_0}. We shall now introduce a suitable dual basis. When doing so, let $\widetilde{Q}^* = \widetilde{Q}^*(x, D_x)$ be the adjoint of the operator $\widetilde{Q} = \widetilde{Q}(x, D_x)$ acting on the space $H_{\Phi_0}(\mathbf{C}^n)$. Here the closed densely defined quadratic operator $\widetilde{Q}$ is equipped with the domain $\{u \in H_{\Phi_0}(\mathbf{C}^n); \widetilde{Q}u \in H_{\Phi_0}(\mathbf{C}^n)\}$. According to the discussion in [13], p.426, we have $\widetilde{Q}^* = T\overline{q}^w T^{-1}$. Here the Weyl symbol of $\overline{q}^w$ is the quadratic form $X \mapsto \overline{q(X)}$, which has a non-negative real part, and whose restriction to the corresponding singular space, which is equal to S, is elliptic. Let

$f_1, \dots, f_{N_0}$, $f_j \in H_{\Phi_0}(\mathbf{C}^n)$, be the basis for the generalized eigenspace of the adjoint operator $\widetilde{Q}^* : H_{\Phi_0}(\mathbf{C}^n) \to H_{\Phi_0}(\mathbf{C}^n)$, associated to the eigenvalue $\overline{\lambda_0}$, which is dual to $e_1, \dots e_{N_0}$. An application of Proposition 2.2 shows that the functions f_j, $1 \leq j \leq N_0$, satisfy

$$f_j \in H_{\Phi_0 - \eta |x|^2}(\mathbf{C}^n), \quad \eta > 0. \tag{3.3}$$

In particular, $f_j \in H_{\Phi_\delta}(\mathbf{C}^n)$, for $\delta > 0$ small enough, and we have

$$\det\left((e_j, f_k)\right) \neq 0, \quad 0 \leq \delta \leq \delta_0, \tag{3.4}$$

for some $\delta_0 > 0$ sufficiently small. Here the scalar product in (3.4) is taken in the space $H_{\Phi_\delta}(\mathbf{C}^n)$.

Let us introduce the operators

$$R_- : \mathbf{C}^{N_0} \to H_{\Phi_\delta}(\mathbf{C}^n)$$

and

$$R_+ : H_{\Phi_\delta}(\mathbf{C}^n) \to \mathbf{C}^{N_0},$$

given by $R_- u_- = \sum_{j=1}^{N_0} u_-(j) e_j$ and $(R_+ u)(j) = (u, f_j)$, with the scalar product taken in the space $H_{\Phi_\delta}(\mathbf{C}^n)$. Arguing as in Section 11 of [8], we obtain that for $z \in \mathrm{neigh}(\lambda_0, \mathbf{C})$, the Grushin operator

$$\begin{pmatrix} \widetilde{Q} - z & R_- \\ R_+ & 0 \end{pmatrix} : \mathcal{D}(\widetilde{Q}) \times \mathbf{C}^{N_0} \to H_{\Phi_\delta}(\mathbf{C}^n) \times \mathbf{C}^{N_0} \tag{3.5}$$

is bijective. Here $\mathcal{D}(\widetilde{Q}) = \{u \in H_{\Phi_\delta}(\mathbf{C}^n);\, (1 + |x|^2) u \in L^2_{\Phi_\delta}(\mathbf{C}^n)\}$.

Continuing to follow [8], we shall now restore the semiclassical parameter $h > 0$ and consider the operators

$$R_{-,h} = \mathcal{O}(1) : \mathbf{C}^{N_0} \to H_{\Phi_\delta,h}(\mathbf{C}^n), \quad R_{+,h} = \mathcal{O}(1) : H_{\Phi_\delta,h} \to \mathbf{C}^{N_0}, \tag{3.6}$$

given by

$$R_{-,h} u_- = \sum_{j=1}^{N_0} u_-(j) e_{j,h}, \quad \left(R_{+,h} u\right)(j) = (u, f_{j,h}). \tag{3.7}$$

Here the scalar product in the definition of $R_{+,h}$ is taken in the space $H_{\Phi_\delta,h}(\mathbf{C}^n)$, and

$$e_{j,h}(x) = h^{-n/2} e_j\left(\frac{x}{\sqrt{h}}\right), \quad f_{j,h}(x) = h^{-n/2} f_j\left(\frac{x}{\sqrt{h}}\right).$$

With $\widetilde{Q} = \widetilde{Q}(x, hD_x)$, we shall now consider the semiclassical Grushin problem, given by

$$\left(\widetilde{Q} - hz\right) u + R_{-,h} u_- = v, \quad R_{+,h} v = v_+. \tag{3.8}$$

Here z varies in a sufficiently small but fixed neighborhood of the eigenvalue λ_0. At this point, we are exactly in the same situation as described in Section 11 of [8] (Proposition 11.5), and arguing exactly as in that paper, we see that for each $(v, v_+) \in H_{\Phi_\delta,h}(\mathbf{C}^n) \times \mathbf{C}^{N_0}$, the problem (3.8) has a unique solution $(u, u_-) \in H_{\Phi_\delta,h}(\mathbf{C}^n) \times \mathbf{C}^{N_0}$ such that $(1 + |x|^2) u \in L^2_{\Phi_\delta,h}(\mathbf{C}^n)$. Furthermore, for every $k \in \mathbf{R}$ fixed, the following a priori estimate holds,

$$|| (h + |x|^2)^{1-k} u || + h^{-k} |u_-| \leq \mathcal{O}(1) \left(|| (h + |x|^2)^{-k} v || + h^{1-k} |v_+| \right). \tag{3.9}$$

Here the norms are taken in the space $L^2_{\Phi_\delta,h}(\mathbf{C}^n)$.

The estimate (3.9) can subsequently be localized, and we see that the result of Proposition 11.6 of [8] can be applied to our situation as it stands, since the proof of Proposition 11.6 in [8] only relies on the ellipticity of the quadratic operator $\widetilde{Q}$ acting on $H_{\Phi_\delta,h}(\mathbf{C}^n)$, for $\delta > 0$ small enough but fixed, together with the decay estimates given in Proposition 2.2 and in (3.3). We therefore obtain the following result, which summarizes the discussion in this section.

Proposition 3.1 *Let $\chi_0 \in C_0^\infty(\mathbf{C}^n)$ be fixed, such that $\chi_0 = 1$ near $x = 0$, and let $k \in \mathbf{R}$ be fixed. Then for $z \in \mathrm{neigh}(\lambda_0, \mathbf{C})$, we have the following estimate for the problem* (3.8)*, valid for all $h > 0$ sufficiently small,*

$$\begin{aligned} &|| (h + |x|^2)^{1-k} \chi_0 u || + h^{-k} |u_-| \\ &\qquad \leq \mathcal{O}(1) \left(|| (h + |x|^2)^{-k} \chi_0 v || + h^{1-k} |v_+| + h^{1/2} || 1_K u || \right). \end{aligned} \tag{3.10}$$

Here K is a fixed neighborhood of $\mathrm{supp}(\nabla \chi_0)$ *and 1_K stands for the characteristic function of this set. The norms in the estimate* (3.10) *are taken in the space $L^2_{\Phi_\delta,h}(\mathbf{C}^n)$.*

Remark. When deriving the estimate (3.10), following [8], we replace the functions $f_{j,h}$ in the definition of $R_{+,h}$ by $\chi(x/R\sqrt{h}) f_{j,h}(x)$, where $\chi \in C_0^\infty(\mathbf{C}^n)$, and $R > 0$ is sufficiently large fixed.

3.2 Localization and exterior estimates

The purpose of this subsection is to study a globally well-posed Grushin problem for the operator P introduced in (1.13). When doing so, we shall be concerned with the action of P, after an FBI-Bargmann transformation, on a suitable weighted space of holomorphic functions on $\mathbf{C}^n$. We shall therefore first proceed to recall the definition and properties of this space, constructed and introduced in [11].

In Proposition 2 of [11], it was shown that for all $0 < \varepsilon \leq \varepsilon_0$, $0 < \delta \leq \delta_0$, with $\varepsilon_0 > 0$, $\delta_0 > 0$ sufficiently small, there exists a function $G_\varepsilon \in C_0^\infty(\mathbf{R}^{2n}, \mathbf{R})$, supported in a sufficiently small but fixed neighborhood of the origin, such that $G_\varepsilon = \mathcal{O}(\varepsilon)$, $\nabla^2 G_\varepsilon = \mathcal{O}(1)$, and such that for some $C > 1$, $\widetilde{C} > 1$, we have

$$|p_0\left(X + i\delta H_{G_\varepsilon}(X)\right)| \geq \frac{\delta}{\widetilde{C}} \min\left(|X|^2, \varepsilon\right),$$

in the region where $|X| \leq 1/C$. Furthermore, in the region where $|X| \geq \varepsilon^{1/2}$, we have

$$\operatorname{Re}\left(\left(1 - \frac{ic\delta\varepsilon}{|X|^2}\right) p_0\left(X + i\delta H_{G_\varepsilon}(X)\right)\right) \geq \frac{\delta\varepsilon}{\widetilde{C}}, \quad c > 0. \tag{3.11}$$

Here we have also written p_0 for an almost analytic extension of the leading symbol p_0 of P to a tubular neighborhood of $\mathbf{R}^{2n}$, bounded together with all of its derivatives.

Remark. For future reference, we may remark that it follows from the construction of the weight function G_ε in [11], that in the region where $|X|^2 \leq \varepsilon/2$, we have

$$G_\varepsilon(X) = G_0(X') + \mathcal{O}(X^3), \tag{3.12}$$

where the quadratic form G_0 is defined in (2.6), see remark p.1002 in [11].

Associated with the weight function G_ε there is an IR-manifold

$$\Lambda_{\delta,\varepsilon} = \left\{X + i\delta H_{G_\varepsilon}(X); X \in \mathbf{R}^{2n}\right\}, \tag{3.13}$$

and arguing as in [11] (Section 3), we obtain that

$$\kappa_T\left(\Lambda_{\delta,\varepsilon}\right) = \Lambda_{\Phi_{\delta,\varepsilon}} := \left\{(x,\xi) \in \mathbf{C}^{2n}; \xi = \frac{2}{i}\frac{\partial \Phi_{\delta,\varepsilon}}{\partial x}(x)\right\}. \tag{3.14}$$

Here $\Phi_{\delta,\varepsilon} \in C^\infty(\mathbf{C}^n)$ is a strictly plurisubharmonic function given by

$$\Phi_{\delta,\varepsilon}(x) = \text{v.c.}_{(y,\eta)\in\mathbf{C}^n\times\mathbf{R}^n}\left(-\operatorname{Im}\varphi(x,y) - (\operatorname{Im} y)\cdot\eta + \delta G_\varepsilon(\operatorname{Re} y, \eta)\right). \tag{3.15}$$

Uniformly on $\mathbf{C}^n$, we have

$$\Phi_{\delta,\varepsilon}(x) = \Phi_0(x) + \delta G_\varepsilon(\operatorname{Re} x, -\operatorname{Im} x) + \mathcal{O}(\delta^2\varepsilon), \tag{3.16}$$

and in particular,

$$\Phi_{\delta,\varepsilon} - \Phi_0 = \mathcal{O}(\delta\varepsilon). \tag{3.17}$$

We furthermore know that $\Phi_{\delta,\varepsilon}$ agrees with Φ_0 outside a bounded set and that

$$\nabla\left(\Phi_{\delta,\varepsilon} - \Phi_0\right) = \mathcal{O}(\delta\varepsilon^{1/2}), \tag{3.18}$$

with $\nabla^2\Phi_{\delta,\varepsilon} \in L^\infty(\mathbf{C}^n)$, uniformly in δ and ε.

In what follows, similarly to [11] (Section 3), we shall be concerned with the case when

$$\varepsilon = Ah, \tag{3.19}$$

when $A \geq 1$ is sufficiently large but fixed, to be chosen in what follows. As explained in [11] (Section 3), following [8], the h–pseudodifferential operator on the FBI–Bargmann transform side, $\widetilde{P} := TPT^{-1}$, can therefore be defined as a uniformly bounded operator

$$\widetilde{P} = \mathcal{O}(1) : H_{\Phi_{\delta,\varepsilon},h}(\mathbf{C}^n) \to H_{\Phi_{\delta,\varepsilon},h}(\mathbf{C}^n),$$

given, when $u \in H_{\Phi_{\delta,\varepsilon},h}(\mathbf{C}^n)$, by

$$\widetilde{P}u(x) = \frac{1}{(2\pi h)^n} \iint_{\Gamma_{\delta,\varepsilon}(x)} e^{\frac{i}{h}(x-y)\cdot\theta} \psi(x-y) \widetilde{P}\left(\frac{x+y}{2}, \theta\right) u(y)\, dy\, d\theta + Ru. \tag{3.20}$$

Here $\psi \in C_0^\infty(\mathbf{C}^n)$ is such that $\psi = 1$ near 0 and $\Gamma_{\delta,\varepsilon}(x)$ is the contour given by

$$\theta = \frac{2}{i}\frac{\partial \Phi_{\delta,\varepsilon}}{\partial x}\left(\frac{x+y}{2}\right) + it_0\overline{(x-y)}, \quad t_0 > 0.$$

The remainder R in (3.20) satisfies

$$R = \mathcal{O}_A(h^\infty) : L^2(\mathbf{C}^n; e^{-\frac{2\Phi_{\delta,\varepsilon}}{h}} L(dx)) \to L^2(\mathbf{C}^n; e^{-\frac{2\Phi_{\delta,\varepsilon}}{h}} L(dx)).$$

Also, in (3.20) we continue to write $\widetilde{P}$ for an almost holomorphic extension of the full symbol $\widetilde{P} \in S(\Lambda_{\Phi_0}, 1)$ of $\widetilde{P}$, $\widetilde{P} = P \circ \kappa_T^{-1}$, to a tubular neighborhood of Λ_{Φ_0}, bounded together with all of its derivatives.

We shall be concerned with a global Grushin problem for the operator $\widetilde{P}$ in the weighted space $H_{\Phi_{\delta,\varepsilon},h}(\mathbf{C}^n)$. In order to exploit the quadratic Grushin problem for $\widetilde{Q}$, described in subsection 3.1, we shall make use of the observation that there exists a constant $C > 0$ such that in the region of $\mathbf{C}^n$, where

$$|x| \leq \frac{\sqrt{\varepsilon}}{C}, \tag{3.21}$$

the weight function $\Phi_{\delta,\varepsilon}$ is independent of ε, and furthermore, in this region, we have

$$\Phi_{\delta,\varepsilon} = \Phi_\delta(x) + \mathcal{O}(\delta\,|x|^3). \tag{3.22}$$

The equality (3.22) is obtained by a straightforward computation, using (2.15), its analogue for the weight $\Phi_{\delta,\varepsilon}$, given by (3.15), as well as (3.12).

By making a rescaling in ε, we may and will assume in the following that we have $C = 1$ in (3.21). It follows that in the region where $|x| \leq \sqrt{\varepsilon}$, the L^2–norm associated to the quadratic weight function Φ_δ can be replaced by the L^2–norm associated to the full weight $\Phi_{\delta,\varepsilon}$, at the expense of a loss which is

$$\exp\left(\mathcal{O}(1)A^{3/2}h^{1/2}\right) = \mathcal{O}(1),$$

provided that $A \geq 1$ is taken large but fixed, and $h \in (0, h_0]$, with $h_0 > 0$ small enough depending on A. We shall therefore replace the fixed cut-off function χ_0 in Proposition 3.1 by $\chi_0(x/\sqrt{\varepsilon})$, and following Section 11 of [8], this can be achieved by a rescaling argument using the change of variables $x = \sqrt{\varepsilon}\widetilde{x}$. This argument is carried out in detail, see (11.33), in Section 11.3 of [8], and for future reference, we shall record it here.

Lemma 3.2 *Let $\chi_0 \in C_0^\infty(\mathbf{C}^n)$ be fixed, such that $\chi_0 = 1$ near $x = 0$, and let $k \in \mathbf{R}$ be fixed. Then for $z \in \mathrm{neigh}(\lambda_0, \mathbf{C})$, we have the following estimate for the Grushin problem (3.8), valid for $h > 0$ sufficiently small, with $\varepsilon = Ah$,*

$$\begin{aligned} ||\,(h+|x|^2)^{1-k}\chi_0\left(\frac{x}{\sqrt{\varepsilon}}\right)u\,|| + h^{-k}\,|u_-| &\leq \mathcal{O}(1)||\,(h+|x|^2)^{-k}\chi_0\left(\frac{x}{\sqrt{\varepsilon}}\right)v\,|| \\ &+ \mathcal{O}(1)\left(h^{1-k}\,|v_+| + \sqrt{\frac{h}{\varepsilon}}||\,(h+|x|^2)^{1-k}1_K\left(\frac{x}{\sqrt{\varepsilon}}\right)u\,||\right). \end{aligned} \tag{3.23}$$

Here K is a fixed neighborhood of $\mathrm{supp}(\nabla\chi_0)$ and 1_K stands for the characteristic function of this set. The norms in the estimate (3.23) are taken in the space $L^2_{\Phi_\delta,h}(\mathbf{C}^n)$. According to (3.22), all the norms in the estimate (3.23) can be replaced by the norms in the space $L^2_{\Phi_{\delta,\varepsilon},h}(\mathbf{C}^n)$, for each fixed $A \gg 1$, provided that $h \in (0, h_0]$, with $h_0 > 0$ small enough, depending on A.

We now come to study the global Grushin problem for the operator $\widetilde{P} - hz$, for $z \in \mathrm{neigh}(\lambda_0 + p_1(0), \mathbf{C})$, in the weighted space $H_{\Phi_{\delta,\varepsilon},h}(\mathbf{C}^n)$. Here $p_1(0)$ is the value of the subprincipal symbol $p_1(x,\xi)$ of P at the unique doubly characteristic point, $(0,0) \in \mathbf{R}^{2n}$. With the operators $R_{-,h}$ and $R_{+,h}$ introduced in (3.7), let us consider

$$(\widetilde{P} - hz)u + R_{-,h}u_- = v, \quad R_{+,h}u = v_+, \tag{3.24}$$

when $(v, v_+) \in H_{\Phi_{\delta,\varepsilon},h}(\mathbf{C}^n) \times \mathbf{C}^{N_0}$. Writing the first equation in (3.24) in the form

$$(\widetilde{Q} - h(z - p_1(0)))u + R_{-,h}u_- = v + (\widetilde{Q} + hp_1(0) - \widetilde{P})u,$$

and applying Lemma 3.2 with $k = 1/2$, we get, with some constant $C > 0$,

$$\begin{aligned}
&||\,(h + |x|^2)^{1/2}\chi_0\left(\frac{x}{\sqrt{\varepsilon}}\right)u\,|| + h^{-1/2}\,|u_-| \\
&\le C||\,(h + |x|^2)^{-1/2}\chi_0\left(\frac{x}{\sqrt{\varepsilon}}\right)v\,|| + C||\,(h + |x|^2)^{-1/2}\chi_0\left(\frac{x}{\sqrt{\varepsilon}}\right)(\widetilde{P} - \widetilde{Q} - hp_1(0))u\,|| \\
&\qquad + \mathcal{O}(h^{1/2})\,|v_+| + C\sqrt{\frac{h}{\varepsilon}}||\,(h + |x|^2)^{1/2}1_K\left(\frac{x}{\sqrt{\varepsilon}}\right)u\,||.
\end{aligned} \tag{3.25}$$

Here the norms are taken in the space $L^2_{\Phi_{\delta,\varepsilon},h}(\mathbf{C}^n)$, as explained in Lemma 3.2. Now, as was already observed and exploited in [11], see (5.7) in Section 5, we have

$$||\,(h + |x|^2)^{-1/2}\chi_0\left(\frac{x}{\sqrt{\varepsilon}}\right)(\widetilde{P} - \widetilde{Q} - hp_1(0))u\,|| = \mathcal{O}_A(h)||\,u\,||,$$

and therefore, using also that $h + |x|^2 \le \mathcal{O}(\varepsilon)$ in the support of the function

$$x \mapsto 1_K(x/\sqrt{\varepsilon}),$$

we get

$$\begin{aligned}
&h^{1/2}||\,\chi_0\left(\frac{x}{\sqrt{\varepsilon}}\right)u\,|| + h^{-1/2}\,|u_-| \\
&\quad \le \mathcal{O}(h^{-1/2})||\,v\,|| + \mathcal{O}_A(h)||\,u\,|| + \mathcal{O}(h^{1/2})\,|v_+| + \mathcal{O}(h^{1/2})||\,1_K\left(\frac{x}{\sqrt{\varepsilon}}\right)u\,||.
\end{aligned} \tag{3.26}$$

It follows from (3.26) upon squaring that

$$\begin{aligned}
&h||\,\chi_0\left(\frac{x}{\sqrt{\varepsilon}}\right)u\,||^2 + h^{-1}\,|u_-|^2 \\
&\qquad \le \frac{\mathcal{O}(1)}{h}||\,v\,||^2 + \mathcal{O}_A(h^2)||\,u\,||^2 + \mathcal{O}(h)\,|v_+|^2 + \mathcal{O}(h)||\,1_K\left(\frac{x}{\sqrt{\varepsilon}}\right)u\,||^2.
\end{aligned} \tag{3.27}$$

The estimate (3.27) will be instrumental in obtaining the global well-posedness of the Grushin problem (3.24).

When deriving an a priori estimate for the problem (3.24) away from an $\mathcal{O}(\sqrt{\varepsilon})$–neighborhood of the doubly characteristic point $x = 0 \in \mathbf{C}^n$, we shall proceed very much in the spirit of Section 6 in [11]. Let $\widetilde{p}_0$ be an almost holomorphic continuation of the leading symbol of $\widetilde{P}$, bounded together with all of its derivatives in a tubular neighborhood of Λ_{Φ_0}. To simplify the notation, we shall write here $p := \widetilde{p}_0$. According to (3.11), we know that

$$\operatorname{Re}\left(\left(1 - ic\frac{\delta\varepsilon}{|x|^2}\right) p\left(x, \frac{2}{i}\frac{\partial\Phi_{\delta,\varepsilon}(x)}{\partial x}\right)\right) \geq \frac{\delta\varepsilon}{\widetilde{C}}, \quad |x| \geq \sqrt{\varepsilon}. \tag{3.28}$$

Following Section 6 of [11], we shall now switch to rescaled variables. Set

$$x = \sqrt{\varepsilon}\widetilde{x}. \tag{3.29}$$

In the new coordinates, the IR-manifold $\Lambda_{\Phi_{\delta,\varepsilon}}$ in (3.14) becomes replaced by the manifold

$$\Lambda_{\widetilde{\Phi}_{\delta,\varepsilon}} = \left\{\left(\tilde{x}, \frac{2}{i}\frac{\partial\widetilde{\Phi}_{\delta,\varepsilon}(\widetilde{x})}{\partial\widetilde{x}}\right) : \tilde{x} \in \mathbf{C}^n\right\}, \tag{3.30}$$

with

$$\widetilde{\Phi}_{\delta,\varepsilon}(\widetilde{x}) = \frac{1}{\varepsilon}\Phi_{\delta,\varepsilon}(\sqrt{\varepsilon}\widetilde{x}).$$

We notice that $\nabla^2\widetilde{\Phi}_{\delta,\varepsilon} \in L^\infty(\mathbf{C}^n)$ uniformly in $\varepsilon \in (0, \varepsilon_0]$, $\delta \in (0, \delta_0]$, and that along $\Lambda_{\widetilde{\Phi}_{\delta,\varepsilon}}$, we have

$$\widetilde{\xi} = -\operatorname{Im}\widetilde{x} + \mathcal{O}(\delta).$$

Let us consider the $\widetilde{h}$–pseudodifferential operator,

$$P_\varepsilon := \frac{1}{\varepsilon}p^w(x, hD_x) = \frac{1}{\varepsilon}p^w\left(\sqrt{\varepsilon}\left(\widetilde{x}, \widetilde{h}D_{\widetilde{x}}\right)\right), \quad \widetilde{h} = \frac{h}{\varepsilon} = \frac{1}{A}, \tag{3.31}$$

with the Weyl symbol given by

$$p_\varepsilon(\widetilde{x}, \widetilde{\xi}) = \frac{1}{\varepsilon}p\left(\sqrt{\varepsilon}(\widetilde{x}, \widetilde{\xi})\right). \tag{3.32}$$

It follows from (3.28) that along the manifold $\Lambda_{\widetilde{\Phi}_{\delta,\varepsilon}}$, the symbol (3.32) satisfies the following estimate,

$$\operatorname{Re}\left(\left(1 - ic\frac{\delta}{|\widetilde{x}^2|}\right) p_\varepsilon\left(\widetilde{x}, \frac{2}{i}\frac{\partial\widetilde{\Phi}_{\delta,\varepsilon}(\widetilde{x})}{\partial\widetilde{x}}\right)\right) \geq \frac{\delta}{\widetilde{C}}, \tag{3.33}$$

in the region where $|\widetilde{x}| \geq 1$.

Associated with the IR-manifold $\Lambda_{\widetilde{\Phi}_{\delta,\varepsilon}}$ is the weighted space $H_{\widetilde{\Phi}_{\delta,\varepsilon},\widetilde{h}}(\mathbf{C}^n)$, where we notice that

$$\frac{\widetilde{\Phi}_{\delta,\varepsilon}(\widetilde{x})}{\widetilde{h}} = \frac{\Phi_{\delta,\varepsilon}(x)}{h}.$$

The map $u(x) \mapsto \widetilde{u}(\widetilde{x}) = \varepsilon^{n/2} u(\sqrt{\varepsilon}\widetilde{x})$ then takes the space $H_{\Phi_{\delta,\varepsilon},h}(\mathbf{C}^n)$ unitarily onto the space $H_{\widetilde{\Phi}_{\delta,\varepsilon},\widetilde{h}}(\mathbf{C}^n)$.

Let now $\chi(\widetilde{x}) \in C_b^\infty(\mathbf{C}^n;[0,1])$ be such that $\chi = 1$ for large $|\widetilde{x}|$, and with $\operatorname{supp}\chi$ contained in the set where $|\widetilde{x}| \geq 1$. Let us set

$$m(\widetilde{x}) = 1 - ic\frac{\delta}{|\widetilde{x}|^2}.$$

Assume also that the spectral parameter $z \in \mathbf{C}$ satisfies $|z| \leq C$, for some fixed $C > 0$. An application of Proposition 3 of [11], as in (6.15) in [11], shows that the scalar product

$$\left(\chi m(P_\varepsilon - \widetilde{h}z)\widetilde{u}, \widetilde{u}\right)_{\widetilde{\Phi}_{\delta,\varepsilon},\widetilde{h}} \tag{3.34}$$

is equal to

$$\int \chi(\widetilde{x}) m(\widetilde{x}) p_\varepsilon\left(\widetilde{x}, \frac{2}{i}\frac{\partial \widetilde{\Phi}_{\delta,\varepsilon}(\widetilde{x})}{\partial \widetilde{x}}\right) |\widetilde{u}(\widetilde{x})|^2 e^{-2\widetilde{\Phi}_{\delta,\varepsilon}(\widetilde{x})/\widetilde{h}} L(d\widetilde{x}) + \mathcal{O}(\widetilde{h}) || \widetilde{u} ||^2_{\widetilde{\Phi}_{\delta,\varepsilon},\widetilde{h}}.$$

Thus,

$$\begin{aligned} \operatorname{Re} &\left(\chi m(P_\varepsilon - \widetilde{h}z)\widetilde{u}, \widetilde{u}\right)_{\widetilde{\Phi}_{\delta,\varepsilon},\widetilde{h}} \\ &= \int \chi(\widetilde{x}) \operatorname{Re}\left(m(\widetilde{x}) p_\varepsilon\left(\widetilde{x}, \frac{2}{i}\frac{\partial \widetilde{\Phi}_{\delta,\varepsilon}(\widetilde{x})}{\partial \widetilde{x}}\right)\right) |\widetilde{u}(\widetilde{x})|^2 e^{-2\widetilde{\Phi}_{\delta,\varepsilon}(\widetilde{x})/\widetilde{h}} L(d\widetilde{x}) \\ &\qquad + \mathcal{O}(\widetilde{h}) || \widetilde{u} ||^2_{\widetilde{\Phi}_{\delta,\varepsilon},\widetilde{h}}, \end{aligned}$$

and using that (3.33) holds near the support of χ, we get, by an application of the Cauchy-Schwarz inequality,

$$\begin{aligned} &\int \chi(\widetilde{x}) |\widetilde{u}(\widetilde{x})|^2 e^{-2\widetilde{\Phi}_{\delta,\varepsilon}(\widetilde{x})/\widetilde{h}} L(d\widetilde{x}) \\ &\qquad \leq \mathcal{O}(1) || \chi(P_\varepsilon - \widetilde{h}z)\widetilde{u} ||_{\widetilde{\Phi}_{\delta,\varepsilon},\widetilde{h}} || \widetilde{u} ||_{\widetilde{\Phi}_{\delta,\varepsilon},\widetilde{h}} + \mathcal{O}(\widetilde{h}) || \widetilde{u} ||^2_{\widetilde{\Phi}_{\delta,\varepsilon},\widetilde{h}}. \end{aligned}$$

Coming back to the original variable $x = \sqrt{\varepsilon}\widetilde{x}$ and using that

$$|| \, \chi(P_\varepsilon - \widetilde{h}z)\widetilde{u} \, ||_{\widetilde{\Phi}_{\delta,\varepsilon},\widetilde{h}} = \frac{1}{\varepsilon} || \, \chi\left(\frac{\cdot}{\sqrt{\varepsilon}}\right) (p^w(x, hD_x) - hz) \, u \, ||_{\Phi_{\delta,\varepsilon},h},$$

we obtain that

$$\varepsilon \int \chi\left(\frac{x}{\sqrt{\varepsilon}}\right) |u(x)|^2 \, e^{-2\Phi_{\delta,\varepsilon}(x)/h} \, L(dx)$$
$$\leq \mathcal{O}(1) || \, \chi\left(\frac{\cdot}{\sqrt{\varepsilon}}\right) (p^w(x, hD_x) - hz) u \, ||_{\Phi_{\delta,\varepsilon},h} \, || \, u \, ||_{\Phi_{\delta,\varepsilon},h} + \mathcal{O}(h) || \, u \, ||^2_{\Phi_{\delta,\varepsilon},h}.$$

An application of (3.24) then gives,

$$\varepsilon \int \chi\left(\frac{x}{\sqrt{\varepsilon}}\right) |u(x)|^2 \, e^{-2\Phi_{\delta,\varepsilon}(x)/h} \, L(dx) \leq \mathcal{O}(1) || \, v \, ||_{\Phi_{\delta,\varepsilon},h} \, || \, u \, ||_{\Phi_{\delta,\varepsilon},h}$$
$$+ \mathcal{O}(1) || \, \chi\left(\frac{\cdot}{\sqrt{\varepsilon}}\right) R_{-,h} u_- \, ||_{\Phi_{\delta,\varepsilon},h} \, || \, u \, ||_{\Phi_{\delta,\varepsilon},h} + \mathcal{O}(h) || \, u \, ||^2_{\Phi_{\delta,\varepsilon},h}.$$

Using Proposition 2.2, together with (3.7) and (3.17), we easily see that

$$|| \, \chi\left(\frac{\cdot}{\sqrt{\varepsilon}}\right) R_{-,h} u_- \, ||_{\Phi_{\delta,\varepsilon},h} = \mathcal{O}\left(\left(\frac{h}{\varepsilon}\right)^\infty\right) |u_-| \, . \tag{3.35}$$

Recalling that $\varepsilon = Ah$, we obtain the following exterior estimate,

$$h \int \chi\left(\frac{x}{\sqrt{Ah}}\right) |u(x)|^2 \, e^{-2\Phi_{\delta,\varepsilon}(x)/h} \, L(dx) \leq \mathcal{O}(1) || \, v \, ||_{\Phi_{\delta,\varepsilon},h} \, || \, u \, ||_{\Phi_{\delta,\varepsilon},h}$$
$$+ \mathcal{O}(A^{-\infty}) \, |u_-| \; || \, u \, ||_{\Phi_{\delta,\varepsilon},h} + \mathcal{O}\left(\frac{h}{A}\right) || \, u \, ||^2_{\Phi_{\delta,\varepsilon},h}. \tag{3.36}$$

The estimates (3.27) and (3.36) are the main results established in this subsection.

3.3 End of the proof of Theorem 1.1

In this subsection, we shall glue together the estimates (3.27) and (3.36), in order to show the well-posedness of the global Grushin problem (3.24). Applying the exterior estimate (3.36) to estimate the last term occurring in the right hand side of (3.27) and adding the estimates (3.27) and (3.36), we obtain that

$$
\begin{aligned}
h\|u\|^2 + h^{-1}|u_-|^2 \leq \frac{\mathcal{O}(1)}{h}\|v\|^2 + \mathcal{O}(1)\|v\|\,\|u\| \\
+ \mathcal{O}(h)\,|v_+|^2 + \mathcal{O}(A^{-\infty})\,|u_-|\,\|u\| + \left(\mathcal{O}_A(h^2) + \mathcal{O}\left(\frac{h}{A}\right)\right)\|u\|^2. \quad (3.37)
\end{aligned}
$$

Here we have also used that we arrange, as we may, that $\chi + \chi_0^2 \geq 1$ on $\mathbf{C}^n$. Now

$$
\mathcal{O}(1)\|v\|\,\|u\| + \mathcal{O}(A^{-\infty})\,|u_-|\,\|u\| \leq \frac{\mathcal{O}(1)}{h}\|v\|^2 + \mathcal{O}(A^{-\infty})h^{-1}\,|u_-|^2 + \frac{h}{2}\|u\|^2,
$$

and it follows that

$$
\begin{aligned}
\frac{h^2}{2}\|u\|^2 + |u_-|^2 \leq \mathcal{O}(1)\|v\|^2 + \mathcal{O}(h^2)\,|v_+|^2 \\
+ \mathcal{O}(A^{-\infty})\,|u_-|^2 + \left(\mathcal{O}_A(h^3) + \mathcal{O}\left(\frac{h^2}{A}\right)\right)\|u\|^2.
\end{aligned}
$$

Taking the parameter A sufficiently large but fixed, and then restricting the attention to the interval $h \in (0, h_0]$, for some $h_0 > 0$ small enough depending on A, we obtain that

$$
h\|u\| + |u_-| \leq \mathcal{O}(1)\|v\| + \mathcal{O}(h)\,|v_+|\,. \quad (3.38)
$$

Here the norms throughout are taken in the space $H_{\Phi_{\delta,\varepsilon},h}(\mathbf{C}^n)$, and according to (3.17), the weight function $\Phi_{\delta,\varepsilon}$ can be replaced by the standard quadratic weight Φ_0, at the expense of an $\mathcal{O}(1)$–loss. The Grushin operator

$$
\widetilde{\mathcal{P}}(z;h) = \begin{pmatrix} \widetilde{P} - hz & R_{-,h} \\ R_{+,h} & 0 \end{pmatrix} : H_{\Phi_0,h}(\mathbf{C}^n) \times \mathbf{C}^{N_0} \to H_{\Phi_0,h}(\mathbf{C}^n) \times \mathbf{C}^{N_0} \quad (3.39)
$$

is therefore injective. On the other hand, being a finite rank perturbation of the Fredholm operator

$$
\begin{pmatrix} \widetilde{P} - hz & 0 \\ 0 & 0 \end{pmatrix} : H_{\Phi_0,h}(\mathbf{C}^n) \times \mathbf{C}^{N_0} \to H_{\Phi_0,h}(\mathbf{C}^n) \times \mathbf{C}^{N_0},
$$

the operator in (3.39) is also Fredholm, and furthermore, as observed in the introduction, the index is zero. It follows that the Grushin operator in (3.39) is invertible, so that the problem (3.24) is well-posed, for $z \in \operatorname{neigh}(\lambda_0 + p_1(0), \mathbf{C})$.

The inverse of the operator in (3.39) is of the form

$$\widetilde{\mathcal{E}}(z;h) = \begin{pmatrix} E(z;h) & E_+(z;h) \\ E_-(z;h) & E_{-+}(z;h) \end{pmatrix} : H_{\Phi_0,h}(\mathbf{C}^n)\times\mathbf{C}^{N_0} \to H_{\Phi_0,h}(\mathbf{C}^n)\times\mathbf{C}^{N_0}, \quad (3.40)$$

and it follows from (3.38) that

$$E(z;h) = \mathcal{O}\left(\frac{1}{h}\right) : H_{\Phi_0,h}(\mathbf{C}^n) \to H_{\Phi_0,h}(\mathbf{C}^n),$$

with

$$E_+(z;h) = \mathcal{O}(1) : \mathbf{C}^{N_0} \to H_{\Phi_0,h}(\mathbf{C}^n), \quad E_-(z;h) = \mathcal{O}(1) : H_{\Phi_0,h}(\mathbf{C}^n) \to \mathbf{C}^{N_0},$$

and

$$E_{-+}(z;h) = \mathcal{O}(h) : \mathbf{C}^{N_0} \to \mathbf{C}^{N_0}.$$

Furthermore, let us recall from [25] that hz, with $z \in \mathrm{neigh}(\lambda_0 + p_1(0), \mathbf{C})$, is an eigenvalue of P precisely when the determinant of $E_{-+}(z;h)$ vanishes.

In Section 11.5 of [8], the action of the Grushin operator in (3.39), after applying the inverse FBI–Bargmann transformation, was studied in detail, on spaces of functions of the form

$$\left(a(x;h)e^{i\Phi(x)/h}, u_-\right),$$

where $a(x;h)$ is a symbol and the quadratic form Φ has been introduced in (2.32). It was deduced there that $E_{-+}(z;h)$ has an asymptotic expansion in half–integer powers of h, with a certain additional structure. A complete asymptotic expansion for the determinant of $E_{-+}(z;h)$ was subsequently obtained and it was shown that it is a classical symbol of order 0, and complete asymptotic expansions for the zeros of the determinant were obtained using Puiseux series. That discussion goes through without any changes in the present situation, and therefore, repeating the arguments of Section 11.5 of [8] as they stand, we obtain that the eigenvalues of $h^{-1}P$ in a sufficiently small but fixed neighborhood of $\lambda_0 + p_1(0)$ have complete asymptotic expansions in powers of h^{1/N_0}, of the form

$$\lambda(h) = \lambda_0 + p_1(0) + c_1 h^{1/N_0} + c_2 h^{2/N_0} + \dots.$$

On the other hand, from the main result of [11] and the discussion in the introduction, we know that for all $h > 0$ small enough, the spectrum of P in the disc $D(0, Ch)$, is contained in the union of the regions

$$D\left(h(\lambda_0 + p_1(0)), \frac{h}{\widetilde{C}}\right),$$

where λ_0 is an eigenvalue of $q(x, D_x)$ with $|\lambda_0 + p_1(0)| < C$, and $\widetilde{C} > 0$ is a sufficiently large constant. The statement of Theorem 1.1 follows and this completes the proof.

4 Proof of Theorem 1.2

We shall begin this section by explaining that it is actually sufficient to establish Theorem 1.2 in the special case when $m = 1$. Indeed, when assuming that Theorem 1.2 has already been proved when $m = 1$, we may consider an order function $m \geq 1$ as in (1.10) such that $m \in S(m)$; and a symbol $P(x, \xi; h)$ satisfying the associated assumptions of Theorem 1.2. Then, one can choose a symbol $\widetilde{p_0} \in S(1)$ with a non-negative real part $\mathrm{Re}\, \widetilde{p_0} \geq 0$ which is elliptic near infinity in the symbol class $S(1)$; and such that $\widetilde{p_0} = p_0$ on a large compact set containing $p_0^{-1}(0)$ where p_0 stands for the principal symbol of $P(x, \xi; h)$. This is for instance the case when taking $\chi_0 \in C_0^\infty(\mathbf{R}^{2n}; [0, 1])$ such that $\chi_0 = 1$ near $p_0^{-1}(0)$ and setting

$$\widetilde{p_0} = \chi_0 p_0 + (1 - \chi_0).$$

Defining also the symbols

$$\widetilde{p_j} = \chi_0 p_j + (1 - \chi_0) \in S(1),$$

when $j \geq 1$, we may choose $\chi \in C_0^\infty(\mathbf{R}^{2n}, [0, 1])$ such that $\chi = 1$ near $p_0^{-1}(0)$ and $\chi_0 = 1$ near $\mathrm{supp}\, \chi$. By setting $P = P^w(x, hD_x; h)$ and $\tilde{P} = \tilde{P}^w(x, hD_x; h)$, where

$$\tilde{P}(x, \xi; h) \sim \sum_{j=0}^{+\infty} \tilde{p}_j(x, \xi) h^j,$$

in the symbol class $S(1)$; and using L^2–norms throughout, we deduce from the semiclassical elliptic regularity that

$$\begin{aligned} & h^{\frac{2k_0}{2k_0+1}} |z|^{\frac{1}{2k_0+1}} \|u\| \\ \leq\ & h^{\frac{2k_0}{2k_0+1}} |z|^{\frac{1}{2k_0+1}} \|\chi^w(x, hD_x)u\| + h^{\frac{2k_0}{2k_0+1}} |z|^{\frac{1}{2k_0+1}} \|(1-\chi)^w(x, hD_x)u\| \\ \leq\ & h^{\frac{2k_0}{2k_0+1}} |z|^{\frac{1}{2k_0+1}} \|\chi^w(x, hD_x)u\| + \mathcal{O}(h^{\frac{2k_0}{2k_0+1}} |z|^{\frac{1}{2k_0+1}}) \|(P-z)u\| + \mathcal{O}(h^\infty)\|u\|, \end{aligned}$$

when $|z| \leq C_0$, for $0 < C_0 \ll 1$, since the principal symbol p_0 of the operator P is elliptic near the support of the function $1 - \chi$. By using that Theorem 1.2 is valid when $m=1$, we may apply it to the operator $\widetilde{P}$ to get that if z is as in Theorem 1.2,

$$h^{\frac{2k_0}{2k_0+1}}|z|^{\frac{1}{2k_0+1}}\|\chi^w(x,hD_x)u\| \leq \mathcal{O}(1)\|(\widetilde{P}-z)\chi^w(x,hD_x)u\| \qquad (4.1)$$
$$\leq \mathcal{O}(1)\|(P-z)\chi^w(x,hD_x)u\| + \mathcal{O}(h^\infty)\|u\|,$$

since $(\widetilde{P}-P)\chi^w(x,hD_x) = \mathcal{O}(h^\infty)$ in $\mathcal{L}(L^2)$ when $h \to 0^+$. We get that

$$h^{\frac{2k_0}{2k_0+1}}|z|^{\frac{1}{2k_0+1}}\|\chi^w(x,hD_x)u\| \leq \mathcal{O}(1)\|(P-z)u\| + \mathcal{O}(1)\|[P,\chi^w(x,hD_x)]u\|$$
$$+\mathcal{O}(h^\infty)\|u\|. \quad (4.2)$$

When estimating the commutator term in the right hand side of (4.2), we take $\widetilde{\chi} \in C_0^\infty(\mathbf{R}^{2n},[0,1])$ such that $\widetilde{\chi}=1$ near $p_0^{-1}(0)$ and $\chi=1$ near $\mathrm{supp}\,\widetilde{\chi}$. Then, by using that

$$[P,\chi^w(x,hD_x)]\widetilde{\chi}^w(x,hD_x) = \mathcal{O}(h^\infty),$$

in $\mathcal{L}(L^2)$, together with the fact that p_0 is elliptic near the support of $1-\widetilde{\chi}$, we get that

$$h^{\frac{2k_0}{2k_0+1}}|z|^{\frac{1}{2k_0+1}}\|\chi^w(x,hD_x)u\| \leq \mathcal{O}(1)\|(P-z)u\| + \mathcal{O}(h^\infty)\|u\|, \qquad (4.3)$$

which in view of previous estimates completes the proof of the reduction to the case when $m=1$. In what follows, we shall therefore be concerned exclusively with the case when $m=1$.

Consider p_0 a symbol in the class $S(1)$ independent of the semiclassical parameter with a non-negative real part

$$\mathrm{Re}\ p_0 \geq 0.$$

We assume that all the hypothesis (1.15), (1.16), (1.17), (1.24) and (1.25) are fulfilled; and study the operator $p_0^w(x,hD_x)$ defined by the h-Weyl quantization of the symbol $p_0(x,\xi)$ with the following choice for the normalization of the Weyl quantization

$$p_0^w(x,hD_x)u(x) = \int_{\mathbf{R}^{2n}} e^{2\pi i(x-y).\xi} p_0\Big(\frac{x+y}{2},h\xi\Big)u(y)dyd\xi. \qquad (4.4)$$

This normalization differs from the one considered in (1.14); but, of course, it is completely equivalent, after a rescaling of the semiclassical parameter, to prove Theorem 1.2 with the normalizations (1.14) or (4.4). This choice is for convenience only. Writing

$$p_0(X_j+Y) = q_j(Y) + \mathcal{O}(Y^3),$$

when $Y \to 0$; where q_j denotes the quadratic approximation which begins the Taylor expansion of the symbol p_0 at the doubly characteristic point X_j and recalling (1.26) and (1.27); we notice that the following intersections of kernels are zero

$$\Big(\bigcap_{l=0}^{k_0}\textrm{Ker}\big[\textrm{Re } F_j(\textrm{Im } F_j)^l\big]\Big)\cap\mathbf{R}^{2n}=\{0\}, \tag{4.5}$$

for any $1 \leq j \leq N$. One can deduce, as in [20], that these properties imply that the following sums of k_0+1 non-negative quadratic forms

$$\sum_{l=0}^{k_0}\textrm{Re } q_j\big((\textrm{Im } F_j)^l X\big), \tag{4.6}$$

when $1 \leq j \leq N$, are all positive definite. Indeed, let $X_0 \in \mathbf{R}^{2n}$ be such that

$$\sum_{l=0}^{k_0}\textrm{Re } q_j\big((\textrm{Im } F_j)^l X_0\big)=0.$$

The non-negativity of the quadratic form Re q_j implies that for all $l=0,...,k_0$,

$$\textrm{Re } q_j\big((\textrm{Im } F_j)^l X_0\big)=0. \tag{4.7}$$

By denoting Re $q_j(X;Y)$ the polar form associated to Re q_j, we deduce from the Cauchy-Schwarz inequality, (1.3) and (4.7) that for all $l=0,...,k_0$ and $Y \in \mathbf{R}^{2n}$,

$$\begin{aligned}\big|\textrm{Re } q_j\big(Y;(\textrm{Im } F_j)^l X_0\big)\big|^2 &= \big|\sigma\big(Y,\textrm{Re } F_j(\textrm{Im } F_j)^l X_0\big)\big|^2 \\ &\leq \textrm{Re } q_j(Y)\textrm{Re } q_j\big((\textrm{Im } F_j)^l X_0\big)=0.\end{aligned}$$

It follows that for all $l=0,...,k_0$ and $Y \in \mathbf{R}^{2n}$,

$$\sigma\big(Y,\textrm{Re } F_j(\textrm{Im } F_j)^l X_0\big)=0,$$

which implies that for all $l=0,...,k_0$,

$$\textrm{Re } F_j(\textrm{Im } F_j)^l X_0=0,$$

since σ is non-degenerate. We finally obtain from (4.5) that $X_0=0$, which proves that the quadratic forms (4.6) are positive definite. We then deduce from Proposition 2.0.1 in [20] that there exist real-valued weight functions

$$g_j \in S\big(1,\langle X\rangle^{-\frac{2}{2k_0+1}}dX^2\big), \tag{4.8}$$

and positive constants $c_{1,j}$ and $c_{2,j}$ such that for all $X \in \mathbf{R}^{2n}$,

$$\text{Re } q_j(X) + c_{1,j} H_{\text{Im}q_j} \ g_j(X) + 1 \geq c_{2,j} \langle X \rangle^{\frac{2}{2k_0+1}} \geq c_{2,j} |X|^{\frac{2}{2k_0+1}}, \tag{4.9}$$

where $H_{\text{Im}q_j}$ denotes the Hamilton vector field of Im q_j. We use here the usual notation $S\big(\tilde{m}_h, \tilde{M}_h^{-2} dX^2\big)$, where $\tilde{m}_h$ and $\tilde{M}_h$ are positive functions depending on the semiclassical parameter h, to stand for the symbol class

$$S\big(\tilde{m}_h, \tilde{M}_h^{-2} dX^2\big) = \Big\{ a_h \in C^\infty(\mathbf{R}^{2n}, \mathbf{C}) : \forall \alpha \in \mathbf{N}^{2n}, \exists C_\alpha > 0, \\ \forall X \in \mathbf{R}^{2n}, \forall\ 0 < h \leq 1,\ |\partial_X^\alpha a_h(X)| \leq C_\alpha \tilde{m}_h(X) \tilde{M}_h(X)^{-|\alpha|} \Big\}.$$

Setting

$$g_{j,h}(X) = g_j\Big(\frac{X}{\sqrt{h}}\Big), \tag{4.10}$$

for $0 < h \leq 1$; it follows from (4.9) and the homogeneity properties of the quadratic form q_j that for all $X \in \mathbf{R}^{2n}$ and $0 < h \leq 1$,

$$h\text{Re } q_j\Big(\frac{X}{\sqrt{h}}\Big) + c_{1,j} h (H_{\text{Im}q_j} \ g_j)\Big(\frac{X}{\sqrt{h}}\Big) + h = \text{Re } q_j(X) \\ + c_{1,j} h (H_{\text{Im}q_j} \ g_{j,h})(X) + h \geq c_{2,j} h^{\frac{2k_0}{2k_0+1}} |X|^{\frac{2}{2k_0+1}}, \tag{4.11}$$

since

$$(H_{\text{Im}q_j} \ g_{j,h})(X) = \Big\{ \text{Im } q_j(X), g_j\Big(\frac{X}{\sqrt{h}}\Big) \Big\} = \Big\{ h\text{Im } q_j\Big(\frac{X}{\sqrt{h}}\Big), g_j\Big(\frac{X}{\sqrt{h}}\Big) \Big\} \\ = h \Big\{ \text{Im } q_j\Big(\frac{X}{\sqrt{h}}\Big), g_j\Big(\frac{X}{\sqrt{h}}\Big) \Big\} = \{\text{Im } q_j, g_j\}\Big(\frac{X}{\sqrt{h}}\Big) = (H_{\text{Im}q_j} \ g_j)\Big(\frac{X}{\sqrt{h}}\Big),$$

where $\{p, q\}$ stands for the Poisson bracket

$$\{p, q\} = \frac{\partial p}{\partial \xi} . \frac{\partial q}{\partial x} - \frac{\partial p}{\partial x} . \frac{\partial q}{\partial \xi}.$$

Since $p_0 \in S(1)$, it follows from (1.17) that there exists $c_3 \geq 1$ such that for all $1 \leq j \leq N$ and $X \in \mathbf{R}^{2n}$,

$$|p_0(X)| \leq c_3 |X - X_j|^2 \tag{4.12}$$

and

$$|p_0(X) - z| \geq \frac{|z|}{2} \text{ when } |X - X_j|^2 \leq \frac{|z|}{2c_3}. \tag{4.13}$$

Recalling the assumption (1.24), one can find a positive constant $c_4 > 0$ such that for all $1 \leq j \leq N$ and $|X| \leq c_4$,

$$r_j(X) \in \Gamma, \tag{4.14}$$

where r_j are the symbols defined in (1.23), and Γ is a closed angular sector with vertex at 0 included in the right open half-plane

$$\Gamma \setminus \{0\} \subset \big\{z \in \mathbf{C} : \text{Re } z > 0\big\}.$$

One may assume that

$$0 < c_4 < \inf_{\substack{j,k=1,\dots,N \\ j \neq k}} |X_j - X_k|. \tag{4.15}$$

One can therefore find a positive constant c_5 such that

$$\forall\ 1 \leq j \leq N,\ \forall |X| \leq c_4,\ |\text{Im } r_j(X)| \leq c_5 \text{Re } r_j(X). \tag{4.16}$$

Let ψ be a $C_0^\infty(\mathbf{R}^{2n}, [0,1])$ function such that

$$\psi(X) = 1,\ \text{when } |X| \leq \frac{c_4}{2};\ \text{and supp } \psi \subset \big\{X \in \mathbf{R}^{2n} : |X| \leq c_4\big\}. \tag{4.17}$$

Setting

$$\tilde{r}_j = \psi r_j, \tag{4.18}$$

and recalling the well-known inequality

$$|f'(x)|^2 \leq 2f(x)\|f''\|_{L^\infty(\mathbf{R})}, \tag{4.19}$$

fulfilled by any non-negative smooth function f with a bounded second derivative, we deduce from (4.16) and (4.17) that there exists a positive constant c_6 such that for all $1 \leq j \leq N$ and $X \in \mathbf{R}^{2n}$,

$$|\nabla \text{Re } \tilde{r}_j(X)| \leq c_6 \sqrt{\text{Re } \tilde{r}_j(X)} \tag{4.20}$$

and

$$|c_5 \nabla \text{Re } \tilde{r}_j(X) - \nabla \text{Im } \tilde{r}_j(X)| \leq c_6 \sqrt{c_5 \text{Re } \tilde{r}_j(X) - \text{Im } \tilde{r}_j(X)}.$$

It follows that

$$\begin{aligned} |\nabla \text{Im } \tilde{r}_j(X)| &\leq |c_5 \nabla \text{Re } \tilde{r}_j(X) - \nabla \text{Im } \tilde{r}_j(X)| + c_5 |\nabla \text{Re } \tilde{r}_j(X)| \\ &\leq c_6 \sqrt{c_5 \text{Re } \tilde{r}_j(X) - \text{Im } \tilde{r}_j(X)} + c_5 c_6 \sqrt{\text{Re } \tilde{r}_j(X)}, \end{aligned} \tag{4.21}$$

for all $X \in \mathbf{R}^{2n}$. We deduce from (1.23), (4.11), (4.17) and (4.18) that for all $|Y| \leq \frac{c_4}{2}$ and $0 < h \leq 1$,

$$\begin{aligned}\text{Re } p_0(X_j+Y) + c_{1,j}h\big\{\text{Im } p_0(X_j+Y), g_{j,h}(Y)\big\} - \text{Re } \tilde{r}_j(Y)\\ - c_{1,j}h(H_{\text{Im}\tilde{r}_j}\ g_{j,h})(Y) + h \geq c_{2,j}h^{\frac{2k_0}{2k_0+1}}|Y|^{\frac{2}{2k_0+1}}. \quad (4.22)\end{aligned}$$

Since from (4.8), (4.10) and (4.21),

$$\begin{aligned}h|(H_{\text{Im}\tilde{r}_j}\ g_{j,h})(X)| &\lesssim h|\nabla \text{Im } \tilde{r}_j(X)||\nabla g_{j,h}(X)| \lesssim \sqrt{h}|\nabla \text{Im } \tilde{r}_j(X)|\\ &\lesssim c_6\sqrt{h}\sqrt{c_5 \text{Re } \tilde{r}_j(X) - \text{Im } \tilde{r}_j(X)} + c_5c_6\sqrt{h}\sqrt{\text{Re } \tilde{r}_j(X)},\end{aligned}$$

it follows from (4.22) that there exists a positive constant c_7 such that for all $|Y| \leq \frac{c_4}{2}$ and $0 < h \leq 1$,

$$\begin{aligned}\text{Re } p_0(X_j+Y) + c_{1,j}h\big\{\text{Im } p_0(X_j+Y), g_{j,h}(Y)\big\} + 2h + c_7\text{Re } \tilde{r}_j(Y)\\ + c_7\big(c_5\text{Re } \tilde{r}_j(Y) - \text{Im } \tilde{r}_j(Y)\big) \geq c_{2,j}h^{\frac{2k_0}{2k_0+1}}|Y|^{\frac{2}{2k_0+1}}. \quad (4.23)\end{aligned}$$

Notice from (1.19), (1.23), (4.16), (4.17) and (4.18), that for all $|Y| \leq \frac{c_4}{2}$,

$$\begin{aligned}c_7\text{Re } \tilde{r}_j(Y) + c_7\big(c_5\text{Re } \tilde{r}_j(Y) - \text{Im } \tilde{r}_j(Y)\big) &\leq (2c_5+1)c_7\text{Re } \tilde{r}_j(Y)\\ \leq (2c_5+1)c_7\big(\text{Re } q_j(Y) + \text{Re } \tilde{r}_j(Y)\big) &= (2c_5+1)c_7\text{Re } p_0(X_j+Y).\end{aligned}$$

It follows that for all $|Y| \leq \frac{c_4}{2}$ and $0 < h \leq 1$,

$$\begin{aligned}\big(1+(2c_5+1)c_7\big)\text{Re } p_0(X_j+Y) + c_{1,j}h\big\{\text{Im } p_0(X_j+Y), g_{j,h}(Y)\big\} + 2h\\ \geq c_{2,j}h^{\frac{2k_0}{2k_0+1}}|Y|^{\frac{2}{2k_0+1}}. \quad (4.24)\end{aligned}$$

Let $C_0 \geq 1$ be a fixed constant. Then, by introducing the real-valued weight function

$$g_h(X) = \sum_{j=1}^{N} c_{1,j}\psi\big(2(X-X_j)\big)g_{j,h}(X-X_j), \quad (4.25)$$

where ψ is the function defined in (4.17); and noticing from (4.8) and (4.10) that

$$h\Big(\sum_{j=1}^{N} c_{1,j}g_{j,h}(X-X_j)\Big)H_{\text{Im}p_0}\big[\psi\big(2(X-X_j)\big)\big] = \mathcal{O}(h),$$

we deduce from (1.15), (1.16), (4.15) and (4.24) that there exist some positive constant c_8, c_9 and h_0 such that for all $X \in \mathbf{R}^{2n}$ and $0 < h \leq h_0$,

$$\text{Re } p_0(X) + h(H_{\text{Im}p_0}\ g_h)(X) + c_8 h \geq c_9 h^{\frac{2k_0}{2k_0+1}} \min\left[C_0^{\frac{1}{2}}, (4c_3)^{\frac{1}{2}}\delta(X)\right]^{\frac{2}{2k_0+1}}, \quad (4.26)$$

where δ stands for the distance to the set $(\text{Re } p_0)^{-1}(0)$. Since $\text{Re } p_0 \geq 0$, we may also assume according to (4.8), (4.9), (4.10) and (4.25) that

$$\sup_{X \in \mathbf{R}^{2n}} |g_h(X)| \leq \frac{3}{4\pi}, \quad (4.27)$$

for all $0 < h \leq h_0$. Let z be in $\mathbf{C}$ and $0 < h \leq h_0$. We shall use a multiplier method inspired by the one used by F. Hérau, J. Sjöstrand and C. Stolk in [8]. By using the Wick quantization whose definition and properties are recalled in Section A, one can write that

$$\begin{aligned} & \text{Re}\big([p_0(\sqrt{h}X) - z]^{\text{Wick}}u, [2 - g_h(\sqrt{h}X)]^{\text{Wick}}u\big) \quad (4.28)\\ = {} & \text{Re}\big([2 - g_h(\sqrt{h}X)]^{\text{Wick}}[p_0(\sqrt{h}X) - z]^{\text{Wick}}u, u\big) \\ = {} & \big(\text{Re}\big([2 - g_h(\sqrt{h}X)]^{\text{Wick}}[p_0(\sqrt{h}X) - z]^{\text{Wick}}\big)u, u\big). \end{aligned}$$

since real Hamiltonians get quantized in the Wick quantization by formally selfadjoint operators on $L^2(\mathbf{R}^n)$. Notice also from (4.8), (4.10) and (4.25) that

$$g_h(\sqrt{h}X) \in S(1, dX^2), \quad (4.29)$$

uniformly with respect to the parameter $0 < h \leq h_0$. We deduce from symbolic calculus in the Wick quantization (A.10) that

$$\begin{aligned} & \text{Re}\big([2 - g_h(\sqrt{h}X)]^{\text{Wick}}[p_0(\sqrt{h}X) - z]^{\text{Wick}}\big) \quad (4.30)\\ = {} & \Big[\big(\ 2 - g_h(\sqrt{h}X)\big)\big(\text{Re } p_0(\sqrt{h}X) - \text{Re } z\big) + \frac{\sqrt{h}}{4\pi}\nabla\big(g_h(\sqrt{h}X)\big).(\nabla \text{Re } p_0)(\sqrt{h}X) \\ & + \frac{1}{4\pi}h(H_{\text{Im}p_0}\ g_h)(\sqrt{h}X)\Big]^{\text{Wick}} + S_h, \end{aligned}$$

with $\|S_h\|_{\mathcal{L}(L^2)} = \mathcal{O}(h)$. Since from (4.19) and (4.29), we have

$$\Big|\frac{\sqrt{h}}{4\pi}\nabla\big(g_h(\sqrt{h}X)\big).(\nabla \text{Re } p_0)(\sqrt{h}X)\Big| \lesssim \sqrt{h}\sqrt{\text{Re } p_0(\sqrt{h}X)} \leq \text{Re } p_0(\sqrt{h}X) + \mathcal{O}(h),$$

it follows from (4.27) that there exists a positive constant c_{10} such that for all $X \in \mathbf{R}^{2n}$ and $0 < h \leq h_0$,

$$\big(2-g_h(\sqrt{h}X)\big)\big(\text{Re } p_0(\sqrt{h}X)-\text{Re } z\big)+\frac{\sqrt{h}}{4\pi}\nabla\big(g_h(\sqrt{h}X)\big).(\nabla\text{Re } p_0)(\sqrt{h}X)$$
$$+\frac{1}{4\pi}h(H_{\text{Im}p_0}\ g_h)(\sqrt{h}X) \geq$$
$$\frac{1}{4\pi}\text{Re } p_0(\sqrt{h}X)+\frac{1}{4\pi}h(H_{\text{Im}p_0}\ g_h)(\sqrt{h}X)-c_{10}h-\frac{9}{4}\max(0,\text{Re } z).$$

It follows from (4.26) that for all $X \in \mathbf{R}^{2n}$ and $0 < h \leq h_0$,

$$\begin{aligned}&\big(2-g_h(\sqrt{h}X)\big)\big(\text{Re } p_0(\sqrt{h}X)-\text{Re } z\big)+\frac{\sqrt{h}}{4\pi}\nabla\big(g_h(\sqrt{h}X)\big).(\nabla\text{Re } p_0)(\sqrt{h}X)\\&+\frac{1}{4\pi}h(H_{\text{Im}p_0}\ g_h)(\sqrt{h}X) \geq -\Big(\frac{1}{4\pi}c_8+c_{10}\Big)h-\frac{9}{4}\max(0,\text{Re } z)\\&+\frac{1}{4\pi}c_9h^{\frac{2k_0}{2k_0+1}}\min\big[C_0^{\frac{1}{2}},(4c_3)^{\frac{1}{2}}\delta(\sqrt{h}X)\big]^{\frac{2}{2k_0+1}}.\end{aligned}$$

We then obtain that for all $X \in \mathbf{R}^{2n}$ and $0 < h \leq h_0$,

$$\begin{aligned}&\big(2-g_h(\sqrt{h}X)\big)\big(\text{Re } p_0(\sqrt{h}X)-\text{Re } z\big)+\frac{\sqrt{h}}{4\pi}\nabla\big(g_h(\sqrt{h}X)\big).(\nabla\text{Re } p_0)(\sqrt{h}X)\\&+\frac{1}{4\pi}h(H_{\text{Im}p_0}\ g_h)(\sqrt{h}X) \geq \frac{1}{8\pi}c_9h^{\frac{2k_0}{2k_0+1}}|z|^{\frac{1}{2k_0+1}}\\&+\frac{1}{4\pi}c_9h^{\frac{2k_0}{2k_0+1}}\Big(\min\big[C_0^{\frac{1}{2}},(4c_3)^{\frac{1}{2}}\delta(\sqrt{h}X)\big]^{\frac{2}{2k_0+1}}-|z|^{\frac{1}{2k_0+1}}\Big)\\&+\frac{9}{4}\Big(\frac{1}{18\pi}c_9h^{\frac{2k_0}{2k_0+1}}|z|^{\frac{1}{2k_0+1}}-\max(0,\text{Re } z)\Big)-\Big(\frac{1}{4\pi}c_8+c_{10}\Big)h.\end{aligned}$$

Considering the set

$$\Omega_{C,h}=\Big\{z \in \mathbf{C} : \text{Re } z \leq \frac{1}{18\pi}c_9h^{\frac{2k_0}{2k_0+1}}|z|^{\frac{1}{2k_0+1}},\ Ch \leq |z| \leq C_0\Big\}, \tag{4.31}$$

where $C \gg 1$ is a large constant whose value will be chosen later, and $\varphi \in C_0^\infty(\mathbf{R},[0,1])$ such that

$$\varphi(X)=1 \text{ when } |X| \leq \frac{1}{4c_3}, \text{ and supp } \varphi \subset \Big\{X \in \mathbf{R} : |X| \leq \frac{1}{3c_3}\Big\}, \tag{4.32}$$

we notice that for all $X \in \mathbf{R}^{2n}$, $0 < h \leq h_0$, $C \geq 1$ and $z \in \Omega_{C,h}$,

$$\frac{1}{4\pi} c_9 h^{\frac{2k_0}{2k_0+1}} \Big(\min \big[C_0^{\frac{1}{2}}, (4c_3)^{\frac{1}{2}} \delta(\sqrt{h}X) \big]^{\frac{2}{2k_0+1}} - |z|^{\frac{1}{2k_0+1}} \Big)$$
$$+ \frac{9}{4} \Big(\frac{1}{18\pi} c_9 h^{\frac{2k_0}{2k_0+1}} |z|^{\frac{1}{2k_0+1}} - \max(0, \mathrm{Re}\ z) \Big) \geq -\frac{1}{4\pi} c_9 h^{\frac{2k_0}{2k_0+1}} |z|^{\frac{1}{2k_0+1}} \varphi\Big(\frac{\delta(\sqrt{h}X)^2}{|z|} \Big).$$

By noticing now that one can find a $C_0^\infty(\mathbf{R}^{2n}, [0,1])$ function Φ such that

$$\Phi(X) = 1 \text{ when } |X| \leq \frac{1}{\sqrt{4c_3}}, \text{ and supp } \Phi \subset \Big\{ X \in \mathbf{R}^{2n} : |X| \leq \frac{1}{\sqrt{3c_3}} \Big\}; \tag{4.33}$$

verifying for all $X \in \mathbf{R}^{2n}$, $0 < h \leq h_0$ and $z \in \Omega_{C,h}$,

$$\varphi\Big(\frac{\delta(\sqrt{h}X)^2}{|z|} \Big) \leq \sum_{j=1}^N \Phi\Big(\frac{\sqrt{h}X - X_j}{\sqrt{|z|}} \Big),$$

we get that for all $X \in \mathbf{R}^{2n}$, $0 < h \leq h_0$, $C \geq 1$ and $z \in \Omega_{C,h}$,

$$\big(2 - g_h(\sqrt{h}X)\big)\big(\mathrm{Re}\ p_0(\sqrt{h}X) - \mathrm{Re}\ z\big)$$
$$+ \frac{\sqrt{h}}{4\pi} \nabla\big(g_h(\sqrt{h}X)\big).(\nabla \mathrm{Re}\ p_0)(\sqrt{h}X) + \frac{1}{4\pi} h (H_{\mathrm{Im} p_0}\ g_h)(\sqrt{h}X)$$
$$\geq \frac{1}{8\pi} c_9 h^{\frac{2k_0}{2k_0+1}} |z|^{\frac{1}{2k_0+1}} - \Big(\frac{1}{4\pi} c_8 + c_{10} \Big) h - \frac{1}{4\pi} c_9 h^{\frac{2k_0}{2k_0+1}} |z|^{\frac{1}{2k_0+1}} \sum_{j=1}^N \Phi\Big(\frac{\sqrt{h}X - X_j}{\sqrt{|z|}} \Big).$$

It follows from (4.28), (4.30) and (A.4) that there exist some positive constants c_{11} and c_{12} such that for all $0 < h \leq h_0$, $C \geq 1$, $z \in \Omega_{C,h}$ and $u \in \mathcal{S}(\mathbf{R}^n)$,

$$\mathrm{Re}\big([p_0(\sqrt{h}X) - z]^{\mathrm{Wick}} u, [2 - g_h(\sqrt{h}X)]^{\mathrm{Wick}} u \big) + c_{11} h \|u\|_{L^2}^2$$
$$+ c_{11} h^{\frac{2k_0}{2k_0+1}} |z|^{\frac{1}{2k_0+1}} \sum_{j=1}^N \Big(\Phi\Big(\frac{\sqrt{h}X - X_j}{\sqrt{|z|}} \Big)^{\mathrm{Wick}} u, u \Big) \geq c_{12} h^{\frac{2k_0}{2k_0+1}} |z|^{\frac{1}{2k_0+1}} \|u\|_{L^2}^2. \tag{4.34}$$

Recalling (4.29) and (4.31), we deduce from the Cauchy-Schwarz inequality and (A.5) that there exist some positive constants c_{13}, c_{14} and c_{15} such that for all $0 < h \leq h_0$, $C \geq c_{13}$, $z \in \Omega_{C,h}$ and $u \in \mathcal{S}(\mathbf{R}^n)$,

$$c_{15} h^{\frac{2k_0}{2k_0+1}} |z|^{\frac{1}{2k_0+1}} \sum_{j=1}^N \Big\| \Phi\Big(\frac{\sqrt{h}X - X_j}{\sqrt{|z|}} \Big)^{\mathrm{Wick}} u \Big\|_{L^2}$$
$$+ \big\| p_0(\sqrt{h}X)^{\mathrm{Wick}} u - zu \big\|_{L^2} \geq c_{14} h^{\frac{2k_0}{2k_0+1}} |z|^{\frac{1}{2k_0+1}} \|u\|_{L^2}. \tag{4.35}$$

Since from (4.33), we have

$$\Phi\Big(\frac{\sqrt{h}X-X_j}{\sqrt{|z|}}\Big)\in S\Big(1,\frac{|z|}{h}dX^2\Big), \tag{4.36}$$

when $1\leq j\leq N$, we notice from (4.31), (A.8) and (A.9) that

$$\Big\|\Phi\Big(\frac{\sqrt{h}X-X_j}{\sqrt{|z|}}\Big)^{\mathrm{Wick}}u\Big\|_{L^2}=\Big\|\Phi\Big(\frac{\sqrt{h}X-X_j}{\sqrt{|z|}}\Big)^{w}u\Big\|_{L^2}+\mathcal{O}\Big(\frac{1}{C}\Big)\|u\|_{L^2}$$

and

$$\|p_0(\sqrt{h}X)^{\mathrm{Wick}}u-zu\|_{L^2}=\|p_0(\sqrt{h}X)^{w}u-zu\|_{L^2}+\mathcal{O}(h)\|u\|_{L^2}.$$

We deduce from (4.35) that there exist some positive constants c_{16} and c_{17} such that for all $0<h\leq h_0$, $C\geq c_{17}$, $z\in\Omega_{C,h}$ and $u\in\mathcal{S}(\mathbf{R}^n)$,

$$c_{15}h^{\frac{2k_0}{2k_0+1}}|z|^{\frac{1}{2k_0+1}}\sum_{j=1}^{N}\Big\|\Phi\Big(\frac{\sqrt{h}X-X_j}{\sqrt{|z|}}\Big)^{w}u\Big\|_{L^2}+\|p_0(\sqrt{h}X)^{w}u-zu\|_{L^2}\geq c_{16}h^{\frac{2k_0}{2k_0+1}}|z|^{\frac{1}{2k_0+1}}\|u\|_{L^2}. \tag{4.37}$$

We shall now study the quantity

$$\sum_{j=1}^{N}\Big\|\Phi\Big(\frac{\sqrt{h}X-X_j}{\sqrt{|z|}}\Big)^{w}u\Big\|_{L^2}.$$

To do so, we shall establish an a priori estimate similar to the one proved in [8] (Proposition 4.1), namely that for all $0<h\leq h_0$, $C\geq c_{17}$, $z\in\Omega_{C,h}$ and $u\in\mathcal{S}(\mathbf{R}^n)$,

$$\frac{1}{|z|}\|p_0(\sqrt{h}X)^{w}u-zu\|_{L^2}+\mathcal{O}\Big(\sqrt{\frac{h}{|z|}}\Big)\|u\|_{L^2}\gtrsim\sum_{j=1}^{N}\Big\|\Phi\Big(\frac{\sqrt{h}X-X_j}{\sqrt{|z|}}\Big)^{w}u\Big\|_{L^2}. \tag{4.38}$$

In [8], this estimate is proved on the FBI transform side. By using similar arguments, namely a second microlocalization, we shall prove this estimate directly without any use of the FBI transform.

Let Ψ be a $C_0^\infty(\mathbf{R}^{2n},[0,1])$ function such that

$$\Psi=1 \text{ when } |Y|\leq\frac{1}{\sqrt{3c_3}};\ \text{and supp }\Psi\subset\Big\{Y\in\mathbf{R}^{2n}:|Y|\leq\frac{1}{\sqrt{2c_3}}\Big\}.$$

We notice from (4.12) and (4.13) that the symbols

$$\frac{1}{|z|}p_0(\sqrt{|z|}Y+X_j)\Psi(Y),$$

where $1 \leq j \leq N$, are uniformly bounded together with all their derivatives with respect to the parameter z when z belongs to $\Omega_{C,h}$; and that these symbols are elliptic

$$\Big|\frac{1}{|z|}p_0(\sqrt{|z|}Y+X_j)\Psi(Y)-\frac{z}{|z|}\Big| \geq \frac{1}{2},$$

on the set

$$\Big\{Y \in \mathbf{R}^{2n} : |Y| \leq \frac{1}{\sqrt{3c_3}}\Big\}.$$

When quantizing these symbols in the $\tilde{h}$-Weyl quantization with the new semiclassical parameter

$$\tilde{h}=\frac{h}{|z|}, \tag{4.39}$$

we deduce from (4.33) and this ellipticity property that for all $0 < h \leq h_0$, $C \geq c_{17}$, $z \in \Omega_{C,h}$ and $u \in \mathcal{S}(\mathbf{R}^n)$,

$$\big\|\Phi(\sqrt{\tilde{h}}Y)^w u\big\|_{L^2} \leq \mathcal{O}(1)\Big\|\frac{1}{|z|}p_0(\sqrt{|z|}\sqrt{\tilde{h}}Y+X_j)^w u-\frac{z}{|z|}u\Big\|_{L^2}+\mathcal{O}(\tilde{h})\|u\|_{L^2}.$$

We recall from (4.31) that

$$\tilde{h}=\frac{h}{|z|} \leq \frac{1}{C} \ll 1,$$

where the large constant $C \gg 1$ appearing in (4.31) remains to be chosen. One can then deduce from (4.39) and the symplectic invariance property of the Weyl quantization (Theorem 18.5.9 in [12]) while using the following affine symplectic transformation

$$X \mapsto X-\frac{1}{\sqrt{h}}X_j,$$

that for all $0 < h \leq h_0$, $C \geq c_{18}$, $z \in \Omega_{C,h}$ and $u \in \mathcal{S}(\mathbf{R}^n)$,

$$\Big\|\Phi\Big(\frac{\sqrt{h}X-X_j}{\sqrt{|z|}}\Big)^w u\Big\|_{L^2} \leq \mathcal{O}(1)\frac{1}{|z|}\|p_0(\sqrt{h}X)^w u-zu\|_{L^2}+\mathcal{O}\Big(\frac{h}{|z|}\Big)\|u\|_{L^2},$$

where c_{18} is a large positive constant and h_0 a new positive constant with $0 < h_0 \ll 1$. This proves the estimate (4.38). We can next conclude as follows. Noticing from

(4.31) that

$$\mathcal{O}\Big(\sqrt{\frac{h}{|z|}}\Big) = \mathcal{O}\Big(\frac{1}{\sqrt{C}}\Big)$$

and

$$h^{\frac{2k_0}{2k_0+1}}|z|^{\frac{1}{2k_0+1}}\mathcal{O}\Big(\frac{1}{|z|}\Big) = \mathcal{O}\Big(\frac{1}{C^{\frac{2k_0}{2k_0+1}}}\Big),$$

we deduce from (4.37) and (4.38) that there exist some positive constants c_0 and $\tilde{c}_0$ such that for all $0 < h \leq h_0$, $C \geq c_0$, $z \in \Omega_{C,h}$ and $u \in \mathcal{S}(\mathbf{R}^n)$,

$$\big\|p_0(\sqrt{h}X)^w u - zu\big\|_{L^2} \geq \tilde{c}_0 h^{\frac{2k_0}{2k_0+1}}|z|^{\frac{1}{2k_0+1}}\|u\|_{L^2}. \tag{4.40}$$

Finally, we deduce from the symplectic invariance property of the Weyl quantization (Theorem 18.5.9 in [12]) while using the linear symplectic transformation

$$(x,\xi) \mapsto (h^{-\frac{1}{2}}x, h^{\frac{1}{2}}\xi),$$

that we have for all $0 < h \leq h_0$, $C \geq c_0$, $z \in \Omega_{C,h}$ and $u \in \mathcal{S}(\mathbf{R}^n)$,

$$\big\|p_0^w(x,hD_x)u - zu\big\|_{L^2} \geq \tilde{c}_0 h^{\frac{2k_0}{2k_0+1}}|z|^{\frac{1}{2k_0+1}}\|u\|_{L^2}. \tag{4.41}$$

Recalling that $m = 1$ and noticing from the asymptotic expansion (1.11) and the Calderón-Vaillancourt Theorem that

$$\|P^w(x,hD_x;h) - p_0^w(x,hD_x)\|_{\mathcal{L}(L^2)} = \mathcal{O}(h),$$

when $h \to 0$, we finally obtain by possibly increasing the value of the positive constant $c_0 > 0$ that for all $0 < h \leq h_0$, $C \geq c_0$, $z \in \Omega_{C,h}$ and $u \in \mathcal{S}(\mathbf{R}^n)$,

$$\big\|P^w(x,hD_x;h)u - zu\big\|_{L^2} \geq \frac{\tilde{c}_0}{2} h^{\frac{2k_0}{2k_0+1}}|z|^{\frac{1}{2k_0+1}}\|u\|_{L^2}, \tag{4.42}$$

where h_0 is a new positive constant such that $0 < h_0 \ll 1$. This ends the proof of Theorem 1.2.

A Appendix on Wick calculus

The purpose of this section is to recall the definition and basic properties of the Wick quantization that we need for the proof of Theorem 1.2. We follow here the presentation of the Wick quantization given by N. Lerner in [14, 15, 16] and refer the reader to his works for the proofs of the results recalled below.

The main property of the Wick quantization is its property of positivity, i.e., that non-negative Hamiltonians define non-negative operators

$$a \geq 0 \Rightarrow a^{\mathrm{Wick}} \geq 0.$$

We recall that this is not the case for the Weyl quantization and refer to [14] for an example of non-negative Hamiltonian defining an operator which is not non-negative.

Before defining properly the Wick quantization, we first need to recall the definition of the wave packets transform of a function $u \in \mathcal{S}(\mathbf{R}^n)$,

$$Wu(y,\eta) = (u, \varphi_{y,\eta})_{L^2(\mathbf{R}^n)} = 2^{n/4} \int_{\mathbf{R}^n} u(x) e^{-\pi(x-y)^2} e^{-2i\pi(x-y).\eta} dx, \ (y,\eta) \in \mathbf{R}^{2n}.$$

where

$$\varphi_{y,\eta}(x) = 2^{n/4} e^{-\pi(x-y)^2} e^{2i\pi(x-y).\eta}, \ x \in \mathbb{R}^n,$$

and $x^2 = x_1^2 + ... + x_n^2$. With this definition, one can check (See Lemma 2.1 in [14]) that the mapping $u \mapsto Wu$ is continuous from $\mathcal{S}(\mathbf{R}^n)$ to $\mathcal{S}(\mathbf{R}^{2n})$, isometric from $L^2(\mathbf{R}^n)$ to $L^2(\mathbf{R}^{2n})$ and that we have the reconstruction formula

$$\forall u \in \mathcal{S}(\mathbf{R}^n), \forall x \in \mathbf{R}^n, \ u(x) = \int_{\mathbf{R}^{2n}} Wu(y,\eta) \varphi_{y,\eta}(x) dy d\eta. \tag{A.1}$$

We denote by Σ_Y the operator defined in the Weyl quantization by the symbol

$$p_Y(X) = 2^n e^{-2\pi|X-Y|^2}, \ Y = (y,\eta) \in \mathbf{R}^{2n},$$

by using the same normalization

$$(a^w u)(x) = \int_{\mathbf{R}^{2n}} e^{2i\pi(x-y).\xi} a\Big(\frac{x+y}{2}, \xi\Big) u(y) dy d\xi, \tag{A.2}$$

as in [14]. This operator is a rank-one orthogonal projection

$$\big(\Sigma_Y u\big)(x) = Wu(Y)\varphi_Y(x) = (u, \varphi_Y)_{L^2(\mathbf{R}^n)} \varphi_Y(x),$$

and we define the Wick quantization of any $L^\infty(\mathbf{R}^{2n})$ symbol a as

$$a^{\mathrm{Wick}} = \int_{\mathbf{R}^{2n}} a(Y) \Sigma_Y dY. \tag{A.3}$$

More generally, one can extend this definition when the symbol a belongs to $\mathcal{S}'(\mathbf{R}^{2n})$ by defining the operator a^{Wick} for any u and v in $\mathcal{S}(\mathbf{R}^n)$ by

$$< a^{\text{Wick}}u, \overline{v} >_{\mathcal{S}'(\mathbf{R}^n),\mathcal{S}(\mathbf{R}^n)}=< a(Y), (\Sigma_Y u, v)_{L^2(\mathbf{R}^n)} >_{\mathcal{S}'(\mathbf{R}^{2n}),\mathcal{S}(\mathbf{R}^{2n})},$$

where $< \cdot, \cdot >_{\mathcal{S}'(\mathbf{R}^n),\mathcal{S}(\mathbf{R}^n)}$ denotes the duality bracket between the spaces $\mathcal{S}'(\mathbf{R}^n)$ and $\mathcal{S}(\mathbf{R}^n)$. The Wick quantization is a positive quantization

$$a \geq 0 \Rightarrow a^{\text{Wick}} \geq 0. \tag{A.4}$$

In particular, real Hamiltonians get quantized in this quantization by formally self-adjoint operators and one has (See Proposition 3.2 in [14]) that $L^\infty(\mathbf{R}^{2n})$ symbols define bounded operators on $L^2(\mathbf{R}^n)$ such that

$$\|a^{\text{Wick}}\|_{\mathcal{L}(L^2(\mathbf{R}^n))} \leq \|a\|_{L^\infty(\mathbf{R}^{2n})}. \tag{A.5}$$

According to Proposition 3.3 in [14], the Wick and Weyl quantizations of a symbol a are linked by the following identities

$$a^{\text{Wick}} = \tilde{a}^w, \tag{A.6}$$

with

$$\tilde{a}(X) = \int_{\mathbf{R}^{2n}} a(X+Y)e^{-2\pi|Y|^2} 2^n dY,\ X \in \mathbf{R}^{2n}, \tag{A.7}$$

and

$$a^{\text{Wick}} = a^w + r(a)^w, \tag{A.8}$$

where $r(a)$ stands for the symbol

$$r(a)(X) = \int_0^1 \int_{\mathbf{R}^{2n}} (1-\theta)a''(X+\theta Y)Y^2 e^{-2\pi|Y|^2} 2^n dY d\theta,\ X \in \mathbf{R}^{2n}. \tag{A.9}$$

We also recall the following composition formula obtained in the proof of Proposition 3.4 in [14],

$$a^{\text{Wick}} b^{\text{Wick}} = \Big[ab - \frac{1}{4\pi}a'.b' + \frac{1}{4i\pi}\{a,b\}\Big]^{\text{Wick}} + S, \tag{A.10}$$

with $\|S\|_{\mathcal{L}(L^2(\mathbf{R}^n))} \leq d_n \|a\|_{L^\infty} \gamma_2(b)$, when $a \in L^\infty(\mathbf{R}^{2n})$ and b is a smooth symbol satisfying

$$\gamma_2(b) = \sup_{\substack{X \in \mathbf{R}^{2n},\\ T \in \mathbf{R}^{2n}, |T|=1}} |b^{(2)}(X)T^2| < +\infty.$$

The term d_n appearing in the previous estimate stands for a positive constant depending only on the dimension n; and the notation $\{a,b\}$ denotes the Poisson bracket

$$\{a,b\} = \frac{\partial a}{\partial \xi}.\frac{\partial b}{\partial x} - \frac{\partial a}{\partial x}.\frac{\partial b}{\partial \xi}.$$

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