SHARP HYPOELLIPTIC ESTIMATES FOR SOME KINETIC EQUATIONS

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dedicated to our friend Yoshinori Morimoto, on his sixtieth birthday

ABSTRACT. We provide a simple overview of some hypoellipticity results with sharp indices for a class of kinetic equations and we outline a general strategy based on some geometrical properties.

CONTENTS

1. EXAMPLES OF NONSELFADJOINT EQUATIONS

1.1. **Uncertainty relations.** With $D_x = \frac{1}{2i\pi}\frac{d}{dx}$(self-adjoint), and ix(skew-adjoint), we have, with $L^2(\mathbb{R})$ dot-products,

$$2\operatorname{Re}\langle D_x u, ixu\rangle = \langle D_x u, ixu\rangle + \langle ixu, D_x u\rangle = \langle(-ixD_x + D_x ix)u, u\rangle$$

$$2\operatorname{Re}\langle D_x u, ixu\rangle = \langle [D_x, ix]u, u\rangle = \frac{1}{2\pi}\|u\|^2 \Longrightarrow \frac{1}{4\pi}\|u\|^2 \le \|D_x u\|\|xu\|$$

and $\frac{1}{4\pi}$ is the largest constant (check the equality with $e^{-\pi x^2/2}$). As a result, we have

$$\|\frac{h}{2i\pi}\frac{du}{dx}\|\|xu\| \ge \frac{h}{4\pi}\|u\|^2 \quad i.e. \quad \Delta\xi_j\Delta x_j \ge \hbar/2,$$

which are the uncertainty relations. Note also from the previous computations that, with $J = J^*$, $K^* = -K$, we have

$$2\operatorname{Re}\langle Ju, Ku\rangle = \langle Ju, Ku\rangle + \langle Ku, Ju\rangle = \langle(K^*J + J^*K)u, u\rangle$$

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that is $2\operatorname{Re}\langle Ju, Ku\rangle = \langle [J,K]u, u\rangle$. The uncertainty relations are based upon the non-commutation of the operators D_x, ix which are such that

$$[D_x, 2i\pi x] = \mathrm{Id}\,.$$

1.2. **Harmonic oscillator, Coulomb potential, Hardy's inequality.** The one-dimensional Harmonic Oscillator is $D_x^2 + x^2$ and we have with $L^2(\mathbb{R})$ norms and dot-product

$$\langle (D_x^2+x^2)u, u\rangle = \| \underbrace{(D_x - ix)}_{\substack{\text{annihilation}\\ \text{operator}}} u\|^2 + \frac{1}{2\pi}\|u\|^2,$$

because $D_x^2 + x^2 = \underbrace{(D_x+ix)}_{\substack{\text{creation}\\ \text{operator}}}(D_x - ix) + \frac{1}{2\pi}$ and in n dimensions, we get

$$\sum_{1\le j\le n} \pi(D_{x_j}^2 + x_j^2) = \frac{n}{2} + \pi \sum_{1\le j\le n} C_j C_j^* \Longrightarrow \inf \pi(|D_x|^2 + |x|^2) = \frac{n}{2}$$

at the ground state $\phi_0 = e^{-\pi|x|^2} 2^{n/4}$ which solves

$$(D_j - ix_j)\phi_0 = \frac{1}{2i\pi}(\partial_j + 2\pi x_j)\phi_0 = 0.$$

We see as well that $C^\alpha \phi_0 = C_1^{\alpha_1}\dots C_n^{\alpha_n}\phi_0$ is an eigenvector for the n-dimensional harmonic oscillator with eigenvalue $\frac{n}{2} + |\alpha|$, so that we find a discrete spectrum $\frac{n}{2} + \mathbb{N}$ for that operator.

The study of nonselfadjoint operators may also be useful to determine lowerbounds for selfadjoint operators: we have

$$\sum_{1\le j\le n} \|(D_j - i\phi_j)u\|^2 = \langle |D|^2 u, u\rangle + \langle |\phi|^2 u, u\rangle - \frac{1}{2\pi}\langle (\operatorname{div}\phi)u, u\rangle. \tag{1.1}$$

Thus with $\phi = \mu x/|x|$, we get $|D|^2 + \mu^2 \ge \frac{\mu}{2\pi}\frac{(n-1)}{|x|}$ with $\mu = \frac{e^2 m 4\pi}{h^2(n-1)}$, so that

$$\frac{h^2|D|^2}{2m} - \frac{e^2}{|x|} = \frac{h^2|D|^2}{2m} - \frac{\mu h^2}{2\pi 2m}\frac{(n-1)}{|x|} \ge -\frac{\mu^2 h^2}{2m} = -\frac{e^4 m^2 16\pi^2 h^2}{h^4(n-1)^2 2m}$$

which gives

$$\frac{h^2|D|^2}{2m} - \frac{e^2}{|x|} \ge -\frac{m e^4 8\pi^2}{(n-1)^2 h^2} > -\infty, \tag{1.2}$$

and provides a stability result for the unbounded Hamiltonian $|\xi|^2/2m - e^2/|x|$ (here with the best constant). Using again (1.1), we get with $\phi = \nu \frac{x}{2\pi|x|^2}$,

$$|D|^2 + \frac{\nu^2}{4\pi^2|x|^2} \ge \frac{\nu(n-2)}{4\pi^2|x|^2},$$

i.e. $(-\Delta) \ge |x|^{-2} \underbrace{\nu(n-2-\nu)}_{\text{largest at } \nu = \frac{n-2}{2}}$ and thus

$$(-\Delta) \ge \big(\frac{n-2}{2}\big)^2 \frac{1}{|x|^2} \tag{1.3}$$

which is Hardy's inequality.

1.3. **Kolmogorov equation, Fokker-Planck equations.** In a 1934 *Annals of Mathematics* two-page paper (written in German) *"zur Theorie der Brownschen Bewegung"*[12], A.N. Kolmogorov proposed a model for the 1-dimensional Brownian motion with the equation

$$\mathcal{K}u \equiv \frac{\partial u}{\partial t} - y\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial y^2} = f, \quad x = \text{position}, \; y = \text{speed}. \tag{1.4}$$

Introducing the (divergence-free) real vector fields $X_0 = \partial_t - y\partial_x$, $X_1 = \partial_y$ we note that

$$\mathcal{K} = X_0 + X_1^* X_1,$$

and that the tangent space is equal to the Lie algebra generated by X_0, X_1 since $\partial_x = [X_0, X_1]$. The operator is micro-hypoelliptic: we have with C^∞ wave-front-sets, $WFu = WF\mathcal{K}u$. The hypoellipticity follows from a 1967 Hörmander's result [11].

One may ask various questions to elaborate on this result: what is the loss of derivatives with respect to the elliptic case ? What type of a priori estimates can be used to prove hypoellipticity ? Everything can be computed explicitely using the flow of X_0: using the change of variables

$$\begin{cases} t &= s \\ x &= x_1 - s x_2 \\ y &= x_2 \end{cases} \quad \text{we get} \quad \begin{cases} \frac{\partial}{\partial s} &= \frac{\partial}{\partial t} - y\frac{\partial}{\partial x} = X_0 \\ \frac{\partial}{\partial x_1} &= \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x_2} &= -t\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \end{cases} \quad \text{and } X_1 = s\partial_{x_1} + \partial_{x_2},$$

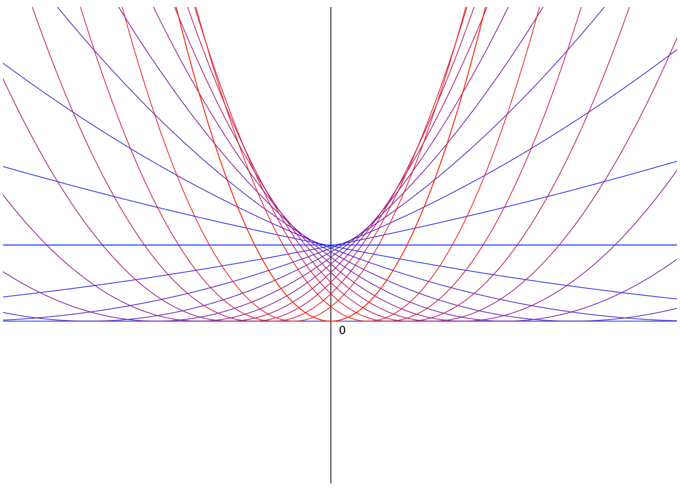
so that

$$\mathcal{K} = \partial_s - (s\partial_{x_1} + \partial_{x_2})^2 = \underbrace{iD_s}_{\text{skew}} + \underbrace{(D_2 + sD_1)^2}_{\text{selfadjoint and } \geq 0}, \tag{1.5}$$

with $D_s = i^{-1}\partial_s$, $D_1 = i^{-1}\partial_{x_1}$ and $D_2 = i^{-1}\partial_{x_2}$. It is easy to solve explicitely that ODE with parameters: using a Fourier transform with respect to the x_1, x_2 variables and we have to deal with

$$\widetilde{\mathcal{K}} = \frac{d}{ds} + (\xi_2 + s\xi_1)^2.$$

It is interesting to look at the family of parabolas $s \mapsto (\xi_2 + s\xi_1)^2$ for $\xi_1^2 + \xi_2^2 = 1$



and to check as a good graphic way to explain hypoellipticity that, although the minimum of these functions is always 0 (except for $\xi_2 = \pm 1, \xi_1 = 0$), the *envelope* has a positive minimum. We have for $\xi_1 \neq 0$

$$\widetilde{\mathcal{K}} = \frac{d}{ds} + {\xi_1}^2 (\xi_2/\xi_1 + s)^2 = i\big(D_\sigma - i\lambda\sigma^2\big), \qquad \sigma = s + \xi_2/\xi_1, \lambda = {\xi_1}^2,$$

and we get the standard subelliptic estimate in $L^2(\mathbb{R}_s)$

$$\|\widetilde{\mathcal{K}} v\| \gtrsim \lambda^{1/3} \|v\| = |\xi_1|^{2/3} \|v\|.$$

Moreover $\operatorname{Re}\langle \widetilde{\mathcal{K}} v, v\rangle = \|(\xi_2 + s\xi_1) v\|^2$ so that, with $L^2(\mathbb{R}_{t,x_1,x_2})$ norms, we get

$$\|u\| + \|\mathcal{K} u\| \gtrsim \||D_1|^{2/3} u\| + \|(D_2 + sD_1) u\|,$$

which is an optimal estimate w.r.t. D_1.

The Fokker-Planck equations read

$$\mathcal{P} = \underbrace{v \cdot \partial_x - \nabla_x V \cdot \partial_v}_{\substack{\text{propagation} \\ \text{skew-adjoint} \\ \text{divergence-free vector field}}} \underbrace{-\Delta_v + \frac{|v|^2}{4} - \frac{d}{2}}_{\substack{\text{harmonic oscillator} \\ \text{self-adjoint } \geq 0 \\ \text{missing the } x \text{ directions}}}$$

With $\mathcal{P} = X_0 + \sum_{1 \leq j \leq d} C_j C_j^*$, we expect that a suitable assumption on the potential $V(x)$ will ensure that the iterated brackets of $X_0, C_1, \dots, C_d$ have some ellipticity property (here the creation operators are the $C_j = \frac{d}{idv_j} + iv_j/2$). Following the heuristics of the previous example, we expect that this equation will be governed by the behaviour of the symbol of the selfadjoint part along the integral curves of the propagator X_0.

2. Pseudodifferential techniques

2.1. **Wick quantization.** The so-called Wick quantization is one of the tool which is quite useful in the investigation of subellipticity properties of differential and pseudodifferential equations. Let us first shortly review that tool. For $X = (x, \xi), Y = (y, \eta) \in \mathbb{R}^{2n}$, we define

$$\Pi(X, Y) = e^{-\frac{\pi}{2}|X-Y|^2} e^{-i\pi[X,Y]} \text{ with } \quad [X, Y] = [(x, \xi), (y, \eta)] = \xi \cdot y - \eta \cdot x. \tag{2.1}$$

We define also for $u \in L^2(\mathbb{R}^n)$,

$$(Wu)(y, \eta) = \langle u, \varphi_{y,\eta} \rangle_{L^2(\mathbb{R}^n)}, \quad \varphi_{y,\eta}(x) = 2^{n/4} e^{-\pi|x-y|^2} e^{2i\pi(x - \frac{y}{2})\cdot\eta}. \tag{2.2}$$

We have $W : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^{2n})$ isometric, not onto, and

$$W^* W = \mathrm{Id}_{L^2(\mathbb{R}^n)} : \quad \text{reconstruction formula, } W \text{ isometric,} \tag{2.3}$$

$$W W^* = \Pi_0 : \quad \text{projection operator onto } \operatorname{ran} W \text{ with kernel } \Pi. \tag{2.4}$$

Let a be a Hamiltonian: we can use the Weyl quantization with the formula

$$(a^w u)(x) = \iint e^{2i\pi\langle x-y, \xi\rangle} a(\frac{x+y}{2}, \xi) u(y) dy d\xi \tag{2.5}$$

or the Wick quantization as given by

$$a^{\mathrm{Wick}} = W^* a W, \tag{2.6}$$

according to the commutative diagram

$$\begin{array}{ccc} L^2(\mathbb{R}^{2n}) & \xrightarrow[\text{(multiplication by } a)]{a} & L^2(\mathbb{R}^{2n}) \\ W \uparrow & & \downarrow W^* \\ L^2(\mathbb{R}^n) & \xrightarrow[a^{\mathrm{Wick}}]{} & L^2(\mathbb{R}^n) \end{array}$$

If a is a semi-classical symbol in S^1_{sc}, i.e. such that

$$|(\partial_x^\alpha \partial_\xi^\beta a)(x,\xi,h)| \le C_{\alpha\beta} h^{-1+\frac{|\alpha|+|\beta|}{2}},$$

then $a^{\mathrm{Wick}} - a^w \in \mathcal{B}(L^2(\mathbb{R}^n))$ so the change of quantization is harmless if we expect to prove some subelliptic estimates.

2.2. **Subellipticity for pseudodifferential equations.** We consider an evolution equation

$$D_t + iq^w(t,x,D_x), \quad 0 \le q \in S^1_{sc}.$$

We replace it by $D_t + iq(t,x,\xi)^{\mathrm{Wick}} = W^*\big(D_t + iq\big)W$, with W acting in the x-variable only, and we apply the isometric W to get a somehow equivalent evolution equation

$$D_t + i\Pi_0 q \Pi_0, \quad \Pi_0 = WW^* \quad \text{is a Toeplitz operator.}$$

We start with the study of the ODE $D_t + iq$. It is quite easy, but we need a

Lemma 2.1. *Let $k \ge 1, \delta > 0, C > 0$, I be an interval of $\mathbb{R}$, $f : I \longrightarrow \mathbb{R}$ such that*

$$\inf_{t\in I} |f^{(k)}(t)| \ge \delta.$$

Then for all $h > 0$,

$$\big|\{t \in I, |f(t)| \le Ch^k\}\big| \le h\alpha(C/\delta, k).$$

The proof is by induction on k and we note that the conclusion can be fulfilled for k non-integer for some f merely continuous (e.g. fractional powers).

Theorem 2.2. *Let $q \in S^1_{sc}$ real-valued such that $q = 0 \Longrightarrow d_{x,\xi} q = 0$ (e.g. $q \ge 0$). Then, if $|\partial_t^k q| h \ge \delta > 0$ we have*

$$\|D_t u + iq^w(t,x,D_x)u\| \gtrsim h^{-\frac{1}{k+1}} \|u\|$$

This is a subelliptic estimate and we describe here an extension of a method used by F. Treves [19] to handle this type of estimate.

Proof, step 1: use the reduction to $D_t + i\Pi_0 q\Pi_0$, where $\Pi_0 = WW^*$ is the Toeplitz operator introduced above.

Proof, step 2: use Lemma 2.1 on the Lebesgue measure to handle the ODE $D_t + iq$.

Proof, step 3: since for $\Phi = Wu$, we have

$$\overbrace{D_t\Phi + iq\Phi}^{\text{known from step 2}} = D_t\Phi + iq\Pi_0\Phi = D_t\Phi + i\Pi_0 q\Pi_0\Phi + i(I-\Pi_0)q\Phi =$$
$$= \underbrace{D_t\Phi + i\Pi_0 q\Pi_0\Phi}_{\substack{\mathcal{L}\Phi:\text{ under scope}\\ \text{with step 1}}} + i \underbrace{[q,\Pi_0]\Phi}_{\text{a commutator term}},$$

we simply need to handle that commutator.

2.3. **Commutator argument.** The unwanted term here is, with $\Phi = Wu$, the term

$$\|[q, \Pi_0]\Phi\|^2 \leq \iint |q'_{x,\xi}|^2 |\Phi|^2 dx d\xi + C\|\Phi\|^2.$$

We can control $|q'_{x,\xi}|^2$ by $C|q|$ since $q = 0 \Longrightarrow d_{x,\xi} q = 0$: in fact, the metric

$$g = \frac{dx^2 + d\xi^2}{\lambda(t,x,\xi)}, \quad \lambda(t,x,\xi) = 1 + |q| + |d_{x,\xi} q|^2$$

is such that $q \in S(\lambda, g)$, $\frac{\lambda}{1+|q|} \sim 1$ and the energy method will provide for free a term $\langle |q|\Phi, \Phi\rangle$.

3. A kinetic equation

3.1. **Presentation.** Let $0 \leq f(t,x,v)$ probability density, $x \in \mathbb{R}^d, v \in \mathbb{R}^d, t \geq 0$. Boltzmann equation reads

$$\underbrace{\partial_t f + (v \cdot \nabla_x) f}_{\text{transport}} = \underbrace{Q(f,f)(t,x,v)}_{\substack{\text{Collision term with some} \\ \text{negativity properties}}}, \tag{3.1}$$

$$Q(f,f) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - v_*, \sigma)\Big\{ f(v'_*) f(v') - f(v_*) f(v) \Big\} d\sigma dv_* \tag{3.2}$$

with

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma.$$

Conservation of momentum reads : $v + v_* = v' + v'_*$, and conservation of kinetic energy as $|v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2$. The *cross-section* $B(z,\sigma)$ depends only on $|z|$ and $\cos\theta = \langle \frac{z}{|z|}, \sigma\rangle$ and

$$B(v - v_*, \sigma) = \Phi(|v - v_*|) b(\cos\theta), \quad \cos\theta = \langle \frac{v - v_*}{|v - v_*|}, \sigma\rangle$$

$$\Phi(|v - v_*|) = |v - v_*|^{\frac{\gamma-5}{\gamma-1}}, \quad \underset{\theta \to 0}{b(\cos\theta) \sim} \kappa \theta^{-2-2\alpha}, \quad \kappa > 0$$

$$0 < \alpha = \frac{1}{\gamma - 1} < 1, \quad b(\cos\theta) \text{ is not integrable on } \mathbb{S}^2.$$

We have

$$\|(-\tilde{\Delta}_v)^{\alpha/2} f\|^2 \lesssim \langle -Q(f,f), f\rangle + \|f\|^2$$

and the subelliptic properties of the Boltzmann equation are closely related to the properties of the equation

$$\mathcal{P}u \equiv \partial_t u + v \cdot \nabla_x u + \sigma(t,x,v)(-\tilde{\Delta}_v)^{\alpha} u = f, \ (t,x,v) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n, \tag{3.3}$$

where σ denotes a positive $C^\infty(\mathbb{R}^{2n+1})$ function

$$\sigma(t,x,v) > 0, \ (t,x,v) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n. \tag{3.4}$$

Here $(-\tilde{\Delta}_v)^{\sigma}$ stands for the Fourier multiplier with symbol

$$F(\eta) = |\eta|^{2\sigma} w(\eta) + |\eta|^2 \big(1 - w(\eta)\big), \ \eta \in \mathbb{R}^n, \tag{3.5}$$

with $|\cdot|$ being the Euclidean norm and $w \in C^\infty(\mathbb{R}^n)$, $0 \le w \le 1$, $w(\eta) = 1$ if $|\eta| \ge 2$ and $w(\eta) = 0$ if $|\eta| \le 1$. The article [15] by N. Lerner, Y. Morimoto and K. Pravda-Starov proves the following result.

Theorem 3.1. *Let $\mathcal{P}$ be the operator defined in (3.3), K be a compact subset of $\mathbb{R}^{2n+1}$ and $s \in \mathbb{R}$. Then, there exists a positive constant $C_{K,s} > 0$ such that for all $u \in C_0^\infty(K)$,*

$$\big\|(1 + |D_t|^{\frac{2\sigma}{2\sigma+1}} + |D_x|^{\frac{2\sigma}{2\sigma+1}} + |D_v|^{2\sigma})u\big\|_s \le C_{K,s}\big(\|Pu\|_s + \|u\|_s\big), \tag{3.6}$$

with $\|\cdot\|_s$ being the $H^s(\mathbb{R}^{2n+1})$ Sobolev norm.

3.2. **Sketch of proof.** We provide some elements of proof in the simplified setting when the function $\sigma = 1$ is constant and $1/2 < \alpha < 1$. We refer the reader to [15] for a proof in the general case. We begin by performing a Fourier transform with respect to (x, v),

$$P = \partial_t - \xi \cdot \partial_\eta + |\eta|^{2\alpha}$$

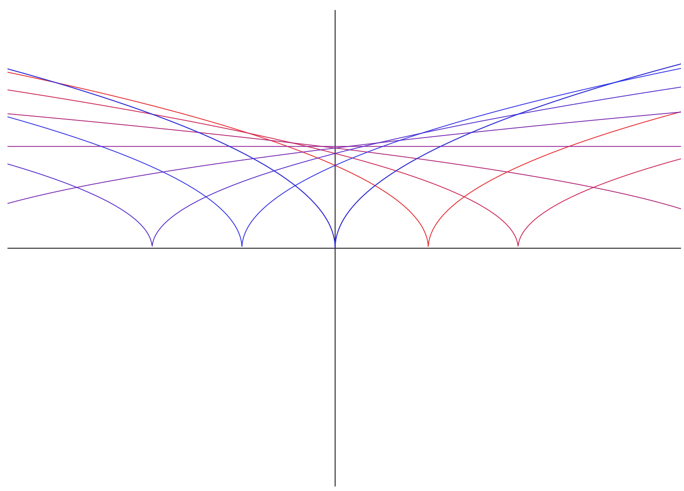
following the flow of $\partial_t - \xi \cdot \partial_\eta$ which is divergence-free, we get

$$\begin{cases} s &= t \\ x_1 &= \xi \\ x_2 &= \eta + t\xi \end{cases} \qquad \begin{cases} t &= s \\ \xi &= x_1 \\ \eta &= x_2 - sx_1 \end{cases}$$

so that

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial s} + x_1 \frac{\partial}{\partial x_2}, \quad \xi \cdot \frac{\partial}{\partial \eta} = x_1 \cdot \frac{\partial}{\partial x_2}, \quad \partial_t - \xi \cdot \partial_\eta = \partial_s$$

and $P = \partial_s + |x_2 - sx_1|^{2\alpha}$. Following the heuristics of picture of page 3, we draw the family of curves $s \mapsto |x_2 - sx_1|^{2\alpha}$ for $x_1^2 + x_2^2 = 1$ and their envelope. We have



Family of curves $\quad s \mapsto |x_2 - sx_1|^{2\alpha} \quad$ for $x_1^2 + x_2^2 = 1$

$$|x_1|^{\frac{2\alpha}{2\alpha+1}} \int |u(s)|^2 ds = |x_1|^{\frac{2\alpha}{2\alpha+1}} \int_{|x_2 - sx_1|^{2\alpha} > |x_1|^{\frac{2\alpha}{2\alpha+1}}} |u(s)|^2 ds + |x_1|^{\frac{2\alpha}{2\alpha+1}} \int_{|x_2 - sx_1|^{2\alpha} \le |x_1|^{\frac{2\alpha}{2\alpha+1}}} |u(s)|^2 ds.$$

We use a fractional version of Lemma 2.1 :

$$\left|\{s, |x_2 - sx_1|^{2\alpha} \le |x_1|^{\frac{2\alpha}{2\alpha+1}}\}\right| \le 2|x_1|^{-\frac{2\alpha}{2\alpha+1}}$$

since

$$|x_2 - sx_1| \le |x_1|^{\frac{1}{2\alpha+1}} \Longrightarrow |x_2/x_1 - s| \le |x_1|^{\frac{1}{2\alpha+1}-1=-\frac{2\alpha}{2\alpha+1}}$$

and this gives

$$|x_1|^{\frac{2\alpha}{2\alpha+1}} \|u\|^2 \le \int |x_2 - sx_1|^{2\alpha} |u(s)|^2 ds + |x_1|^{\frac{2\alpha}{2\alpha+1}} 2 |x_1|^{-\frac{2\alpha}{2\alpha+1}} \sup_{s\in\mathbb{R}} |u(s)|^2 \le \operatorname{Re}\langle Pu, u\rangle + 2 \sup_{s\in\mathbb{R}} |u(s)|^2.$$

A direct computation gives

$$2\operatorname{Re}\langle Pu, H(T-s)u\rangle \ge |u(T)|^2 \Longrightarrow 2\|Pu\|\|u\| \ge \sup_{s\in\mathbb{R}} |u(s)|^2$$

and thus

$$|x_1|^{\frac{2\alpha}{2\alpha+1}} \|u\|^2 \le 5\|Pu\|\|u\| \Longrightarrow |x_1|^{\frac{2\alpha}{2\alpha+1}} \|u\| \lesssim \|Pu\| \quad \text{(integrals w.r.t. } s).$$

We get

$$c_0 \||\xi|^{\frac{2\alpha}{2\alpha+1}} u\| \lesssim \|(\partial_t - \xi\cdot\partial_\eta + |\eta|^{2\alpha})u\|.$$

By using

$$\|(\partial_t - \xi\cdot\partial_\eta + |\eta|^{2\alpha})u\|^2 = \|(\partial_t - \xi\cdot\partial_\eta)u\|^2 + 2\operatorname{Re}\langle(\partial_t - \xi\cdot\partial_\eta)u, |\eta|^{2\alpha}u\rangle + \||\eta|^{2\alpha}u\|^2,$$

we obtain for any $\beta \ge 1$,

$$(1+\beta)\|(\partial_t - \xi\cdot\partial_\eta + |\eta|^{2\alpha})u\|^2 + \|u\|^2 \gtrsim \iiint |u|^2 \big(\beta|\xi|^{\frac{4\alpha}{2\alpha+1}} + |\eta|^{4\alpha} + \underbrace{2\alpha\xi\cdot\frac{\eta}{|\eta|}|\eta|^{2\alpha-1}}_{\text{bad term}} + 1\big) dt d\eta d\xi.$$

However, Hölder's inequality implies

$$|\xi||\eta|^{2\alpha-1} = \big(|\xi|^{\frac{4\alpha}{2\alpha+1}}\big)^{\frac{2\alpha+1}{4\alpha}} \big(|\eta|^{4\alpha}\big)^{\frac{2\alpha-1}{4\alpha}} \le \frac{2\alpha+1}{4\alpha}|\xi|^{\frac{4\alpha}{2\alpha+1}} + \frac{2\alpha-1}{4\alpha}|\eta|^{4\alpha},$$

when $1/2 < \alpha < 1$. For $\beta \ge \frac{2\alpha+1}{4\alpha}$, we get

$$\||\eta|^{2\alpha}u\| + \||\xi|^{\frac{2\alpha}{2\alpha+1}}u\| \lesssim \|(\partial_t - \xi\cdot\partial_\eta + |\eta|^{2\alpha})u\|,$$

which is essentially the result of Theorem 3.1.

3.3. **Geometry of the characteristics and a conjecture.** We would like to end this note with an outline of what we believe could be a general strategy for handling kinetic equations in a somewhat geometrical approach. We consider an operator

$$\mathcal{L} = X_0 + Q, \quad X_0^* = -X_0, \quad Q \geq 0, \tag{3.7}$$

so that X_0 is the skew-adjoint part (e.g. a divergence-free vector field) and Q is the self-adjoint part (e.g. a Laplacean in some of the variables). An obvious thing to do is to calculate

$$\mathrm{Re}\langle \mathcal{L}u, u\rangle = \langle Qu, u\rangle \geq \|Eu\|^2, \quad E \text{ partially elliptic.} \tag{3.8}$$

Of course it is not enough, even in the simplest models, such as the Kolmogorov operator (1.4). The bicharacteristic curves of $iX_0 = a^w$, a real-valued, are the integral curves of H_a, the Hamiltonian vector field of a: we have

$$H_a = \frac{\partial a}{\partial \xi}\cdot\frac{\partial}{\partial x} - \frac{\partial a}{\partial x}\cdot\frac{\partial}{\partial \xi},$$

so that

$$\dot\gamma(t; x, \xi) = H_a\big(\gamma(t; x, \xi)\big), \quad \gamma(0; x, \xi) = (x, \xi).$$

Let $\mu(x, \xi)$ be a positive function defined on the phase space; we define $l_\mu(x, \xi)$ as

$$\textit{Lebesgue measure}\Big(\{t, q\big(\gamma(t, x, \xi)\big) \leq \mu(x, \xi)\}\Big) = l_\mu(x, \xi), \tag{3.9}$$

where q is the symbol of the operator Q. Let $\nu(x, \xi)$ be defined by

$$\nu(x, \xi) = \sup\{\mu(x, \xi) \geq 0, \sup_{(x,\xi)\in\mathbb{R}^{2n}} \mu(x, \xi) l_\mu(x, \xi) < +\infty\}. \tag{3.10}$$

We note that $\nu(x, \xi) \geq 0$ and also that for $q\big(\gamma(t, x, \xi)\big)$ polynomial in the t variable with degree k and (positive) leading coefficient $\lambda(x, \xi)$, then we have

$$l_\mu(x, \xi) \leq \alpha(k)\mu(x, \xi)^{1/k}\lambda(x, \xi)^{-1/k}.$$

As a result, we have with $\mu = \lambda^{1/k+1}$,

$$\mu(x, \xi) l_\mu(x, \xi) \leq \lambda(x, \xi)^{\frac{1}{k+1}}\alpha(k)\mu(x, \xi)^{1/k}\lambda(x, \xi)^{-1/k} = \alpha(k),$$

and thus in that case

$$\nu(x, \xi) \geq \lambda(x, \xi)^{\frac{1}{k+1}}.$$

Conjecture 3.2. *We consider an operator $\mathcal{L}$ given by (3.7), ν be given by (3.10). Then, there exists a constant $c_0 > 0$ such that, for all $u \in \mathscr{S}(\mathbb{R}^n)$,*

$$\|\mathcal{L}u\|_{L^2(\mathbb{R}^n)} \geq c_0\|\nu^w u\|_{L^2(\mathbb{R}^n)}. \tag{3.11}$$

Since ν given by (3.10) may not be smooth, we may replace $\|\nu^w u\|_{L^2(\mathbb{R}^n)}$ in the above inequality by $\sup_{\mathscr{S}(\mathbb{R}^{2n})\ni\tilde\nu, \tilde\nu\leq\nu} \|\tilde\nu^w u\|_{L^2(\mathbb{R}^n)}$.

Conjecture 3.3 (Another formulation)**.** *We consider an operator $\mathcal{L}$ given by (3.7). Let $0 \leq \nu \in \mathscr{S}(\mathbb{R}^{2n})$ such that*

$$\sup_{(x,\xi)\in\mathbb{R}^{2n}} \nu(x, \xi)\Big|\{t, q\big(\gamma(t, x, \xi)\big) \leq \nu(x, \xi)\}\Big| < +\infty \tag{3.12}$$

where $|A|$ stands for the Lebesgue measure of A. Then, there exists a constant $c_0 > 0$ such that, for all $u \in \mathscr{S}(\mathbb{R}^n)$,

$$\|\mathcal{L}u\|_{L^2(\mathbb{R}^n)} \geq c_0 \|\nu^w u\|_{L^2(\mathbb{R}^n)}. \tag{3.13}$$

Some comments are in order. Theorem 2.2 is a particular case of that conjecture with

$$\mathcal{L} = \partial_t + q_\lambda^w(t, x, D_x), \quad q_\lambda(t, x, \xi) \geq 0,$$

where q_λ is a semi-classical symbol of order 1, i.e.

$$\sup_{\substack{(t,x,\xi)\in\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n \\ \lambda\geq 1}} |(\partial_x^\alpha \partial_\xi^\beta q_\lambda)(t, x, \xi)\lambda^{-1+\frac{|\alpha|+|\beta|}{2}}| = C_{\alpha\beta}.$$

The choice of $\nu = \lambda^{\frac{1}{k+1}}$ in the conjecture follows from the hypothesis (in the Theorem) $|\partial_t^k q| \geq \delta\lambda$ and the discussion before the statement of the conjecture. Note also that the estimate (3.11) appears in this case as a subelliptic estimate with loss of $k/k+1$ derivatives with respect to the elliptic case ($k = 0$). Theorem 3.1 is also an indirect consequence of that conjecture, since via a straightening of the vector field $\partial_t + v \cdot \nabla_x$ and some subsequent microlocalization, we are reduced to a semi-classical problem of a rather similar nature that in Theorem 3.1, but with a fractional $k = 2\alpha$. The most important point in our view in that the assumption of the conjecture tolerates that q can vanish along the integral curves of H_a, but that the Lebesgue measure of the set where q is small, say smaller than μ, should be bounded above by a constant times $1/\mu$. It is also important to notice that this type of assumption, valid for polynomial of degree k, is also relevant for non-integer k, as it is the case for Theorem 3.1.

Summing-up, the main point of this conjecture is that we claim that the hypoellipticity properties of an operator (3.7) are governed by the values of the symbol q of the diffusion part Q along the characteristic curves of the transport part: using the notations above, we may check

$$\mathbb{R} \ni t \mapsto q(\gamma(t; x, \xi)) \in \mathbb{R}_+$$

and take the envelope of these curves when (x, ξ) varies. If that envelope stays positive away from zero, then hypoellipticity follows. More precisely, the computation of the minimum of that envelope provides the index of hypoellipticity, e.g. the loss of derivatives with respect to the elliptic case.

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