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DE L'ANALYSE MICROLOCALE DE QUELQUES CLASSES D'ÉQUATIONS AUX DÉRIVÉES PARTIELLES

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Avant propos

Depuis le début des années 70, l'étude microlocale des équations aux dérivées partielles doublement caractéristiques jouit d'une solide tradition en analyse [**23, 24, 25, 26, 60, 63, 73, 75, 91, 109, 110, 113**]. Ces dernières années, ce domaine a connu un nouvel intérêt et de récents développements inspirés par l'étude de certains problèmes non-autoadjoints à caractéristiques doubles. Ce regain d'intérêt trouve quelques unes de ses sources dans l'étude du comportement asymptotique des solutions d'équations d'évolution associées à des opérateurs non-autoadjoints

$$\begin{cases} (\partial_t + P)u(t,x) = 0, \\ u(t,\cdot)|_{t=0} = u_0 \in L^2(\mathbb{R}^n). \end{cases}$$

Cette problématique est particulièrement importante en théorie cinétique et dans l'étude des problèmes de retour à l'équilibre en physique statistique. L'opérateur de Kramers-Fokker-Planck

$$P = -\Delta_v + \frac{|v|^2}{4} - \frac{n}{2} + v \cdot \partial_x - \nabla_x V(x) \cdot \partial_v, \quad (x,v) \in \mathbb{R}^{2n}, \tag{0.1}$$

associé à un potentiel régulier $V \in C^\infty(\mathbb{R}^n, \mathbb{R})$, est un exemple d'opérateur cinétique non-autoadjoint dont l'étude spectrale et pseudospectrale initiée par Hérau, Sjöstrand et Stolk [**70**] a été l'une des principales sources d'inspiration pour la théorie de l'espace singulier et l'étude générale des opérateurs pseudodifférentiels doublement caractéristiques developpées dans le premier chapitre de ce manuscrit.

Les premiers exemples d'opérateurs doublement caractéristiques sont donnés par les opérateurs différentiels quadratiques

$$Q(x, D_x) = \sum_{|\alpha+\beta|=2} q_{\alpha,\beta} x^\alpha D_x^\beta, \quad x \in \mathbb{R}^n,$$

où $q_{\alpha,\beta} \in \mathbb{C}$, $D_{x_j} = i^{-1}\partial_{x_j}$, et $\alpha, \beta \in \mathbb{N}^n$. Dans le cas elliptique, le spectre de ces opérateurs est connu depuis plusieurs décennies et a été décrit explicitement par Sjöstrand [**113**] et Boutet de Monvel [**23**]. Les opérateurs quadratiques jouent un rôle central dans l'analyse des propriétés des opérateurs pseudodifférentiels à caractéristiques doubles. C'est notamment le cas concernant les résultats généraux d'hypoellipticité avec perte d'une dérivée

$$\|u\|_{(s+m-1)} \leq C_K(\|Pu\|_{(s)} + \|u\|_{(s+m-2)}), \quad u \in C_0^\infty(K), \ K \Subset \mathbb{R}^n, \tag{0.2}$$

où $\|\cdot\|_{(s)}$ désigne la norme de Sobolev H^s. Étant donné $P = p^w(x, D_x)$ un opérateur pseudodifférentiel classique d'ordre m dont le symbole de Weyl

$$p(x,\xi) = p_m(x,\xi) + p_{m-1}(x,\xi) + p_{m-2}(x,\xi) + \ldots,$$

admet un point de caractéristique double

$$p_m(X_0) = \nabla p_m(X_0) = 0, \quad X_0 = (x_0, \xi_0) \in \mathbb{R}^{2n},$$

il est naturel pour étudier les propriétés de cet opérateur de considérer la forme quadratique q qui commence le développement de Taylor du symbole principal p_m au point X_0. Génériquement [**73, 113**], lorsque le symbole principal s'annule exactement au deuxième ordre au point doublement caractéristique, i.e., lorsque la forme quadratique q est elliptique, l'estimation hypoelliptique (0.2) est satisfaite pour tout compact $K \Subset \mathbb{R}^n$, si et seulement si l'opposé du symbole sous-principal $-p_{m-1}(X_0)$ évalué au point doublement caractéristique évite le spectre de l'approximation quadratique du symbole principal au point doublement caractéristique $q^w(x,D_x)$. Ce résultat met en exergue le lien entre les propriétés des opérateurs pseudodifférentiels doublement caractéristiques et celles de leurs approximations quadratiques, et souligne le fait que l'analyse microlocale de ces opérateurs à caractéristiques doubles passe au préalable par une bonne compréhension des propriétés notamment spectrales des opérateurs quadratiques. Pour de nombreux modèles cinétiques comme l'opérateur de Kramers-Fokker-Planck, les approximations quadratiques de ces opérateurs

$$q^w(x,D_x) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n),$$

ne satisfont pas à l'hypothèse d'ellipticité

$$(x,\xi) \in \mathbb{R}^{2n},\ q(x,\xi) = 0 \Rightarrow (x,\xi) = (0,0), \tag{0.3}$$

et la théorie microlocale classique pour les opérateurs à caractéristiques doubles est inopérante. Ces modèles cinétiques présentent une certaine dégénérescence au niveau de leurs ensembles doublement caractéristiques. L'étude de la structure microlocale de ce type de dégénérescence est à l'origine de la série de travaux présentée dans le premier chapitre de ce manuscrit. Plus particulièrement, la première partie du chapitre 1 est dédiée à l'étude des propriétés de certaines classes d'opérateurs quadratiques non elliptiques. Pour ce faire, nous introduisons la notion d'espace singulier associé à un opérateur quadratique. L'espace singulier est un sous-espace vectoriel particulier de l'espace des phases

$$S = \Big(\bigcap_{j=0}^{+\infty} \mathrm{Ker}\big[\mathrm{Re}\ F(\mathrm{Im}\ F)^j\big]\Big) \cap \mathbb{R}^{2n} \subset \mathbb{R}^{2n}, \tag{0.4}$$

défini intrinsèquement à partir de l'application hamiltonienne F du symbole de Weyl d'un opérateur quadratique. La théorie développée dans ce manuscrit montre que cette notion d'espace singulier est un objet algébrique simple particulièrement adapté à la description des propriétés des opérateurs quadratiques non elliptiques, et que la structure de ce sous-espace vectoriel permet de caractériser certaines propriétés dynamiques du symbole de ces opérateurs. Pour de nombreux modèles cinétiques comme l'opérateur de Kramers-Fokker-Planck, les approximations quadratiques de ces opérateurs violent l'hypothèse d'ellipticité (0.3) mais satisfont à une hypothèse d'ellipticité partielle le long de leurs espaces singuliers

$$(x,\xi) \in S,\ q(x,\xi) = 0 \Rightarrow (x,\xi) = (0,0). \tag{0.5}$$

La première partie du chapitre 1 étudie les propriétés spectrales et sous-elliptiques de la classe des opérateurs quadratiques accrétifs satisfaisant à cette hypothèse d'ellipticité partielle. Nous montrons que ces opérateurs partiellement elliptiques ont un spectre discret ayant une structure identique à celle des opérateurs quadratiques elliptiques

$$\sigma(q^w(x,D_x))=\Big\{\sum_{\substack{\lambda\in\sigma(F),\\ -i\lambda\in\mathbb{C}_+\cup(\Sigma(q|_S)\setminus\{0\})}}\big(r_\lambda+2k_\lambda\big)(-i\lambda):\ k_\lambda\geq 0\Big\}.$$

La notion d'espace singulier permet également de caractériser les propriétés sous-elliptiques des opérateurs quadratiques et de démontrer qu'ils sont microlocalement régularisants dans toutes les directions de l'espace des phases données par l'orthogonal symplectique de l'espace singulier

$$S^{\sigma\perp}=\{X\in\mathbb{R}^{2n}:\ \forall Y\in S,\ \sigma(X,Y)=0\}.$$

La seconde partie du chapitre 1 a trait à l'étude spectrale et pseudo-spectrale d'opérateurs pseudodifférentiels non-autoadjoints semi-classiques dont les approximations quadratiques aux points doublement caractéristiques satisfont à l'hypothèse d'ellipticité partielle (0.5). Étudier les propriétés pseudospectrales d'un opérateur revient à étudier les lignes de niveau de la norme de sa résolvante. Pour des opérateurs non-autoadjoints, il s'agit d'un problème non trivial, et ce même lorsque le spectre de ces opérateurs est connu. En effet, il n'y a aucun contrôle a priori de la résolvante d'un opérateur non-autoadjoint par son spectre, et la résolvante d'un tel opérateur peut exploser en norme dans des régions non bornées de l'ensemble résolvant très éloignées du spectre. Ces phénomènes de non contrôle de la résolvante sont liés à la possible très forte instabilité du spectre des opérateurs non-autoadjoints sous l'effet de petites perturbations. Les travaux de thèse de l'auteur ont montré que ce genre d'instabilités spectrales apparaît pour tout opérateur quadratique elliptique non normal dans des régions de l'ensemble résolvant à la géométrie caractéristique. L'analyse de l'équation de Kramers-Fokker-Planck [**70**] a également mis en évidence une géométrie particulière régissant les propriétés spectrales et pseudospectrales de l'opérateur (0.1) au voisinage d'un point doublement caractéristique. La deuxième partie du premier chapitre de ce manuscrit propose de généraliser ces résultats et de décrire précisément les propriétés spectrales et les phénomèmes pseudospectraux se produisant au voisinage d'un point de caractéristique double pour des classes générales d'opérateurs pseudodifférentiels doublement caractéristiques. Nous montrons en particulier que la géométrie propre à ces phénomènes pseudospectraux est déterminée par les propriétés sous-elliptiques des approximations quadratiques de ces opérateurs pseudodifférentiels doublement caractéristiques

$$\|\langle(x,D_x)\rangle^{2(1-\delta)}u\|_{L^2}\leq C(\|q^w(x,D_x)u\|_{L^2}+\|u\|_{L^2}),$$

où $\langle(x,D_x)\rangle^2=1+|x|^2+|D_x|^2$, et qu'elle dépend directement de la perte de dérivées maximale $0\leq\delta<1$ par rapport au cas elliptique satisfaite par les approximations quadratiques de ces opérateurs.

Le second chapitre de ce manuscrit regroupe plusieurs travaux sur l'hypoellipticité et la non résolubilité de certaines classes d'opérateurs différentiels ou pseudodifférentiels non-autoadjoints. Nous présentons plusieurs méthodes microlocales utilisant des techniques par multiplicateur ou d'états cohérents pour démontrer des estimations hypoelliptiques pour des opérateurs cinétiques modèles pour les linéarisations des équations de Landau et Boltzmann, et pour établir des estimations sous-elliptiques en régularité limitée pour une classe d'opérateurs pseudodifférentiels violant la condition (P) mais satisfaisant la condition $(\overline{\Psi})$ adjointe de la condition de

Nirenberg-Trèves. Nous donnons également des exemples d'opérateurs différentiels faiblement hyperboliques à coefficients réels dont la non résolubilité est conséquence d'une violation d'une version quasi-homogène de la condition $(\overline{\Psi})$.

Le troisième et dernier chapitre de ce manuscrit propose une étude microlocale de l'équation de Boltzmann

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = Q(f,f), \\ f|_{t=0} = f_0. \end{cases}$$

Cette équation proposée par Boltzmann en 1872 pour décrire les propriétés d'un gaz dilué lorsque les seules interactions entre les particules sont données par les collisions binaires inter-moléculaires, est l'une des équations majeures de la physique statistique. L'équation de Boltzmann régit l'évolution de la densité de particules contenue dans un gaz $f = f(t,x,v) \geq 0$ à un temps t, à la position $x \in \mathbb{R}^d$ et à la vitesse $v \in \mathbb{R}^d$. Le terme non linéaire $Q(f,f)$ correspond à l'opérateur de collision de Boltzmann dérivant de l'opérateur bilinéaire

$$Q(g,f) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} B(v - v_*, \sigma)\big(g'_* f' - g_* f\big) d\sigma dv_*, \qquad d \geq 2,$$

où l'on use des notations standard en théorie cinétique $f'_* = f(t,x,v'_*)$, $f' = f(t,x,v')$, $f_* = f(t,x,v_*)$, $f = f(t,x,v)$. Dans cette expression, v, v_* et v', v'_* correspondent aux vitesses d'une paire de particules respectivement avant et après collision. Ces vitesses sont liées par les relations

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \quad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma,$$

où le paramètre $\sigma \in \mathbb{S}^{d-1}$ appartient à la sphère unité, qui correspondent physiquement à des collisions binaires élastiques préservant la quantité de mouvement et l'énergie cinétique

$$v + v_* = v' + v'_*, \quad |v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2.$$

Pour un gaz monoatomique, un modèle standard de section efficace $B(v - v_*, \sigma)$ est donné par une fonction positive dépendant séparément de la vitesse relative avant collision $|v - v_*|$ et de l'angle de déviation θ défini par le produit scalaire

$$\cos\theta = k \cdot \sigma, \quad k = \frac{v - v_*}{|v - v_*|}.$$

Plus précisément, on suppose que la section efficace a la structure suivante

$$B(v - v_*, \sigma) = |v - v_*|^\gamma b\Big(\frac{v - v_*}{|v - v_*|} \cdot \sigma\Big), \quad \gamma \in]-d, +\infty[, \tag{0.6}$$

et l'on qualifie les molécules de maxwelliennes lorsque le paramètre $\gamma = 0$. Le second terme composant la section efficace est un facteur angulaire singulier en zéro

$$(\sin\theta)^{d-2} b(\cos\theta) \underset{\theta \to 0_+}{\approx} \theta^{-1-2s}, \tag{0.7}$$

où[1] $0 < s < 1$ est un paramètre physique correspondant à des particules interagissant selon un potentiel sphérique inter-moléculaire répulsif de la forme

$$\phi(\rho) = \frac{1}{\rho^r}, \quad r = \frac{1}{s} > 1.$$

Cette singularité n'est pas intégrable en zéro

$$\int_0^{\frac{\pi}{2}} (\sin\theta)^{d-2} b(\cos\theta) d\theta = +\infty,$$

et joue un rôle majeur quant aux propriétés qualitatives des solutions de l'équation de Boltzmann en particulier pour que le phénomène de régularisation opère. En effet, les propriétés régularisantes de l'équation de Boltzmann sont uniquement dues à la grande quantité de collisions rasantes pour lesquelles l'angle de déviation est presque nul $\theta \sim 0$.

On considère la linéarisation de l'équation de Boltzmann

$$f = \mu_d + \sqrt{\mu_d} g,$$

autour de la distribution maxwellienne

$$\mu_d(v) = (2\pi)^{-\frac{d}{2}} e^{-\frac{|v|^2}{2}}, \quad v \in \mathbb{R}^d.$$

D'après la conservation de l'énergie cinétique, cette distribution est un état d'équilibre $Q(\mu_d, \mu_d) = 0$ et l'opérateur linéarisé de Boltzmann

$$\mathscr{L} g = -\mu_d^{-1/2} Q(\mu_d, \mu_d^{1/2} g) - \mu_d^{-1/2} Q(\mu_d^{1/2} g, \mu_d),$$

permet de réduire l'équation de Boltzmann au problème de Cauchy suivant pour la fluctuation autour de la distribution maxwellienne

$$\begin{cases} \partial_t g + v \cdot \nabla_x g + \mathscr{L} g = \mu_d^{-1/2} Q(\sqrt{\mu_d} g, \sqrt{\mu_d} g), \\ g|_{t=0} = g_0. \end{cases}$$

Il y a de cela plus de quarante ans Cercignani [**27**] mettait en évidence que pour des molécules maxwelliennes, l'opérateur linéarisé de Boltzmann se comportait comme un opérateur diffusif fractionnaire. À travers le temps, ces premiers résultats se sont transformés en la conjecture suggérant que l'opérateur linéarisé de Boltzmann se comportait essentiellement comme un laplacien fractionnaire en la variable de vitesse [**6, 8, 126**]:

$$f \mapsto Q(\mu_d, f) \sim -(-\Delta_v)^s f + \text{ termes d'ordre inférieur},$$

où $0 < s < 1$ est le paramètre structurel de la singularité (0.7). Le chapitre 3 de ce manuscrit propose une analyse dans l'espace des phases des propriétés diffusives de l'opérateur linéarisé de Boltzmann pour des molécules maxwelliennes. On considère tout d'abord le cas où l'opérateur linéarisé de Boltzmann agit sur des fonctions à symétrie radiale. Dans ce cas, nous montrons que l'opérateur linéarisé de Boltzmann est un opérateur pseudodifférentiel

$$\mathscr{L} f = l^w(v, D_v) f, \quad f \in \mathscr{S}_r(\mathbb{R}^d),$$

défini par un symbole admettant un développement complet dans une classe de symboles standard jouissant d'un bon calcul symbolique. Plus précisément, à des

[1] La notation $a \approx b$ signifie que le ratio a/b est borné inférieurement et supérieurement par des constantes strictement positives.

termes d'ordre inférieur bornés sur $L^2(\mathbb{R}^d)$ près, l'opérateur linéarisé de Boltzmann agissant sur des fonctions à symétrie radiale se réduit à l'oscillateur harmonique fractionnaire en la variable de vitesse

$$c_0\Big(1-\Delta_v+\frac{|v|^2}{4}\Big)^s,$$

où $0<s<1$ est le paramètre structurel de la singularité (0.7). Dans le cas de l'espace physique tridimensionnel $d=3$ où l'opérateur linéarisé de Boltzmann n'agit plus nécessairement sur des fonctions à symétrie radiale, nous montrons que l'opérateur linéarisé de Boltzmann est égal à l'opérateur linéarisé de Landau fractionnaire

$$\mathscr{L}=a(\mathcal{H},\Delta_{\mathbb{S}^2})\mathscr{L}_L^s,$$

où $0<s<1$ est le paramètre apparaissant dans la condition (0.7). L'opérateur linéarisé de Landau est un opérateur différentiel anisotrope correspondant à la linéarisation

$$\begin{aligned}\mathscr{L}_L g=-\mu_3^{-1/2}Q_L(\mu_3,\mu_3^{1/2}g)-\mu_3^{-1/2}Q_L(\mu_3^{1/2}g,\mu_3)=2\Big(-\Delta_v+\frac{|v|^2}{4}-\frac{3}{2}\Big)g-\Delta_{\mathbb{S}^2}g\\+\Big[\Delta_{\mathbb{S}^2}-2\Big(-\Delta_v+\frac{|v|^2}{4}-\frac{3}{2}\Big)\Big]\mathbb{P}_1 g+\Big[-\Delta_{\mathbb{S}^2}-2\Big(-\Delta_v+\frac{|v|^2}{4}-\frac{3}{2}\Big)\Big]\mathbb{P}_2 g,\end{aligned}$$

de l'opérateur de collision de Landau

$$\begin{aligned}Q_L(g,f)=\nabla_v\cdot\Big(\int_{\mathbb{R}^3}(|v-v_*|^2\,\mathrm{Id}-(v-v_*)\otimes(v-v_*))\\\times\big(g(v_*)(\nabla_v f)(v)-(\nabla_v g)(v_*)f(v)\big)dv_*\Big).\end{aligned}$$

Ce résultat permet d'expliciter l'anisotropie des propriétés diffusives de l'opérateur linéarisé de Boltzmann et d'obtenir des estimations coercives optimales dans le cas de molécules maxwelliennes et non-maxwelliennes. Il dévoile aussi plus avant les liens inhérents existant entre les équations de Boltzmann et de Landau en accord avec des résultats précédents présentant l'opérateur de Landau comme l'opérateur de Boltzmann limite dans le cas où le paramètre $s=1$.

La dernière partie du chapitre 3 propose une étude du problème de Cauchy associé à l'équation de Boltzmann spatialement homogène. Cette étude montre que pour de petites fluctuations initiales à symétrie radiale autour de la distribution maxwellienne, l'équation de Boltzmann spatialement homogène possède les mêmes propriétés régularisantes que l'équation d'évolution associée à l'oscillateur harmonique fractionnaire

$$\begin{cases}\partial_t f+\mathcal{H}^s f=0, & \mathcal{H}=-\Delta_v+\frac{|v|^2}{4},\\ f|_{t=0}=f_0\in L^2(\mathbb{R}^d),\end{cases}$$

où $0<s<1$ est le paramètre apparaissant dans la condition (0.7). Ce résultat met en évidence un phénomème de régularisation dans la classe de Gelfand-Shilov $S_{1/2s}^{1/2s}(\mathbb{R}^d)$ pour tout temps $t>0$, qui correspond à la régularité Gevrey $G^{1/2s}(\mathbb{R}^d)$ de la fluctuation et de sa transformée de Fourier.

Contents

CHAPTER 1

Spectral properties and resolvent bounds for pseudodifferential operators with double characteristics

1. Foreword

There has been recently a renewed interest in the analysis of the spectra and resolvents of non-selfadjoint operators with double characteristics. This interest finds some of its grounds in the study of the long-time behavior of evolution equations associated to non-selfadjoint operators

$$\begin{cases} (\partial_t + P)u(t,x) = 0, \\ u(t,\cdot)|_{t=0} = u_0 \in L^2(\mathbb{R}^n). \end{cases}$$

This is for instance the case in the analysis of kinetic equations and the study of the trend to equilibrium in statistical physics. The Kramers-Fokker-Planck operator

$$P = -\Delta_v + \frac{|v|^2}{4} - \frac{n}{2} + v \cdot \partial_x - \nabla_x V(x) \cdot \partial_v, \quad (x,v) \in \mathbb{R}^{2n},$$

with a smooth potential $V \in C^\infty(\mathbb{R}^n, \mathbb{R})$, is an example of non-selfadjoint kinetic operator whose spectral and pseudospectral analysis by Hérau, Sjöstrand and Stolk [**70**] has been a starting point of the present work on pseudodifferential operators with double characteristics.

The study of doubly characteristic operators has a long and distinguished tradition in the analysis of partial differential equations [**23, 24, 25, 26, 60, 63, 73, 75, 91, 109, 110, 113**]. The simplest examples of such operators are given by quadratic differential operators

$$Q(x, D_x) = \sum_{|\alpha+\beta|=2} q_{\alpha,\beta} x^\alpha D_x^\beta, \quad x \in \mathbb{R}^n,$$

with $q_{\alpha,\beta} \in \mathbb{C}$, $D_{x_j} = i^{-1}\partial_{x_j}$, and $\alpha, \beta \in \mathbb{N}^n$. In the elliptic case, the spectrum of these operators has been understood and described explicitly for some time [**23, 113**]. Quadratic operators are playing a basic role in the analysis of partial differential operators with double characteristics. This is for instance the case for general results about hypoellipticity with loss of one derivative

$$\|u\|_{(s+m-1)} \leq C_K(\|Pu\|_{(s)} + \|u\|_{(s+m-2)}), \quad u \in C_0^\infty(K), \ K \Subset \mathbb{R}^n, \tag{1.1}$$

where $\|\cdot\|_{(s)}$ stands for the Sobolev norm H^s. Given $P = p^w(x, D_x)$ a classical pseudodifferential operator of order m whose Weyl symbol

$$p(x,\xi) = p_m(x,\xi) + p_{m-1}(x,\xi) + p_{m-2}(x,\xi) + \dots,$$

admits a doubly characteristic point

$$p_m(X_0) = \nabla p_m(X_0) = 0, \quad X_0 = (x_0, \xi_0) \in \mathbb{R}^{2n},$$

it is natural to consider the quadratic form q which begins the Taylor expansion of the principal symbol p_m at X_0, in order to investigate the properties of the operator P.

Generically [**73, 113**], when the principal symbol vanishes exactly at the second order at this doubly characteristic point, i.e., when the quadratic form q is elliptic, the hypoelliptic estimate (1.1) holds for any compact $K \Subset \mathbb{R}^n$, if and only if the opposite of the subprincipal symbol $-p_{m-1}(X_0)$ evaluated at the doubly characteristic point avoids the spectrum of the quadratic approximation of the principal symbol at the doubly characteristic point $q^w(x, D_x)$.

This particular result emphasizes that a comprehensive understanding of the properties of quadratic operators allows to account for the features of general doubly characteristic operators. It also accounts for the structure of the present chapter. The first part of this chapter is dedicated to the analysis of non-elliptic quadratic operators, when the second deals with the study of general pseudodifferential operators whose quadratic approximations at the doubly characteristic set may fail ellipticity. More precisely, the first part of this chapter is concerned in studying the spectral and subelliptic properties of non-elliptic quadratic operators. We introduce the notion of singular space associated to a quadratic operator. The singular space is a particular linear subspace in the phase space intrinsically defined from the Weyl symbol of the quadratic operator. We develop an abstract theory showing that this notion of singular space is a particularly relevant algebraic tool in order to describe sharply the properties of non-elliptic quadratic operators. The second part of the chapter deals with semiclassical pseudodifferential operators with double characteristics. Building on the results obtained for quadratic operators, we study the spectral and pseudospectral properties of these operators around their double characteristics. Studying the pseudospectral properties refers to studying resolvent bounds. In the non-selfadjoint case, this is a non-trivial problem, and this even when the spectrum of the operator is fully known. Indeed, there is no a priori control of the resolvent growth by the spectrum in the non-selfadjoint case.

The study of pseudospectrum has a recent tradition coming originally from numerical analysis where, for certain problems of science and engineering involving non-selfadjoint operators, it has been noticed that the predictions suggested by the spectral analysis do not match with the numerical simulations. This indicates that in some cases, the only knowledge of the spectrum of an operator is not enough to understand sufficiently its action. To supplement this lack of information contained in the spectrum, some new subsets of the complex plane called pseudospectra were defined. The main idea about the definition of these new subsets is that it is interesting to study not only the points where the resolvent of an operator is not defined, i.e. the spectrum, but also where this resolvent may be large in norm. This explains the following definition of the ε-pseudospectrum of a matrix or an operator A,

$$\sigma_\varepsilon(A) = \Big\{z \in \mathbb{C},\ \|(A - z)^{-1}\| \geq \frac{1}{\varepsilon}\Big\}, \quad \varepsilon > 0,$$

with the convention that $\|(A - z)^{-1}\| = +\infty$ for any point z belonging to the spectrum of the operator $\sigma(A)$. There exists an abundant literature about this notion of pseudospectrum. We refer the reader to [**117, 118**] and all the references therein.

Studying the pseudospectra of an operator is studying the level lines of the norm of its resolvent. What is interesting in studying such level lines is that it gives some information about the spectral stability of the operator. Indeed, the pseudospectra can be defined in an equivalent way in term of the spectrum of perturbations of the operator. For instance, we have for any $A \in M_n(\mathbb{C})$,

$$\sigma_\varepsilon(A) = \{z \in \mathbb{C},\ z \in \sigma(A+B) \text{ for some } B \in M_n(\mathbb{C}) \text{ with } \|B\| \leq \varepsilon\}.$$

It follows that a complex number z belongs to the ε-pseudospectrum of a matrix A if and only if it belongs to the spectrum of one of its perturbations $A+B$ with $\|B\| \leq \varepsilon$. More generally, if A is a closed unbounded linear operator with a dense domain on a complex Hilbert space H, the same description holds [**111**],

$$\sigma_\varepsilon(A) = \bigcup_{B \in \mathcal{L}(H),\ \|B\|_{\mathcal{L}(H)} \leq \varepsilon} \sigma(A+B),$$

where $\mathcal{L}(H)$ stands for the set of bounded linear operators on H. This second description emphasizes the interest of studying such subsets when we aim at determining numerically the spectrum of an operator. Indeed, the discretization of the operator and inevitable round-off errors generate some perturbations of the initial operator. Eventually, algorithms for eigenvalues computing determine the eigenvalues of a perturbation of the initial operator, i.e. a value in a ε-pseudospectrum of the initial operator but not necessarily a spectral one. This explains why it is important in some numerical computations to understand if the spectrum lies more or less deeply inside the ε-pseudospectra.

The study of pseudospectrum is non-trivial only for non-selfadjoint operators, or more precisely for non-normal operators. Indeed, the classical formula

$$\forall z \notin \sigma(A), \quad \|(A-z)^{-1}\| = \frac{1}{\operatorname{dist}(z, \sigma(A))}, \tag{1.2}$$

emphasizes that the resolvent of a normal operator cannot blow up far from its spectrum. This ensures the stability of the spectrum under small perturbations

$$\sigma_\varepsilon(A) = \{z \in \mathbb{C} :\ \operatorname{dist}(z, \sigma(A)) \leq \varepsilon\}. \tag{1.3}$$

However, the formula (1.2) does not hold anymore for non-normal operators and the resolvent of such operators may become very large in norm far from the spectrum. This implies that the spectra of these operators may become very unstable under small perturbations. The rotated harmonic oscillator is an example of elliptic quadratic operator whose spectrum is very unstable under small perturbations

$$P = -\partial_x^2 + e^{i\pi/4}x^2.$$

This feature may be illustrated by computing numerically the spectrum of the matrix discretization

$$((P\Psi_i, \Psi_j)_{L^2(\mathbb{R})})_{1 \leq i,j \leq N},$$

with $N = 100$, where $(\Psi_j)_{j \geq 1}$ stands for the $L^2(\mathbb{R})$ Hermite basis. The black dots correspond to the numerically computed eigenvalues. For low energies, they are in perfect agreement with the theoretical eigenvalues

$$\sigma(P) = \{e^{i\pi/8}(2n+1) :\ n \geq 0\},$$

regularly spaced out onto the half-line $e^{i\pi/8}\mathbb{R}_+$. However, this is no more true for the high energies. For these energies, some spectral instabilities lead to the computation of spurious eigenvalues far away from the half-line $e^{i\pi/8}\mathbb{R}_+$. The geometry of the pseudospectra for the rotated harmonic oscillator is described in the works [**22, KPS16**]. These theoretical results confirm the numerical computation (Fig. 1) and emphasize that the norm of the resolvent $(P-z)^{-1}$ exhibits rapid growth far from the

FIGURE 1. Numerical computation of some level lines $\|(P-z)^{-1}\| = \varepsilon^{-1}$ and eigenvalues of the rotated harmonic oscillator. The right column gives the corresponding values of $\log_{10}\varepsilon$.

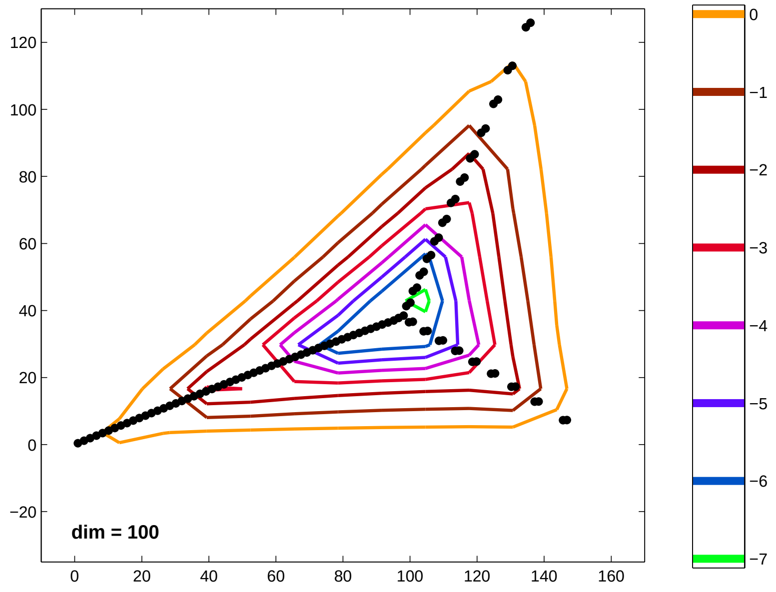
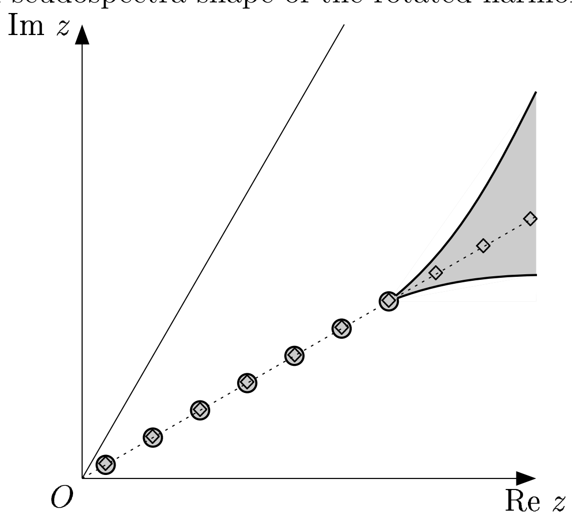


FIGURE 2. Pseudospectra shape of the rotated harmonic oscillator.



spectrum in a region of the resolvent set whose geometry may be exactly determined. These phenomena of strong instabilities for high energies are not peculiar to the rotated harmonic. They were shown to be the typical behavior of any non-normal elliptic quadratic operator [**KPS17, KPS20, KPS22**]. More precisely, the resolvent of any non-normal elliptic quadratic operator

$$\|(q(x, hD_x) - z)^{-1}\| = \mathcal{O}(h^{-\infty}),$$

is shown to grow rapidy, as the semiclassical parameter $h \to 0$, when the spectral parameter z lies inside the range of the symbol q. This is linked to some properties of microlocal non-solvability of non-normal elliptic quadratic operators and to violations of the adjoint condition to the Nirenberg-Trèves condition (Ψ):

> Condition $(\overline{\Psi})$: A symbol p is said to satisfy the condition $(\overline{\Psi})$ when the imaginary part $\mathrm{Im}(ap)$ is not allowed to change sign from + to - along the oriented bicharacteristics of the real part $\mathrm{Re}(ap)$ for any $0 \neq a \in C^\infty$

which allow the construction of semiclassical quasimodes [**35, 75, 132, 133, KPS15, KPS17**]. Similar types of spectral instabilities were shown to occur for general pseudodifferential operators around a doubly characteristic point when the quadratic approximations of these operators at the doubly characteristic set are non-normal [**KPS18**]. Starting from these early insights, the present work aims at accounting for the typical phenomena occurring around the doubly characteristic set and at providing a sharp description of the spectral and pseudospectral properties of general pseudodifferential operators around a doubly characteristic point.

2. Quadratic differential operators

2.1. Definition of the singular space associated to a quadratic operator. Quadratic operators are pseudodifferential operators defined in the Weyl quantization

$$q^w(x, D_x)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi} q\Big(\frac{x+y}{2}, \xi\Big) u(y) dy d\xi, \tag{1.4}$$

by symbols $q(x, \xi)$, with $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$, $n \geq 1$, which are complex-valued quadratic forms

$$\begin{array}{rcl} q : \mathbb{R}^n_x \times \mathbb{R}^n_\xi & \rightarrow & \mathbb{C} \\ (x, \xi) & \mapsto & q(x, \xi). \end{array}$$

These operators are non-selfadjoint differential operators with simple and fully explicit expression since the Weyl quantization of the quadratic symbol $x^\alpha \xi^\beta$, with $(\alpha, \beta) \in \mathbb{N}^{2n}$, $|\alpha + \beta| = 2$, is the differential operator

$$\frac{x^\alpha D_x^\beta + D_x^\beta x^\alpha}{2}, \quad D_x = i^{-1}\partial_x.$$

We know from [**76**] (p.425-426) that the maximal closed realization of the operator $q^w(x, D_x)$ with domain

$$D(q) = \{u \in L^2(\mathbb{R}^n) : \ q^w(x, D_x)u \in L^2(\mathbb{R}^n)\},$$

coincides with the graph closure of its restriction to the Schwartz space

$$q^w(x, D_x) : \mathscr{S}(\mathbb{R}^n) \rightarrow \mathscr{S}(\mathbb{R}^n).$$

When the quadratic symbol has a non-negative real part $\mathrm{Re}\ q \geq 0$, the operator is maximally accretive

$$\mathrm{Re}(q^w(x, D_x)u, u)_{L^2} \geq 0, \quad u \in D(q).$$

Associated to a quadratic symbol q is the numerical range $\Sigma(q)$ defined as the closure in the complex plane of all its values

$$\Sigma(q) = \overline{q(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)}. \tag{1.5}$$

Another classical object associated to a quadratic symbol is the notion of Hamilton map, also known as the fundamental matrix. The Hamilton map $F \in M_{2n}(\mathbb{C})$ associated to the quadratic form q is the unique map defined by the identity

$$q((x, \xi); (y, \eta)) = \sigma((x, \xi), F(y, \eta)), \quad (x, \xi) \in \mathbb{R}^{2n}, \ (y, \eta) \in \mathbb{R}^{2n}, \tag{1.6}$$

where $q\big(\cdot;\cdot\big)$ stands for the polarized form associated to the quadratic form q and σ is the canonical symplectic form on $\mathbb{R}^{2n}$,

$$\sigma((x,\xi),(y,\eta))=\xi\cdot y-x\cdot\eta,\quad (x,\xi)\in\mathbb{R}^{2n},\ (y,\eta)\in\mathbb{R}^{2n}. \tag{1.7}$$

It follows from the definition that the real and imaginary parts of the Hamilton map F,

$$\text{Re } F=\frac{1}{2}(F+\overline{F}),\quad \text{Im } F=\frac{1}{2i}(F-\overline{F}),$$

are the Hamilton maps associated respectively to the real and imaginary parts of the quadratic symbol q. A Hamilton map is always skew-symmetric with respect to the symplectic form. This is a consequence of the properties of skew-symmetry of the symplectic form, and symmetry of the polarized form

$$\forall X,Y\in\mathbb{R}^{2n},\quad \sigma(X,FY)=q(X;Y)=q(Y;X)=\sigma(Y,FX)=-\sigma(FX,Y). \tag{1.8}$$

Hitrik and the author introduced in [**KPS2**] the notion of singular space. The singular space of a quadratic symbol q is defined as the linear subspace in the phase space given by the following infinite intersection of kernels

$$S=\Big(\bigcap_{j=0}^{+\infty}\text{Ker}\big[\text{Re } F(\text{Im } F)^j\big]\Big)\cap\mathbb{R}^{2n}, \tag{1.9}$$

where Re F, Im F are respectively the real and imaginary parts of the Hamilton map associated to q. By definition, the singular space enjoys the stability properties

$$(\text{Re } F)S=\{0\},\quad (\text{Im } F)S\subset S. \tag{1.10}$$

Equivalently, the singular space may be defined as the following finite intersection of kernels

$$S=\Big(\bigcap_{j=0}^{2n-1}\text{Ker}\big[\text{Re } F(\text{Im } F)^j\big]\Big)\cap\mathbb{R}^{2n}, \tag{1.11}$$

since the Cayley-Hamilton theorem implies that all the vectors

$$(\text{Im } F)^kX\in\text{Span}(X,...,(\text{Im } F)^{2n-1}X),\quad X\in\mathbb{R}^{2n},\ k\geq 0,$$

belong to the vector space spanned by X, ..., $(\text{Im } F)^{2n-1}X$.

As discussed in the following sections, the singular space is playing a basic role in understanding the properties of non-elliptic quadratic operators. It is interesting to notice that the computation of a singular space reduces to a fairly simple linear algebraic calculation and, as we shall see in the following, that the partition induced by its structure provides relevant information about the dynamical properties of the symbol q.

Let us consider the case when the quadratic symbol q has a non-negative real part Re $q\geq 0$. In this case, we shall see that the singular space may be defined in a third equivalent way as the subset in the phase space where all the Poisson brackets $H_{\text{Im}q}^k\text{Re } q$ are vanishing

$$S=\{X\in\mathbb{R}^{2n}:\ H_{\text{Im}q}^k\text{Re } q(X)=0,\ k\geq 0\}. \tag{1.12}$$

This dynamical definition shows that the singular space is exactly the set of points $X_0\in\mathbb{R}^{2n}$ where the function

$$t\mapsto\text{Re } q(e^{tH_{\text{Im}q}}X_0), \tag{1.13}$$

vanishes at the infinite order at $t=0$. As we shall see in the following, the partition defined by the structure of the singular space allows to distinguish different regions in the phase space according to the vanishing order of the function (1.13).

Lastly, when q is a quadratic form with a non-negative real part $\operatorname{Re} q \geq 0$ satisfying a condition of ellipticity on its singular space

$$(x,\xi) \in S,\ q(x,\xi)=0 \Rightarrow (x,\xi)=0,$$

this singular space may be described directly in terms of the eigenspaces of its Hamilton map associated to its real eigenvalues. In this case, the set of real eigenvalues of the Hamilton map F may be written as

$$\sigma(F) \cap \mathbb{R} = \{\lambda_1, ..., \lambda_r, -\lambda_1, ..., -\lambda_r\},$$

with $\lambda_j \neq 0$, $\lambda_j \neq \pm\lambda_k$ if $j \neq k$. The singular space is then equal to the direct sum of the symplectically orthogonal spaces

$$S = S_{\lambda_1} \oplus^{\sigma\perp} S_{\lambda_2} \oplus^{\sigma\perp} ... \oplus^{\sigma\perp} S_{\lambda_r}, \tag{1.14}$$

where S_{λ_j} stands for the symplectic space

$$S_{\lambda_j} = (\operatorname{Ker}(F-\lambda_j) \oplus \operatorname{Ker}(F+\lambda_j)) \cap \mathbb{R}^{2n}. \tag{1.15}$$

Thus, in this case, the singular space is equal to zero $S=\{0\}$ if and only if the Hamilton map F has no real eigenvalues.

A first example of quadratic operator is given by the Kramers-Fokker-Planck operator

$$K = -\Delta_v + \frac{v^2}{4} - \frac{1}{2} + v\partial_x - \nabla_x V(x)\partial_v, \quad (x,v) \in \mathbb{R}^2,$$

with a quadratic potential

$$V(x) = \frac{1}{2}ax^2, \quad a \in \mathbb{R}^*.$$

This operator writes as

$$K = q^w(x,v,D_x,D_v) - \frac{1}{2}, \tag{1.16}$$

where

$$q(x,v,\xi,\eta) = \eta^2 + \frac{1}{4}v^2 + i(v\xi - ax\eta),$$

is a non-elliptic complex-valued quadratic form with a non-negative real part. The Hamilton map of the symbol

$$q(x,v,\xi,\eta) = \sigma((x,v,\xi,\eta), F(x,v,\xi,\eta)),$$

is given by

$$F = \begin{pmatrix} 0 & \frac{1}{2}i & 0 & 0 \\ -\frac{1}{2}ai & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{1}{2}ai \\ 0 & -\frac{1}{4} & -\frac{1}{2}i & 0 \end{pmatrix}.$$

A simple algebraic computation shows that

$$\operatorname{Ker}(\operatorname{Re} F) \cap \operatorname{Ker}(\operatorname{Re} F \operatorname{Im} F) \cap \mathbb{R}^4 = \{0\}.$$

The singular space of the symbol q is therefore equal to zero

$$S = \{0\}.$$

This implies that zero is the only point in the phase space where the function (1.13) vanishes at the infinite order at $t = 0$. As we shall see in the following, this structure of the singular space implies the following partition of the phase space

$$\mathbb{R}^4 = \big[\mathbb{R}^4 \setminus (\mathrm{Ker}(\mathrm{Re}\ F) \cap \mathbb{R}^4)\big] \sqcup \big[(\mathrm{Ker}(\mathrm{Re}\ F) \cap \mathbb{R}^4) \setminus \{0\}\big] \sqcup \{0\},$$

where the microlocal region $\mathbb{R}^4 \setminus (\mathrm{Ker}(\mathrm{Re}\ F) \cap \mathbb{R}^4)$ corresponds to the points where the function (1.13) does not vanish at $t = 0$, whereas the microlocal region

$$(\mathrm{Ker}(\mathrm{Re}\ F) \cap \mathbb{R}^4) \setminus \{0\},$$

corresponds to the points where the function (1.13) vanishes exactly at the second order at $t = 0$.

2.2. Spectral and smoothing properties of quadratic operators. Since the classical works by Sjöstrand [**113**] and Boutet de Monvel [**23**], a complete description for the spectrum of elliptic quadratic operators is known and has played an important role in the analysis of partial differential operators with double characteristics. Elliptic quadratic operators are quadratic operators whose Weyl symbols satisfy to the global ellipticity condition

$$(x,\xi) \in \mathbb{R}^{2n},\ q(x,\xi) = 0 \Rightarrow (x,\xi) = 0. \tag{1.17}$$

When this ellipticity condition holds, the numerical range $\Sigma(q)$ has a specific shape [**113**]. In dimension $n \geq 2$, one may always find a non-zero complex number $z \in \mathbb{C}^*$ such that $\mathrm{Re}(zq)$ is a positive definite quadratic form; whereas in dimension $n = 1$, the same result holds true if in addition we assume that $\Sigma(q) \neq \mathbb{C}$. The numerical range of an elliptic quadratic form can therefore take only two shapes: $\mathbb{C}$ or a closed angular sector with a vertex in 0 and an angle strictly less than π. Elliptic quadratic operators define Fredholm operators [**73, 113**],

$$q^w(x,D_x) + z : D(q) \to L^2(\mathbb{R}^n), \tag{1.18}$$

where $D(q)$ is equal to the Hilbert space

$$B = \{u \in L^2(\mathbb{R}^n) :\ x^\alpha D_x^\beta u \in L^2(\mathbb{R}^n),\ |\alpha+\beta| \leq 2\}, \tag{1.19}$$

equipped with the norm

$$\|u\|_B^2 = \sum_{|\alpha+\beta|\leq 2} \|x^\alpha D_x^\beta u\|^2_{L^2(\mathbb{R}^n)}.$$

The Fredholm index is independent of z. It is equal to 0 when $\Sigma(q) \neq \mathbb{C}$, so it is always equal to 0 when $n \geq 2$; whereas in dimension $n = 1$, it can take the values -2, 0 or 2. The exact description of the spectrum of an elliptic quadratic operator

$$q^w(x,D_x) : B \to L^2(\mathbb{R}^n),$$

whose numerical range satisfies $\Sigma(q) \neq \mathbb{C}$, was given independently by Sjöstrand and Boutet de Monvel in the 1970s [**23, 113**]. This spectrum is only composed of eigenvalues with finite algebraic multiplicity

$$\sigma(q^w(x,D_x)) = \Big\{ \sum_{\substack{\lambda\in\sigma(F),\\ -i\lambda\in\Sigma(q)\setminus\{0\}}} (r_\lambda + 2k_\lambda)(-i\lambda) :\ k_\lambda \in \mathbb{N}\Big\}, \tag{1.20}$$

where F is the Hamilton map of the quadratic form q and r_λ is the dimension of the space of generalized eigenvectors of F in $\mathbb{C}^{2n}$ associated to the eigenvalue $\lambda \in \mathbb{C}$. Much more recently [**70**], the very same description of the spectrum was shown to hold as well for non-elliptic quadratic operators when their symbols have non-negative real parts $\textrm{Re } q \geq 0$ and satisfy to the following condition of subelliptic type

$$\exists 0 < \varepsilon \leq 1, \exists c > 0,\ c|X|^2 \leq \textrm{Re } q(X) + \varepsilon H_{\textrm{Im}q}^2 \textrm{Re } q(X) \leq \frac{1}{c}|X|^2,\ X \in \mathbb{R}^{2n}. \tag{1.21}$$

When this condition holds, it means that the real part of the symbol might not be positive definite. However, the iterated Poisson bracket $H_{\textrm{Im}q}^2 \textrm{Re } q$ must be positive at these non-zero points where the non-negative quadratic form $\textrm{Re } q$ vanishes

$$X_0 \in \mathbb{R}^{2n},\ X_0 \neq 0, \quad \textrm{Re } q(X_0) = 0 \Rightarrow H_{\textrm{Im}q}^2 \textrm{Re } q(X_0) > 0.$$

The Kramers-Fokker-Planck operator with quadratic potential (1.16) is an example of such a non-elliptic quadratic operator satisfying the subelliptic condition

$$\textrm{Re } q + \varepsilon H_{\textrm{Im}q}^2 \textrm{Re } q = (1 - 2a\varepsilon)\Big(\eta^2 + \frac{v^2}{4}\Big) + 2\varepsilon\Big(\xi^2 + a^2\frac{x^2}{4}\Big) \gg 0,$$

when $0 < \varepsilon \ll 1$. Following this breakthrough, the seminal questions at the origin of the joint work with Hitrik [**KPS2**] were to understand further under which general assumptions the description of the spectrum known in the elliptic case (1.20) still holds. In order to weaken the condition (1.21), we have introduced the notion of singular space. More specifically, we study in [**KPS2**] the class of quadratic operators $q^w(x, D_x)$ whose symbols have non-negative real parts $\textrm{Re } q \geq 0$ and satisfy to a condition of partial ellipticity on the phase space, namely a condition of ellipticity on the singular space

$$(x, \xi) \in S,\ q(x, \xi) = 0 \Rightarrow (x, \xi) = 0. \tag{1.22}$$

This condition of partial ellipticity is obviously weaker than the global ellipticity condition (1.17) since $S \subset \mathbb{R}^{2n}$, and this strictly as soon as $S \neq \mathbb{R}^{2n}$, that is, when the real part of the symbol is not identically equal to zero. According to the dynamical definition of the singular space (1.12), the condition of subelliptic type (1.21) implies that the singular space is reduced to zero. Thus, the condition (1.22) holds as well trivially in this case.

A first geometric feature is the fact that the singular space S associated to a complex-valued quadratic form q has always a symplectic structure[1] when the symbol q is elliptic on S. This feature allows to decompose the phase space into the direct sum of the two symplectically orthogonal spaces

$$\mathbb{R}^{2n} = S \oplus^{\sigma\perp} S^{\sigma\perp},$$

where $S^{\sigma\perp}$ is the orthogonal complement of the singular space in $\mathbb{R}^{2n}$ with respect to the symplectic form

$$S^{\sigma\perp} = \{X \in \mathbb{R}^{2n} :\ \forall Y \in S,\ \sigma(X, Y) = 0\}.$$

By choosing some symplectic coordinates onto the symplectic spaces S and $S^{\sigma\perp}$,

$$X = X' + X'', \quad X = (x, \xi) \in \mathbb{R}^{2n}, \quad X' = (x', \xi') \in S^{\sigma\perp}, \quad X'' = (x'', \xi'') \in S,$$

[1] In the sense that the restriction of the symplectic form $\sigma|_S$ to the singular space is non-degenerate.

the first result in [**KPS2**] shows that the contraction semigroup generated by the accretive quadratic operator $q^w(x,D_x)$,

$$\left\{ \begin{array}{l} \dfrac{\partial u}{\partial t}(t,x) + q^w(x,D_x)u(t,x) = 0, \\ u(t,\cdot)|_{t=0} = u_0 \in L^2(\mathbb{R}^n), \end{array} \right. \tag{1.23}$$

is smoothing in every direction of the orthogonal complement of the singular space $S^{\sigma\perp}$.

THEOREM 1.1. ([**KPS2**], Hitrik, KPS) *Let $q : \mathbb{R}^n_x \times \mathbb{R}^n_\xi \to \mathbb{C}$, $n \geq 1$, be a complex-valued quadratic form with a non-negative real part* $\mathrm{Re}\, q \geq 0$. *When the quadratic symbol q is elliptic on its singular space*

$$(x,\xi) \in S,\ q(x,\xi) = 0 \Rightarrow (x,\xi) = 0,$$

or more generally when its singular space S has a symplectic structure, then for all $t > 0$, $N \in \mathbb{N}$ and $u \in L^2(\mathbb{R}^n)$,

$$(1 + |x'|^2 + |D_{x'}|^2)^N e^{-tq^w(x,D_x)} u \in L^2(\mathbb{R}^n), \tag{1.24}$$

if (x',ξ') are some linear symplectic coordinates on the symplectic space $S^{\sigma\perp}$.

When the singular space is equal to zero, the assumptions of Theorem 1.1 trivially hold and the contraction semigroup is smoothing

$$e^{-tq^w(x,D_x)} u \in \mathscr{S}(\mathbb{R}^n), \quad u \in L^2(\mathbb{R}^n),$$

for any positive time $t > 0$.

The next result shows that the description of the spectrum known in the elliptic case (1.20) extends to non-elliptic quadratic operators whose symbols have a non-negative real part and enjoy ellipticity on their singular spaces.

THEOREM 1.2. ([**KPS2**], Hitrik, KPS) *Let $q : \mathbb{R}^n_x \times \mathbb{R}^n_\xi \to \mathbb{C}$, $n \geq 1$, be a complex-valued quadratic form with a non-negative real part* $\mathrm{Re}\, q \geq 0$. *When the quadratic symbol q is elliptic on its singular space*

$$(x,\xi) \in S,\ q(x,\xi) = 0 \Rightarrow (x,\xi) = 0,$$

the spectrum of the quadratic operator $q^w(x,D_x)$ is only composed by eigenvalues with finite algebraic multiplicity

$$\sigma(q^w(x,D_x)) = \Big\{ \sum_{\substack{\lambda \in \sigma(F), \\ -i\lambda \in \mathbb{C}_+ \cup (\Sigma(q|_S)\setminus\{0\})}} \big(r_\lambda + 2k_\lambda\big)(-i\lambda) : k_\lambda \in \mathbb{N} \Big\}, \tag{1.25}$$

where F is the Hamilton map of the quadratic form q, r_λ is the dimension of the space of generalized eigenvectors of F in $\mathbb{C}^{2n}$ associated to the eigenvalue $\lambda \in \mathbb{C}$,

$$\Sigma(q|_S) = \overline{q(S)}, \quad \mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Re}\, z > 0\}.$$

The following theorem extends the result obtained in [**KPS21**] by providing some necessary and sufficient conditions for the exponential decay in time of the contraction semigroup generated by non-elliptic quadratic operators with symplectic singular spaces.

THEOREM 1.3. ([**KPS2**], Hitrik, KPS) *Let* $q : \mathbb{R}^n_x \times \mathbb{R}^n_\xi \rightarrow \mathbb{C}$, $n \geq 1$, *be a complex-valued quadratic form with a non-negative real part* $\textrm{Re } q \geq 0$. *When the quadratic symbol* q *is elliptic on its singular space*

$$(x,\xi) \in S,\ q(x,\xi)=0 \Rightarrow (x,\xi)=0,$$

or more generally when its singular space S *has a symplectic structure, then the following assertions are equivalent:*

- (i) *The contraction semigroup generated by the operator* $q^w(x,D_x)$ *decays exponentially in time*

$$\exists M>0, \exists a>0, \forall t \geq 0, \quad \|e^{-tq^w(x,D_x)}\|_{\mathcal{L}(L^2)} \leq Me^{-at}$$

- (ii) *The real part of the symbol* q *is not identically equal to zero*

$$\exists (x_0,\xi_0) \in \mathbb{R}^{2n}, \quad \textrm{Re } q(x_0,\xi_0) \neq 0$$

- (iii) *The singular space is distinct from the phase space* $S \neq \mathbb{R}^{2n}$

We recall that the only condition (ii) is not sufficient in general for getting the exponential decay in time of the contraction semigroup. Indeed, let us consider the case when $q(x,\xi)=x^2$. The quadratic operator $q^w(x,D_x)$ is the operator of multiplication by x^2 generating the contraction semigroup

$$e^{-tq^w(x,D_x)}u = e^{-tx^2}u, \quad t \geq 0,\ u \in L^2(\mathbb{R}^n),$$

whose norm is identically equal to 1,

$$\|e^{-tq^w(x,D_x)}\|_{\mathcal{L}(L^2)} = 1, \quad t \geq 0.$$

PROOF. The first part in the proofs of these theorems is purely algebraic. Under the assumptions of these theorems, the singular space has a symplectic structure allowing to decompose the phase space into the direct sum of the two symplectically orthogonal spaces

$$\mathbb{R}^{2n} = S \oplus^{\sigma\perp} S^{\sigma\perp}, \quad \dim_{\mathbb{R}} S^{\sigma\perp} = 2n', \quad \dim_{\mathbb{R}} S = 2n'', \quad n = n'+n''.$$

We choose some symplectic coordinates onto the symplectic spaces S and $S^{\sigma\perp}$,

$$X = X'+X'',\ X=(x,\xi) \in \mathbb{R}^{2n},\ X'=(x',\xi') \in S^{\sigma\perp},\ X''=(x'',\xi'') \in S. \tag{1.26}$$

The stability properties (1.10) holding true as well by duality for the space $S^{\sigma\perp}$,

$$(\textrm{Re } F)S^{\sigma\perp} \subset S^{\sigma\perp}, \quad (\textrm{Im } F)S^{\sigma\perp} \subset S^{\sigma\perp},$$

imply that the variables in S and $S^{\sigma\perp}$ may be tensorized. The quadratic form q may be written as the sum of a quadratic form defined on S and another one defined on $S^{\sigma\perp}$,

$$q = q|_S + q|_{S^{\sigma\perp}}, \tag{1.27}$$

where the first quadratic form $q|_S$ is purely imaginary-valued, i.e., equal to

$$q|_S = i\tilde{q}|_S,$$

with $\tilde{q}|_S$ a real-valued quadratic form, whereas the second quadratic form $q|_{S^{\sigma\perp}}$ is a complex-valued quadratic form with a non-negative real part $\textrm{Re } q|_{S^{\sigma\perp}} \geq 0$. When the assumption of ellipticity on the singular space holds

$$(x,\xi) \in S,\ q(x,\xi)=0 \Rightarrow (x,\xi)=0,$$

this means that $\tilde{q}|_S$ is an elliptic real-valued quadratic form, i.e., that there exists $\varepsilon_0 \in \{\pm 1\}$ such that $\varepsilon_0 \tilde{q}|_S$ is a positive definite quadratic form. Up to a new choice of symplectic coordinates, any positive definite quadratic form can be reduced to a harmonic oscillator. In such coordinates, the restriction of the quadratic form $q|_S$ on the singular space reduces to the normal form

$$q|_S(X'') = i\varepsilon_0 \sum_{j=1}^{n''} \lambda_j(\xi_j''^2 + x_j''^2), \quad X'' = (x'', \xi'') \in S, \tag{1.28}$$

with $\lambda_j > 0$. This normal form partly accounts for the structure of the spectrum in Theorem 1.2. Regarding the second quadratic form $q|_{S^{\sigma\perp}}$, its real part fails in general to be positive definite unless the real part of q is. However, this quadratic form $q|_{S^{\sigma\perp}}$ enjoys specific dynamical features. Indeed, the definition of the singular space implies the following properties:

(i) The average of the real part of $q|_{S^{\sigma\perp}}$ by the flow generated by the Hamilton vector field associated to its imaginary part $H_{\mathrm{Im} q|_{S^{\sigma\perp}}}$,

$$\langle \mathrm{Re}\ q|_{S^{\sigma\perp}} \rangle_T(X') = \frac{1}{2T} \int_{-T}^{T} (\mathrm{Re}\ q|_{S^{\sigma\perp}})(e^{tH_{\mathrm{Im} q|_{S^{\sigma\perp}}}} X')dt \gg 0, \quad T > 0, \tag{1.29}$$

is a positive definite quadratic form on the symplectic space $S^{\sigma\perp}$

(ii) The sum of the non-negative quadratic forms

$$\sum_{j=0}^{2n'-1} (\mathrm{Re}\ q|_{S^{\sigma\perp}})((\mathrm{Im}\ F|_{S^{\sigma\perp}})^j X') \gg 0, \tag{1.30}$$

is positive definite on the symplectic space $S^{\sigma\perp}$

Studying the quadratic form $q|_{S^{\sigma\perp}}$ reduces to studying the case when the singular space is equal to zero. Thus, we may assume from now that q is a quadratic symbol with a non-negative real part $\mathrm{Re}\ q \geq 0$ and a zero singular space $S = 0$. In this case, the quadratic forms

$$\langle \mathrm{Re}\ q \rangle_T(X) = \frac{1}{2T} \int_{-T}^{T} (\mathrm{Re}\ q)(e^{tH_{\mathrm{Im} q}} X)dt \gg 0, \quad T > 0 \tag{1.31}$$

and

$$\sum_{j=0}^{2n-1} \mathrm{Re}\ q((\mathrm{Im}\ F)^j X) \gg 0, \tag{1.32}$$

are positive definite. Algebraic computations show that we have the following partition of the phase space

$$\mathbb{R}^{2n} = \{0\} \sqcup \Lambda_0 \sqcup ... \sqcup \Lambda_{2n-1}, \tag{1.33}$$

where the microlocal region

$$\begin{aligned}\Lambda_j &= \{X \in \mathbb{R}^{2n} : X \in \mathrm{Ker}(\mathrm{Re}\ F) \cap ... \cap \mathrm{Ker}[\mathrm{Re}\ F(\mathrm{Im}\ F)^{j-1}], X \notin \mathrm{Ker}[\mathrm{Re}\ F(\mathrm{Im}\ F)^j]\} \\ &= \{X \in \mathbb{R}^{2n} :\ \mathrm{Re}\ q(X) = ... = \mathrm{Re}\ q((\mathrm{Im}\ F)^{j-1}X) = 0,\ \mathrm{Re}\ q((\mathrm{Im}\ F)^j X) > 0\},\end{aligned}$$

corresponds to the points $X_0 \in \mathbb{R}^{2n}$ where the function

$$t \mapsto \mathrm{Re}\ q(e^{tH_{\mathrm{Im} q}} X_0),$$

vanishes at $t=0$ exactly at the order $2j$,

$$\Lambda_j=\{X\in\mathbb{R}^{2n}:\ \textrm{Re }q(X)=H_{\textrm{Im}q}\textrm{Re }q(X)=...=H_{\textrm{Im}q}^{2j-1}\textrm{Re }q(X)=0,\\ H_{\textrm{Im}q}^{2j}\textrm{Re }q(X)>0\}.$$

The rest of the proof is analytic. It relies on some techniques of complex deformations of the phase space developed by Sjöstrand, and closely follows the analysis of the Kramers-Fokker-Planck equation led by Hérau, Sjöstrand and Stolk in [**70**].

In order to take advantage of the positive definiteness of the quadratic form (1.31), we consider the real-valued quadratic form

$$G(X)=\int_{\mathbb{R}}k_T(t)\textrm{Re }q(e^{tH_{\textrm{Im}q}}X)dt,$$

with $k_T(t)=k\big(\frac{t}{2T}\big)$, where $k\in C(\mathbb{R}\backslash\{0\})$ is the odd function satisfying

$$k(t)=0\textrm{ for }|t|\geq\frac{1}{2},\quad k'(t)=-1\textrm{ for }0<|t|<\frac{1}{2}.$$

It solves the equation

$$H_{\textrm{Im}q}G=-\textrm{Re }q+\langle\textrm{Re }q\rangle_T.$$

Associated to G is a linear IR-manifold

$$\Lambda_{\varepsilon G}=e^{i\varepsilon H_G}(\mathbb{R}^{2n})\subset\mathbb{C}^{2n},\quad 0<\varepsilon\ll 1,$$

on which the quadratic symbol q becomes elliptic, since the real part of the quadratic form

$$\widetilde{q}_\varepsilon(X)=q(e^{i\varepsilon H_G}X),$$

is positive definite

$$\textrm{Re }\widetilde{q}_\varepsilon(X)=\textrm{Re }q(X)+\varepsilon H_{\textrm{Im}q}G(X)+\mathcal{O}(\varepsilon^2|X|^2)\\ =(1-\varepsilon)\textrm{Re }q+\varepsilon\langle\textrm{Re }q\rangle_T+\mathcal{O}(\varepsilon^2|X|^2)\gg 0,$$

when $0<\varepsilon\ll 1$. A complete description for the spectrum of the elliptic quadratic operator $\tilde{q}_\varepsilon(x,D_x)$ is known thanks to Sjöstrand's result [**113**] (see also [**23**]):

$$\sigma(\widetilde{q}_\varepsilon^w(x,D_x))=\Big\{\sum_{\substack{\lambda\in\sigma(\tilde{F}_\varepsilon),\\ -i\lambda\in\Sigma(\tilde{q}_\varepsilon)\backslash\{0\}}}(\tilde{r}_{\varepsilon,\lambda}+2k_\lambda)(-i\lambda):\ k_\lambda\in\mathbb{N}\Big\},$$

when $0<\varepsilon\ll 1$. The Hamilton maps of the quadratic forms q and $\tilde{q}_\varepsilon$ are isospectral

$$\tilde{F}_\varepsilon=e^{-i\varepsilon H_G}Fe^{i\varepsilon H_G}.$$

This implies that the spectrum of the operator $\widetilde{q}_\varepsilon^w(x,D_x)$ is actually independent of the parameter ε when $0<\varepsilon\ll 1$. The core of the proof is then to establish that the quadratic operator $q^w(x,D_x)$ has discrete spectrum, and more specifically that the operators $q^w(x,D_x)$ and $\widetilde{q}_\varepsilon^w(x,D_x)$, with $0<\varepsilon\ll 1$, share the same spectrum, eigenvectors and generalized eigenvectors. To that end, we study the contraction semigroups generated by these operators as Fourier integral operators on the FBI-Bargmann side

$$e^{-tQ_0}:H_{\Phi_0}(\mathbb{C}^n)\rightarrow H_{\Phi_0}(\mathbb{C}^n),\quad e^{-tQ_0}:H_{\tilde{\Phi}_\varepsilon}(\mathbb{C}^n)\rightarrow H_{\tilde{\Phi}_\varepsilon}(\mathbb{C}^n),$$

with

$$H_\Phi(\mathbb{C}^n)=\textrm{Hol}(\mathbb{C}^n)\cap L^2\big(\mathbb{C}^n,e^{-2\Phi(x)}L(dx)\big),$$

where $L(dx)$ stands for the Lebesgue measure in $\mathbb{C}^n$. The key point is then to check that the algebraic feature (1.31) implies that these contraction semigroups are compact smoothing operators

$$e^{-tQ_0} : H_{\Phi_0}(\mathbb{C}^n) \to H_{\Phi_0 - \alpha(t)|x|^2}(\mathbb{C}^n), \quad e^{-tQ_0} : H_{\tilde{\Phi}_\varepsilon}(\mathbb{C}^n) \to H_{\tilde{\Phi}_\varepsilon - \alpha_\varepsilon(t)|x|^2}(\mathbb{C}^n),$$

with $\alpha(t) > 0$, $\alpha_\varepsilon(t) > 0$, when $t > 0$. This proves that the quadratic operator $q^w(x, D_x)$ has discrete spectrum, and additional semigroup arguments allow to check that the operators $q^w(x, D_x)$ and $\tilde{q}_\varepsilon^w(x, D_x)$, with $0 < \varepsilon \ll 1$, have the same spectrum, eigenvectors and generalized eigenvectors. $\Box$

2.3. Subelliptic properties of quadratic operators.

2.3.1. *The scalar case.* The Kramers-Fokker-Planck operator

$$K = q^w(x, v, D_x, D_v) - \frac{1}{2} = -\Delta_v + \frac{v^2}{4} - \frac{1}{2} + v\partial_x - \nabla_x V(x)\partial_v, \quad (x, v) \in \mathbb{R}^2,$$

with a quadratic potential

$$V(x) = \frac{1}{2}ax^2, \quad a \in \mathbb{R}^*,$$

is an example of non-elliptic quadratic operator enjoying subelliptic properties [**64**],

$$\exists C > 0, \forall u \in \mathscr{S}(\mathbb{R}^2), \quad \|\Lambda_x^{2/3}u\|_{L^2}^2 + \|\Lambda_v^2 u\|_{L^2}^2 \leq C(\|Ku\|_{L^2}^2 + \|u\|_{L^2}^2), \tag{1.34}$$

with

$$\Lambda_x^2 = -\Delta_x + \frac{x^2}{4}, \quad \Lambda_v^2 = -\Delta_v + \frac{v^2}{4}.$$

This global subelliptic estimate is sharp. Both indices $2/3$ for the subelliptic estimates in the (x, ξ) variables, and 2 for the elliptic estimates in the (v, η) variables are optimal. This index $2/3$ may be explained in term of the structure of the singular space associated to the quadratic symbol

$$q(x, v, \xi, \eta) = \eta^2 + \frac{1}{4}v^2 + i(v\xi - ax\eta).$$

We recall that the singular space of this symbol is equal to zero $S = 0$, and that the structure of the singular space implies the following partition of the phase space

$$\mathbb{R}^4 = \big[\mathbb{R}^4 \setminus (\mathrm{Ker}(\mathrm{Re}\ F) \cap \mathbb{R}^4)\big] \sqcup \big[(\mathrm{Ker}(\mathrm{Re}\ F) \cap \mathbb{R}^4) \setminus \{0\}\big] \sqcup \{0\},$$

according to the vanishing order of the function

$$t \mapsto \mathrm{Re}\ q(e^{tH_{\mathrm{Im} q}}X_0), \quad X_0 \in \mathbb{R}^4. \tag{1.35}$$

As we shall see in the next theorem, the index $2/3$ directly relates to the maximal finite vanishing order of the function (1.35) over the phase space. For the Kramers-Fokker-Planck operator with quadratic potential, this maximal finite vanishing order is equal to 2 and occurs in the microlocal region $(\mathrm{Ker}(\mathrm{Re}\ F) \cap \mathbb{R}^4) \setminus \{0\} \subset \mathbb{R}^4$.

The next result shows that any quadratic operator whose symbol has a non-negative real part and a zero singular space enjoys global subelliptic estimates of the type

$$\|\langle (x, D_x)\rangle^{2(1-\delta)}u\|_{L^2} \lesssim \|q^w(x, D_x)u\|_{L^2} + \|u\|_{L^2}, \tag{1.36}$$

where $\langle (x, D_x)\rangle^2 = 1 + |x|^2 + |D_x|^2$, with a sharp loss of derivatives $0 \leq \delta < 1$ with respect to the elliptic case (case $\delta = 0$), which may be explicitly derived from the structure of the singular space.

THEOREM 1.4. ([**KPS24**], KPS) *Let $q : \mathbb{R}^n_x \times \mathbb{R}^n_\xi \to \mathbb{C}$, $n \geq 1$, be a complex-valued quadratic form with a non-negative real part* Re $q \geq 0$ *and a zero singular space $S = 0$. Let $0 \leq k_0 \leq 2n-1$ be the smallest integer*[2] *satisfying*

$$\Big(\bigcap_{j=0}^{k_0} \mathrm{Ker}\big[\mathrm{Re}\ F(\mathrm{Im}\ F)^j\big]\Big) \cap \mathbb{R}^{2n} = \{0\}. \tag{1.37}$$

Then, the quadratic operator $q^w(x, D_x)$ satisfies the global subelliptic estimate

$$\exists C > 0, \forall u \in D(q), \quad \|\langle(x, D_x)\rangle^{2/(2k_0+1)}u\|_{L^2} \leq C(\|q^w(x, D_x)u\|_{L^2} + \|u\|_{L^2}). \tag{1.38}$$

The case when $k_0 = 0$ corresponds to the elliptic case when the symbol q has a positive definite real part. According to the previous discussion about the possible shapes of the numerical range, any elliptic symbol with $\Sigma(q) \neq \mathbb{C}$ may be reduced to this case. For the Kramers-Fokker-Planck operator with quadratic potential, the integer k_0 is equal to 1,

$$\mathrm{Ker}(\mathrm{Re}\ F) \cap \mathrm{Ker}(\mathrm{Re}\ F\ \mathrm{Im}\ F) \cap \mathbb{R}^4 = \{0\}.$$

This accounts for the index 2/3 in the global subelliptic estimate (1.34).

In the general case, let us emphasize that these estimates are global with the sharp loss of $\delta = 2k_0/(2k_0+1)$ derivatives. This loss is in agreement with general results about microlocal subellipticity with optimal loss of derivatives [**74, 75**]. Indeed, the assumptions of Theorem 1.4 imply the following partition of the phase space

$$\mathbb{R}^{2n} = \{0\} \sqcup \Lambda_0 \sqcup ... \sqcup \Lambda_{k_0}, \tag{1.39}$$

where the microlocal region

$$\Lambda_j = \{X \in \mathbb{R}^{2n} :\ \mathrm{Re}\ q(X) = H_{\mathrm{Im}q}\mathrm{Re}\ q(X) = ... = H^{2j-1}_{\mathrm{Im}q}\mathrm{Re}\ q(X) = 0, \\ H^{2j}_{\mathrm{Im}q}\mathrm{Re}\ q(X) > 0\},$$

corresponds to the points $X_0 \in \mathbb{R}^{2n}$ where the function

$$t \mapsto \mathrm{Re}\ q(e^{tH_{\mathrm{Im}q}}X_0),$$

vanishes at $t = 0$, exactly at the order $2j$. The maximal finite vanishing order of this function is therefore equal to $2k_0$. Roughly speaking, in each microlocal region Λ_j, the operator microlocally reduces to the subelliptic model with large parameter $\lambda \geq 1$,

$$D_t + i\lambda t^{2j},$$

near $X_0 = (x_0, \xi_0) \in \Lambda_j$. This comes from the non-negativeness of the real part Re $q \geq 0$ and the averaging property (1.31), which imply that the operator is of principal type and satisfies to the condition (P) in all these points of finite type. The large parameter λ is of the order of the gain

$$\lambda \sim \langle(x_0, \xi_0)\rangle^2,$$

in the natural symbolic calculus associated to quadratic operators

$$q \in S\Big(\langle(x,\xi)\rangle^2, \frac{dx^2 + d\xi^2}{\langle(x,\xi)\rangle^2}\Big), \quad \text{i.e.} \quad \forall \alpha, \beta \in \mathbb{N}^n, \quad |\partial_x^\alpha \partial_\xi^\beta q(x,\xi)| \lesssim \langle(x,\xi)\rangle^{2-|\alpha|-|\beta|}.$$

[2] Such an integer exists according to the definition (1.11).

The standard subelliptic estimate [**86**] (Section 1.4) with sharp loss of derivatives

$$\|D_t u + i\lambda t^{2j} u\|_{L^2} \gtrsim \lambda^{\frac{1}{2j+1}} \|u\|_{L^2}, \tag{1.40}$$

then accounts for the sharpness of the loss $\delta = 2k_0/(2k_0+1)$ in the global subelliptic estimate (1.38). The next result shows that Theorem 1.4 may be refined as follows:

THEOREM 1.5. ([**KPS24**], KPS) *Let $q : \mathbb{R}^n_x \times \mathbb{R}^n_\xi \to \mathbb{C}$, $n \geq 1$, be a complex-valued quadratic form with a non-negative real part* $\mathrm{Re}\ q \geq 0$. *Let $0 \leq k_0 \leq 2n-1$ be the smallest integer*[3] *satisfying*

$$S = \Big(\bigcap_{j=0}^{k_0} \mathrm{Ker}\big[\mathrm{Re}\ F(\mathrm{Im}\ F)^j\big]\Big) \cap \mathbb{R}^{2n}. \tag{1.41}$$

When the quadratic symbol q is elliptic on its singular space

$$(x,\xi) \in S,\ q(x,\xi) = 0 \Rightarrow (x,\xi) = 0,$$

or more generally when its singular space S has a symplectic structure, then the quadratic operator $q^w(x,D_x)$ is subelliptic in any direction of the space $S^{\sigma\perp}$ in the sense that

$$\exists C > 0, \forall u \in D(q),\ \|\langle(x',D_{x'})\rangle^{2/(2k_0+1)} u\|_{L^2} \leq C(\|q^w(x,D_x)u\|_{L^2} + \|u\|_{L^2}), \tag{1.42}$$

if (x',ξ') are some linear symplectic coordinates on the symplectic space $S^{\sigma\perp}$.

When the singular space is equal to zero, we recover the result stated in Theorem 1.4. As in Theorem 1.1, we notice that regularizing properties only hold in the microlocal directions given by the linear subspace $S^{\sigma\perp}$.

Before giving some insights for the proof of these two theorems, we provide some examples of subelliptic quadratic operators with zero singular space showing that the integer $0 \leq k_0 \leq 2n-1$, related to the loss of derivatives $\delta = 2k_0/(2k_0+1)$, can actually take any value in the set $\{0, ..., 2n-1\}$, when $n \geq 1$:

- Case $k_0 = 0$: Any quadratic symbol q with $\mathrm{Re}\ q \gg 0$ a positive definite real part
- Case $k_0 = 1$:

$$q(x,\xi) = \xi_2^2 + x_2^2 + i(x_2\xi_1 - x_1\xi_2) + \sum_{j=3}^{n}(\xi_j^2 + x_j^2)$$

- Case $k_0 = 2p$, with $1 \leq p \leq n-1$:

$$q(x,\xi) = \xi_1^2 + x_1^2 + i(\xi_1^2 + 2x_2\xi_1 + \xi_2^2 + 2x_3\xi_2 + ... + \xi_p^2 + 2x_{p+1}\xi_p + \xi_{p+1}^2) + \sum_{j=p+2}^{n}(\xi_j^2 + x_j^2)$$

- Case $k_0 = 2p+1$, with $1 \leq p \leq n-1$:

$$q(x,\xi) = x_1^2 + i(\xi_1^2 + 2x_2\xi_1 + \xi_2^2 + 2x_3\xi_2 + ... + \xi_p^2 + 2x_{p+1}\xi_p + \xi_{p+1}^2) + \sum_{j=p+2}^{n}(\xi_j^2 + x_j^2)$$

[3] Such an integer exists according to the definition (1.11).

We notice that the global subelliptic estimate

$$\|\langle(x,D_x)\rangle^{2/(4n-1)}u\|_{L^2} \lesssim \|q^w(x,D_x)u\|_{L^2} + \|u\|_{L^2},$$

fulfilled for instance by the quadratic operator $q^w(x,D_x)$ whose symbol

$$q(x,\xi) = x_1^2 + i(\xi_1^2 + 2x_2\xi_1 + \xi_2^2 + 2x_3\xi_2 + ... + \xi_{n-1}^2 + 2x_n\xi_{n-1} + \xi_n^2),$$

has a very degenerate real part, well-emphasizes some non-trivial interactions between the selfadjoint and skew-selfadjoint parts of the operator. Indeed, the imaginary part has no sign property and a naive approach purely based on the accretivity of the operator only allows to get a control of the x_1 variable

$$\|x_1u\|_{L^2}^2 = \mathrm{Re}(q^w(x,D_x)u,u)_{L^2} \leq \|q^w(x,D_x)u\|_{L^2}\|u\|_{L^2} \leq \|q^w(x,D_x)u\|_{L^2}^2 + \|u\|_{L^2}^2.$$

Starting from any of the previous examples of quadratic symbol q with a non-negative real part and a zero singular space, other examples of quadratic forms with a non-negative real part and a non-trivial symplectic singular space may be built up by adding to q a purely imaginary-valued quadratic form in other symplectic variables $(\tilde{x},\tilde{\xi})$,

$$q(x,\xi) + i\tilde{q}(\tilde{x},\tilde{\xi}).$$

In this case, the singular space is given by the space $S = \{(x,\tilde{x},\xi,\tilde{\xi}) : x = \xi = 0\}$.

Proof. For simplicity, we consider first the case when the singular space is equal to zero $S = 0$. The proof of Theorem 1.4 is a proof by multiplier and the core of this proof is the construction of the multiplier symbol. This construction is technical and relies on the identity

$$H_{\mathrm{Im}q}^{2j}\mathrm{Re}\ q(X) = c_j\mathrm{Re}\ q((\mathrm{Im}\ F)^jX) > 0, \quad c_j > 0,$$

holding for points in the microlocal region Λ_j. With $0 \leq k_0 \leq 2n-1$ the smallest integer satisfying

$$\Big(\bigcap_{j=0}^{k_0}\mathrm{Ker}\big[\mathrm{Re}\ F(\mathrm{Im}\ F)^j\big]\Big) \cap \mathbb{R}^{2n} = \{0\},$$

the algebraic feature (1.32) reduces to

$$\sum_{j=0}^{k_0}\mathrm{Re}\ q((\mathrm{Im}\ F)^jX) \gg 0.$$

By taking advantage of the positive definiteness of this quadratic form, we build up a real-valued bounded symbol

$$g \in S(1, \langle X\rangle^{-\frac{2}{2k_0+1}}dX^2) \quad \text{i.e.} \quad \forall\alpha\in\mathbb{N}^{2n}, \quad |\partial_X^\alpha g(X)| \lesssim \langle X\rangle^{-\frac{|\alpha|}{2k_0+1}},$$

satisfying to the following global estimate on the phase space

$$\exists c_1,c_2>0, \forall X\in\mathbb{R}^{2n}, \quad \mathrm{Re}\ q(X) + c_1H_{\mathrm{Im}q}g(X) + 1 \geq c_2\langle X\rangle^{\frac{2}{2k_0+1}}. \tag{1.43}$$

The boundedness of this symbol is important to deal with a bounded multiplier. In the case when the condition (1.21) holds, a natural choice for the multiplier would be to take for g the quadratic symbol $H_{\mathrm{Im}q}\mathrm{Re}\ q$. However, this symbol is unbounded and needs to be suitably weighted and microlocalized in order to design a consistent multiplier. In the general case, the global construction of the symbol g is microlocally designed according to partition (1.39).

A key instrumental step in the proof the global subelliptic estimate with loss of $2k_0/(2k_0+1)$ derivatives

$$\|\langle(x,D_x)\rangle^{2/(2k_0+1)}u\|_{L^2} \leq C(\|q^w(x,D_x)u\|_{L^2}+\|u\|_{L^2}), \tag{1.44}$$

is to establish first the weaker subelliptic estimate

$$\|\langle(x,D_x)\rangle^{1/(2k_0+1)}u\|_{L^2} \leq C(\|q^w(x,D_x)u\|_{L^2}+\|u\|_{L^2}). \tag{1.45}$$

The estimate (1.44) is then derived from (1.45) through a commutator argument. In order to prove the estimate (1.45), we use some elements of Wick calculus and define the multiplier through the Wick quantization. A short exposition of the Wick calculus is given in Appendix (Section 1). Standard Wick symbolic calculus allows to write for any $0<\varepsilon\ll 1$,

$$\underbrace{\mathrm{Re}(q^{\mathrm{Wick}}u,(1-\varepsilon g)^{\mathrm{Wick}}u)_{L^2}}_{\substack{\text{Term bounded in modulus as}\\ |\cdot|\lesssim\|q^w(x,D_x)u\|^2_{L^2}+\|u\|^2_{L^2}}} = (\underbrace{[(1-\varepsilon g)\mathrm{Re}\ q+\frac{\varepsilon}{4\pi}H_{\mathrm{Im}q}\ g]}_{\gtrsim\langle X\rangle^{\frac{2}{2k_0+1}}}{}^{\mathrm{Wick}}u,u)_{L^2}$$
$$+\underbrace{\frac{\varepsilon}{4\pi}([\nabla g.\nabla\mathrm{Re}\ q]^{\mathrm{Wick}}u,u)_{L^2}+\mathcal{O}(\|u\|^2_{L^2})}_{\substack{\text{Remainder terms bounded in modulus as}\\ |\cdot|\lesssim\|q^w(x,D_x)u\|^2_{L^2}+\|u\|^2_{L^2}}}$$

and to derive the estimate (1.45) from the positivity property of the Wick quantization

$$([(1-\varepsilon g)\mathrm{Re}\ q+\frac{\varepsilon}{4\pi}H_{\mathrm{Im}q}\ g]^{\mathrm{Wick}}u,u)_{L^2} \gtrsim \|\langle(x,D_x)\rangle^{1/(2k_0+1)}u\|^2_{L^2}.$$

Then, the proof of Theorem 1.5 follows from the same lines after taking advantage of the variables tensorization (1.27). $\square$

2.3.2. *Overdetermined systems of quadratic operators.* In the scalar case, the concept of singular space has proven to be a simple and relevant algebraic tool for deriving the sharp subelliptic properties of quadratic operators. This notion may be extended to overdetermined systems of quadratic operators for studying their global subellipticity

$$\|\langle(x,D_x)\rangle^{2(1-\delta)}u\|_{L^2} \lesssim \sum_{j=1}^N \|q_j^w(x,D_x)u\|_{L^2}+\|u\|_{L^2}, \quad 0\leq\delta<1, \tag{1.46}$$

possibly arising thanks to non-trivial interactions between the different operators composing the system. In the following statement, the intersection of kernels

$$\bigcap_{\substack{j=1,...,N,\\ (l_1,...,l_k)\in\{1,...,N\}^k}} \mathrm{Ker}(\mathrm{Re}\ F_j\mathrm{Im}\ F_{l_1}...\mathrm{Im}\ F_{l_k}),$$

is understood as

$$\bigcap_{j=1,...,N} \mathrm{Ker}\ \mathrm{Re}\ F_j,$$

when $k=0$.

THEOREM 1.6. ([**KPS25**], KPS) *Let $q_j : \mathbb{R}^n_x \times \mathbb{R}^n_\xi \to \mathbb{C}$, $n \geq 1$, $1 \leq j \leq N$, be a system of complex-valued quadratic forms with non-negative real parts* $\textrm{Re } q_j \geq 0$. *Assume that there exists an integer $k_0 \geq 0$ such that*

$$\Big(\bigcap_{0\leq k\leq k_0}\ \bigcap_{\substack{j=1,\dots,N,\\(l_1,\dots,l_k)\in\{1,\dots,N\}^k}} \textrm{Ker}(\textrm{Re } F_j \textrm{Im } F_{l_1}\dots\textrm{Im } F_{l_k})\Big)\cap\mathbb{R}^{2n}=\{0\}, \tag{1.47}$$

with F_j the Hamilton map of the quadratic form q_j. Then, the overdetermined system of quadratic operators $(q_j^w)_{1\leq j\leq N}$ is subelliptic with a loss of $\delta = 2k_0/(2k_0+1)$ derivatives, i.e., $\exists C > 0$,

(1.48) $\forall u \in D(q_1)\cap\dots\cap D(q_N)$,

$$\|\langle(x,D_x)\rangle^{2/(2k_0+1)}u\|_{L^2} \leq C\Big(\sum_{j=1}^N \|q_j^w(x,D_x)u\|_{L^2}+\|u\|_{L^2}\Big).$$

Theorem 1.6 emphasizes non-trivial interactions between the different operators composing the system, which cannot be derived from the result known in the scalar case. Indeed, setting

$$q_j(x,\xi)=x_1^2+\xi_1^2+i(\xi_1^2+x_{j+1}\xi_1),\quad \tilde{q}_j(x,\xi)=x_1^2+\xi_1^2+i(\xi_1^2+\xi_{j+1}\xi_1),$$

with $1\leq j\leq n-1$, $(x,\xi)\in\mathbb{R}^{2n}$, $n\geq 2$, the singular space of the quadratic form

$$\sum_{j=1}^{n-1}(\lambda_j q_j+\tilde{\lambda}_j\tilde{q}_j),\quad\text{with}\quad \lambda_j,\tilde{\lambda}_j\in\mathbb{R},\quad \sum_{j=1}^{n-1}(\lambda_j+\tilde{\lambda}_j)>0,$$

is given by the non-zero vector subspace

$$S=\Big\{(x,\xi)\in\mathbb{R}^{2n}:\ x_1=\xi_1=\sum_{j=1}^{n-1}(\lambda_j x_{j+1}+\tilde{\lambda}_j\xi_{j+1})=0\Big\}.$$

We cannot deduce any result about the subellipticity of the scalar operator

$$\sum_{j=1}^{n-1}(\lambda_j q_j^w(x,D_x)+\tilde{\lambda}_j\tilde{q}_j^w(x,D_x)),$$

in order to prove the subellipticity of the overdetermined system composed by the $2n-2$ quadratic operators $(q_j^w,\tilde{q}_k^w)_{1\leq j,k\leq n-1}$. However, denoting F_j and $\tilde{F}_j$ the Hamilton maps of the quadratic forms q_j and $\tilde{q}_j$, one may easily check that

$$\textrm{Ker Re } F_j\cap\textrm{Ker}(\textrm{Re } F_j\textrm{Im } F_j)\cap\mathbb{R}^{2n}=\{(x,\xi)\in\mathbb{R}^{2n}: x_1=\xi_1=x_{j+1}=0\},$$

$$\textrm{Ker Re } \tilde{F}_j\cap\textrm{Ker}(\textrm{Re } \tilde{F}_j\textrm{Im } \tilde{F}_j)\cap\mathbb{R}^{2n}=\{(x,\xi)\in\mathbb{R}^{2n}: x_1=\xi_1=\xi_{j+1}=0\}.$$

We deduce from Theorem 1.6 the global subelliptic estimate with a loss of 2/3 derivatives

$$\|\langle(x,D_x)\rangle^{2/3}u\|_{L^2}\lesssim\sum_{j=1}^{n-1}(\|q_j^w(x,D_x)u\|_{L^2}+\|\tilde{q}_j^w(x,D_x)u\|_{L^2})+\|u\|_{L^2}.$$

Setting now

$$q_1(x,\xi)=x_1^2+\xi_1^2+i(x_2\xi_1-x_1\xi_2+x_3\xi_2-x_2\xi_3),\quad q_2(x,\xi)=i(x_3\xi_1-x_1\xi_3),$$

with $(x,\xi) = (x_1,x_2,x_3,\xi_1,\xi_2,\xi_3) \in \mathbb{R}^6$. The subellipticity of the system (q_1^w, q_2^w) may be derived from the scalar case (Theorem 1.4). Indeed, let us consider the quadratic form $q = q_1 + \mu q_2$, for $\mu \in \mathbb{R}$, with Hamilton map F. Some algebraic computations show

$$\mathrm{Ker}(\mathrm{Re}\ F) = \{(x,\xi) \in \mathbb{R}^6 :\ x_1 = \xi_1 = 0\},$$

$$\mathrm{Ker}(\mathrm{Re}\ F) \cap \mathrm{Ker}(\mathrm{Re}\ F \mathrm{Im}\ F) = \{(x,\xi) \in \mathbb{R}^6 :\ x_1 = \xi_1 = x_2 + \mu x_3 = \xi_2 + \mu\xi_3 = 0\},$$

$$\begin{aligned}&\mathrm{Ker}(\mathrm{Re}\ F) \cap \mathrm{Ker}(\mathrm{Re}\ F \mathrm{Im}\ F) \cap \mathrm{Ker}\big(\mathrm{Re}\ F(\mathrm{Im}\ F)^2\big)\\ &= \{(x,\xi) \in \mathbb{R}^6 :\ x_1 = \xi_1 = x_2+\mu x_3 = \xi_2+\mu\xi_3 = -\mu x_2+x_3 = -\mu\xi_2+\xi_3 = 0\} = \{0\}.\end{aligned}$$

Theorem 1.4 establishes the subellipticity with a loss of 4/5 derivatives

$$\begin{aligned}\|\langle(x,D_x)\rangle^{2/5}u\|_{L^2} &\lesssim \|q^w(x,D_x)u\|_{L^2} + \|u\|_{L^2}\\ &\lesssim \|q_1^w(x,D_x)u\|_{L^2} + \|q_2^w(x,D_x)u\|_{L^2} + \|u\|_{L^2},\end{aligned} \tag{1.49}$$

whereas Theorem 1.6 provides a sharper subelliptic estimate with a loss of 2/3 derivatives

$$\|\langle(x,D_x)\rangle^{2/3}u\|_{L^2} \lesssim \|q_1^w(x,D_x)u\|_{L^2} + \|q_2^w(x,D_x)u\|_{L^2} + \|u\|_{L^2}, \tag{1.50}$$

since

$$\mathrm{Ker}(\mathrm{Re}\ F_1) = \{(x,\xi) \in \mathbb{R}^6 :\ x_1 = \xi_1 = 0\},$$

$$\mathrm{Ker}(\mathrm{Re}\ F_1 \mathrm{Im}\ F_1) = \{(x,\xi) \in \mathbb{R}^6 :\ x_2 = \xi_2 = 0\},$$

$$\mathrm{Ker}(\mathrm{Re}\ F_1 \mathrm{Im}\ F_2) = \{(x,\xi) \in \mathbb{R}^6 :\ x_3 = \xi_3 = 0\},$$

where F_1 and F_2 are the Hamilton maps of the quadratic symbols q_1 and q_2. This result shows that the estimate (1.49) is sharp for the quadratic operator q^w, but not for the system of the two quadratic operators (q_1^w, q_2^w). Obviously, more complex interactions between the different operators composing the system may be emphasized when considering operators with different real parts.

The subellipticity of overdetermined systems of pseudodifferential operators was studied by Bolley, Camus and Nourrigat in the work [**20**], where is established the microlocal subellipticity for overdetermined systems of pseudodifferential operators with real principal symbols satisfying the Hörmander-Kohn condition. This condition refers to the existence of an elliptic iterated commutator of the operators composing the system. Regarding overdetermined systems of non-selfadjoint pseudodifferential operators, the greatest achievements about microlocal subellipticity and maximal hypoellipticity were obtained by Nourrigat in [**104, 105**] by the mean of nilpotent groups representations.

In the case of a system of non-selfadjoint quadratic operators $(q_j^w)_{1\le j\le N}$, the natural extension of the Hörmander-Kohn condition is to request the ellipticity of an iterated commutator of the real parts $((\mathrm{Re}\ q_j)^w)_{1\le j\le N}$ and imaginary parts $((\mathrm{Im}\ q_j)^w)_{1\le j\le N}$ of the operators composing the system. In the scalar case, this natural condition holds according to the partition of the phase space (1.39). In the system case, the situation is more complicated and the link between the algebraic condition (1.47) and the existence of an elliptic iterated commutator of the operators composing the system is less obvious to highlight explicitly. More specifically, the algebraic condition (1.47) implies the positive definiteness of the quadratic form

$$\sum_{k=0}^{k_0} \sum_{\substack{j=1,...,N,\\(l_1,...,l_k)\in\{1,...,N\}^k}} \textrm{Re } q_j(\textrm{Im } F_{l_1}...\textrm{Im } F_{l_k}X) \gg 0.$$

This property implies that for any non-zero point $X_0 \in \mathbb{R}^{2n}$, there exist $0 \leq k \leq k_0$, $j \in \{1,...,N\}$ and $(l_1,...,l_k) \in \{1,...,N\}^k$ such that

$$\textrm{Re } q_j(\textrm{Im } F_{l_1}...\textrm{Im } F_{l_k}X_0) > 0.$$

By considering the minimal non-negative integer k with this property, we may check that there is no elliptic iterated commutator of order less or equal to $2k-1$ at X_0 of the type

$$[P_1,[P_2,[P_3,[...,[P_r,P_{r+1}]...]]]],$$

with $r \leq 2k-1$, $P_l = \textrm{Re } q_s^w$ or $P_l = \textrm{Im } q_s^w$, where at least one P_l is equal to Re q_s^w. We may also check that the non-zero term

$$\textrm{Re } q_j(\textrm{Im } F_{l_1}...\textrm{Im } F_{l_k}X_0) > 0,$$

actually appears when expanding the Weyl symbol at X_0 of the $2k^{\textrm{th}}$ iterated commutator

$$[\textrm{Im } q_{l_k}^w,[\textrm{Im } q_{l_k}^w,[\textrm{Im } q_{l_{k-1}}^w,[\textrm{Im } q_{l_{k-1}}^w,[...,[\textrm{Im } q_{l_1}^w,[\textrm{Im } q_{l_1}^w,\textrm{Re } q_j^w]]]...] \\ = (-1)^k(H^2_{\textrm{Im}q_{l_k}}...H^2_{\textrm{Im}q_{l_1}}\textrm{Re } q_j)^w.$$

However, contrary to the scalar case, there may be also other non-zero terms in this expansion and it is not clear if this natural commutator associated to the term

$$\textrm{Re } q_j(\textrm{Im } F_{l_1}...\textrm{Im } F_{l_k}X_0) > 0,$$

is actually elliptic at X_0,

$$H^2_{\textrm{Im}q_{l_k}}...H^2_{\textrm{Im}q_{l_1}}\textrm{Re } q_j(X_0) \overset{?}{\neq} 0.$$

Though it may be difficult to determine exactly at each point which specific commutator is elliptic, it is very likely that condition (1.47) ensures that the Hörmander-Kohn condition is fulfilled at any non-zero point of the phase space, and that these associated elliptic commutators are all of order less or equal to $2k_0$. It is actually what the loss of derivatives appearing in the estimate (1.48) suggests, and this in agreement with the optimal loss of derivatives obtained in [**20**] (Theorem 1.1) for $2k_0$ commutators

$$\delta = 1 - \frac{1}{2k_0+1} = \frac{2k_0}{2k_0+1}.$$

In conclusion, Theorem 1.6 provides an explicit and simple algebraic condition for proving global subelliptic estimates for systems of quadratic operators. It is possible that some of these global subelliptic estimates for systems of quadratic operators may also be derived from the results of microlocal subellipticity and maximal hypoellipticity proven in [**20, 104, 105**], but checking the assumptions of these results turns out to be quite difficult in practice.

2.4. Exponential return to equilibrium. We consider the evolution equations associated with accretive non-selfadjoint quadratic operators

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) + q^w(x,D_x)u(t,x) = 0, \\ u(t,\cdot)|_{t=0} = u_0 \in L^2(\mathbb{R}^n), \end{cases} \tag{1.51}$$

and address the problem of the exponential return to equilibrium for these systems.

The following result makes explicit the structure of the bottom of the spectrum and the existence of a ground state of exponential type:

THEOREM 1.7. ([**KPS12**], Ottobre, Pavliotis, KPS) *Let $q : \mathbb{R}^n_x \times \mathbb{R}^n_\xi \to \mathbb{C}$, $n \geq 1$, be a complex-valued quadratic form with a non-negative real part* $\mathrm{Re}\, q \geq 0$*, and a zero singular space $S = 0$. Then, the first eigenvalue in the bottom of the spectrum of the quadratic operator $q^w(x,D_x)$ given by*

$$\mu_0 = \sum_{\substack{\lambda \in \sigma(F) \\ -i\lambda \in \mathbb{C}_+}} -i\lambda r_\lambda, \tag{1.52}$$

has algebraic multiplicity 1 and the eigenspace

$$\mathrm{Ker}(q^w(x,D_x) - \mu_0) = \mathbb{C}u_0,$$

is spanned by a ground state of exponential type

$$u_0(x) = e^{-a(x)} \in \mathscr{S}(\mathbb{R}^n),$$

where a is a complex-valued quadratic form on $\mathbb{R}^n$ whose real part is positive definite $\mathrm{Re}\, a \gg 0$*. Furthermore, the spectral gap of the quadratic operator*

$$\sigma\big(q^w(x,D_x)\big) \setminus \{\mu_0\} \subset \{z \in \mathbb{C} : \mathrm{Re}\, z \geq \mathrm{Re}\, \mu_0 + \tau_0\},$$

is exactly given by the positive rate

$$\tau_0 = 2 \min_{\substack{\lambda \in \sigma(F) \\ \mathrm{Im}\, \lambda > 0}} \mathrm{Im}\, \lambda > 0. \tag{1.53}$$

Here F stands for the Hamilton map of the quadratic form q and r_λ is the dimension of the space of generalized eigenvectors of F in $\mathbb{C}^{2n}$ associated to the eigenvalue λ.

The next result establishes the property of exponential return to equilibrium and provides an exact formula for the optimal rate of convergence:

THEOREM 1.8. ([**KPS12**], Ottobre, Pavliotis, KPS) *Let $q : \mathbb{R}^n_x \times \mathbb{R}^n_\xi \to \mathbb{C}$, $n \geq 1$, be a complex-valued quadratic form with a non-negative real part* $\mathrm{Re}\, q \geq 0$*, and a zero singular space $S = 0$. By using the notations introduced in* (1.52) *and* (1.53)*, we consider the operator*

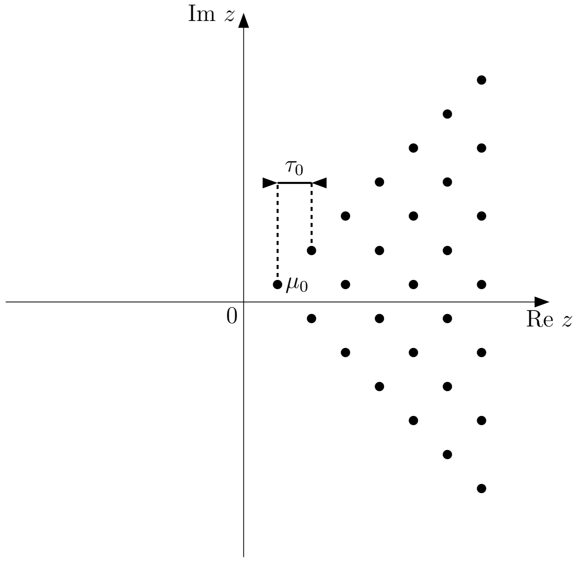
$$Q = q^w(x,D_x) - \mu_0.$$

Then, for all $0 \leq \tau < \tau_0$, there exists a positive constant $C > 0$ such that

$$\forall t \geq 0, \quad \|e^{-tQ} - \Pi_0\|_{\mathcal{L}(L^2)} \leq Ce^{-\tau t},$$

where Π_0 is the rank-one spectral projection associated with the simple eigenvalue zero of the operator Q and $\|\cdot\|_{\mathcal{L}(L^2)}$ stands for the norm of bounded operators on $L^2(\mathbb{R}^n)$.

FIGURE 3. The first eigenvalue in the bottom of the spectrum is not necessarily real.



We consider now the particular case when in addition, the quadratic operator is real in the sense that $q^w(x, D_x)u$ is a real-valued function whenever u is a real-valued function. Under this assumption, the first eigenvalue μ_0 is necessarily real and the quadratic form $a \gg 0$ defining the ground state

$$u_0(x) = e^{-a(x)} \in \mathscr{S}(\mathbb{R}^n), \quad q^w(x, D_x)u_0 = \mu_0 u_0,$$

is positive definite. The adjoint operator $q^w(x, D_x)^*$ is the quadratic operator

$$\overline{q}^w(x, D_x),$$

which also satisfies the assumptions of both Theorems 1.7 and 1.8. In this case, μ_0 is the first eigenvalue in the bottom of the spectrum for both quadratic operators $q^w(x, D_x)$ and $q^w(x, D_x)^*$. We know from Theorem 1.7 that the eigenspaces associated with this eigenvalue are one-dimensional with ground states of exponential type. We assume further that the two operators have the same ground state

$$\operatorname{Ker}(q^w(x, D_x) - \mu_0) = \operatorname{Ker}(q^w(x, D_x)^* - \mu_0) = \mathbb{C}u_0 \subset \mathscr{S}(\mathbb{R}^n), \tag{1.54}$$

with $u_0(x) = e^{-a(x)}$, $x \in \mathbb{R}^n$, where $a \gg 0$ is a positive definite quadratic form on $\mathbb{R}^n$. Under these assumptions, the rank-one spectral projection Π_0 is orthogonal:

THEOREM 1.9. ([**KPS12**], Ottobre, Pavliotis, KPS) *Let $q : \mathbb{R}^n_x \times \mathbb{R}^n_\xi \to \mathbb{C}$, $n \geq 1$, be a complex-valued quadratic form with a non-negative real part* $\operatorname{Re} q \geq 0$*, and a zero singular space $S = 0$. Assume that the quadratic operator $q^w(x, D_x)$ is real and satisfies to assumption*

$$\operatorname{Ker}(q^w(x, D_x) - \mu_0) = \operatorname{Ker}(q^w(x, D_x)^* - \mu_0).$$

By using the notations introduced in (1.52) *and* (1.53)*, we consider the operator*

$$Q = q^w(x, D_x) - \mu_0.$$

Then, for all $0 \leq \tau < \tau_0$, *there exists a positive constant* $C > 0$ *such that*

$$\forall t \geq 0, \forall u \in L^2(\mathbb{R}^n), \quad \|e^{-tQ}u - c_u u_0\|_{L^2(\mathbb{R}^n)} \leq Ce^{-\tau t}\|u\|_{L^2(\mathbb{R}^n)},$$

where c_u *is the* $L^2(\mathbb{R}^n)$ *scalar product of* u *and* $u_0/\|u_0\|^2_{L^2(\mathbb{R}^n)}$,

$$c_u = \|u_0\|^{-2}_{L^2(\mathbb{R}^n)}(u, u_0)_{L^2(\mathbb{R}^n)}.$$

Proof. Under the assumptions of Theorem 1.8, the quadratic operator $q^w(x, D_x)$ enjoys the global subelliptic estimate

$$\|\langle(x, D_x)\rangle^{2/(2k_0+1)}u\|_{L^2} \leq C(\|q^w(x, D_x)u\|_{L^2} + \|u\|_{L^2}),$$

where $0 \leq k_0 \leq 2n - 1$ is the smallest integer satisfying

$$\Big(\bigcap_{j=0}^{k_0} \mathrm{Ker}\big[\mathrm{Re}\ F(\mathrm{Im}\ F)^j\big]\Big) \cap \mathbb{R}^{2n} = \{0\}.$$

This estimate is shown to extend as

$$\exists C > 0, \forall u \in D(q), \forall \nu \in \mathbb{R}, \quad \|\langle(x, D_x)\rangle^{2/(2k_0+1)}u\|^2_{L^2} \leq C(\|q^w(x, D_x)u - i\nu u\|^2_{L^2} + \|u\|^2_{L^2}). \tag{1.55}$$

Then, some functional analysis arguments allow to derive from this estimate the localization of the spectrum

$$\exists c, C > 0, \quad \{z \in \mathbb{C} : \mathrm{Re}\ z \geq -1/2,\ \mathrm{Re}\ z + 1 \leq c|z + 1|^{\frac{1}{2k_0+1}}\} \cap \sigma(q^w(x, D_x)) = \emptyset, \tag{1.56}$$

together with the following resolvent estimate

$$\|(q^w(x, D_x) - z)^{-1}\| \leq C|z + 1|^{-\frac{1}{2k_0+1}}, \tag{1.57}$$

for all $z \in \mathbb{C}$ such that $\mathrm{Re}\ z \geq -1/2$, $\mathrm{Re}\ z + 1 \leq c|z + 1|^{\frac{1}{2k_0+1}}$. We introduce two complex contours γ and $\tilde{\gamma}$. Both contours are given by the curve

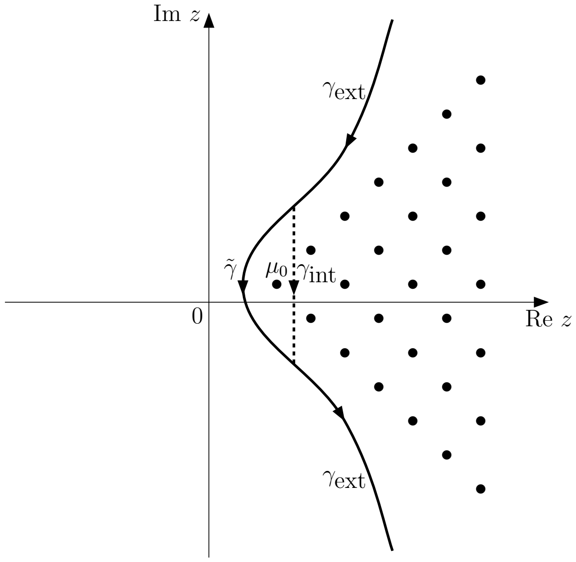
$$\mathrm{Re}\ z = \frac{1}{C}|\mathrm{Im}\ z|^{1/(2k_0+1)},$$

with $C > 0$, in the region where $\mathrm{Re}\ z > b$, with $b = \mathrm{Re}\ \mu_0 + \tau$. In the region where $\mathrm{Re}\ z \leq b$, the contour $\gamma = \gamma_{\mathrm{int}} \cup \gamma_{\mathrm{ext}}$ is given by the equation $\mathrm{Re}\ z = b$, while $\tilde{\gamma}$ joins the two points $b + iC^{2k_0+1}b^{2k_0+1}$ and $b - iC^{2k_0+1}b^{2k_0+1}$, further to the left so that $\tilde{\gamma}$ is entirely to the left of the spectrum of the quadratic operator $q^w(x, D_x)$, while γ only has the first eigenvalue μ_0 to its left. The resolvent estimate (1.57) allows to derive the following identities

$$e^{-tq^w(x,D_x)} = \frac{1}{2\pi i}\int_{\tilde{\gamma}} e^{-tz}(z - q^w(x, D_x))^{-1}dz$$
$$= e^{-\mu_0 t}\Pi_0 + \frac{1}{2\pi i}\int_{\gamma} e^{-tz}(z - q^w(x, D_x))^{-1}dz$$

and after further estimates to establish Theorem 1.8. ∎

FIGURE 4



2.4.1. *Example 1 : The Kramers-Fokker-Planck operator with quadratic potential.* The spectrum of the Kramers-Fokker-Planck operator

$$K = -\Delta_v + \frac{v^2}{4} - \frac{1}{2} + v\partial_x - \nabla_x V(x)\partial_v, \quad (x,v) \in \mathbb{R}^2,$$

with the quadratic potential

$$V(x) = \frac{1}{2}ax^2, \quad a \in \mathbb{R}^*,$$

is given by

$$\left\{(2k_1+1)\frac{\lambda_1}{i} + (2k_2+1)\frac{\lambda_2}{i} - \frac{1}{2}, \quad k_1, k_2 \geq 0\right\},$$

where

$$\lambda_1 = \frac{i + i\sqrt{1-4a}}{4}, \quad \lambda_2 = \frac{i - i\sqrt{1-4a}}{4}, \quad \text{when } a > 0,$$

$$\lambda_1 = \frac{-i + i\sqrt{1-4a}}{4}, \quad \lambda_2 = \frac{i + i\sqrt{1-4a}}{4}, \quad \text{when } a < 0.$$

When $a > 0$, the lowest eigenvalue of the Kramers-Fokker-Planck operator is $\tilde{\mu}_0 = 0$, whereas when $a < 0$, this lowest eigenvalue is equal to

$$\tilde{\mu}_0 = \frac{\sqrt{1-4a}}{2} - \frac{1}{2} > 0.$$

The spectral gap $\tau_0 > 0$ is equal to

$$\frac{\sqrt{1-4a}-1}{2}, \quad \frac{1-\sqrt{1-4a}}{2}, \quad \frac{1}{2},$$

when respectively

$$a < 0, \quad 0 < a \leq 1/4, \quad a > 1/4.$$

The lowest eigenvalue is always real. This is consistent with the fact that the Kramers-Fokker-Planck operator is real. Further calculations allow to determine explicitly the ground state $Ku_0 = \tilde{\mu}_0 u_0$. When $a > 0$, it is the usual Maxwellian

$$u_0(x,v) = e^{-\frac{1}{4}ax^2-\frac{v^2}{4}} = e^{-\frac{1}{2}(\frac{v^2}{2}+V(x))} \in \mathscr{S}(\mathbb{R}^2),$$

whereas when $a < 0$, the ground state is given by

$$u_0(x,v) = e^{\frac{a}{4}\sqrt{1-4a}x^2-axv-\frac{\sqrt{1-4a}}{4}v^2} \in \mathscr{S}(\mathbb{R}^2).$$

The assumption

$$\operatorname{Ker}\big(q^w(x,D_x)-\mu_0\big) = \operatorname{Ker}\big(q^w(x,D_x)^*-\mu_0\big) = \mathbb{C}u_0 \subset \mathscr{S}(\mathbb{R}^2),$$

holds true only if $a > 0$. Theorems 1.7 and 1.8 therefore apply whenever $a \neq 0$, whereas Theorem 1.9 only applies when $a > 0$.

2.4.2. *Example 2 : Chain of oscillators.* This example comes from the series of works [**48, 49, 50, 66, 67**]. It is a model describing a chain of two oscillators coupled with two heat baths at each side. The particles are described by their respective position and velocity $(x_j, y_j) \in \mathbb{R}^{2d}$. For each oscillator, the particles are submitted to an external force derived from a real-valued potential $V_j(x_j)$ and a coupling between the two oscillators derived from a real-valued potential $V_c(x_2-x_1)$. We denote the full potential

$$V(x) = V_1(x_1)+V_2(x_2)+V_c(x_2-x_1), \quad x=(x_1,x_2) \in \mathbb{R}^{2d},$$

$y=(y_1,y_2) \in \mathbb{R}^{2d}$ the velocities and $z=(z_1,z_2) \in \mathbb{R}^{2d}$ the variables describing the state of the particles in each of the heat baths. In each bath, the particles are submitted to a coupling with the nearest oscillator, a force given by the friction coefficient γ and a thermal diffusion at the temperature T_j. We denote w_1, w_2 two standard d-dimensional Brownian motions and $w=(w_1,w_2)$. The system of equations describing this model is given by

$$\left\{\begin{aligned} dx_1 &= y_1 dt\\ dx_2 &= y_2 dt\\ dy_1 &= -\partial_{x_1}V(x)dt + z_1 dt\\ dy_2 &= -\partial_{x_2}V(x)dt + z_2 dt\\ dz_1 &= -\gamma z_1 dt + \gamma x_1 dt - \sqrt{2\gamma T_1}dw_1\\ dz_2 &= -\gamma z_2 dt + \gamma x_2 dt - \sqrt{2\gamma T_2}dw_2. \end{aligned}\right. \tag{1.58}$$

Setting $T_1 = \alpha_1 h/2$, $T_2 = \alpha_2 h/2$, the corresponding equation for the density of particles is

$$\begin{aligned} h\partial_t f + \frac{\gamma}{2}\alpha_1(-h\partial_{z_1})\Big(h\partial_{z_1}+\frac{2}{\alpha_1}(z_1-x_1)\Big)f& \\ + \frac{\gamma}{2}\alpha_2(-h\partial_{z_2})\Big(h\partial_{z_2}+\frac{2}{\alpha_2}(z_2-x_2)\Big)f + \Big(yh\partial_x - (\nabla_x V(x)-z)h\partial_y\Big)f &= 0. \end{aligned} \tag{1.59}$$

Define

$$\Phi(x,y,z) = V(x)+\frac{y^2}{2}+\frac{z^2}{2}-zx, \quad \mathcal{M}_\alpha = \frac{1}{C}e^{-\frac{2\Phi}{\alpha h}},$$

with $\alpha > 0$, $C > 0$. When the temperatures are the same $\alpha = \alpha_1 = \alpha_2$, the function $\mathcal{M}_\alpha$ is the Maxwellian of the process. In the general case when temperatures may be different, the function $\mathcal{M}_\alpha$ is used to define the weighted space $L^2(e^{-\frac{2\Phi}{\alpha h}}dxdydz)$.

In order to work in the flat L^2 space, the unknown is changed as $f = \mathcal{M}_\alpha^{1/2} u$. Then, the new equation for the unknown u is

$$h\partial_t u + \frac{\gamma}{2}\alpha_1\Big(- h\partial_{z_1} + \frac{1}{\alpha}(z_1 - x_1)\Big)\Big(h\partial_{z_1} + \big(\frac{2}{\alpha_1} - \frac{1}{\alpha}\big)(z_1 - x_1)\Big)u + \tag{1.60}$$
$$\frac{\gamma}{2}\alpha_2\Big(-h\partial_{z_2}+\frac{1}{\alpha}(z_2-x_2)\Big)\Big(h\partial_{z_2}+\Big(\frac{2}{\alpha_2}-\frac{1}{\alpha}\Big)(z_2-x_2)\Big)u+\Big(yh\partial_x-(\nabla_x V(x)-z)h\partial_y\Big)u = 0.$$

We consider the case when external potentials are quadratic. For simplicity, we may assume that $h = 1$, $\gamma = 2$, $d = 1$ and take

$$V_1(x_1) = \frac{1}{2}ax_1^2, \quad V_2(x_2) = \frac{1}{2}bx_2^2, \quad V_c(x_1 - x_2) = \frac{1}{2}c(x_1 - x_2)^2, \tag{1.61}$$

with $a, b, c \in \mathbb{R}$. The equation (1.60) writes as

$$\partial_t u + q^w(X, D_X)u - 2u = 0, \quad X = (x, y, z) \in \mathbb{R}^6,$$

where $q^w(X, D_X)$ is the quadratic operator with symbol

$$\begin{aligned} q = \alpha_1\zeta_1^2 + \alpha_2\zeta_2^2 + \beta_1(z_1 - x_1)^2 + \beta_2(z_2 - x_2)^2 + i\big[2\delta_1\zeta_1(z_1 - x_1) + 2\delta_2\zeta_2(z_2 - x_2) \\ + y_1\xi_1 + y_2\xi_2 - \eta_1\big((a + c)x_1 - cx_2 - z_1\big) - \eta_2\big(- cx_1 + (b + c)x_2 - z_2\big)\big], \end{aligned}$$

with

$$\beta_1 = \frac{\alpha_1}{\alpha}\Big(\frac{2}{\alpha_1} - \frac{1}{\alpha}\Big), \ \beta_2 = \frac{\alpha_2}{\alpha}\Big(\frac{2}{\alpha_2} - \frac{1}{\alpha}\Big), \ \delta_1 = \frac{\alpha_1}{\alpha} - 1, \ \delta_2 = \frac{\alpha_2}{\alpha} - 1,$$

where the notations ξ, η, ζ stand respectively for the dual variables of x, y, z. The condition

$$\alpha \geq \frac{1}{2}\max(\alpha_1, \alpha_2),$$

appearing in [**66**] exactly ensures that this quadratic symbol has a non-negative real part $\text{Re } q \geq 0$. The work [**66**] studies the case with identical temperatures $\alpha = \alpha_1 = \alpha_2$. In this case, the operator enjoys a supersymmetric structure. The supersymmetry is a particularly convenient structure which simplifies considerably the analysis of the splitting between the two smallest real parts of the eigenvalues and the analysis of the tunneling effect. When the temperatures are different, it was shown that the operator may fail this supersymmetric structure for certain (non-quadratic) external potentials [**67**]. However, this supersymmetry still holds in the case of quadratic external potentials with different temperatures. Indeed, let us consider the case with possibly different temperatures $\alpha_1 \neq \alpha_2$ and assume that

$$\alpha > \frac{1}{2}\max(\alpha_1, \alpha_2), \quad \alpha_1 > 0, \quad \alpha_2 > 0.$$

Some algebraic computations show that the Hamilton map satisfies to

$$\text{Ker}(\text{Re } F) \cap \mathbb{R}^{12} = \{(x, y, z, \xi, \eta, \zeta) \in \mathbb{R}^{12} : \ \zeta = 0, \ x = z\},$$

$$\text{Ker}(\text{Re } F)\cap\text{Ker}(\text{Re } F\text{Im } F)\cap\mathbb{R}^{12} = \{(x, y, z, \xi, \eta, \zeta) \in \mathbb{R}^{12} : \ y = \eta = \zeta = 0, \ x = z\},$$

$$\begin{aligned} &\text{Ker}(\text{Re } F) \cap \text{Ker}(\text{Re } F\text{Im } F) \cap \text{Ker}\big(\text{Re } F(\text{Im } F)^2\big) \cap \mathbb{R}^{12} \\ &= \{y = \xi = \eta = \zeta = 0, \ x = z, \ (a + c - 1)x_1 - cx_2 = 0, -cx_1 + (b + c - 1)x_2 = 0\}. \end{aligned}$$

When the condition

$$(a + c - 1)(b + c - 1) - c^2 \neq 0, \tag{1.62}$$

holds, the singular space is equal to zero after intersecting exactly k_0+1 kernels with here $0 \leq k_0 = 2 \leq 11$. The condition (1.62) corresponds exactly to the assumption $V(x) - x^2/2$ is a Morse function required in the work [**66**] (Lemma 6.1) to ensure that the needed dynamical conditions hold. The other conditions $\partial_x^\alpha V_j(x) = \mathcal{O}(1)$, when $|\alpha| \geq 2$, for $j = 1, 2, c$; and $|\nabla_x V(x) - x| \geq 1/C$ when $|x| \geq C$, are also satisfied for quadratic potentials fulfilling condition (1.62). When this condition holds, Theorems 1.7 and 1.8 apply both for the operator q^w and its adjoint $(q^w)^* = \overline{q}^w$. This implies in particular that the first eigenvalues of these operators have algebraic multiplicity 1 with eigenspaces

$$\text{Ker}(q^w - \mu_0) = \mathbb{C}\mathcal{M}_{\alpha_1,\alpha_2}, \quad \text{Ker}((q^w)^* - \tilde{\mu}_0) = \mathbb{C}\tilde{\mathcal{M}}_{\alpha_1,\alpha_2}, \tag{1.63}$$

spanned by ground states of exponential type

$$\mathcal{M}_{\alpha_1,\alpha_2}(x,y,z) = e^{-a(x,y,z)} \in \mathscr{S}(\mathbb{R}^6), \ \tilde{\mathcal{M}}_{\alpha_1,\alpha_2}(x,y,z) = e^{-\tilde{a}(x,y,z)} \in \mathscr{S}(\mathbb{R}^6), \tag{1.64}$$

where $a, \tilde{a}$ are complex-valued quadratic forms on $\mathbb{R}^6$ whose real parts are positive definite. The operator

$$q^w(X,D_X)u = 2u + \alpha_1\Big(-\partial_{z_1} + \frac{1}{\alpha}(z_1 - x_1)\Big)\Big(\partial_{z_1} + \Big(\frac{2}{\alpha_1} - \frac{1}{\alpha}\Big)(z_1 - x_1)\Big)u +$$
$$\alpha_2\Big(-\partial_{z_2} + \frac{1}{\alpha}(z_2 - x_2)\Big)\Big(\partial_{z_2} + \Big(\frac{2}{\alpha_2} - \frac{1}{\alpha}\Big)(z_2 - x_2)\Big)u + \Big(y\partial_x - (\nabla_x V(x) - z)\partial_y\Big)u,$$

and its adjoint are real. It implies that the quadratic forms $a, \tilde{a}$ are positive definite. Furthermore, the first eigenvalues $\mu_0, \tilde{\mu}_0$ are necessarily real and equal $\mu_0 = \tilde{\mu}_0$. This proves the existence of a Maxwellian in the general case when the temperatures may be different $\alpha_1 \neq \alpha_2$. We deduce from [**67**] (Theorem 1.2) that the operator $q^w - \mu_0$ also enjoys a supersymmetric structure in the case of different temperatures.

3. Resolvent bounds for doubly characteristic pseudodifferential operators

As discussed in the previous section, studying the long-time behavior of the evolution problem associated to a non-selfadjoint operator

$$\begin{cases} (\partial_t + P)u(t,x) = 0, \\ u(t,\cdot)|_{t=0} = u_0 \in L^2(\mathbb{R}^n), \end{cases}$$

requires to study the spectrum of this operator, but also to establish sharp bounds for its resolvent, since there is no a priori control of the resolvent growth by the spectrum in the non-selfadjoint case. The semiclassical Kramers-Fokker-Planck operator

$$P = -h^2\Delta_v + \frac{v^2}{4} - \frac{n}{2}h + v \cdot h\partial_x - \nabla_x V(x) \cdot h\partial_v, \quad (x,v) \in \mathbb{R}^{2n},$$

with general (not necessarily quadratic) potential $V \in C^\infty(\mathbb{R}^n, \mathbb{R})$ is an example of non-elliptic non-selfadjoint kinetic operator whose spectral and pseudospectral properties have been thoroughly studied by Hérau, Sjöstrand and Stolk [**70**]. More generally, these authors establish some accurate resolvent estimates and provide a precise description of the spectrum near the imaginary axis for doubly characteristic pseudodifferential operators of Kramers-Fokker-Planck type:

Let $m \geq 1$ be a smooth order function

$$\exists C_0 \geq 1, N_0 > 0, \quad m(X) \leq C_0 \langle X - Y\rangle^{N_0} m(Y), \quad X, Y \in \mathbb{R}^{2n}, \tag{1.65}$$

and $S(m)$ the symbol class

$$S(m, dX^2) = \{p \in C^\infty(\mathbb{R}^{2n}, \mathbb{C}) : \ \forall \alpha \in \mathbb{N}^{2n}, \exists C_\alpha > 0, \forall X \in \mathbb{R}^{2n}, \\ |\partial_X^\alpha p(X)| \leq C_\alpha m(X)\}.$$

Let P be the semiclassical pseudodifferential operator

$$P = p_0^w(x, hD_x)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi} p_0\Big(\frac{x+y}{2}, h\xi\Big) u(y) dy d\xi, \quad 0 < h \leq 1,$$

whose Weyl symbol $p_0(x, \xi) \in S(m)$ has a non-negative real part $\mathrm{Re}\, p_0 \geq 0$, and finitely many characteristic points

$$p_0^{-1}(\{0\}) = \{X_1, X_2, ..., X_N\}.$$

All the characteristic points are assumed to be doubly characteristic

$$p_0(X_j) = \nabla p_0(X_j) = 0, \quad 1 \leq j \leq N,$$

and the quadratic approximations of the symbol at these points

$$p_0(X_j + Y) = q_j(Y) + \mathcal{O}(Y^3), \quad \text{when } Y \to 0,$$

are required to satisfy the condition of subelliptic type (1.21),

$$\exists 0 < \varepsilon_j \leq 1, c_j > 0, \\ c_j |X|^2 \leq \mathrm{Re}\ q_j(X) + \varepsilon_j H^2_{\mathrm{Im} q_j} \mathrm{Re}\ q_j(X) \leq \frac{1}{c_j}|X|^2, \quad X \in \mathbb{R}^{2n}. \tag{1.66}$$

Notice that the sign assumption $\mathrm{Re}\, p_0 \geq 0$ implies that these quadratic forms have non-negative real parts $\mathrm{Re}\ q_j \geq 0$. When equipped with the domain

$$H(m) = (m^w(x, hD_x))^{-1}(L^2(\mathbb{R}^n)),$$

with $0 < h \ll 1$, the operator P is a closed densely defined operator on $L^2(\mathbb{R}^n)$.

Under additional assumptions of subellipticity at infinity, the first result in [**70**] provides a localization of the spectrum of P in a h-neighborhood of the doubly characteristic point $z = 0$. Let $\Omega \subset \mathbb{C}$ be a fixed neighborhood of the union of the spectra of the quadratic operators associated to the quadratic approximations of the symbol at doubly characteristic points

$$\bigcup_{j=1}^N \sigma(q_j^w(x, D_x)) \subset \Omega.$$

We recall that the description of the spectrum (1.20) holds true when the quadratic approximations satisfy to the subelliptic condition (1.66). Then, for any $C > 0$, there exist some positive constants $0 < h_0 \leq 1$, $C_0 > 0$ such that for all $0 < h \leq h_0$, $|z| \leq C$, $z \notin \Omega$,

$$h\|u\|_{L^2} \leq C_0 \|Pu - hzu\|_{L^2}, \quad u \in \mathscr{S}(\mathbb{R}^n). \tag{1.67}$$

This resolvent estimate shows that the spectrum of P in any h-ball of the doubly characteristic point $z = 0$, is localized in a h-neighborhood of the union of the spectra of its quadratic approximations at doubly characteristic points. The second result in [**70**] shows that sharp resolvent estimates may also be derived in a particular parabolic region of the resolvent set outside any h-ball centered in zero. More specifically, it is shown that there exist some positive constants $c, C_0 > 0$ such that, for any $C \geq 1$, there exists $h_0 > 0$ so that for all $0 < h \leq h_0$, $\text{Re}\, z \leq c|z|^{1/3}h^{2/3}$, $|z| \geq Ch$,

$$h^{2/3}|z|^{1/3}\|u\|_{L^2} \leq C_0\|Pu - zu\|_{L^2}, \quad u \in \mathscr{S}(\mathbb{R}^n). \tag{1.68}$$

Lastly, when the operator satisfies the additional assumption

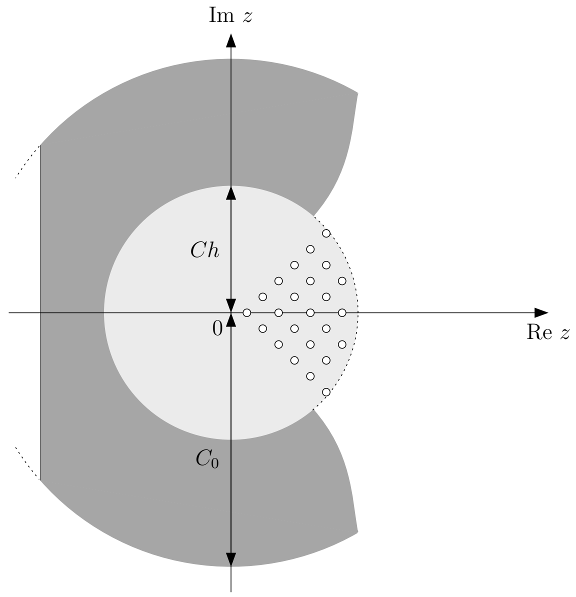
$$u \in L^2(\mathbb{R}^n),\ (P+1)u \in \mathscr{S}(\mathbb{R}^n) \Rightarrow u \in \mathscr{S}(\mathbb{R}^n),$$

this pseudospectral picture is completed by the following result about the spectrum: for any $C > 0$, when $0 < h \ll 1$, the spectrum of P in the h-ball $D(0, Ch)$ is discrete with eigenvalues satisfying to the semiclassical expansions

$$\lambda_{j,k}(h) \sim h(\mu_{j,k} + h^{1/N_{j,k}}\mu_{j,k,1} + h^{2/N_{j,k}}\mu_{j,k,2} + ...),$$

where the leading terms $\mu_{j,k}$ are the eigenvalues of the quadratic operator $q_j^w(x, D_x)$ located in the fixed ball $D(0, C)$, and $N_{j,k}$ is the dimension of the corresponding generalized eigenspace. These results provide a sharp picture of the spectral and pseudospectral properties of pseudodifferential operators of Kramers-Fokker-Planck type around a doubly characteristic point. Drawing the inspiration from this analysis, the

FIGURE 5. The estimate $h^{2/3}|z|^{1/3}\|u\|_{L^2} \leq C_0\|Pu - zu\|_{L^2}$ is fulfilled when z belongs to the dark grey region of the figure, whereas the estimate $h\|u\|_{L^2} \leq C_0\|Pu - zu\|_{L^2}$ is fulfilled in the light grey one.



starting point of the two works with Hitrik [**KPS3, KPS4**] was to understand how this spectral and pseudospectral picture may be extended to more general classes of doubly characteristic operators.

In these works, we study semiclassical pseudodifferential operators

$$P = P^w(x, hD_x, h)u(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi}P\Big(\frac{x+y}{2}, h\xi, h\Big)u(y)dyd\xi, \quad 0 < h \leq 1,$$

whose Weyl symbols $P(x,\xi,h)$ admit semiclassical asymptotic expansions

$$P(x,\xi;h) \sim \sum_{j=0}^{+\infty} p_j(x,\xi)h^j, \quad p_j \in S(m), \tag{1.69}$$

with a principal symbol having a non-negative real part

$$\text{Re}\, p_0 \geq 0, \tag{1.70}$$

and finitely many characteristic points

$$p_0^{-1}(\{0\}) = \{X_1, X_2, ..., X_N\}. \tag{1.71}$$

As before, all the characteristic points are assumed to be doubly characteristic

$$p_0(X_j) = \nabla p_0(X_j) = 0, \quad 1 \leq j \leq N, \tag{1.72}$$

and q_j stands for the quadratic approximation of the principal symbol

$$p_0(X_j + Y) = q_j(Y) + \mathcal{O}(Y^3), \quad \text{when } Y \to 0,$$

at the doubly characteristic point X_j. Contrary to the Kramers-Fokker-Planck operator, these operators are not allowed to fail ellipticity both microlocally and at infinity. In order to focus further on the doubly characteristic features, the assumptions at infinity are simplified by requesting the ellipticity of the real part of the principal symbol

$$\exists C > 0, \forall |X| \geq C, \quad \text{Re}\, p_0(X) \geq \frac{m(X)}{C}. \tag{1.73}$$

This assumption ensures that for $0 < h \ll 1$, the analytic family of operators

$$P - z : H(m) \to L^2(\mathbb{R}^n),$$

is Fredholm of index 0. An application of the analytic Fredholm theory allows to conclude that the spectrum of P in a small but fixed neighborhood of the doubly characteristic point $z = 0$, is discrete and consists of eigenvalues with finite algebraic multiplicity.

By elaborating on the singular space theory described in the previous section, we may weaken the assumptions on the quadratic approximations at the doubly characteristic points and extend the resolvent estimate (1.67) as follows:

Theorem 1.10. ([**KPS3**], Hitrik, KPS) *Let $P = P^w(x, hD_x, h)$ be a semiclassical pseudodifferential operator whose Weyl symbol $P(x,\xi,h)$ satisfies to the assumptions* (1.69)*,* (1.70)*,* (1.71)*,* (1.72) *and* (1.73)*. When all the quadratic approximations q_j are elliptic on their associated singular spaces*

$$X \in S_j, \ q_j(X) = 0 \Rightarrow X = 0, \quad 1 \leq j \leq N, \tag{1.74}$$

then, for any constant $C > 1$ and fixed neighborhood $\Omega_j \subset \mathbb{C}$ of the spectrum[4] *of the quadratic operator $q_j^w(x, D_x)$,*

$$\sigma(q_j^w(x, D_x)) \subset \Omega_j,$$

there exist some positive constants $0 < h_0 \leq 1$, $C_0 > 0$ such that for all $0 < h \leq h_0$, $|z| \leq C$ satisfying

$$z - p_1(X_j) \notin \Omega_j, \quad 1 \leq j \leq N,$$

[4]This spectrum is described in Theorem 1.2.

we have

$$h\|u\|_{L^2} \leq C_0\|(P-hz)u\|_{L^2}, \quad u \in \mathscr{S}(\mathbb{R}^n), \tag{1.75}$$

where $p_1(X_j)$ *stands for the value of the subprincipal symbol at the doubly characteristic point* X_j.

Let us recall that the condition (1.74) weakens the condition (1.66), since the singular space is equal to zero when the latter holds. We also notice that the localization of the spectrum depends on the properties of both the principal and subprincipal symbols. More precisely, the spectrum of P in any h-ball of the doubly characteristic point $z=0$, is localized in a h-neighborhood of the union of the spectra of its quadratic approximations shifted by the value of the subprincipal symbol at these doubly characteristic points

$$\bigcup_{j=1}^{N}\{p_1(X_j)+\sigma(q_j^w(x,D_x))\}.$$

As for the Kramers-Fokker-Planck operator, this pseudospectral picture in any h-ball of the doubly characteristic point $z=0$, may be completed by the following result about the spectrum:

THEOREM 1.11. ([**KPS4**], Hitrik, KPS) *Let* $P=P^w(x,hD_x,h)$ *be a semiclassical pseudodifferential operator whose Weyl symbol* $P(x,\xi,h)$ *satisfies to the assumptions* (1.69), (1.70), (1.71), (1.72) *and* (1.73). *When all the quadratic approximations* q_j *are elliptic on their associated singular spaces*

$$X \in S_j, \ q_j(X)=0 \Rightarrow X=0, \quad 1 \leq j \leq N, \tag{1.76}$$

then, for any $C>0$, *there exists* $0<h_0\leq 1$ *such that for all* $0<h\leq h_0$, *the spectrum of the operator* P *in the open ball* $D(0,Ch)$ *is given by eigenvalues satisfying to the semiclassical expansions*

$$z_{j,k} \sim h(\lambda_{j,k}+p_1(X_j)+h^{1/N_{j,k}}\lambda_{j,k,1}+h^{2/N_{j,k}}\lambda_{j,k,2}+\ldots), \quad 1 \leq j \leq N, \tag{1.77}$$

where $\lambda_{j,k}$ *are the eigenvalues of the quadratic operator* $q_j^w(x,D_x)$ *located in the fixed ball* $D(0,C)$, *and* $N_{j,k}$ *is the dimension of the corresponding generalized eigenspace.*

Lastly, in order to generalize the resolvent estimate (1.68), we need to introduce the remainder terms

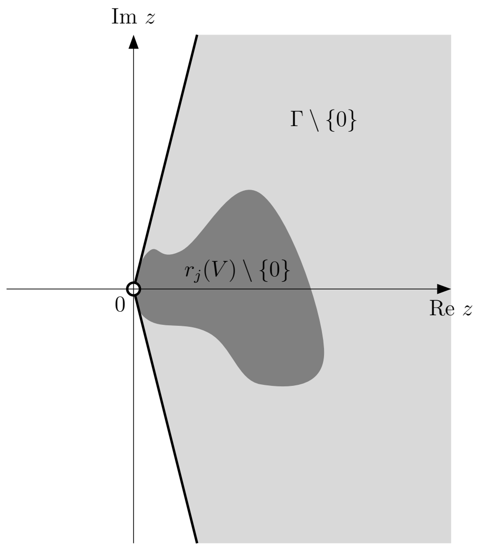
$$r_j(Y)=p_0(X_j+Y)-q_j(Y), \quad 1 \leq j \leq N, \tag{1.78}$$

and make a technical geometrical assumption about the ranges of these symbols. We assume the existence of a closed angular sector Γ with vertex at 0 and a neighborhood V of the origin in $\mathbb{R}^{2n}$ such that

$$r_j(V)\setminus\{0\} \subset \Gamma\setminus\{0\} \subset \{z \in \mathbb{C} : \text{Re } z>0\}, \quad 1 \leq j \leq N. \tag{1.79}$$

The resolvent estimate (1.68) and the geometry of the parabolic region where it holds, directly relate to the subelliptic properties of the quadratic approximations at doubly characteristic points:

FIGURE 6. Range of the symbol r_j.



THEOREM 1.12. ([**KPS4**], Hitrik, KPS) *Let $P = P^w(x, hD_x, h)$ be a semiclassical pseudodifferential operator whose Weyl symbol $P(x, \xi, h)$ satisfies to the assumptions* (1.69)*,* (1.70)*,* (1.71)*,* (1.72)*,* (1.73) *and* (1.79)*. Assume that all the quadratic forms q_j have zero singular spaces $S_j = \{0\}$. Let $0 \leq k_j \leq 2n-1$ be the smallest integers such that*

$$\Big(\bigcap_{l=0}^{k_j} \mathrm{Ker}\big[\mathrm{Re}\ F_j (\mathrm{Im}\ F_j)^l\big]\Big) \cap \mathbb{R}^{2n} = \{0\}, \tag{1.80}$$

where F_j is the Hamilton map of q_j. Then, for any constant $c_0 > 0$ sufficiently small, there exist positive constants $0 < h_0 \leq 1$, $C \geq 1$, $C_0 > 0$ such that for all $0 < h \leq h_0$, $u \in \mathscr{S}(\mathbb{R}^n)$, $z \in \Omega_h$,

$$h^{\frac{2k_0}{2k_0+1}} |z|^{\frac{1}{2k_0+1}} \|u\|_{L^2} \leq C_0 \|Pu - zu\|_{L^2}, \tag{1.81}$$

with $0 \leq k_0 = \max_{j=1,\dots,N} k_j \leq 2n-1$,

$$\Omega_h = \Big\{ z \in \mathbb{C} :\ \mathrm{Re}\ z \leq \frac{1}{C} h^{\frac{2k_0}{2k_0+1}} |z|^{\frac{1}{2k_0+1}},\ Ch \leq |z| \leq c_0 \Big\}.$$

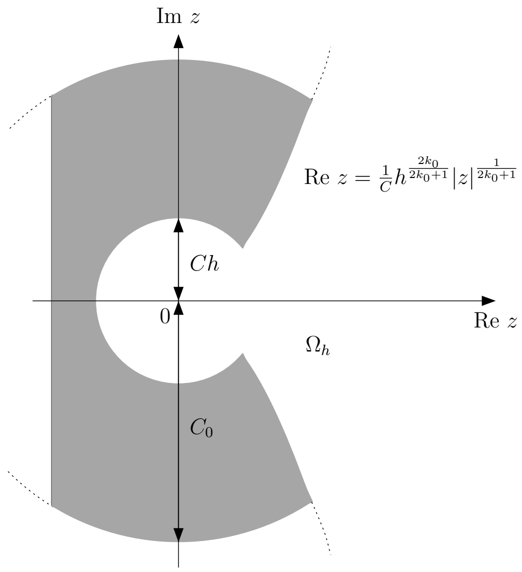
The term $h^{\frac{2k_0}{2k_0+1}} |z|^{\frac{1}{2k_0+1}}$ increases when the spectral parameter z moves away from the origin in the region where $Ch \leq |z| \leq c_0$. When the spectral parameter is of magnitude h, we recover the semiclassical estimate (1.75). Let us emphazise that the resolvent estimate

$$(P - z)^{-1} = \mathcal{O}(h^{-\frac{2k_0}{2k_0+1}} |z|^{-\frac{1}{2k_0+1}}) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n),$$

and the geometry of the parabolic region where it holds, directly relate to the maximal loss of $2k_0/(2k_0+1)$ derivatives appearing in the global subelliptic estimates enjoyed by all the quadratic approximations of the operator at doubly characteristic points

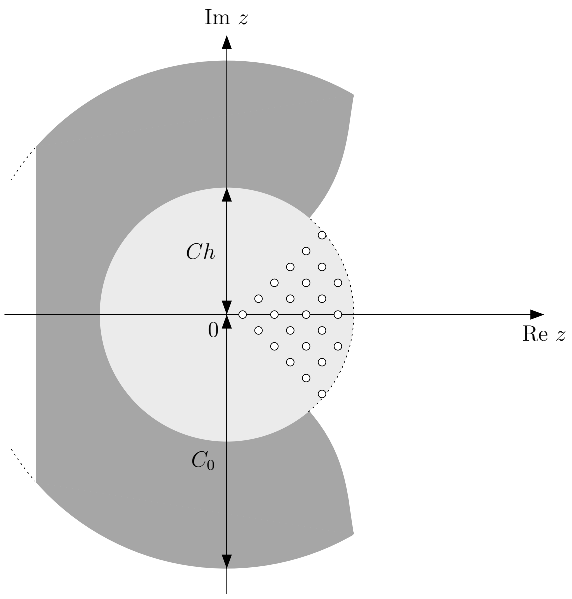
$$\|\langle (x, D_x) \rangle^{2/(2k_0+1)} u\|_{L^2} \lesssim \|q_j^w(x, D_x) u\|_{L^2} + \|u\|_{L^2}, \quad 1 \leq j \leq N.$$

FIGURE 7. Set Ω_h.



These results show that the singular space theory allows to sharply account for

FIGURE 8. The estimate $h^{\frac{2k_0}{2k_0+1}}|z|^{\frac{1}{2k_0+1}}\|u\|_{L^2} \leq C_0\|Pu - zu\|_{L^2}$ is fulfilled when z belongs to the dark grey region of the figure, whereas the estimate $h\|u\|_{L^2} \leq C_0\|Pu - zu\|_{L^2}$ is fulfilled in the light grey one.



the spectral and pseudospectral properties of pseudodifferential operators around a doubly characteristic point.

PROOF. The proofs of Theorems 1.10 and 1.11 follow the analysis of the Kramers-Fokker-Planck equation led by Hérau, Sjöstrand and Stolk [**70**], and rely on some techniques of compact complex deformations of the phase space. The key point in generalizing this machinery is to take advantage of the geometrical features (1.27), (1.28), (1.29), (1.30) enjoyed by the quadratic approximations of the doubly characteristic operator. By elaborating on these algebraic properties, we may build up a $C_0^\infty(\mathbb{R}^{2n},\mathbb{R})$ weight function G_ε supported in a neighborhood of the doubly characteristic set

$$\text{supp } G_\varepsilon \subset \{X \in \mathbb{R}^{2n} : \ \text{dist}(X, p_0^{-1}(0)) \leq 1/C_1\}, \quad C_1 > 1, \ 0 < \varepsilon \ll 1,$$

and satisfying to the estimates

(i) $G_\varepsilon = \mathcal{O}(\varepsilon)$, $\quad \partial^2 G_\varepsilon = \mathcal{O}(1)$
(ii) $\nabla G_\varepsilon = \mathcal{O}(\text{dist}(X, p_0^{-1}(0)))$ when $\text{dist}(X, p_0^{-1}(0)) \leq \varepsilon^{1/2}$
(iii) $\nabla G_\varepsilon = \mathcal{O}(\varepsilon^{1/2})$ when $\text{dist}(X, p_0^{-1}(0)) \geq \varepsilon^{1/2}$

This bounded weight function is devised in order to get second order ellipticity on a compact complex deformation of the phase space near the doubly characteristic set for an analytic extension of the principal symbol

$$|\widetilde{p}_0(X + i\delta H_{G_\varepsilon}(X))| \geq \frac{\delta}{C_2}\min[\text{dist}(X, p_0^{-1}(0))^2, \varepsilon], \quad C_2 > 1, \ 0 < \delta \ll 1,$$

when $\text{dist}(X, p_0^{-1}(0)) \leq 1/C$, and full ellipticity outside of a $\varepsilon^{1/2}$-neighborhood of the doubly characteristic set

$$\text{Re}\Big(\Big(1 - iC_3\frac{\delta\varepsilon}{\text{dist}(X, p_0^{-1}(0))^2}\Big)\widetilde{p}_0(X + i\delta H_{G_\varepsilon}(X))\Big) \geq \frac{\delta\varepsilon}{C_2}, \quad C_3 \neq 0,$$

when $\text{dist}(X, p_0^{-1}(0)) \geq \varepsilon^{1/2}$. The resolvent estimate (1.75) is microlocally derived successively in a tiny neighborhood of the doubly characteristic set

$$\text{dist}(X, p_0^{-1}(0)) \leq \varepsilon^{1/2},$$

with $\varepsilon = Ah$, $A \gg 1$, and then in the exterior region $\text{dist}(X, p_0^{-1}(0)) \geq \varepsilon^{1/2}$, while Theorem 1.11 is proven by solving a Grushin problem. For Theorem 1.12, we mostly rely on the subelliptic properties of the quadratic approximations of the doubly characteristic operator

$$\|\langle(x, D_x)\rangle^{2/(2k_j+1)}u\|_{L^2} \lesssim \|q_j^w(x, D_x)u\|_{L^2} + \|u\|_{L^2}, \quad 1 \leq j \leq N.$$

Indeed, starting from the construction of the real-valued symbol

$$g_j \in S(1, \langle X\rangle^{-\frac{2}{2k_j+1}}dX^2),$$

performed in the quadratic case (1.43),

$$\exists c_{1,j}, c_{2,j} > 0, \forall X \in \mathbb{R}^{2n}, \quad \text{Re } q_j(X) + c_{1,j}H_{\text{Im}q_j}g_j(X) + 1 \geq c_{2,j}\langle X\rangle^{\frac{2}{2k_j+1}},$$

we may devise a real-valued weight function g_h compactly supported in the phase space satisfying

$$\text{Re } p_0(X) + h(H_{\text{Im}p_0}\ g_h)(X) + c_1 h \geq c_2 h^{\frac{2k_0}{2k_0+1}} \min\big[c_3, \text{dist}(X, p_0^{-1}(0))\big]^{\frac{2}{2k_0+1}}, \tag{1.82}$$

with $c_1, c_2, c_3 > 0$, when $0 < h \ll 1$. Building on this estimate, standard microlocal techniques allow to derive the semiclassical resolvent estimate (1.81). $\square$

The picture drawn so far for the pseudospectral properties of pseudodifferential operators around a doubly characteristic point, has been recently completed by Viola [**127, 128**]. In these works, the author studies the case when the spectral parameter z enters deeper into the numerical range and may grow slightly more rapidly than the semiclassical parameter h outside of the parabolic region Ω_h. His result shows that polynomial resolvent bounds still hold in a larger $h(\log\log h^{-1})^{1/n}$-neighborhood of the doubly characteristic point. More precisely, under the assumptions of Theorem 1.12 with a single doubly characteristic point $X_1 \in \mathbb{R}^{2n}$, Viola shows that, for any $\rho > 0$, there exist positive constants $C_0, C_1 > 0$ such that the resolvent

$$(P-z)^{-1} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n),$$

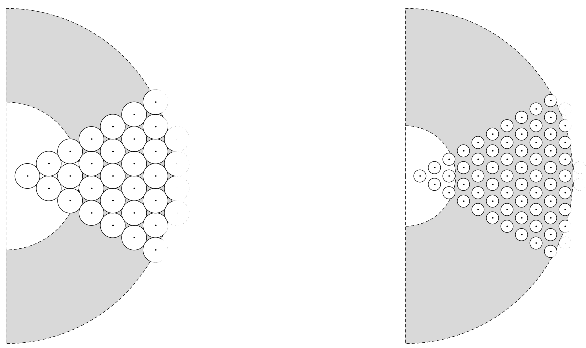
exists and satisfies to the bound

$$\|(P-z)^{-1}\|_{\mathcal{L}(L^2)} = \mathcal{O}(h^{-1-\rho}),$$

when $0 < h \ll 1$, as long as the spectral parameter obeys

$$|z| \leq \frac{1}{C_0} h\Big(\log\log\frac{1}{h}\Big)^{1/n}, \quad \operatorname{dist}(z, \sigma(q_1^w(x,D_x))) \geq h e^{-\frac{1}{C_1}(\log\log\frac{1}{h})^{1/n}}.$$

The next figure[5] is an illustration of a typical region in the complex plane where this resolvent estimate holds, for decreasing value of h.



The circles surrounding the spectral values of the quadratic operator $q_1^w(x,D_x)$ correspond to the forbidden region

$$\operatorname{dist}(z, \sigma(q_1^w(x,D_x))) < h e^{-\frac{1}{C_1}(\log\log\frac{1}{h})^{1/n}}.$$

By coming back to the resolvent estimate (1.81), we notice that the estimate

$$h^{\frac{2k_0}{2k_0+1}}\|u\|_{L^2} \leq C_0\|Pu - zu\|_{L^2}, \tag{1.83}$$

holds true at the boundary of the parabolic set Ω_h, when

$$\operatorname{Re} z \leq c_1 h^{\frac{2k_0}{2k_0+1}}, \quad \Big||\operatorname{Im} z| - \frac{c_0}{2}\Big| \leq c_1,$$

with $0 < c_1 \ll 1$. By using semigroups techniques, this resolvent estimate was improved by Sjöstrand [**114**] as

$$|\operatorname{Re} z|\|u\|_{L^2} \leq C_0\|Pu - zu\|_{L^2}, \tag{1.84}$$

[5]Courtesy of Joe Viola.

when

$$-c_1 \leq \text{Re } z \leq -h^{\frac{2k_0}{2k_0+1}}, \quad \Big||\text{Im } z| - \frac{c_0}{2}\Big| \leq c_1,$$

and

$$h^{\frac{2k_0}{2k_0+1}} \|u\|_{L^2} \leq C_0 \exp\Big(\frac{C_0}{h}(\text{Re } z)_+^{\frac{2k_0+1}{2k_0}}\Big) \|Pu - zu\|_{L^2}, \tag{1.85}$$

when

$$-h^{\frac{2k_0}{2k_0+1}} \leq \text{Re } z \leq c_1 \Big(h \log \frac{1}{h}\Big)^{\frac{2k_0}{2k_0+1}}, \quad \Big||\text{Im } z| - \frac{c_0}{2}\Big| \leq c_1.$$

For $\text{Re } z \sim h^{\frac{2k_0}{2k_0+1}}$, we recover the estimate (1.83). Furthermore, this result shows that the spectral parameter may enter logarithmically deeper into the numerical range outside of the parabolic region Ω_h,

$$\text{Re } z \sim \Big(h \log \frac{1}{h}\Big)^{\frac{2k_0}{2k_0+1}},$$

while keeping a polynomial resolvent bound

$$\|(P - z)^{-1}\|_{\mathcal{L}(L^2)} = \mathcal{O}(h^{-\frac{2k_0}{2k_0+1} - \rho_0}), \quad \rho_0 > 0.$$

It would be most interesting to study if this logarithmic incursion inside the numerical range holds all along the boundary of the parabolic set Ω_h, when $Ch \leq |z| \leq c_0$. Besides, the exponent $2k_0/(2k_0 + 1)$ for the semiclassical parameter h is known to be sharp and to be directly related to the subelliptic properties of the quadratic approximations of the doubly characteristic operator (Theorem 1.12). It would be most interesting to investigate the sharpness of the exponent $2k_0/(2k_0 + 1)$ for the logarithmic improvement in the term

$$\Big(h \log \frac{1}{h}\Big)^{\frac{2k_0}{2k_0+1}},$$

and to study if it may be related to critical indices for the possible Gevrey hypoellipticity of the quadratic approximations of the doubly characteristic operator. Another direction of current investigation is to study further the pseudospectral properties of the operator P in the h-neighborhood of the set

$$\bigcup_{j=1}^{N} \{p_1(X_j) + \sigma(q_j^w(x, D_x))\}.$$

First results in this direction in a joint work with Parmeggiani indicate that we can refine the pseudospectral picture in this particular part of the resolvent set by relating it to some results of hypoellipticity with big loss of derivatives [**KPS14**].

4. Works in progress and perspectives

In some applications, the operator depends explicitly on a set of parameters and it is interesting to try to maximize the rate of return to equilibrium τ_0,

$$\forall\ 0 \leq \tau < \tau_0, \exists C > 0, \forall t \geq 0, \quad \|e^{-tP} - \Pi_0\|_{\mathcal{L}(L^2)} \leq Ce^{-\tau t},$$

by tuning the parameters in order to lead the system the most rapidly to equilibrium. For a general pseudodifferential operator with a doubly characteristic point, a strategy to address this problem is to consider first the same optimization problem by substituting to the doubly characteristic operator, its quadratic approximation at the doubly characteristic point. In the quadratic case, we notice from Theorem 1.7 that this optimization problem is reduced to a simpler maximization problem in numerical linear algebra thanks to the formula

$$\tau_0 = 2 \min_{\substack{\lambda \in \sigma(F) \\ \mathrm{Im}\ \lambda > 0}} \mathrm{Im}\ \lambda > 0,$$

where F is Hamilton map of the quadratic approximation. The dependence of the Hamilton map with respect to the parameters may be derived explicitly and this simple maximization problem may be solved numerically in order to determine the right asymptotics of the parameters which optimize the rate of return to equilibrium. When these asymptotics are disclosed, a semiclassical problem may be set up and the theory developped in this chapter may be used in order to get sharp quantitative estimates for the rate of return to equilibrium for the original pseudodifferential operator in these asymptotics. In a work in progress, this strategy is implemented for studying the Fokker-Planck operator associated to a finite-dimensional Markovian approximation of the non-Markovian generalized Langevin equation in $\mathbb{R}^n$,

$$\ddot{x} = -\nabla_x V(x) - \int_0^t \gamma(t-s)\dot{x}(s)ds + F(t), \tag{1.86}$$

where $V(x)$ is a smooth confining potential and $F(t)$ a mean zero stationary Gaussian process with autocorrelation function γ in accordance to the fluctuation-dissipation theorem

$$\langle F(t) \otimes F(s) \rangle = \beta^{-1}\gamma(t-s)I_n,$$

where $\beta > 0$ stands for the inverse temperature and I_n the identity matrix.

CHAPTER 2

Solvability and hypoelliptic estimates for some classes of non-selfadjoint operators

1. Anisotropic hypoelliptic estimates for Landau-type operators

This section presents the work [**KPS1**] about the hypoellipticity of a particular class of inhomogeneous kinetic equations whose study is motivated by the linearization of the Landau equation

$$\partial_t f + v \cdot \nabla_x f = Q_L(f, f), \qquad x, v \in \mathbb{R}^n, \tag{2.1}$$

around the Maxwellian equilibrium distribution

$$\mu_n(v) = (2\pi)^{-\frac{n}{2}} e^{-\frac{|v|^2}{2}}.$$

The Landau collision operator is defined as

$$\begin{aligned} Q_L(g, f) = \nabla_v \cdot \Big(\int_{\mathbb{R}^n} & a(v - v_*)(g(t, x, v_*)(\nabla_v f)(t, x, v) \\ & - (\nabla_v g)(t, x, v_*) f(t, x, v)) dv_* \Big), \end{aligned} \tag{2.2}$$

where $a = (a_{i,j})_{1 \le i,j \le n}$ stands for the non-negative symmetric matrix

$$a(v) = (|v|^2 \operatorname{Id} - v \otimes v)|v|^\gamma \in M_n(\mathbb{R}), \quad -n < \gamma < +\infty. \tag{2.3}$$

As discussed in Chapter 3 (Section 4), the linearization of the Landau equation around the Maxwellian equilibrium distribution

$$f = \mu_n + \sqrt{\mu_n} g,$$

reduces to the equation for the fluctuation

$$\partial_t g + v \cdot \nabla_x g + \mathscr{L}_L g = \mu_n^{-1/2} Q_L(\sqrt{\mu_n} g, \sqrt{\mu_n} g), \tag{2.4}$$

with the linearized operator

$$\mathscr{L}_L g = -\mu_n^{-1/2} Q_L(\mu_n, \mu_n^{1/2} g) - \mu_n^{-1/2} Q_L(\mu_n^{1/2} g, \mu_n).$$

The linearized Landau operator is known [**36, 61, 71**] to be an accretive unbounded symmetric operator on $L^2(\mathbb{R}^n_v)$ (acting in the velocity variable),

$$(\mathscr{L}_L g, g)_{L^2(\mathbb{R}^n_v)} \ge 0,$$

satisfying

$$(\mathscr{L}_L g, g)_{L^2(\mathbb{R}^n_v)} = 0 \Leftrightarrow g = \mathbf{P} g, \tag{2.5}$$

where $\mathbf{P}$ is the L^2-orthogonal projection onto the space of collisional invariants

$$\mathcal{N} = \operatorname{Span}\big\{\mu_n^{1/2}, v_1 \mu_n^{1/2}, ..., v_n \mu_n^{1/2}, |v|^2 \mu_n^{1/2}\big\}.$$

Furthermore, the linearized Landau operator enjoys specific coercive estimates [**61, 100, 101**]:

$$\exists C_\gamma > 0, \quad (\mathscr{L}_L g, g)_{L^2(\mathbb{R}^n_v)} \geq C_\gamma |||g - \mathbf{P}g|||^2_\gamma, \tag{2.6}$$

where $|||\cdot|||_\gamma$ is the anisotropic norm

$$|||g|||^2_\gamma = \|\langle v\rangle^{\frac{\gamma}{2}} \Pi_v \nabla_v g\|^2_{L^2(\mathbb{R}^n_v)} + \|\langle v\rangle^{1+\frac{\gamma}{2}}(1-\Pi_v)\nabla_v g\|^2_{L^2(\mathbb{R}^n_v)} + \|\langle v\rangle^{1+\frac{\gamma}{2}} g\|^2_{L^2(\mathbb{R}^n_v)},$$

with

$$\langle v\rangle = \sqrt{1+|v|^2}, \qquad \Pi_v \nabla_v g = \Big(\frac{v}{|v|}\cdot\nabla_v\Big)\frac{v}{|v|}.$$

For Maxwellian molecules $\gamma = 0$, the linearized Landau operator is given by

$$\mathscr{L}_L f = (n-1)\Big(-\Delta_v + \frac{|v|^2}{4} - \frac{n}{2}\Big) f - \Delta_{\mathbb{S}^{n-1}} f + \Big[\Delta_{\mathbb{S}^{n-1}} - (n-1)\Big(-\Delta_v + \frac{|v|^2}{4} - \frac{n}{2}\Big)\Big]\mathbb{P}_1 f$$
$$+ \Big[-\Delta_{\mathbb{S}^{n-1}} - (n-1)\Big(-\Delta_v + \frac{|v|^2}{4} - \frac{n}{2}\Big)\Big]\mathbb{P}_2 f, \qquad f \in \mathscr{S}(\mathbb{R}^n),$$

see Chapter 3, Proposition 3.9, where

$$\Delta_{\mathbb{S}^{n-1}} = \frac{1}{2}\sum_{\substack{1\leq j,k\leq n\\ j\neq k}} (v_j\partial_k - v_k\partial_j)^2,$$

stands for the Laplace-Beltrami operator on the unit sphere $\mathbb{S}^{n-1}$ and $\mathbb{P}_k$ the orthogonal projections onto the Hermite basis described in Appendix (Section 2). In the 3-dimensional case, the linearized Landau operator with Maxwellian molecules is a pseudodifferential operator

$$\mathscr{L}_L f = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^6} e^{i(v-y)\cdot\xi} l\Big(\frac{v+y}{2}, \xi\Big) f(y)\,dy\,d\xi,$$

whose Weyl symbol is anisotropic and satisfies to

$$l(v,\xi) = 2\Big(|\xi|^2 + \frac{|v|^2}{4} - \frac{3}{2}\Big) + |v\wedge\xi|^2 - \frac{3}{2} \bmod \mathbf{S}^{-\infty}(\mathbb{R}^6), \tag{2.7}$$

with $\mathbf{S}^{-\infty}(\mathbb{R}^{2n}) = \cap_{m\in\mathbb{R}} \mathbf{S}^m(\mathbb{R}^{2n})$, where $\mathbf{S}^m(\mathbb{R}^{2n})$, $m \in \mathbb{R}$, is the symbol class

$$\forall(\alpha,\beta)\in\mathbb{N}^{2n}, \exists C_{\alpha\beta} > 0, \forall(v,\xi)\in\mathbb{R}^{2n}, \quad |\partial_v^\alpha\partial_\xi^\beta a(v,\xi)| \leq C_{\alpha,\beta}\langle(v,\xi)\rangle^{2m-|\alpha|-|\beta|}.$$

In the work [**KPS1**], we consider the class of linear Landau-type operators

$$P = iv\cdot D_x + D_v\cdot\lambda(v)D_v + (v\wedge D_v)\cdot\mu(v)(v\wedge D_v) + F(v), \quad x, v \in \mathbb{R}^3, \tag{2.8}$$

that is

$$P = i\sum_{j=1}^3 v_j D_{x_j} + \sum_{j=1}^3 D_{v_j}\lambda(v)D_{v_j} + (v_2 D_{v_3} - v_3 D_{v_2})\mu(v)(v_2 D_{v_3} - v_3 D_{v_2})$$
$$+ (v_3 D_{v_1} - v_1 D_{v_3})\mu(v)(v_3 D_{v_1} - v_1 D_{v_3}) + (v_1 D_{v_2} - v_2 D_{v_1})\mu(v)(v_1 D_{v_2} - v_2 D_{v_1}) + F(v),$$

with $D_x = i^{-1}\partial_x$, $D_v = i^{-1}\partial_v$ and $\gamma \in [-3, 1]$. The diffusion is given by smooth positive functions λ, μ and F satisfying for all $\alpha \in \mathbb{N}^3$,

$$\exists C_\alpha > 0, \forall v \in \mathbb{R}^3, \tag{2.9}$$
$$|\partial_v^\alpha\lambda(v)| + |\partial_v^\alpha\mu(v)| \leq C_\alpha\langle v\rangle^{\gamma-|\alpha|}, \quad |\partial_v^\alpha F(v)| \leq C_\alpha\langle v\rangle^{\gamma+2-|\alpha|},$$

$$\exists C > 0, \forall v \in \mathbb{R}^3, \quad \lambda(v) \geq C\langle v\rangle^\gamma, \quad \mu(v) \geq C\langle v\rangle^\gamma, \quad F(v) \geq C\langle v\rangle^{\gamma+2}. \tag{2.10}$$

The evolution equations associated to linear Landau-type operators are kinetic models for the analysis of the linearized spatially inhomogeneous Landau equation with general molecules

$$\partial_t + v \cdot \nabla_x + \mathscr{L}_L.$$

Indeed, they are natural extensions for general molecules of the phase space structure known in the Maxwellian case (2.7). Furthermore, their accretivity is consistent with the coercive estimates satisfied by the linearized Landau operator (2.6),

$$\text{(2.11)} \qquad \operatorname{Re}(Pu, u)_{L^2} = \|\lambda(v)^{\frac{1}{2}} D_v u\|_{L^2}^2 + \|\mu(v)^{\frac{1}{2}}(v \wedge D_v)u\|_{L^2}^2 + \|F(v)^{\frac{1}{2}} u\|_{L^2}^2 \geq 0,$$

when $u \in \mathscr{S}(\mathbb{R}^6_{x,v})$, thanks to the anisotropic diffusion due to the presence of the cross product term $v \wedge D_v$.

By denoting (ξ, η) the dual variables of (x, v), we notice that the diffusion of linear Landau-type operators only occurs in the variables (v, η), but not in the other directions, and that the cross product term $v \wedge D_v$ improves this diffusion in the directions of the phase space where the variables v and η are orthogonal. On the other hand, there is a lack of diffusion in the spatial derivative D_x. However, we prove that the regularization process in both space and velocity variables still occurs. This phenomenon of hypoellipticity is due to non-trivial interactions between the diffusive and transport parts of these operators.

The next result provides sharp global hypoelliptic estimates both in the spatial and velocity derivatives in a Sobolev scale whose structure is related to the anisotropies of the diffusion and some iterated commutators.

THEOREM 2.1. ([**KPS1**], Hérau, KPS) *Let P be the linear Landau-type operator defined in* (2.8)*. Then, there exists a positive constant $C > 0$ such that for all $u \in \mathscr{S}(\mathbb{R}^6_{x,v})$,*

$$\text{(2.12)} \quad \begin{aligned} &\|\langle v\rangle^{\gamma+2} u\|_{L^2}^2 + \|\langle v\rangle^{\gamma} |D_v|^2 u\|_{L^2}^2 + \|\langle v\rangle^{\gamma} |v \wedge D_v|^2 u\|_{L^2}^2 \\ &\quad + \|\langle v\rangle^{\gamma/3} |D_x|^{2/3} u\|_{L^2}^2 + \|\langle v\rangle^{\gamma/3} |v \wedge D_x|^{2/3} u\|_{L^2}^2 \leq C(\|Pu\|_{L^2}^2 + \|u\|_{L^2}^2), \end{aligned}$$

respectively for all $u \in \mathscr{S}(\mathbb{R}^7_{t,x,v})$,

$$\text{(2.13)} \quad \begin{aligned} &\|\langle v\rangle^{\gamma+2} u\|_{L^2}^2 + \|\langle v\rangle^{\gamma} |D_v|^2 u\|_{L^2}^2 + \|\langle v\rangle^{\gamma} |v \wedge D_v|^2 u\|_{L^2}^2 \\ &\quad + \|\langle v\rangle^{\gamma/3} |D_x|^{2/3} u\|_{L^2}^2 + \|\langle v\rangle^{\gamma/3} |v \wedge D_x|^{2/3} u\|_{L^2}^2 \leq C(\|\partial_t u + Pu\|_{L^2}^2 + \|u\|_{L^2}^2), \end{aligned}$$

where the notation $\|\cdot\|_{L^2}$ stands for the $L^2(\mathbb{R}^6_{x,v})$-norm, respectively for the $L^2(\mathbb{R}^7_{t,x,v})$-norm.

The terms controlled in these estimates are sharp and have an anisotropic structure similar to the diffusion term. More specifically, the presence of the two cross products $v \wedge D_v$ and $v \wedge D_x$ in

$$\|\langle v\rangle^{\gamma} |v \wedge D_v|^2 u\|_{L^2}^2 + \|\langle v\rangle^{\gamma/3} |v \wedge D_x|^{2/3} u\|_{L^2}^2,$$

improves the regularity estimates provided by the terms

$$\|\langle v\rangle^{\gamma} |D_v|^2 u\|_{L^2}^2 + \|\langle v\rangle^{\gamma/3} |D_x|^{2/3} u\|_{L^2}^2,$$

in the directions of the phase space where either, v and D_v, or v and D_x are orthogonal. The anisotropic feature and the different indices are optimal. Indeed, these hypoelliptic estimates split up into two parts. For instance, the first part of the estimate (2.12),

$$\|\langle v\rangle^{\gamma+2}u\|_{L^2}^2+\|\langle v\rangle^{\gamma}|D_v|^2u\|_{L^2}^2+\|\langle v\rangle^{\gamma}|v\wedge D_v|^2u\|_{L^2}^2\leq C(\|Pu\|_{L^2}^2+\|u\|_{L^2}^2), \tag{2.14}$$

is purely provided by the diffusion term of the linear Landau-type operator. Its left-hand-side has the very same anisotropic structure and asymptotic growth as the diffusion term

$$D_v\cdot\lambda(v)D_v+(v\wedge D_v)\cdot\mu(v)(v\wedge D_v)+F(v).$$

On the other hand, the most interesting result in Theorem 2.1 is the anisotropic regularity estimate in the spatial derivative D_x,

$$\|\langle v\rangle^{\gamma/3}|D_x|^{2/3}u\|_{L^2}^2+\|\langle v\rangle^{\gamma/3}|v\wedge D_x|^{2/3}u\|_{L^2}^2\leq C(\|Pu\|_{L^2}^2+\|u\|_{L^2}^2). \tag{2.15}$$

This second estimate is also optimal in term of the index $2/3$ appearing in its left-hand-side. Indeed, the optimality of this index $2/3$ is suggested by general results about microlocal hypoellipticity with optimal loss of derivatives established in [**20**] (Corollary 1.3) or [**54**]. Let us recall the general result about microlocal hypoellipticity proved by Bolley, Camus and Nourrigat in [**20**] (Theorem 1.1, Corollary 1.3):

Let $(A_j)_{1\leq j\leq l}$ be a system of properly supported classical pseudodifferential operators on an open subset Ω of $\mathbb{R}^n$ of arbitrary real orders $m_1,\cdots,m_l$. Suppose that $A_j-A_j^*$ has order m_j-1 for all $1\leq j\leq l$. Let $(x_0,\xi_0)\in T^*(\Omega)\setminus 0$ be such that there is a commutator of length r, $Y=(\mathrm{ad}\,A_{i_1})\cdots(\mathrm{ad}\,A_{i_{r-1}})A_{i_r}$ which is elliptic of order $m_{i_1}+\cdots+m_{i_r}-r+1$ at (x_0,ξ_0). Then the following implication holds for all $s\in\mathbb{R}$: If $u\in\mathcal{D}'(\Omega)$ and $A_ju\in H^{s-m_j}(x_0,\xi_0)$, $j=1,\cdots,l$, then $u\in H^{s-1+1/r}(x_0,\xi_0)$. As a corollary, one obtains that if all the m_j are equal then $\Sigma_1^l A_j^*A_j$ is hypoelliptic at (x_0,ξ_0) with loss of $2(1-r^{-1})$ derivatives.

When each A_j is a real vector field, this is a microlocal version of the celebrated Hörmander's theorem on the hypoellipticity of sums of squares [**72**]. A simpler proof of the Hörmander's theorem, but with less precise information about the loss of derivatives, was given by Kohn [**80**], whereas optimal estimates for the loss of derivatives were obtained, in the case of real vector fields, by Rothschild and Stein [**112**].

The linear Landau-type operators are non-selfadjoint operators for which these general results of hypoellipticity do not apply. We derive the hypoellipticity of linear Landau-type operators from non-trivial mixing interactions between their diffusion and transport parts, in particular from the ellipticity of commutators of length 3 of their diffusion and transport parts. This explains that the optimal loss of derivatives expected in this case is $2(1-1/3)=4/3$. The order 2 associated to the diffusion term and the regularity estimate with respect to the velocity derivative D_v must therefore be substituted by an order $2-4/3=2/3$ in the regularity estimate with respect to the spatial derivative D_x. The anisotropic structure of the hypoelliptic estimate (2.15) directly relates to the anisotropic structure of the elliptic commutators of length 3.

Proof. The Kohn's method is the simplest and most flexible way for proving hypoellipticity. However, it does not provide sharp loss of derivatives. In order to obtain the optimal loss of derivatives, more subtle microlocal and geometric methods are needed. Here, the proof of Theorem 2.1 relies on a general method by multiplier which allows to prove hypoellipticity with optimal loss of 4/3 derivatives. This method was used by Hérau, Sjöstrand and Stolk in their work on the

Kramers-Fokker-Planck equation [**70**]. This approach was also extended to get optimal hypoelliptic (subelliptic) estimates with loss of $2(1-(2k+1)^{-1})$ derivatives for more degenerate classes of quadratic differential operators [**KPS24, KPS25**].

In order to prove Theorem 2.1, we consider the class of generalized linear Landau-type operators

$$P = iv \cdot D_x + \sum_{j,k=1}^{n} D_{v_j} A_{j,k}(v) D_{v_k} + F(v), \tag{2.16}$$

where $x, v \in \mathbb{R}^n$, $\gamma \in [-3, 1]$. Here $A(v) = (A_{j,k}(v))_{1\leq j,k\leq n}$ stands for a positive definite symmetric matrix with real-valued smooth entries verifying

$$|\partial_v^\alpha A_{j,k}(v)| \lesssim \langle v\rangle^{\gamma+2-|\alpha|}, \quad \alpha \in \mathbb{N}^n,\ 1 \leq j,k \leq n, \tag{2.17}$$

and F is a smooth positive function satisfying

$$F(v) \gtrsim \langle v\rangle^{\gamma+2} \quad \text{and} \quad |\partial_v^\alpha F(v)| \lesssim \langle v\rangle^{\gamma+2-|\alpha|},\ \alpha \in \mathbb{N}^n. \tag{2.18}$$

We assume that the matrix A factorizes smoothly as

$$A(v) = B(v)^T B(v), \tag{2.19}$$

where $B(v)$ is a matrix with real-valued smooth entries verifying

$$|\partial_v^\alpha B_{j,k}(v)| \lesssim \langle v\rangle^{\frac{\gamma}{2}+1-|\alpha|}, \quad \alpha \in \mathbb{N}^n,\ 1 \leq j,k \leq n, \tag{2.20}$$

and $B(v)^T$ denotes its adjoint. Moreover, we assume that there exists $c > 0$ such that

$$A(v)\eta \cdot \eta = |B(v)\eta|^2 \geq c\langle v\rangle^\gamma |\eta|^2, \quad v, \eta \in \mathbb{R}^n. \tag{2.21}$$

The linear Landau-type operators are generalized linear Landau-type operators when taking

$$B(v) = \begin{pmatrix} \sqrt{\lambda(v)} & -v_3\sqrt{\mu(v)} & v_2\sqrt{\mu(v)} \\ v_3\sqrt{\mu(v)} & \sqrt{\lambda(v)} & -v_1\sqrt{\mu(v)} \\ -v_2\sqrt{\mu(v)} & v_1\sqrt{\mu(v)} & \sqrt{\lambda(v)} \end{pmatrix}, \tag{2.22}$$

with λ and μ being the functions defined in (2.9) and (2.10), since

$$|B(v)\eta|^2 = |\sqrt{\lambda(v)}\eta + \sqrt{\mu(v)}v\wedge\eta|^2 = |\sqrt{\lambda(v)}\eta|^2 + |\sqrt{\mu(v)}v\wedge\eta|^2 \geq c\langle v\rangle^\gamma |\eta|^2. \tag{2.23}$$

The Weyl symbol of a generalized linear Landau-type operator writes as

$$iv.\xi + |B(v)\eta|^2 + F(v) + \text{lower order terms}.$$

Setting

$$\tilde{p} = iv.\xi + |B(v)\eta|^2 + F(v),$$

we take advantage of the ellipticity in the variables (v, η) of the real part of this principal symbol

$$\text{Re } \tilde{p} = |B(v)\eta|^2 + F(v).$$

The main point is then to control the ξ variable. This control cannot be derived from the ellipticity of the principal symbol, but it comes from the ellipticity in the ξ variable of the iterated commutator of the operators defined by the real and imaginary parts of the principal symbol

$$[(\text{Im } \tilde{p})^w, [(\text{Re } \tilde{p})^w, (\text{Im } \tilde{p})^w]].$$

The Weyl symbol of this iterated commutator is given by the Poisson brackets

$$-\{\mathrm{Im}\ \tilde{p}, \{\mathrm{Re}\ \tilde{p}, \mathrm{Im}\ \tilde{p}\}\} = \{\mathrm{Im}\ \tilde{p}, \{\mathrm{Im}\ \tilde{p}, \mathrm{Re}\ \tilde{p}\}\} = 2|B(v)\xi|^2.$$

This suggests to introduce the symbol

$$\lambda = \big(1 + |B(v)\xi|^2 + |B(v)\eta|^2 + F(v)\big)^{1/2}, \tag{2.24}$$

which defines an anisotropic Sobolev scale related to the anisotropies of the diffusion and the elliptic iterated Poisson bracket. We aim at establishing sharp hypoelliptic estimates with loss of 4/3 derivatives in this anisotropic Sobolev scale

$$\|(\lambda^{2/3})^w u\|_{L^2}^2 \lesssim \|Pu\|_{L^2}^2 + \|u\|_{L^2}^2. \tag{2.25}$$

A key step is to prove first the weaker hypoelliptic estimate

$$\|(\lambda^{1/3})^w u\|_{L^2}^2 \lesssim \|Pu\|_{L^2}^2 + \|u\|_{L^2}^2. \tag{2.26}$$

The estimate (2.25) is then derived from (2.26) through a commutator argument. For simplicity only, we consider the case when $F(v) = \langle v\rangle^{\gamma+2}$. In order to prove (2.26), the phase space is split in two different regions. In the first microlocal region where the diffusion is strong enough to control the weight $\lambda^{2/3}$,

$$\mathrm{Re}\ \tilde{p} \gtrsim \lambda^{2/3},$$

the hypoelliptic estimate (2.26) purely relies on the diffusion of the generalized linear Landau-type operator

$$\mathrm{Re}(Pu, u) = \|B(v)\nabla_v u\|_{L^2}^2 + \big\|\langle v\rangle^{\frac{\gamma}{2}+1} u\big\|_{L^2}^2,$$

whereas in the second microlocal region where the diffusion is weaker than the weight $\lambda^{2/3}$,

$$\mathrm{Re}\ \tilde{p} \lesssim \lambda^{2/3},$$

we need to take advantage of the ellipticity of the commutator's symbol

$$\{\mathrm{Im}\ \tilde{p}, \{\mathrm{Im}\ \tilde{p}, \mathrm{Re}\ \tilde{p}\}\} = 2|B(v)\xi|^2.$$

To that end, we use a multiplier defined by a real-valued symbol g. This multiplier method is designed to produce the good term $H_{\mathrm{Im}\ \tilde{p}} g$. The natural choice is then to take for g the symbol $\{\mathrm{Im}\ \tilde{p}, \mathrm{Re}\ \tilde{p}\}$. This can be achieved after weighting and microlocalizing this symbol in order to give rise to a bounded operator

$$g = \frac{\{\mathrm{Im}\ \tilde{p}, \mathrm{Re}\ \tilde{p}\}}{2\lambda^{4/3}} \psi\left(\frac{\mathrm{Re}\ \tilde{p}}{\lambda^{2/3}}\right) = -\frac{B(v)\xi \cdot B(v)\eta}{\lambda^{4/3}} \psi\left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}}\right), \tag{2.27}$$

where ψ is a $C_0^\infty(\mathbb{R}, [0,1])$ cutoff function satisfying $\psi = 1$ on $[-1,1]$ and $\mathrm{supp}\ \psi \subset [-2,2]$. This symbol actually gives rise to a bounded operator since

$$g \in S(1, dv^2 + d\eta^2),$$

uniformly with respect to the parameter $\xi \in \mathbb{R}^n$. However, because of the anisotropy of the generalized linear Landau-type operator which accounts for the difference between the lower and upper bounds in the estimate

$$\langle v\rangle^\gamma |\eta|^2 \lesssim |B(v)\eta|^2 \lesssim \langle v\rangle^{\gamma+2}|\eta|^2,$$

the symbol g belongs to a gainless symbol class. To handle this difficulty, we use some elements of Wick calculus and define the multiplier through the Wick quantization. A short exposition of the Wick calculus is given in Appendix (Section 1). By denoting

$\|\cdot\|_{L^2}$ the $L^2(\mathbb{R}^n_v)$ norm and working on the Fourier side in the position variable, we may write for any $0<\varepsilon\ll 1$,

$$\operatorname{Re}(Pu,(1-\varepsilon g^{\mathrm{Wick}})u)=\underbrace{\|B(v)\nabla_v u\|^2_{L^2}+\big\|\langle v\rangle^{\frac{\gamma}{2}+1}u\big\|^2_{L^2}}_{\text{Diffusion terms}}-\underbrace{\varepsilon\operatorname{Re}\big(iv\cdot\xi u,g^{\mathrm{Wick}}u\big)}_{\substack{\text{Interesting term when}\\ |B(v)\eta|^2+F(v)\lesssim\lambda^{2/3}}}$$
$$\underbrace{-\varepsilon\operatorname{Re}\Big(\sum_{j,k=1}^n D_{v_j}A_{j,k}(v)D_{v_k}u,g^{\mathrm{Wick}}u\Big)-\varepsilon\operatorname{Re}\big(\langle v\rangle^{\gamma+2}u,g^{\mathrm{Wick}}u\big)}_{\text{Remainder terms}},$$

with some remainder terms controlled by the diffusion

$$\Big|\Big(\sum_{j,k=1}^n D_{v_j}A_{j,k}(v)D_{v_k}u,g^{\mathrm{Wick}}u\Big)\Big|+|(\langle v\rangle^{\gamma+2}u,g^{\mathrm{Wick}}u)|\lesssim\|B(v)\nabla_v u\|^2_{L^2}+\big\|\langle v\rangle^{\frac{\gamma}{2}+1}u\big\|^2_{L^2}.$$

Indeed, we have

$$\begin{aligned}-\varepsilon\operatorname{Re}(i\xi\cdot vu,g^{\mathrm{Wick}}u)&=-\varepsilon\operatorname{Re}(i\xi\cdot v^{\mathrm{Wick}}u,g^{\mathrm{Wick}}u)\\&=-\varepsilon(\operatorname{Re}(g^{\mathrm{Wick}}(i\xi\cdot v)^{\mathrm{Wick}})u,u)=\varepsilon\frac{1}{4\pi}(\{\xi\cdot v,g\}^{\mathrm{Wick}}u,u),\end{aligned}$$

where

$$\{\xi\cdot v,g\}=\underbrace{\frac{|B(v)\xi|^2}{\lambda^{4/3}}\psi\left(\frac{|B(v)\eta|^2+\langle v\rangle^{\gamma+2}}{\lambda^{2/3}}\right)}_{\text{Good term}}+\text{Remainder terms},$$

with some remainder terms uniformly controlled w.r.t. ξ by the diffusion

$$|\text{Remainder terms}|\lesssim 1+|B(v)\eta|^2+\langle v\rangle^{\gamma+2}.$$

The good term provides a control of the weight $\lambda^{2/3}$,

$$\frac{|B(v)\xi|^2}{\lambda^{4/3}}\psi\left(\frac{|B(v)\eta|^2+\langle v\rangle^{\gamma+2}}{\lambda^{2/3}}\right)\gtrsim\lambda^{2/3},$$

in the second microlocal region where

$$\operatorname{Re}\tilde{p}=|B(v)\eta|^2+\langle v\rangle^{\gamma+2}\lesssim\lambda^{2/3}.$$

The proof of the hypoelliptic estimate (2.26) is then completed by using some properties of the Wick calculus and other standard microlocal techniques. $\square$

An alternative proof of Theorem 2.1 was recently given by Alexandre [**5**]. This proof avoids the use of pseudodifferential calculus and relies on arguments originally introduced by Bouchut [**21**] and Perthame [**108**]. This second approach has the advantage to be simpler and to require less regularity for the coefficients of the operator. The phase space method is more complicated, but also much more general. It accounts for the structure of the left-hand-side terms in the estimates (2.12), (2.13), and can be easily adapted to other kinetic models as shown for instance in the recent work [**68**] with a transport part involving an external potential.

Regarding the Kramers-Fokker-Planck operator for which this phase space method was originally introduced [**70**], there are still many open questions about its hypoelliptic properties

$$K=-\Delta_v+\frac{v^2}{4}-\frac{n}{2}+v\cdot\partial_x-\nabla_xV(x)\cdot\partial_v,\quad (x,v)\in\mathbb{R}^{2n},$$

for a general potential $V\in C^\infty(\mathbb{R}^n,\mathbb{R})$, and some conjectures about its link with the Witten Laplacian

$$\Delta^{(0)}_{\frac{\Phi}{2}}=-\Delta+\frac{1}{4}|\nabla\Phi|^2-\frac{1}{2}\Delta\Phi,$$

where

$$\Phi(x,v)=\frac{v^2}{2}+V(x).$$

The work [**64**] emphasizes that the properties of the Kramers-Fokker-Planck operator and those of the Witten Laplacian are closely related. Indeed, the property

$$(1+K)^{-1}\text{ is a compact operator}\Rightarrow(1+\Delta^{(0)}_{\frac{\Phi}{2}})^{-1}\text{ is a compact operator},$$

holds for any smooth potential $V\in C^\infty(\mathbb{R}^n,\mathbb{R})$, whereas the reverse implication was shown to hold [**69**] for general elliptic potential satisfying

$$|\partial_x^\alpha V(x)|\leq C_\alpha\langle x\rangle^{2\mu-|\alpha|},\quad \frac{1}{C}\langle x\rangle^{2\mu}\leq 1+|V(x)|\leq C\langle x\rangle^{2\mu},\quad \mu\geq 1.$$

A key progress in analyzing the compactness of the resolvent of the Kramers-Fokker-Planck operator was later made in [**64**], where the global hypoelliptic estimate

$$\exists C>0,\forall u\in\mathscr{S}(\mathbb{R}^{2n}),\quad \|\Lambda^{1/4}u\|^2_{L^2}\leq C\big(\|Ku\|^2+\|u\|^2\big),$$

with

$$\Lambda^2=1+\Delta^{(0)}_{\frac{\Phi}{2}},$$

is established for potentials $V\in C^\infty(\mathbb{R}^n,\mathbb{R})$ satisfying

$$\exists M,C\geq 1,\forall\alpha\in\mathbb{N}^n,|\alpha|\geq 1,\exists C_\alpha>0,\forall x\in\mathbb{R}^n,\quad |\partial_x^\alpha V(x)|\leq C_\alpha\langle\nabla V(x)\rangle\leq C\langle x\rangle^M,$$

and the coercivity condition

$$\exists M,C\geq 1,\forall x\in\mathbb{R}^n,\quad \langle\nabla V(x)\rangle\geq\frac{1}{C}\langle x\rangle^{1/M}.$$

This global hypoelliptic estimate implies that the Kramers-Fokker-Planck operator has a compact resolvent if and only if the Witten Laplacian enjoys the same property. In view of this result, Helffer and Nier conjecture that the equivalence of compactness

$$(1+K)^{-1}\text{ is a compact operator}\Leftrightarrow(1+\Delta^{(0)}_{\frac{\Phi}{2}})^{-1}\text{ is a compact operator},$$

holds for any smooth potential $V\in C^\infty(\mathbb{R}^n,\mathbb{R})$. So far, this conjecture is largely left open and it is not known if it holds for some potentials whose Hessian is not dominated at infinity by the gradient. This is for instance the case of the polynomial potential $V_-(x_1,x_2)=-x_1^2x_2^2$, for which the resolvent of the Witten Laplacian is compact, but the compactness of the resolvent of the Kramers-Fokker-Planck operator is still open. It would be most interesting to tackle this Helffer-Nier conjecture for general classes of potentials as polynomial potentials and in its full generality.

2. Hypoelliptic estimates for a linear model of the non-cutoff Boltzmann equation

This section presents the results of the works [**KPS5, KPS10**] about the hypoelliptic properties of a linear model of the Boltzmann equation without angular cutoff. As discussed in Chapter 3 (Section 1), the non-cutoff Boltzmann operator

$$Q(g,f)=\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}}B(v-v_*,\sigma)\big(g'_*f'-g_*f\big)d\sigma dv_*,\qquad d\geq 2,$$

with

$$v'=\frac{v+v_*}{2}+\frac{|v-v_*|}{2}\sigma,\quad v'_*=\frac{v+v_*}{2}-\frac{|v-v_*|}{2}\sigma,\quad \sigma\in\mathbb{S}^{d-1},$$

whose cross section

$$B(v-v_*,\sigma)=|v-v_*|^\gamma b\Big(\frac{v-v_*}{|v-v_*|}\cdot\sigma\Big),\quad \gamma\in]-d,+\infty[,$$

satisfies to the assumption[1]

$$\exists 0<s<1,\quad (\sin\theta)^{d-2}b(\cos\theta)\underset{\theta\to 0_+}{\approx}\theta^{-1-2s}, \tag{2.28}$$

enjoys diffusive properties. It was noticed forty years ago by Cercignani [**27**] that the linearized Boltzmann operator with Maxwellian molecules behaves like a fractional diffusive operator. Over the time, this point of view transformed into the following widespread heuristic conjecture on the diffusive behavior of the Boltzmann operator as a flat fractional Laplacian [**6, 8, 126**]:

$$f\mapsto Q(\mu_d,f)\sim -(-\Delta_v)^s f+\text{ lower order terms},$$

where

$$\mu_d(v)=(2\pi)^{-\frac{d}{2}}e^{-\frac{|v|^2}{2}},\quad v\in\mathbb{R}^d,$$

with $0<s<1$ being the parameter appearing in the singularity assumption (2.28). This conjecture is discussed in Chapter 3. For now, we only consider the linear operator

$$P=\partial_t+v\cdot\nabla_x+a(t,x,v)(-\tilde{\Delta}_v)^s,\quad t\in\mathbb{R},\ x,v\in\mathbb{R}^d, \tag{2.29}$$

which is a simplified model for the non-cutoff spatially inhomogeneous Boltzmann equation

$$\partial_t f+v\cdot\nabla_x f=Q(f,f).$$

Here $0<s<1$ and a denotes a $C_b^\infty(\mathbb{R}^{2d+1})$ function satisfying

$$\exists a_0>0,\forall(t,x,v)\in\mathbb{R}^{2d+1},\quad a(t,x,v)\geq a_0>0. \tag{2.30}$$

The notation $C_b^\infty(\mathbb{R}^{2d+1})$ stands for the space of $C^\infty(\mathbb{R}^{2d+1})$ functions whose derivatives of any order are bounded over $\mathbb{R}^{2d+1}$ and $(-\tilde{\Delta}_v)^s$ is the Fourier multiplier with symbol

$$F(\eta)=|\eta|^{2s}w(\eta)+|\eta|^2(1-w(\eta)),\quad \eta\in\mathbb{R}^d, \tag{2.31}$$

with $|\cdot|$ being the Euclidean norm, w a $C^\infty(\mathbb{R}^d)$ function satisfying $0\leq w\leq 1$, $w(\eta)=1$ if $|\eta|\geq 2$, $w(\eta)=0$ if $|\eta|\leq 1$. This linear model has the particular structure

$$\text{Transport part in the }(t,x)\text{ variables }+\text{ Elliptic part in the }v\text{ variable}$$

[1] The notation $a\approx b$ means a/b is bounded from above and below by fixed positive constants.

We aim at studying the regularizing properties of this linear model and establishing hypoelliptic estimates with optimal loss of derivatives with respect to the t, x, v variables.

When $a = a_0 > 0$ is constant and $s = 1$, this operator relates to the Kolmogorov equation. In a 1934 *Annals of Mathematics* two-page paper (written in German) *"zur Theorie der Brownschen Bewegung"* [**81**], Kolmogorov introduced a model for the one-dimensional Brownian motion with the equation

$$Ku = \frac{\partial u}{\partial t} - v\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial v^2} = f, \quad x = \text{position}, \, v = \text{speed}. \tag{2.32}$$

Introducing the (divergence-free) real vector fields $X_0 = \partial_t - v\partial_x$, $X_1 = \partial_v$, we note that

$$K = X_0 + X_1^* X_1$$

and that the tangent space is equal to the Lie algebra generated by X_0, X_1, since $\partial_x = [X_0, X_1]$. According to a 1967 Hörmander's result [**72, 75**], this operator is micro-hypoelliptic, i.e. with C^∞ wavefront sets, $WFu = WF(Ku)$. To elaborate on this result, one may ask various questions:

- What is the loss of derivatives with respect to the elliptic case ?
- What type of a priori estimates can be used to prove hypoellipticity ?

These questions can be addressed by straightening the flow of the vector field X_0 through the change of variables

$$\begin{cases} t &= \tau \\ x &= x_1 - \tau x_2, \\ v &= x_2 \end{cases} \quad \text{we get} \quad \begin{cases} \frac{\partial}{\partial \tau} &= \frac{\partial}{\partial t} - v\frac{\partial}{\partial x} = X_0 \\ \frac{\partial}{\partial x_1} &= \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x_2} &= -t\frac{\partial}{\partial x} + \frac{\partial}{\partial v} \end{cases} \quad \text{and} \quad X_1 = \tau\partial_{x_1} + \partial_{x_2},$$

so that

$$K = \underbrace{\partial_\tau}_{\text{skew-adjoint}} \underbrace{-(\tau\partial_{x_1} + \partial_{x_2})^2}_{\text{selfadjoint } \geq 0}. \tag{2.33}$$

One may solve explicitly that ODE with parameters. By using the Fourier transform with respect to the x_1, x_2 variables, this reduces to

$$\widetilde{K} = \frac{d}{d\tau} + (\xi_2 + \tau\xi_1)^2.$$

It is interesting to look at the family of parabolas $\tau \mapsto (\xi_2 + \tau\xi_1)^2$, for $\xi_1^2 + \xi_2^2 = 1$, and to check as a good graphic way to explain the property of hypoellipticity that, although the minimum of all these functions is always zero (except for $\xi_2 = \pm 1, \xi_1 = 0$), their envelope has a positive minimum. More specifically, for $\xi_1 \neq 0$, the operator

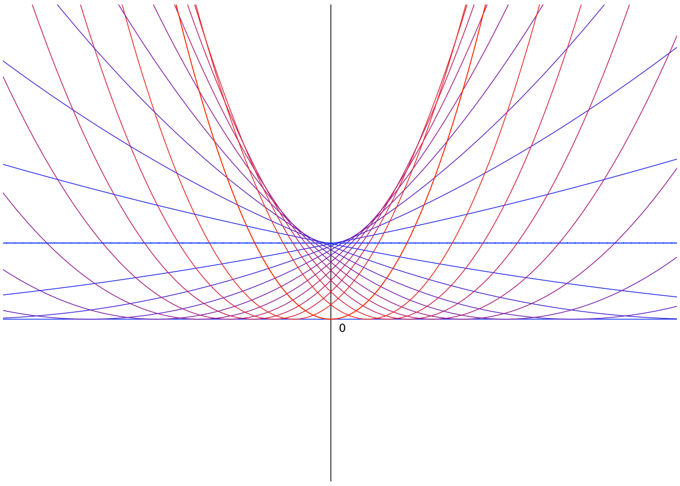
$$\widetilde{K} = \frac{d}{d\tau} + {\xi_1}^2\Big(\tau + \frac{\xi_2}{\xi_1}\Big)^2 = i(D_X - i\lambda X^2), \quad X = \tau + \frac{\xi_2}{\xi_1}, \quad \lambda = {\xi_1}^2, \quad D_X = i^{-1}\partial_X,$$

satisfies to the standard sharp subelliptic estimate

$$\|\widetilde{K}v\|_{L^2} \gtrsim \lambda^{1/3}\|v\|_{L^2} = |\xi_1|^{2/3}\|v\|_{L^2},$$

with $L^2(\mathbb{R}_\tau)$ norms. This subelliptic estimate is derived by splitting up the $L^2(\mathbb{R}_X)$-norm

$$\lambda^{1/3}\int_{\mathbb{R}} |u(X)|^2 dX = \lambda^{1/3}\int_{\{\lambda X^2 > \lambda^{1/3}\}} |u(X)|^2 dX + \lambda^{1/3}\int_{\{\lambda X^2 \leq \lambda^{1/3}\}} |u(X)|^2 dX.$$



By using that

$$|\{X \in \mathbb{R},\ \lambda X^2 \leq \lambda^{1/3}\}| \leq 2\lambda^{-1/3},$$

it follows, with $L^2(\mathbb{R}_X)$-norms, that

$$\begin{aligned}\lambda^{1/3}\|u\|_{L^2}^2 \leq \int_{\mathbb{R}} \lambda X^2 |u(X)|^2 dX + 2\sup_{X\in\mathbb{R}} |u(X)|^2 \\ \leq \operatorname{Re}((D_X - i\lambda X^2)u, -iu)_{L^2} + 2\sup_{X\in\mathbb{R}} |u(X)|^2.\end{aligned}$$

By denoting H the Heaviside function, a direct computation gives

$$\begin{aligned}2\operatorname{Re}((D_X - i\lambda X^2)u, -iH(T-X)u)_{L^2} \geq |u(T)|^2 \\ \Rightarrow 2\|(D_X - i\lambda X^2)u\|_{L^2}\|u\|_{L^2} \geq \sup_{X\in\mathbb{R}} |u(X)|^2\end{aligned}$$

and thus

$$\lambda^{1/3}\|u\|_{L^2}^2 \leq 5\|(D_X - i\lambda X^2)u\|_{L^2}\|u\|_{L^2} \Rightarrow \lambda^{1/3}\|u\|_{L^2} \lesssim \|(D_X - i\lambda X^2)u\|_{L^2}.$$

Finally, we deduce from the accretivity

$$\operatorname{Re}(\widetilde{K}v, v)_{L^2} = \|(\xi_2 + \tau\xi_1)v\|_{L^2}^2,$$

the following sharp estimate with respect to the derivative D_1,

$$\|u\|_{L^2} + \|Ku\|_{L^2} \gtrsim \||D_1|^{2/3}u\|_{L^2} + \|(D_2 + \tau D_1)u\|_{L^2}, \quad D_1 = i^{-1}\partial_{x_1},\ D_2 = i^{-1}\partial_{x_2},$$

with $L^2(\mathbb{R}_{\tau,x_1,x_2})$ norms.

Regarding the kinetic model (2.29), the existence and the C^∞ regularity for the solutions to the Cauchy problem of linear and semi-linear equations associated with this operator were proven in [**97**]. Chen, Li and Xu also investigated the Gevrey hypoelliptic properties of this operator. More specifically, they established in [**33**] (Proposition 2.1) the following hypoelliptic estimate:

Let K be a compact subset of $\mathbb{R}^{2d+1}$. For any $r \geq 0$, there exists a positive constant $C_{K,r} > 0$ such that for any $u \in C_0^\infty(K)$,

$$\|u\|_{r+\delta} \leq C_{K,r}(\|Pu\|_r + \|u\|_r), \tag{2.34}$$

with $\|\cdot\|_r$ standing for the $H^r(\mathbb{R}^{2d+1})$ Sobolev norm and

$$\delta = \max\Big(\frac{s}{4}, \frac{s}{2} - \frac{1}{6}\Big) > 0. \tag{2.35}$$

The notation $C_0^\infty(K)$ stands for the set of $C_0^\infty(\mathbb{R}^{2d+1})$ functions with support in K. This hypoelliptic estimate with loss of

$$\max(2s,1)-\delta>0,$$

derivatives is then a key instrumental ingredient for their investigation of the Gevrey hypoellipticity of the operator P. However, this hypoelliptic estimate (2.34) is not optimal. In [**KPS5**], we establish hypoelliptic estimates with optimal loss of derivatives with respect to the exponent $0<s<1$ of the fractional Laplacian $(-\tilde{\Delta}_v)^s$. More specifically, we show that the operator P is hypoelliptic with a loss of

$$\frac{\max(4s^2,1)}{(2s+1)}>0,$$

derivatives, that is, that the hypoelliptic estimates (2.34) hold with the new positive gain

$$\delta=\frac{2s}{2s+1}>0, \tag{2.36}$$

which improves for any $0<s<1$ the gain provided by (2.35),

$$\frac{2s}{2s+1}>\max\Big(\frac{s}{4},\frac{s}{2}-\frac{1}{6}\Big).$$

Theorem 2.2. ([**KPS5**], Lerner, Morimoto, KPS) *Let P be the operator defined in* (2.29)*, K be a compact subset of $\mathbb{R}^{2d+1}$ and $r\in\mathbb{R}$. Then, there exists a positive constant $C_{K,r}>0$ such that for all $u\in C_0^\infty(K)$,*

$$\|(1+|D_t|^{\frac{2s}{2s+1}}+|D_x|^{\frac{2s}{2s+1}}+|D_v|^{2s})u\|_r\leq C_{K,r}\big(\|Pu\|_r+\|u\|_r\big), \tag{2.37}$$

with $D_t=i^{-1}\partial_t$, $D_x=i^{-1}\partial_x$, $D_v=i^{-1}\partial_v$, $\|\cdot\|_r$ being the $H^r(\mathbb{R}^{2d+1})$ Sobolev norm.

This hypoelliptic estimate (2.37) is optimal in term of the exponents of the derivative terms appearing in the left-hand-side, namely, $2s/(2s+1)$ for the regularity in the time and space variables and $2s$ for the regularity in the velocity variable. Theorem 2.2 is a natural extension for the values of the parameter $0<s<1$ of the well-known optimal hypoelliptic estimates with loss of $4/3$ derivatives known for the Kolmogorov operator, case $s=1$, (see [**21, 31, 108**]),

$$\|(1+|D_t|^{2/3}+|D_x|^{2/3}+|D_v|^2)u\|_r\leq C_{K,r}\big(\|Pu\|_r+\|u\|_r\big).$$

The exponent $2s$ for the regularity in the velocity variable has the same growth as the diffusive part of the kinetic operator (2.29) and the optimality of the exponent $2s/(2s+1)$ for the regularity in the time and space variables may be checked through a simple scaling argument.

As a consequence of these optimal hypoelliptic estimates, we obtain the following result where we write

$$f\in H^s_{\mathrm{loc},(t_0,x_0)}(\mathbb{R}^{2d+1}_{t,x,v}),$$

if there exists an open neighborhood U of the point (t_0,x_0) in $\mathbb{R}^{d+1}$ such that $\phi(t,x)f\in H^s(\mathbb{R}^{2d+1}_{t,x,v})$ for any $\phi\in C_0^\infty(U)$.

COROLLARY 2.3. ([**KPS5**], Lerner, Morimoto, KPS) *Let P be the operator defined in* (2.29) *and $N \in \mathbb{N}$. If $u \in H_{-N}(\mathbb{R}^{2d+1}_{t,x,v})$ and $Pu \in H^r_{\mathrm{loc},(t_0,x_0)}(\mathbb{R}^{2d+1}_{t,x,v})$ with $r \geq 0$, then there exists an integer $k \geq 1$ such that*

$$\frac{1}{\langle v\rangle^k} u \in H^{r+\frac{2s}{2s+1}}_{\mathrm{loc},(t_0,x_0)}(\mathbb{R}^{2d+1}_{t,x,v}),$$

where $\langle v\rangle = (1+|v|^2)^{1/2}$. In particular, if $u \in H_{-N}(\mathbb{R}^{2d+1})$ and $Pu \in H^\infty(\mathbb{R}^{2d+1})$ then $u \in C^\infty(\mathbb{R}^{2d+1})$.

This result allows to recover the C^∞ hypoellipticity proved in [**97**] (Theorem 1.2) with now the optimal loss of derivatives. The equation (2.29) is not a classical pseudodifferential equation since the coefficient v in (2.29) is unbounded and the fractional Laplacian $(-\tilde{\Delta}_v)^s$ is a classical pseudo-differential operator in the velocity variable v but not in all the variables t, x, v. This accounts for parts of the difficulties encountered when studying this kinetic operator in particular when using cutoff functions in the velocity variable. This also accounts for the weight $\langle v\rangle^{-k}$ appearing in the statement of Corollary 2.3.

PROOF. The proof of Theorem 2.2 is relying on microlocal techniques developed by Lerner for proving energy estimates by using the Wick calculus [**87**]. They extend a method used by Trèves [**119**] to handle this type of estimates. The strategy of the proof is the following:

- *Step 1.* Consider the problem as an evolution equation along the characteristic curves of $\partial_t + v \cdot \partial_x$ by straightening this vector field in order, after relabeling the variables, to reduce the study to the analysis of the operator with normal form
$$iD_t + a(t, D_{x_2}, D_{x_1} + tD_{x_2})F(x_2 - tx_1).$$
- *Step 2.* Derive a priori estimates for the one-dimensional first-order differential operator
$$iD_t + a(t, \xi_2, \xi_1 + t\xi_2)F(x_2 - tx_1),$$
depending on the parameters $x_1, x_2, \xi_1, \xi_2 \in \mathbb{R}^d$. This step uses the same approach as the one used above for deriving the subelliptic estimate for the operator $D_X - i\lambda X^2$.
- *Step 3.* Deduce from the a priori estimates satisfied by the one-dimensional first-order differential operator with parameters some a priori estimates for the operator
$$iD_t + \big[a(t, \xi_2, \xi_1 + t\xi_2)F(x_2 - tx_1)\big]^{\mathrm{Wick}},$$
defined by using the Wick quantization the symbol
$$a(t, \xi_2, \xi_1 + t\xi_2)F(x_2 - tx_1).$$
A short exposition of the Wick calculus is given in Appendix (Section 1).
- *Step 4.* Thanks to the link between the Wick and the standard quantizations, control some remainder terms in order to come back to the standard quantization and derive a priori estimates for the original operator
$$iD_t + a(t, D_{x_2}, D_{x_1} + tD_{x_2})F(x_2 - tx_1).$$

□

An alternative proof of Theorem 2.2 was recently given by Alexandre [**4**] for specific positive functions a. This proof relies on averaging regularity type arguments following techniques originally introduced by Bouchut [**21**] and Perthame [**108**].

3. Explicit examples of nonsolvable weakly hyperbolic operators with real coefficients

This section presents the work [**KPS19**] which provides some explicit examples of nonsolvable weakly hyperbolic operators with real coefficients. These examples are given by the following two operators, with $(t,x,y) \in \mathbb{R}^3$,

$$L_1 = \partial_t(\partial_t + y\partial_x) + \partial_y,$$
$$L_2 = \partial_t^2 - H(-y)|y|^k\partial_x^2 + \partial_y, \quad k \geq 1,\ H = \mathbb{1}_{\mathbb{R}_+},$$

where the notation $\mathbb{1}_{\mathbb{R}_+}$ stands for the characteristic function of the set $\mathbb{R}_+$. Both examples are weakly hyperbolic operators in two-space-dimensions. The operator L_1 has affine coefficients and the operator L_2 has coefficients in C^{k-1}. Egorov gave in [**53**] an example of a nonsolvable weakly hyperbolic operator in one-space-dimension with a quite complicated expression. Although these examples are 2-space-dimensional, it is particularly worth noticing the simplicity of their expressions.

Let us recall some results about the solvability of pseudodifferential operators with real principal symbols. Let L be a classical pseudodifferential operator on an open set Ω of $\mathbb{R}^n$ with a real-valued principal symbol a_m. When the doubly characteristic set

$$\Sigma_2 = \{(x,\xi) \in \dot{T}^*(\Omega) :\ a_m(x,\xi) = 0,\ d_\xi a_m(x,\xi) = 0\},$$

is empty, the operator L is of strong-real-principal-type and local solvability with a loss of one derivative holds [**75**] (Theorem 26.1.7). Local solvability with a loss of one derivative holds as well for doubly characteristic pseudodifferential operators with real principal symbols whose subprincipal symbols a^s_{m-1} have non-zero imaginary parts

$$a_m(x,\xi) = 0,\ d_\xi a_m(x,\xi) = 0 \Rightarrow \text{Im } a^s_{m-1}(x,\xi) \neq 0, \tag{2.38}$$

on the doubly characteristic set [**89**] (Theorem 1.1). This is for instance the case of most of the operators of the type

$$AB + C,$$

where A, B, C are smooth real vector fields in $\mathbb{R}^3$ such that A, B and $[A,B]$ are linearly independent, which have been shown to be locally solvable in [**120**]. When the set

$$\tilde{\Sigma}_2 = \{(x,\xi) \in \dot{T}^*(\Omega) :\ a_m(x,\xi) = 0,\ d_\xi a_m(x,\xi) = 0,\ \text{Im } a^s_{m-1}(x,\xi) = 0\},$$

is non-empty, different situations can occur. Local solvability may hold. This is for instance the case of some operators of the type $AB + C$. However, local solvability may also fail as shown by the work of Mendoza and Uhlmann [**92**], which provides a necessary condition for local solvability of a class of doubly characteristic operators: let P be a classical, properly supported operator on an open set X in $\mathbb{R}^n$ whose principal symbol p is real and factorizes microlocally, i.e., near any point in $T^*X \setminus 0$, $p = p_1p_2$ with p_j real valued, C^∞ and homogeneous. Assume that the doubly characteristic set is an involutive submanifold of codimension 2 and that at doubly characteristic points, the Hamilton vector fields H_{p_1}, H_{p_2} and the cone direction are independent. Then, the condition:

> Sub($\mathcal{P}$): The imaginary part of the subprincipal symbol of P does not change sign at the doubly characteristic point ν_0 along either the bicharacteristics of the symbol p_1, or those of p_2

is necessary for the local solvability of the operator P at the doubly characteristic point ν_0. This condition is analogous to the condition ($\mathcal{P}$) of Nirenberg-Trèves [**103**]. More generally, we recall that local solvability for principal-type pseudodifferential operators is equivalent [**38, 75**] to the condition (Ψ):

> Condition (Ψ): The homogeneous principal symbol p satisfies that $\mathrm{Im}(ap)$ does not change sign from - to + along the oriented bicharacteristics of $\mathrm{Re}(ap)$ for any $0 \neq a \in C^\infty$

Here both operators L_1 and L_2 have non-empty set $\tilde{\Sigma}_2$. The operator L_1 is defined in the standard quantization, or in the Weyl quantization, by the symbol

$$p(t,x,y;\tau,\xi,\eta) = -\tau(\tau + y\xi) + i\eta,$$

with the real-valued principal symbol $p_2 = -\tau(\tau + y\xi)$. The doubly characteristic set

$$\begin{aligned}\Sigma_2(L_1) = \{(t,x,y;\tau,\xi,\eta) \in \dot{T}^*(\mathbb{R}^3) :\ y = \tau = 0,\ (\xi,\eta) \neq (0,0)\}\\ \cup \{(t,x,y;\tau,\xi,\eta) \in \dot{T}^*(\mathbb{R}^3) :\ \tau = \xi = 0,\ \eta \neq 0\},\end{aligned}$$

is not empty and is an involutive submanifold of codimension 2 near the point $\nu_0 = (t_0, x_0, 0; 0, 1, 0) \in \Sigma_2(L_1)$, $t_0, x_0 \in \mathbb{R}$,

$$\begin{aligned}\big(T_\nu\Sigma_2(L_1)\big)^\sigma = \{(t,x,y;\tau,\xi,\eta) \in \mathbb{R}^6 :\ x = y = \tau = \xi = 0\}\\ \subset T_\nu\Sigma_2(L_1) = \{(t,x,y;\tau,\xi,\eta) \in \mathbb{R}^6 :\ y = \tau = 0\},\end{aligned}$$

for all ν in a neighbourhood of ν_0 in $\Sigma_2(L_1)$. Setting $q = -\tau$ and $s = \tau + y\xi$, the Hamilton vector fields H_q, H_s and the radial vector field at points in $\Sigma_2(L_1)$ near ν_0,

$$H_q = -\frac{\partial}{\partial t}, \quad H_s = \frac{\partial}{\partial t} - \xi\frac{\partial}{\partial \eta}, \quad r = \xi\frac{\partial}{\partial \xi} + \eta\frac{\partial}{\partial \eta},$$

are independent. The imaginary part of the subprincipal symbol $p_1^s = i\eta$ does change sign at the first order in 0 along the bicharacteristic of the symbol s,

$$\begin{cases} \gamma'(t) = H_s(\gamma(t)), \\ \gamma(0) = \nu_0, \end{cases}$$

since

$$\mathrm{Im}\ p_1^s(\nu_0) = 0, \quad \frac{d}{dt}\big[\mathrm{Im}\ p_1^s(\gamma(t))\big]\big|_{t=0} = \{s, \mathrm{Im}\ p_1^s\}(\gamma(t))\big|_{t=0} = -1 \neq 0.$$

The condition Sub($\mathcal{P}$) is therefore violated and the nonsolvability at $\nu_0 \in \Sigma_2(L_1)$ follows from [**92**] (Theorem 1.2). This induces that the operator L_1 is nonsolvable in any neighbourhood of 0 in $\mathbb{R}^3$. In order to prove the nonsolvability in any neighbourhood of 0 for the operator with C^{k-1} coefficients L_2, we build a quasimode to show that no a priori estimates of the following type could hold:

$$\exists C_0 > 0, \exists N_0 \in \mathbb{N}, \exists V_0 \text{ an open neighbourhood of 0 in } \mathbb{R}^3 \text{ such that}$$
$$\forall u \in C_0^\infty(V_0),\ C_0\|L_2^* u\|_{(k-3)} \geq \|u\|_{(-N_0)}, \tag{2.39}$$

where $\|\cdot\|_{(s)}$ stands for the $H^s(\mathbb{R}^3)$ Sobolev norm. This implies that there do not exist an integer $N_0 \in \mathbb{N}$ and an open neighbourhood V_0 of 0 in $\mathbb{R}^3$ such that for all $f \in H^{N_0}(V_0)$, there exists $u \in H^{-k+3}(\mathbb{R}^3)$ solving

$$L_2 u = f,$$

on V_0 ($L_2 u$ is well defined for $u \in H^{-k+3}(\mathbb{R}^3)$). Indeed, if it was the case, we would have using similar arguments as in [**75**] (Lemma 26.4.5) that for all $v \in C_0^\infty(V_0)$,

$$|(f,v)_{L^2(V_0)}| = |(L_2 u, v)| = |(u, L_2^* v)| \leq \|u\|_{(-k+3)}\|L_2^* v\|_{(k-3)}. \tag{2.40}$$

Define

$$\begin{array}{rcl} T_v : H^{N_0}(V_0) & \rightarrow & \mathbb{C} \\ f & \mapsto & (f,v)_{L^2(V_0)}, \end{array}$$

for v in $C_0^\infty(V_0)$. It follows from the previous estimate that for all f in $H^{N_0}(V_0)$, there exists $u \in H^{-k+3}(\mathbb{R}^3)$ such that

$$\sup_{v \in W} |T_v(f)| \leq \|u\|_{(-k+3)} < +\infty,$$

with $W = \{v \in C_0^\infty(V_0),\ \|L_2^* v\|_{(k-3)} \leq 1\}$. Since T_v is a bounded linear form for v in W, we deduce from the uniform boundedness principle that there exists a positive constant $C_0 > 0$ such that

$$\sup_{v \in W} \|T_v\| \leq C_0 < +\infty.$$

It follows that for all $f \in H^{N_0}(V_0)$ and $v \in C_0^\infty(V_0)$, $\|L_2^* v\|_{(k-3)} \leq 1$,

$$|(f,v)_{L^2(V_0)}| \leq C_0\|f\|_{(N_0)},$$

which induces by homogeneity that for all $f \in H^{N_0}(V_0)$ and $v \in C_0^\infty(V_0)$,

$$|(f,v)_{L^2(V_0)}| \leq C_0\|f\|_{(N_0)}\|L_2^* v\|_{(k-3)}. \tag{2.41}$$

By using that $\|T_v\| = \|v\|_{(-N_0)}$ for all v in $C_0^\infty(V_0)$, we finally obtain from (2.41) that the following estimate

$$\forall v \in C_0^\infty(V_0),\ C_0\|L_2^* v\|_{(k-3)} \geq \|v\|_{(-N_0)},$$

would hold and reach a contradiction.

The operator L_2 is defined in the standard quantization, or in the Weyl quantization, by the symbol

$$p = i\eta + (\theta_k(y)\xi^2 - \tau^2) = i(\eta + i(\tau^2 - \theta_k(y)\xi^2)),$$

where θ_k is the $C^{k-1}(\mathbb{R},\mathbb{R})$ function

$$\theta_k(y) = (-1)^k y^k H(-y), \quad k \geq 1.$$

The principal symbol $p_2 = \theta_k(y)\xi^2 - \tau^2$ is a real C^{k-1} symbol whose doubly characteristic set

$$\Sigma_2(L_2) = \{(t,x,y;\tau,\xi,\eta) \in \dot{T}^*(\mathbb{R}^3) : \ \tau = 0, \ y \in \mathbb{R}_+\} \\ \cup \{(t,x,y;\tau,\xi,\eta) \in \dot{T}^*(\mathbb{R}^3) : \ \tau = \xi = 0\},$$

is not empty. This set contains some points $(t,x,0;0,\pm 1,0) \in \Sigma_2(L_2)$, where the imaginary part of the subprincipal symbol $p_1^s = i\eta$ vanishes. Furthermore, we notice that the symbol p violates a quasi-homogeneous version of the condition (Ψ), since the function $y \mapsto \tau^2 - \theta_k(y)\xi^2$ changes sign from $-$ to $+$, if y increases, whenever $\tau\xi \neq 0$. This property of sign change allows to construct a quasimode

$$u_\lambda(t,x,y) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{i(x\xi+t\tau)}\psi_\lambda(\tau,\xi)\chi_0(\lambda^\mu(y+(\tau\xi^{-1})^{\frac{2}{k}}))e^{-\Phi_1(\tau,\xi,y)}d\tau d\xi, \quad \lambda \geq 1,$$

with a non-negative phase function

$$\Phi_1(\tau,\xi,y) = \int_{-(\tau\xi^{-1})^{\frac{2}{k}}}^{y} (\tau^2 - \theta_k(s)\xi^2)ds \geq 0,$$

showing that no a priori estimates of the type (2.39) could hold. Notice that a violation of a quasi-homogeneous version of the condition (Ψ) does not always imply nonsolvability. A nice example of such an operator violating a quasi-homogeneous version of the condition (Ψ), but satisfying (2.38) so locally solvable, is given in [**34**].

4. Subelliptic estimates for operators with limited smoothness

This section presents the work [**KPS23**] about the subellipticity of a class of non-selfadjoint pseudodifferential operators with limited smoothness.

In this work, we study the sharp regularity for the symbols of a standard class of subelliptic pseudodifferential operators

$$hD_t + iq^w(t,x,hD_x), \tag{2.42}$$

which is needed to derive the global semiclassical subelliptic a priori estimate

$$\exists C>0, \exists\, 0<h_0\leq 1, \forall u \in C_0^\infty(\mathbb{R}_t\times\mathbb{R}_x^n), \forall\, 0<h<h_0, \\ \|hD_tu + iq^w(t,x,hD_x)u\|_{L^2(\mathbb{R}^{n+1})} \geq Ch^{\frac{N}{N+1}}\|u\|_{L^2(\mathbb{R}^{n+1})}. \tag{2.43}$$

The main result in [**KPS23**] provides a proof of this global subelliptic estimate under the smoothness assumption

$$q(t,x,\xi) \in C_b^{2[n/2]+4}(\mathbb{R}_t\times\mathbb{R}_x^n\times\mathbb{R}_\xi^n,\mathbb{R}), \tag{2.44}$$

where $[n/2]$ is the integer part of $n/2$ and C_b^n stands for the space of C^n functions which are bounded as well as all their derivatives up to the order n. This class of operators is quite general since any principal-type operator can be microlocally reduced to the normal form (2.42) after left and right multiplications by elliptic Fourier integral operators. To ensure the subellipticity of this operator, we assume that the symbol

$$p(t,x,\tau,\xi) = \tau + iq(t,x,\xi),$$

satisfies the condition $(\overline{\Psi})$:

> Condition $(\overline{\Psi})$: The symbol p satisfies that $\mathrm{Im}(ap)$ does not change sign from + to - along the oriented bicharacteristics of $\mathrm{Re}(ap)$ for any $0 \neq a \in C^\infty$

For the normal form $p(t,x,\tau,\xi) = \tau + iq(t,x,\xi)$, the condition $(\overline{\Psi})$ is equivalent to the assumption

$$q(t,X) > 0, \quad s > t \Rightarrow q(s,X) \geq 0.$$

This hypothesis means that for all $X \in \mathbb{R}^{2n}$, the function $t \mapsto q(t,X)$ can only change sign in the right sense, i.e. from negative values to positive ones. We also assume that for all $X \in \mathbb{R}^{2n}$, the function $t \mapsto q(t,X)$ only vanishes in a fixed compact set $[-A,A]$, $A > 0$, exactly N times, $N \geq 1$, and that these roots are some Lipschitz functions with respect to the variable X. More precisely, we assume that

$$\exists A > 0, \quad \inf_{|t| \geq A, X \in \mathbb{R}^{2n}} |q(t,X)| > 0, \tag{2.45}$$

$$\exists N \geq 1, \forall t \in [-A,A], \forall X \in \mathbb{R}^{2n}, \quad q(t,X) = e(t,X)\prod_{j=1}^{N}(t - \alpha_j(X)), \tag{2.46}$$

where e is a positive function on $\mathbb{R}^{2n+1}$ satisfying

$$M_0 = \inf_{|t| \leq A, X \in \mathbb{R}^{2n}} e(t,X) > 0, \tag{2.47}$$

and α_j are some real-valued Lipschitz functions on $\mathbb{R}^{2n}$ satisfying

$$\|\alpha_j\|_{L^\infty(\mathbb{R}^{2n})} \leq A, \quad j = 1, ..., N. \tag{2.48}$$

Under additional smoothness assumptions on the function e, the hypothesis (2.46) implies that for all points in the phase space, there exists a non-vanishing iterated Poisson brackets

$$H^l_{\mathrm{Re}p} \operatorname{Im} p(t,x,\tau,\xi) \neq 0, \quad 0 \leq l \leq N.$$

This means that all the points of the numerical range $p(\mathbb{R}^{2n+2})$ are of finite order with an order bounded above by the integer N. Let us underline that the operator (2.42) does not need to satisfy the condition (P):

> Condition (P): The symbol p satisfies that $\operatorname{Im}(ap)$ does not change sign along the bicharacteristics of $\operatorname{Re}(ap)$ for any $0 \neq a \in C^\infty$

as it is the case for the class of pseudodifferential operators studied by Dencker, Sjöstrand and Zworski in [**39**] (Theorem 1.4), where the function q is assumed to keep a sign.

A reading of the chapter 27 in the book by Hörmander [**75**] shows that there exists a complete theory for the microlocal subellipticity of pseudodifferential operators. This is a subtle and complex theory which distinguishes several types of subelliptic models, whether or not, the condition (P) is fulfilled. When the condition (P) holds, the proof of the subellipticity is a sharp and difficult result but its complexity is still relatively reasonable compared to the proof of the general subelliptic case when only the condition $(\overline{\Psi})$ is fulfilled. This is well-emphasized by Hörmander's comments in the Fields medallists' lectures [**77**]:

"For the scalar case, Egorov [**51**] *found necessary and sufficient conditions for subellipticity with less of δ derivatives; the proof of sufficiency was completed in* [**74**]*. A slight modification of the presentation is given in* [**75**]*, but it is still very complicated technically. Another approach which also covers systems operating on scalars has been given by Nourrigat* [**104**] *(see also the book* [**65**] *by Helffer and*

Nourrigat), but it is also far from simple so the study of subelliptic operators may not be in a final form."

As an illustration, the differential operators

$$L = D_t + it^{2k}(D_x + x^{2l}t^{2m+1}\Lambda), \quad (t,x) \in \mathbb{R}^2,\ k,l \geq 1,\ m \geq 0,\ \Lambda \geq 1,$$

are some examples of subelliptic operators violating the condition (P). Despite the simplicity of their expressions and up to your knowledge, no other proof than mimicking the proof of general case is known for proving their subellipticity

$$\forall u \in C_0^\infty(\mathbb{R}^2), \forall \Lambda \geq 1, \quad \|D_t u + it^{2k}(D_x + x^{2l}t^{2m+1}\Lambda)u\|_{L^2} \geq C\Lambda^{\frac{1}{(2k+1)(2l+1)+(2m+1)}}\|u\|_{L^2}.$$

Unfortunately, the subelliptic model (2.42) does not cover the general subelliptic case since there is a strong assumption on the Lipschitz regularity for the roots of the functions $t \mapsto q(t,X)$. However, despite this restiction, the result in [**KPS23**] presents a double interest: the first being to provide an estimate for the regularity needed to obtain such subelliptic estimates, the second to give a proof for the subellipticity of a class of pseudodifferential operators violating the condition (P) which remains relatively reasonable compared to the proof of the general case.

CHAPTER 3

Phase space analysis of the non-cutoff Boltzmann equation

This chapter presents the results obtained in the works [**KPS7, KPS8, KPS9, KPS11**]. We study in [**KPS7**] the spectral and phase space properties of the linearized non-cutoff Kac collision operator. The non-cutoff Kac operator is a kinetic model for the non-cutoff radially symmetric Boltzmann operator with Maxwellian molecules. The linearization of the non-cutoff Kac operator around a Maxwellian distribution is shown to be a function of the harmonic oscillator and to be equal to a fractional power of the harmonic oscillator up to some lower order terms. Related results for the non-cutoff radially symmetric Boltzmann operator are also proven. In [**KPS8**], we study the non-cutoff Boltzmann and Landau collision operators linearized around a normalized Maxwellian distribution. For Maxwellian molecules, we prove that the linearized non-cutoff Boltzmann operator is equal to a fractional power of the linearized Landau operator. This extends the result obtained in [**KPS7**] in the radially symmetric case. Furthermore, we provide the exact anisotropic phase space structure for both the linearized non-cutoff Boltzmann and Landau operators, and display explicitly the sharp anisotropic coercive estimates satisfied by the linearized non-cutoff Boltzmann operator for both Maxwellian and non-Maxwellian molecules. In [**KPS9**], we study the smoothing properties of the Cauchy problem associated to the radially symmetric spatially homogeneous non-cutoff Boltzmann equation with Maxwellian molecules in a close-to-equilibrium framework. This Cauchy problem is shown to enjoy the same Gelfand-Shilov regularizing properties as the Cauchy problem defined by the evolution equation associated to a fractional harmonic oscillator. Related regularizing results for the Cauchy problem associated to the spatially homogeneous Landau equation with Maxwellian molecules are given in [**KPS11**].

1. Preliminaries

1.1. The Boltzmann equation. The Boltzmann equation describes the behaviour of a dilute gas when the only interactions taken into account are binary collisions [**28**]. It reads as the equation

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = Q(f,f), \\ f|_{t=0} = f_0, \end{cases} \tag{3.1}$$

for the density distribution of the particles $f = f(t,x,v) \geq 0$ at time t, having position $x \in \mathbb{R}^d$ and velocity $v \in \mathbb{R}^d$. The Boltzmann equation derived in 1872 is one of the fundamental equations in mathematical physics and, in particular, a cornerstone of statistical physics.

The term appearing in the right-hand-side of this equation $Q(f,f)$ is the so-called Boltzmann collision operator associated to the Boltzmann bilinear operator

$$Q(g,f) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} B(v-v_*, \sigma)(g'_* f' - g_* f) d\sigma dv_*, \qquad d \geq 2, \tag{3.2}$$

with the standard shorthand $f'_* = f(t,x,v'_*)$, $f' = f(t,x,v')$, $f_* = f(t,x,v_*)$, $f = f(t,x,v)$. In this expression, v, v_* and v', v'_* are the velocities in $\mathbb{R}^d$ of a pair of particles respectively before and after the collision. They are connected through the formulas

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \quad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma,$$

where the parameter $\sigma \in \mathbb{S}^{d-1}$ belongs to the unit sphere. Those relations correspond physically to elastic collisions with the conservations of momentum and kinetic energy in the binary collisions

$$v + v_* = v' + v'_*, \quad |v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2,$$

where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^d$. The Boltzmann equation is said to be spatially homogeneous when the density distribution of the particles does not depend on the position variable

$$\begin{cases} \partial_t f = Q(f,f), \\ f|_{t=0} = f_0. \end{cases} \tag{3.3}$$

For monatomic gas, a standard model of cross sections $B(v-v_*,\sigma)$ is given by non-negative functions which depend separately on the relative velocity $|v-v_*|$ and on the deviation angle θ defined through the scalar product in $\mathbb{R}^d$,

$$\cos\theta = k\cdot\sigma, \quad k = \frac{v-v_*}{|v-v_*|}.$$

The cross sections are assumed to be supported on the set where $k\cdot\sigma \geq 0$, i.e. where $0 \leq \theta \leq \frac{\pi}{2}$. More specifically, they are assumed to have the following structure

$$B(v-v_*,\sigma) = \Phi(|v-v_*|)b\Big(\frac{v-v_*}{|v-v_*|}\cdot\sigma\Big), \tag{3.4}$$

with the kinetic factor

$$\Phi(|v-v_*|) = |v-v_*|^\gamma, \quad \gamma \in]-d,+\infty[. \tag{3.5}$$

The molecules are said to be Maxwellian when the parameter $\gamma = 0$. The second term appearing in the cross sections is a factor related to the deviation angle with a singularity

$$(\sin\theta)^{d-2}b(\cos\theta) \underset{\theta\to 0_+}{\approx} \theta^{-1-2s}, \tag{3.6}$$

for[1] some $0 < s < 1$. Notice that this singularity is not integrable

$$\int_0^{\frac{\pi}{2}} (\sin\theta)^{d-2}b(\cos\theta)d\theta = +\infty.$$

This non-integrability plays a major role regarding the qualitative behaviour of the solutions of the Boltzmann equation and this feature is essential for the smoothing effect to be present. Indeed, as first observed by Desvillettes for the Kac equation [**41**], grazing collisions that account for the non-integrability of the angular factor near $\theta = 0$ do induce smoothing effects for the solutions of the non-cutoff Kac equation, or more generally for the solutions of the non-cutoff Boltzmann equation. On the other hand, these solutions are at most as regular as the initial data, see e.g. [**130**],

[1] The notation $a \approx b$ means a/b is bounded from above and below by fixed positive constants.

when the cross section is assumed to be integrable, or after removing the singularity by using a cutoff function (Grad's angular cutoff assumption).

The physical motivation for considering this specific structure of cross sections is derived from particles interacting according to a spherical intermolecular repulsive potential of the form

$$\phi(\rho)=\frac{1}{\rho^r}, \quad r>1,$$

with ρ being the distance between two interacting particles. In the physical 3-dimensional space $\mathbb{R}^3$, the cross section satisfies the above assumptions with

$$s=\frac{1}{r}\in]0,1[, \quad \gamma=1-4s\in]-3,1[.$$

Further details on the physics background and the derivation of the Boltzmann equation may be found in the extensive expositions [**28, 126**].

1.2. The Kac equation. The Kac operator is a one-dimensional collision model for the radially symmetric Boltzmann operator with Maxwellian molecules defined as a finite part integral, see e.g [**KPS7**],

$$K(g,f)=\int_{|\theta|\leq\frac{\pi}{4}}\beta(\theta)\left(\int_{\mathbb{R}}(g'_*f'-g_*f)dv_*\right)d\theta, \tag{3.7}$$

with $f'_*=f(t,x,v'_*)$, $f'=f(t,x,v')$, $f_*=f(t,x,v_*)$, $f=f(t,x,v)$, where the relations between pre and post collisional velocities given by

$$v'=v\cos\theta-v_*\sin\theta, \quad v'_*=v\sin\theta+v_*\cos\theta, \quad v,v_*\in\mathbb{R}, \tag{3.8}$$

follow from the conservation of the kinetic energy in the binary collisions

$$v^2+v_*^2=v'^2+v'^2_*$$

and where the cross section is an even non-negative function satisfying

$$\beta\geq 0, \quad \beta\in L^1_{\text{loc}}(0,1), \quad \beta(-\theta)=\beta(\theta). \tag{3.9}$$

As for the Boltzmann operator, the main assumption concerning the cross-section is the presence of a non-integrable singularity for grazing collisions

$$\beta(\theta)\underset{\theta\to 0}{\approx}|\theta|^{-1-2s}, \tag{3.10}$$

with $0<s<1$.

1.3. The linearized Boltzmann operator. We consider the linearization of the Boltzmann equation

$$f=\mu_d+\sqrt{\mu_d}g,$$

around the Maxwellian equilibrium distribution

$$\mu_d(v)=(2\pi)^{-\frac{d}{2}}e^{-\frac{|v|^2}{2}}, \quad v\in\mathbb{R}^d. \tag{3.11}$$

Since $Q(\mu_d,\mu_d)=0$ by the conservation of the kinetic energy, the Boltzmann operator $Q(f,f)$ can be split into three terms

$$Q(\mu_d+\sqrt{\mu_d}g,\mu_d+\sqrt{\mu_d}g)=Q(\mu_d,\sqrt{\mu_d}g)+Q(\sqrt{\mu_d}g,\mu_d)+Q(\sqrt{\mu_d}g,\sqrt{\mu_d}g),$$

whose linearized part is $Q(\mu_d,\sqrt{\mu_d}g)+Q(\sqrt{\mu_d}g,\mu_d)$. Setting

$$\mathscr{L}g=-\mu_d^{-1/2}Q(\mu_d,\mu_d^{1/2}g)-\mu_d^{-1/2}Q(\mu_d^{1/2}g,\mu_d), \tag{3.12}$$

the original Boltzmann equation (3.1) is reduced to the Cauchy problem for the fluctuation

$$\begin{cases} \partial_t g + v \cdot \nabla_x g + \mathscr{L} g = \mu_d^{-1/2} Q(\sqrt{\mu_d} g, \sqrt{\mu_d} g), \\ g|_{t=0} = g_0. \end{cases} \tag{3.13}$$

The Boltzmann operator is local in the time and position variables and from now on, we consider it as acting only in the velocity variable. This linearized operator is known [**28**] to be an unbounded symmetric operator on $L^2(\mathbb{R}_v^d)$ (acting in the velocity variable) such that its Dirichlet form satisfies

$$(\mathscr{L} g, g)_{L^2(\mathbb{R}_v^d)} \geq 0.$$

Setting

$$\mathbf{P} g = (a + b \cdot v + c|v|^2)\mu_d^{1/2},$$

with $a, c \in \mathbb{R}$, $b \in \mathbb{R}^d$, the L^2-orthogonal projection onto the space of collisional invariants

$$\mathscr{N} = \text{Span}\big\{\mu_d^{1/2}, v_1\mu_d^{1/2}, ..., v_d\mu_d^{1/2}, |v|^2\mu_d^{1/2}\big\}, \tag{3.14}$$

we have

$$(\mathscr{L} g, g)_{L^2(\mathbb{R}^d)} = 0 \Leftrightarrow g = \mathbf{P} g. \tag{3.15}$$

It was noticed forty years ago by Cercignani [**27**] that the linearized Boltzmann operator with Maxwellian molecules behaves like a fractional diffusive operator. Over the time, this point of view transformed into the following widespread heuristic conjecture on the diffusive behavior of the Boltzmann operator as a flat fractional Laplacian [**3, 6, 8, 106, 107, 126**]:

$$f \mapsto Q(\mu_d, f) \sim -(-\Delta_v)^s f + \text{ lower order terms},$$

with $0 < s < 1$ being the parameter appearing in the singularity assumption (3.6). See [**KPS5, 97, 98**] for works related to this simplified model of the non-cutoff Boltzmann equation. This heuristics was the starting point of the series of works [**KPS7, KPS8**] studying the exact phase space structure of the linearized non-cutoff Boltzmann operator and its diffusive properties. The linearized non-cutoff Boltzmann operator enjoys coercive estimates which play a basic role when studying the Cauchy problem [**8, 9, 10, 58**]. For general molecules, sharp coercive estimates in the weighted isotropic Sobolev spaces $H_l^k(\mathbb{R}^d)$ were proven in [**9, 10, 58, 99, 101**]:

$$\|(1-\mathbf{P})g\|^2_{H^s_{\frac{\gamma}{2}}} + \|(1-\mathbf{P})g\|^2_{L^2_{s+\frac{\gamma}{2}}} \lesssim (\mathscr{L} g, g)_{L^2(\mathbb{R}^d)} \lesssim \|(1-\mathbf{P})g\|^2_{H^s_{s+\frac{\gamma}{2}}}, \tag{3.16}$$

where

$$H_l^k(\mathbb{R}^d) = \big\{f \in \mathscr{S}'(\mathbb{R}^d) : \ (1+|v|^2)^{\frac{l}{2}} f \in H^k(\mathbb{R}^d)\big\}, \quad k, l \in \mathbb{R}.$$

These estimates are sharp in this isotropic scale but the Boltzmann operator is a truly anisotropic operator. This accounts in general for the difference between the lower and upper bounds in (3.16). In the recent works [**10, 58, 59**], sharp coercive estimates for the linearized non-cutoff Boltzmann operator were proven independently. In [**10**], these sharp coercive estimates established in the three-dimensional setting $d = 3$ (Theorem 1.1 in [**10**]),

$$|||(1-\mathbf{P})f|||^2_{\gamma} \lesssim (\mathscr{L} f, f)_{L^2} \lesssim |||(1-\mathbf{P})f|||^2_{\gamma}, \quad f \in \mathscr{S}(\mathbb{R}^3), \tag{3.17}$$

involve the anisotropic norm

$$(3.18)\quad |||f|||_{\gamma}^2=\int_{\mathbb{R}_v^3\times\mathbb{R}_{v_*}^3\times\mathbb{S}_\sigma^2}|v-v_*|^\gamma b(\cos\theta)((\mu_3)_*(f'-f)^2+f_*^2(\sqrt{\mu_3'}-\sqrt{\mu_3})^2)dvdv_*d\sigma,$$

whereas in [**58, 59**], coercive estimates involving the anisotropic norms

$$\|f\|_{N^{s,\gamma}}^2=\|f\|_{L^2_{\gamma+2s}}^2+\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\langle v\rangle^{\frac{\gamma+2s+1}{2}}\langle v'\rangle^{\frac{\gamma+2s+1}{2}}\frac{|f(v)-f(v')|^2}{d(v,v')^{d+2s}}\mathbb{1}_{d(v,v')\leq 1}dvdv',$$

where

$$d(v,v')=\sqrt{|v-v'|^2+\frac{1}{4}(|v|^2-|v'|^2)^2},$$

were derived and a model of a fractional geometric Laplacian on a lifted paraboloid in $\mathbb{R}^{d+1}$ was heuristically suggested for interpreting the anisotropic diffusive properties of the Boltzmann collision operator. These coercive estimates are sharp but the diffusion seems difficult to analyze from the phase space view point. In [**KPS8**], these coercive estimates are made explicit and a sharp description of the anisotropy inherent to the diffusive properties of the linearized non-cutoff Boltzmann operator is given.

2. Spectral and phase space analysis of the linearized non-cutoff Kac operator

We consider the non-cutoff Kac collision operator (3.7) whose cross section satisfies to the assumptions (3.9) and (3.10). As before for the Boltzmann equation, we consider the fluctuation around the normalized Maxwellian distribution

$$\mu_1(v)=(2\pi)^{-\frac{1}{2}}e^{-\frac{v^2}{2}},\quad v\in\mathbb{R},$$

by setting

$$f=\mu_1+\sqrt{\mu_1}h.$$

Since $K(\mu_1,\mu_1)=0$ by conservation of the kinetic energy, we may write

$$K(\mu_1+\sqrt{\mu_1}h,\mu_1+\sqrt{\mu_1}h)=K(\mu_1,\sqrt{\mu_1}h)+K(\sqrt{\mu_1}h,\mu_1)+K(\sqrt{\mu_1}h,\sqrt{\mu_1}h)$$

and consider the linearized Kac operator

$$(3.19)\qquad \mathcal{K}h=\mathcal{K}_1h+\mathcal{K}_2h,$$

with

$$(3.20)\qquad \mathcal{K}_1h=-\mu_1^{-1/2}K(\mu_1,\mu_1^{1/2}h),\quad \mathcal{K}_2h=-\mu_1^{-1/2}K(\mu_1^{1/2}h,\mu_1).$$

The first result gives an operator-theoretical formula expressing the first part of the linearized non-cutoff Kac operator as a function of the contraction semigroup generated by the one-dimensional harmonic oscillator

$$(3.21)\qquad \mathcal{H}=-\Delta_v+\frac{v^2}{4}.$$

A reminder on classical notations and formulas for the harmonic oscillator and the Hermite functions is given in Appendix (Section 2).

PROPOSITION 3.1. ([**KPS7**], Lerner, Morimoto, KPS, Xu) *The first part of the linearized non-cutoff Kac operator defined by*

$$\mathcal{K}_1 f = -\mu_1^{-1/2} K(\mu_1, \mu_1^{1/2} f),$$

is equal to

$$\mathcal{K}_1 = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \beta(\theta)\Big[1 - (\sec\theta)^{\frac{1}{2}} \exp(-\mathcal{H}\ln(\sec\theta))\Big]d\theta,$$

where $\mathcal{H}$ is the one-dimensional harmonic oscillator (3.21) *so that*

$$\mathcal{K}_1 = \sum_{k\geq 1}\Big(\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \beta(\theta)(1-(\cos\theta)^k)d\theta\Big)\mathbb{P}_k, \tag{3.22}$$

where the projections $\mathbb{P}_k$ onto the Hermite basis are described in Appendix (Section 2).

The integrals appearing in the formula (3.22) are well-defined since the L^1 singularity at 0 of the function β is erased by the factor $(1-(\cos\theta)^k)$ vanishing at the second order. This first result shows that $\mathcal{K}_1$ is an unbounded nonnegative operator on $L^2(\mathbb{R})$ which is diagonal in the Hermite basis. Furthermore, the domain of the operator $\mathcal{K}_1$ can be taken as

$$\mathcal{D} = \Big\{u \in L^2(\mathbb{R}),\quad \sum_{k\geq 0} k^{2s}\|\mathbb{P}_k u\|_{L^2}^2 < +\infty\Big\} = \{u \in L^2(\mathbb{R}),\ \mathcal{H}^s u \in L^2(\mathbb{R})\}. \tag{3.23}$$

PROOF. The operator $\mathcal{K}_1 = \mathrm{Op}^w a$ is considered as a pseudodifferential operator and its Weyl symbol is computed explicitly. It is noticeable that this computation can be performed explicitly. In addition to the Bobylev formula[2], this calculation only requires the computations of the Fourier transformations of Gaussian terms. The Weyl symbol of the operator $\mathcal{K}_1$ is shown to be a function of the Weyl symbol of the harmonic oscillator

$$a = f\Big(\xi^2 + \frac{v^2}{4}\Big).$$

Furthermore, its specific structure allows to deduce from the Mehler formula [**76**],

$$\exp -t\mathcal{H} = \mathrm{Op}^w\Big(\frac{\exp\big[-2\tanh(\frac{t}{2})(\xi^2+\frac{v^2}{4})\big]}{\cosh(\frac{t}{2})}\Big),$$

the operator-theoretical formula in Proposition 3.1. $\square$

The next proposition provides an operator-theoretical formula expressing the second part of the linearized non-cutoff Kac operator as a function of the spectral projections of the one-dimensional harmonic oscillator:

[2]Formula providing an explicit expression for the Fourier transform of the Boltzmann collision operator.

PROPOSITION 3.2. ([**KPS7**], Lerner, Morimoto, KPS, Xu) *The second part of the linearized non-cutoff Kac operator defined by*

$$\mathcal{K}_2 f = -\mu_1^{-1/2} K(\mu_1^{1/2} f, \mu_1),$$

is equal to

$$\mathcal{K}_2 = -\sum_{l=1}^{+\infty} \Big(\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \beta(\theta)(\sin\theta)^{2l} d\theta \Big) \mathbb{P}_{2l}.$$

Furthermore, there exist some positive constants $c_1, c_2 > 0$ such that

$$0 \le -\mathcal{K}_2 \le c_1 \exp -c_2 \mathcal{H}, \tag{3.24}$$

where $\mathcal{H}$ is the one-dimensional harmonic oscillator (3.21) *and $\mathbb{P}_k$ are the spectral projections onto the Hermite basis described in Appendix (Section 2).*

The L^1 singularity at 0 of the function β is erased by the factor $(\sin\theta)^{2l}$ which vanishes at order $2l \ge 2$. The operator $\mathcal{K}_2$, as well as $\mathcal{H}^N \mathcal{L}_2$ for any $N \in \mathbb{N}$, is a trace class operator on $L^2(\mathbb{R})$.

PROOF. The proof of Proposition 3.2 follows the same lines as the proof of Proposition 3.1 by using generalizations of the Mehler formula proven in [**122**]. ☐

As the first part of the linearized non-cutoff Kac operator, the second part $\mathcal{K}_2$ is also diagonal in the Hermite basis. We therefore obtain the following spectral decomposition of the linearized non-cutoff Kac operator:

PROPOSITION 3.3. ([**KPS7**], Lerner, Morimoto, KPS, Xu) *The linearized non-cutoff Kac operator defined by*

$$\mathcal{K} f = -\mu_1^{-1/2} K(\mu_1, \mu_1^{1/2} f) - \mu_1^{-1/2} K(\mu_1^{1/2} f, \mu_1),$$

is a non-negative unbounded operator on $L^2(\mathbb{R})$ with domain $\mathcal{D}$ defined in (3.23)*. It is diagonal in the Hermite basis*

$$\mathcal{K} = \sum_{k \ge 1} \lambda_k \mathbb{P}_k, \tag{3.25}$$

with a discrete spectrum only composed by the non-negative eigenvalues

$$\lambda_{2k+1} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \beta(\theta)(1 - (\cos\theta)^{2k+1}) d\theta \ge 0, \quad k \ge 0, \tag{3.26}$$

$$\lambda_{2k} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \beta(\theta)(1 - (\cos\theta)^{2k} - (\sin\theta)^{2k}) d\theta \ge 0, \quad k \ge 1, \tag{3.27}$$

satisfying to the asymptotic estimates

$$\lambda_k \approx k^s \quad \text{when } k \to +\infty. \tag{3.28}$$

The lowest eigenvalue zero corresponds to the fact that the Maxwellian distribution μ_1 is an equilibrium

$$\mathcal{K}\mu_1^{1/2} = -\mu_1^{-1/2}K(\mu_1,\mu_1) - \mu_1^{-1/2}K(\mu_1,\mu_1) = 0,$$

by conservation of the kinetic energy. We shall now relate these operator-theoretical properties to the phase space structure of the linearized non-cutoff Kac operator. To that end, we define for any $m \in \mathbb{R}$, the symbol classes $\mathbf{S}^m(\mathbb{R}^{2d})$ as the set of smooth functions $a(v,\xi)$ from $\mathbb{R}^d \times \mathbb{R}^d$ into $\mathbb{C}$ satisfying to the estimates

$$\forall(\alpha,\beta) \in \mathbb{N}^{2d}, \exists C_{\alpha\beta} > 0, \forall(v,\xi) \in \mathbb{R}^{2d}, \quad |\partial_v^\alpha \partial_\xi^\beta a(v,\xi)| \leq C_{\alpha,\beta}\langle(v,\xi)\rangle^{2m-|\alpha|-|\beta|}, \tag{3.29}$$

with $\langle(v,\xi)\rangle = \sqrt{1+|v|^2+|\xi|^2}$. We consider the Weyl quantization of symbols in the class $\mathbf{S}^m(\mathbb{R}^{2d})$,

$$a^w(v,D_v)u = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^{2d}} e^{i(v-y)\cdot\xi} a\Big(\frac{v+y}{2},\xi\Big)u(y)dyd\xi. \tag{3.30}$$

We notice in particular that the Weyl symbol of the d-dimensional harmonic oscillator

$$|\xi|^2 + \frac{|v|^2}{4} \in \mathbf{S}^1(\mathbb{R}^{2d}),$$

is a first order symbol in this symbolic calculus. The symbol class $\mathbf{S}^{-\infty}(\mathbb{R}^{2d})$ denotes the class $\cap_{m\in\mathbb{R}}\mathbf{S}^m(\mathbb{R}^{2d})$. We define the Sobolev space

$$\begin{aligned} B^m(\mathbb{R}^d) &= \{u \in L^2(\mathbb{R}^d),\ \mathcal{H}^m u \in L^2(\mathbb{R}^d)\} \\ &= \Big\{u \in L^2(\mathbb{R}^d),\ \sum_{k\geq 1} k^{2m}\|\mathbb{P}_k u\|_{L^2}^2 < +\infty\Big\}, \quad m \geq 0 \end{aligned} \tag{3.31}$$

and $B^{-m}(\mathbb{R}^d)$ as the dual space of $B^m(\mathbb{R}^d)$. It follows from the general theory of Sobolev spaces attached to a pseudodifferential calculus (see e.g. Section 2.6 in [**88**]) that

$$\forall m \in \mathbb{R}, \quad B^m(\mathbb{R}^d) = \{u \in \mathscr{S}'(\mathbb{R}^d) :\ \forall a \in \mathbf{S}^m(\mathbb{R}^{2d}),\ a^w u \in L^2(\mathbb{R}^d)\}.$$

For definiteness, we now make the following choice for the cross section

$$\beta(\theta) = \frac{|\cos\frac{\theta}{2}|}{|\sin\frac{\theta}{2}|^{1+2s}}, \quad |\theta| \leq \frac{\pi}{4}. \tag{3.32}$$

With that choice, we get a more precise equivalent than in Proposition 3.3,

$$\lambda_k \sim c_0 k^s \quad \text{when } k \to +\infty \text{ with } c_0 = \frac{2^{1+s}}{s}\Gamma(1-s), \tag{3.33}$$

where Γ stands for the Gamma function.

THEOREM 3.4. ([**KPS7**], Lerner, Morimoto, KPS, Xu) *Under the assumption* (3.32), *the linearized non-cutoff Kac operator*

$$\mathcal{K} = l^w(v,D_v),$$

is a pseudodifferential operator whose Weyl symbol $l(v,\xi)$ is real-valued, belongs to the symbol class $\mathbf{S}^s(\mathbb{R}^2)$ with the following asymptotic expansion: there exists a sequence of real numbers $(c_k)_{k\geq 1}$ such that

$$\forall N \geq 1, \quad l(v,\xi) \equiv c_0\Big(1+\xi^2+\frac{v^2}{4}\Big)^s - d_0 + \sum_{k=1}^{N} c_k\Big(1+\xi^2+\frac{v^2}{4}\Big)^{s-k} \quad \text{mod } \mathbf{S}^{s-N-1}(\mathbb{R}^2),$$

where the positive constant $c_0 > 0$ is defined in (3.33) *and*

$$d_0 = \frac{2^{1+s}(2+\sqrt{2})^s}{s} > 0.$$

Proof. These estimates are derived directly on the explicit expression of the Weyl symbol of the linearized non-cutoff Kac operator. $\square$

This result shows that the linearized non-cutoff Kac operator is a pseudodifferential operator whose principal symbol is the same as for the fractional harmonic oscillator

$$c_0\Big(1-\Delta_v+\frac{v^2}{4}\Big)^s.$$

According to standard results about the phase space structure of the powers of positive elliptic pseudodifferential operators (see e.g. Section 4.4 in [**64**]), we notice that the linearized non-cutoff Kac operator is equal to the fractional harmonic oscillator

$$c_0\Big(1-\Delta_v+\frac{v^2}{4}\Big)^s,$$

up to a bounded operator on $L^2(\mathbb{R})$. Let us underline that the fractional power $0<s<1$ of the harmonic oscillator only relates to structure of the singularity (3.10), whereas the different constants $d_0, (c_k)_{k\geq 0}$ appearing in the asymptotic expansion

$$l(v,\xi) \sim c_0\Big(1+\xi^2+\frac{v^2}{4}\Big)^s - d_0 + \sum_{k=1}^{+\infty} c_k\Big(1+\xi^2+\frac{v^2}{4}\Big)^{s-k}, \tag{3.34}$$

may be computed explicitly and depend directly on the exact expression chosen for the angular factor (3.32). This asymptotic expansion provides a complete description of the phase space structure of the linearized non-cutoff Kac operator. The two parts $\mathcal{K}_1$ and $\mathcal{K}_2$ account very differently in the way the linearized non-cutoff Kac operator acts. The first part $\mathcal{K}_1$ is a pseudodifferential operator whose Weyl symbol l_1 accounts for all the asymptotic expansion of the symbol l,

$$l_1(v,\xi) \sim c_0\Big(1+\xi^2+\frac{v^2}{4}\Big)^s - d_0 + \sum_{k=1}^{+\infty} c_k\Big(1+\xi^2+\frac{v^2}{4}\Big)^{s-k},$$

whereas the symbol of the operator $\mathcal{K}_2$ belongs to the symbol class $\mathbf{S}^{-\infty}(\mathbb{R}^2)$. This shows that $\mathcal{K}_2$ is a smoothing operator in any direction of the phase space

$$\|\langle v\rangle^{N_1}\mathcal{K}_2 f\|_{H^{N_2}(\mathbb{R})} \lesssim \|f\|_{L^2(\mathbb{R})},$$

for all $N_1, N_2 \in \mathbb{N}$, $f \in \mathscr{S}(\mathbb{R})$ and that $\mathcal{K}_2$ defines a compact operator on $L^2(\mathbb{R})$.

3. Spectral and phase space analysis of the linearized non-cutoff radially symmetric Boltzmann operator

We consider the linearized non-cutoff Boltzmann operator defined in (3.12) with Maxwellian molecules

$$\mathscr{L}f = -\mu_d^{-1/2} Q(\mu_d, \mu_d^{1/2} f) - \mu_d^{-1/2} Q(\mu_d^{1/2} f, \mu_d),$$

acting on the radially symmetric Schwartz space on $\mathbb{R}^d$ with $d \geq 2$,

$$\mathscr{S}_r(\mathbb{R}^d) = \big\{ f \in \mathscr{S}(\mathbb{R}^d) : \ \forall v \in \mathbb{R}^d, \forall A \in O(d), \ f(v) = f(Av) \big\} = \big\{ f(|v|) \big\}_{\substack{f \text{ even} \\ f \in \mathscr{S}(\mathbb{R})}}, \tag{3.35}$$

where $O(d)$ stands for the orthogonal group of $\mathbb{R}^d$. We recall that the case of Maxwellian molecules corresponds to the case when $\gamma = 0$ in the kinetic factor (3.5) and that the non-negative cross section $b(\cos\theta)$ is assumed to be supported where $\cos\theta \geq 0$ and to satisfy the assumption (3.6). We define the following function

$$\beta(\theta) = |\mathbb{S}^{d-2}| |\sin 2\theta|^{d-2} b(\cos 2\theta) \underset{\theta \to 0}{\approx} |\theta|^{-1-2s}. \tag{3.36}$$

The first result gives an operator-theoretical formula expressing the first part of the linearized non-cutoff radially symmetric Boltzmann operator as a function of the contraction semigroup generated by the d-dimensional harmonic oscillator:

Proposition 3.5. ([**KPS7**], Lerner, Morimoto, KPS, Xu) *When it acts on the function space $\mathscr{S}_r(\mathbb{R}^d)$, the first part of the linearized non-cutoff Boltzmann operator with Maxwellian molecules defined by*

$$\mathscr{L}_1 f = -\mu_d^{-1/2} Q(\mu_d, \mu_d^{1/2} f),$$

is equal to

$$\mathcal{L}_1 = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \beta(\theta) \Big[1 - (\sec\theta)^{\frac{d}{2}} \exp(-\mathcal{H} \ln(\sec\theta)) \Big] d\theta, \tag{3.37}$$

where β is the function defined in (3.36) *and $\mathcal{H} = -\Delta_v + \frac{|v|^2}{4}$ is the d-dimensional harmonic oscillator so that*

$$\mathcal{L}_1 = \sum_{k \geq 1} \Big(\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \beta(\theta)(1 - (\cos\theta)^k) d\theta \Big) \mathbb{P}_k, \tag{3.38}$$

where the projections $\mathbb{P}_k$ onto the Hermite basis are described in Appendix (Section 2).

Proof. Thanks to the similarity between the Bobylev formulas for the Kac operator and the radially symmetric Boltzmann operator, the proof of Proposition 3.5 follows from the same lines as Proposition 3.1. $\square$

As for the first part of the linearized non-cutoff Kac operator, the domain of the operator $\mathcal{L}_1$ can be taken as

$$\begin{aligned} \mathcal{D} &= \Big\{ u \in L^2(\mathbb{R}^d), \ \sum_{k \geq 0} k^{2s} \|\mathbb{P}_k u\|_{L^2}^2 < +\infty \Big\} \\ &= \{ u \in L^2(\mathbb{R}^d), \ \mathcal{H}^s u \in L^2(\mathbb{R}^d) \}. \end{aligned} \tag{3.39}$$

Similarly to the second part of the linearized non-cutoff Kac operator, the next proposition provides an operator-theoretical formula expressing the second part of the linearized non-cutoff radially symmetric Boltzmann operator as a function of the spectral projections of the harmonic oscillator.

PROPOSITION 3.6. ([**KPS7**], Lerner, Morimoto, KPS, Xu) *When it acts on the function space $\mathscr{S}_r(\mathbb{R}^d)$, the second part of the linearized non-cutoff Boltzmann operator with Maxwellian molecules defined by*

$$\mathscr{L}_2 f = -\mu_d^{-1/2} Q(\mu_d^{1/2} f, \mu_d),$$

is equal to

$$\mathcal{L}_2 = -\sum_{l \geq 1} \Big(\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \beta(\theta)(\sin\theta)^{2l} d\theta \Big) \mathbb{P}_{2l}, \tag{3.40}$$

where β is the function defined in (3.36) *and $\mathbb{P}_k$ are the spectral projections onto the Hermite basis described in Appendix (Section 2). Furthermore, there exist some positive constants $c_1, c_2 > 0$ such that*

$$0 \leq -\mathcal{L}_2 \leq c_1 \exp -c_2 \mathcal{H}, \tag{3.41}$$

where $\mathcal{H} = -\Delta_v + \frac{|v|^2}{4}$ is the d-dimensional harmonic oscillator.

PROOF. Thanks to the similarity between the Bobylev formulas for the Kac operator and the radially symmetric Boltzmann operator, the proof of Proposition 3.6 follows from the same lines as Proposition 3.2. □

Collecting the two previous results and using the fact that $\mathbb{P}_{2k+1} f = 0$ when $k \geq 0$, $f \in \mathscr{S}_r(\mathbb{R}^d)$, we recover in the radially symmetric case the spectral diagonalization obtained in [**129**] for the linearized Boltzmann operator:

COROLLARY 3.7. ([**KPS7**], Lerner, Morimoto, KPS, Xu) *When it acts on the function space $\mathscr{S}_r(\mathbb{R}^d)$, the linearized non-cutoff Boltzmann operator with Maxwellian molecules*

$$\mathscr{L} f = -\mu_d^{-1/2} Q(\mu_d, \mu_d^{1/2} f) - \mu_d^{-1/2} Q(\mu_d^{1/2} f, \mu_d),$$

is equal to

$$\mathcal{L} = \sum_{k \geq 1} \Big(\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \beta(\theta)(1 - (\sin\theta)^{2k} - (\cos\theta)^{2k}) d\theta \Big) \mathbb{P}_{2k},$$

where β is the function defined in (3.36) *and $\mathbb{P}_k$ are the spectral projections onto the Hermite basis described in Appendix (Section 2). Furthermore, the estimates*

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \beta(\theta)(1 - (\sin\theta)^{2k} - (\cos\theta)^{2k}) d\theta \approx k^s \quad \textit{when } k \to +\infty, \tag{3.42}$$

are satisfied and imply the following coercive estimates

$$\|\mathcal{H}^{\frac{s}{2}}(1 - \mathbf{P}) f\|_{L^2}^2 \lesssim (\mathscr{L} f, f)_{L^2} \lesssim \|\mathcal{H}^{\frac{s}{2}}(1 - \mathbf{P}) f\|_{L^2}^2, \tag{3.43}$$

for $f \in \mathscr{S}_r(\mathbb{R}^d)$, where $\mathcal{H} = -\Delta_v + \frac{|v|^2}{4}$ is the d-dimensional harmonic oscillator.

The results of Propositions 3.5, 3.6 and Corollary 3.7 (except for (3.42) and (3.43)) hold true as well for the cutoff case when β is integrable. For definiteness, we shall now make the following choice for the cross section

$$\beta(\theta) = |\mathbb{S}^{d-2}||\sin 2\theta|^{d-2}b(\cos 2\theta) = \frac{|\cos\frac{\theta}{2}|}{|\sin\frac{\theta}{2}|^{1+2s}}. \tag{3.44}$$

With that choice, we get as before a more precise equivalent than in Corollary 3.7

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \beta(\theta)(1-(\sin\theta)^{2k}-(\cos\theta)^{2k})d\theta \sim c_0(2k)^s,$$

when $k \to +\infty$, where the positive constant $c_0 > 0$ is defined in (3.33).

Theorem 3.8. ([**KPS7**], Lerner, Morimoto, KPS, Xu) *Under the assumption* (3.44)*, the linearized non-cutoff Boltzmann operator with Maxwellian molecules acting on the radially symmetric function space $\mathscr{S}_r(\mathbb{R}^d)$ is equal to a pseudodifferential operator*

$$\mathscr{L}f = l^w(v, D_v)f, \quad f \in \mathscr{S}_r(\mathbb{R}^d),$$

whose Weyl symbol $l(v,\xi)$ is real-valued, belongs to the symbol class $\mathbf{S}^s(\mathbb{R}^{2d})$ with the following asymptotic expansion: there exists a sequence of real numbers $(c_k)_{k\geq 1}$ such that $\forall N \geq 1$,

$$l(v,\xi) \equiv c_0\Big(1+|\xi|^2+\frac{|v|^2}{4}\Big)^s - d_0 + \sum_{k=1}^{N} c_k\Big(1+|\xi|^2+\frac{|v|^2}{4}\Big)^{s-k} \mod \mathbf{S}^{s-N-1}(\mathbb{R}^{2d}),$$

where $|\cdot|$ is the Euclidean norm, $c_0 > 0$ is the positive constant defined in (3.33) *and*

$$d_0 = \frac{2^{1+s}(2+\sqrt{2})^s}{s} > 0.$$

Proof. The operator $\mathcal{L} = l^w(v, D_v)$ defined in Corollary 3.7, is considered as a pseudodifferential operator and its Weyl symbol is computed explicitly. The estimates of Theorem 3.8 are then derived directly on the explicit expression of this symbol. □

This result shows that when acting on the function space $\mathscr{S}_r(\mathbb{R}^d)$, the linearized non-cutoff Boltzmann operator with Maxwellian molecules is a pseudodifferential operator whose principal symbol is the same as for the fractional harmonic oscillator

$$c_0\Big(1-\Delta_v+\frac{|v|^2}{4}\Big)^s.$$

For Maxwellian molecules, this accounts for the exact diffusive structure of the linearized non-cutoff radially symmetric Boltzmann operator and shows that this operator is equal to the fractional harmonic oscillator

$$c_0\Big(1-\Delta_v^2+\frac{|v|^2}{4}\Big)^s,$$

up to a bounded operator on $L^2(\mathbb{R}^d)$. The phase space structure of the linearized non-cutoff Boltzmann operator was first investigated in [**106, 107**], but these results were somehow controversial (see remarks in [**44, 79**]). In these works, the linearized non-cutoff Boltzmann operator with Maxwellian molecules satisfying the assumption (3.6) with $s = 1/4$, was shown to be a pseudodifferential operator whose symbol in the standard quantization satisfies to the following estimates

$$\exists c_1, c_2 > 0, \quad \text{Re } p(v,\xi) > c_1(|\xi|^2 + |v|^2)^{\frac{1}{4}} - c_2,$$

$$|p(v,\xi)| \lesssim \langle v\rangle^{\frac{1}{2}}\langle\xi\rangle^{\frac{1}{2}}, \quad \forall \alpha, \beta \in \mathbb{N}^3, \ |\alpha| + |\beta| \geq 1, \quad |\partial_v^\alpha \partial_\xi^\beta p(v,\xi)| \lesssim \langle(v,\xi)\rangle^{\frac{1}{2}}.$$

From a microlocal view point, these estimates are of a limited interest since the above estimates only point out that the symbol p belongs to a gainless symbol class without any asymptotic calculus. In the radially symmetric case, the situation is much more favorable since the Weyl symbol of the linearized non-cutoff Boltzmann operator with Maxwellian molecules belongs to $\mathbf{S}^s(\mathbb{R}^{2d})$ which is a standard symbol class enjoying nice symbolic calculus (see Lemma 2.2.18 in [**88**]). Indeed, the function space $\mathbf{S}^m(\mathbb{R}^{2d})$ which writes with Hörmander's convention as

$$S\Big(\langle(v,\xi)\rangle^{2m}, \frac{|dv|^2 + |d\xi|^2}{\langle(v,\xi)\rangle^2}\Big),$$

is a symbol class with gain $\lambda = \langle(v,\xi)\rangle^2$ in the symbolic calculus

$$\underbrace{a_1}_{\in S^{m_1}} \sharp^w \underbrace{a_2}_{\in S^{m_2}} = \underbrace{a_1 a_2}_{\in S^{m_1+m_2}} + \frac{1}{2i} \underbrace{\{a_1, a_2\}}_{\in S^{m_1+m_2-1}} + ...$$

As for the linearized non-cutoff Kac operator, the two operators $\mathcal{L}_1$ and $\mathcal{L}_2$ defined in Propositions 3.5, 3.6 account very differently in the way the operator $l^w(v, D_v)$ acts on functions. The first part $\mathcal{L}_1$ is a pseudodifferential operator whose Weyl symbol accounts for all the asymptotic expansion of the symbol l,

$$l_1(v,\xi) \sim c_0\Big(1 + |\xi|^2 + \frac{|v|^2}{4}\Big)^s - d_0 + \sum_{k=1}^{+\infty} c_k\Big(1 + |\xi|^2 + \frac{|v|^2}{4}\Big)^{s-k},$$

whereas the symbol of the operator $\mathcal{L}_2$ belongs to the class $S^{-\infty}(\mathbb{R}^{2d})$. This shows that $\mathcal{L}_2$ is a smoothing operator in any direction of the phase space

$$\|\langle v\rangle^{N_1}\mathcal{L}_2 f\|_{H^{N_2}(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mathbb{R}^d)},$$

for all $N_1, N_2 \in \mathbb{N}$, $f \in \mathscr{S}(\mathbb{R}^d)$ and that $\mathcal{L}_2$ defines a compact operator on $L^2(\mathbb{R}^d)$.

4. Phase space and functional calculus for the linearized Landau and Boltzmann operators

We consider the non-cutoff Boltzmann collision operator (3.2) whose cross section satisfies to the assumptions (3.4), (3.5) and (3.6). We recall that the physical motivation for considering this specific structure of cross sections is derived from particles interacting according to a spherical intermolecular repulsive potential of the form

$$\phi(\rho) = \rho^{-r}, \quad r > 1,$$

with ρ being the distance between two interacting particles. In the physical 3-dimensional space $\mathbb{R}^3$, the cross section satisfies the assumptions with $s = \frac{1}{r} \in]0,1[$ and $\gamma = 1 - 4s \in] - 3, 1[$. For Coulomb potential $r = 1$, i.e. $s = 1$, the Boltzmann operator is not well defined [**125**]. In this case, the Landau operator is substituted to the Boltzmann operator [**126**] in the equation (3.1). The Landau equation was first written by Landau in 1936 [**82**]. It is similar to the Boltzmann equation

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = Q_L(f, f), \\ f|_{t=0} = f_0, \end{cases} \tag{3.45}$$

with a different collision operator Q_L. Indeed, in the case of long-distance interactions, collisions occur mostly for grazing collisions. When all collisions become concentrated near $\theta = 0$, one obtains by the grazing collision limit asymptotic [**15, 18, 37, 40, 123**] the Landau collision operator

$$Q_L(g, f) = \nabla_v \cdot \Big(\int_{\mathbb{R}^d} a(v - v_*)(g(t, x, v_*)(\nabla_v f)(t, x, v) \\ - (\nabla_v g)(t, x, v_*) f(t, x, v)) dv_* \Big), \tag{3.46}$$

where $a = (a_{i,j})_{1 \leq i,j \leq d}$ stands for the non-negative symmetric matrix

$$a(v) = (|v|^2 \operatorname{Id} - v \otimes v)|v|^\gamma \in M_d(\mathbb{R}), \quad -d < \gamma < +\infty. \tag{3.47}$$

The Landau operator is understood as the limiting Boltzmann operator in the case when $s = 1$ in the singularity assumption (3.6). In the work [**KPS8**], we confirm this feature and prove that for Maxwellian molecules, the linearized non-cutoff Boltzmann operator is actually equal to the fractional linearized Landau operator with exponent exactly given by the singularity parameter $0 < s < 1$.

As for the Boltzmann operator, we consider the linearization of the Landau equation (3.45) by considering the fluctuation

$$f = \mu_d + \sqrt{\mu_d} g,$$

around the Maxwellian equilibrium distribution

$$\mu_d(v) = (2\pi)^{-\frac{d}{2}} e^{-\frac{|v|^2}{2}}.$$

Since $Q_L(\mu_d, \mu_d) = 0$, the collision operator $Q_L(f, f)$ can be split into three terms

$$Q_L(\mu_d + \sqrt{\mu_d} g, \mu_d + \sqrt{\mu_d} g) = Q_L(\mu_d, \sqrt{\mu_d} g) + Q_L(\sqrt{\mu_d} g, \mu_d) + Q_L(\sqrt{\mu_d} g, \sqrt{\mu_d} g),$$

whose linearized part is $Q_L(\mu_d, \sqrt{\mu_d} g) + Q_L(\sqrt{\mu_d} g, \mu_d)$. Setting

$$\mathscr{L}_L g = -\mu_d^{-1/2} Q_L(\mu_d, \mu_d^{1/2} g) - \mu_d^{-1/2} Q_L(\mu_d^{1/2} g, \mu_d),$$

the original Landau equation (3.45) is reduced to the Cauchy problem for the fluctuation

$$\begin{cases} \partial_t g + v \cdot \nabla_x g + \mathscr{L}_L g = \mu_d^{-1/2} Q_L(\sqrt{\mu_d} g, \sqrt{\mu_d} g), \\ g|_{t=0} = g_0. \end{cases} \tag{3.48}$$

As for the Boltzmann operator, the Landau collision operator is local in the time and position variables and from now on, we consider it as acting only in the velocity variable. The linearized operator $\mathscr{L}_L$ is known [**36, 61, 71**] to be an unbounded symmetric operator on $L^2(\mathbb{R}_v^d)$ (acting in the velocity variable) such that its Dirichlet form satisfies

$$(\mathscr{L}_L g, g)_{L^2(\mathbb{R}_v^d)} \geq 0.$$

We also have

$$(\mathscr{L}_L g, g)_{L^2(\mathbb{R}^d)} = 0 \Leftrightarrow g = \mathbf{P} g, \tag{3.49}$$

where $\mathbf{P}$ is the L^2-orthogonal projection onto the space of collisional invariants

$$\mathcal{N} = \text{Span}\{\mu_d^{1/2}, v_1\mu_d^{1/2}, ..., v_d\mu_d^{1/2}, |v|^2\mu_d^{1/2}\} = \text{Span}\Big\{\Psi_0, \Psi_{e_1}, ..., \Psi_{e_d}, \sum_{j=1}^{d}\Psi_{2e_j}\Big\},$$

where $(\Psi_\alpha)_{\alpha\in\mathbb{N}^d}$ stands for the orthonormal basis of $L^2(\mathbb{R}^d)$ composed by the eigenfunctions of the d-dimensional harmonic oscillator

$$\mathcal{H} = -\Delta_v + \frac{|v|^2}{4},$$

described in Appendix (Section 2).

The following result (probably well-known) provides an explicit expression for the linearized Landau operator with Maxwellian molecules:

Proposition 3.9. ([**KPS8**], Lerner, Morimoto, KPS, Xu) *The linearized Landau operator with Maxwellian molecules*

$$\mathscr{L}_L f = -\mu_d^{-1/2} Q_L(\mu_d, \sqrt{\mu_d}\, f) - \mu_d^{-1/2} Q_L(\sqrt{\mu_d}\, f, \mu_d),$$

acting on the Schwartz space $\mathscr{S}(\mathbb{R}^d)$ is equal to

$$\mathscr{L}_L = (d-1)\Big(-\Delta_v + \frac{|v|^2}{4} - \frac{d}{2}\Big) - \Delta_{\mathbb{S}^{d-1}} + \Big[\Delta_{\mathbb{S}^{d-1}} - (d-1)\Big(-\Delta_v + \frac{|v|^2}{4} - \frac{d}{2}\Big)\Big]\mathbb{P}_1 + \Big[-\Delta_{\mathbb{S}^{d-1}} - (d-1)\Big(-\Delta_v + \frac{|v|^2}{4} - \frac{d}{2}\Big)\Big]\mathbb{P}_2,$$

where $\Delta_{\mathbb{S}^{d-1}}$ stands for the Laplace-Beltrami operator on the unit sphere $\mathbb{S}^{d-1}$ and $\mathbb{P}_k$ the orthogonal projections onto the Hermite basis described in Appendix (Section 2).

Proof. Explicit computation of the linearized Landau operator with Maxwellian molecules. $\square$

The Laplace-Beltrami operator on the unit sphere $\mathbb{S}^{d-1}$ is a sum of squares of vector fields in $\mathbb{R}^d$ given by the differential operator

$$\Delta_{\mathbb{S}^{d-1}} = \frac{1}{2}\sum_{\substack{1\le j,k\le d\\ j\neq k}} (v_j\partial_k - v_k\partial_j)^2.$$

In the 3-dimensional case, the Laplace-Beltrami operator on the unit sphere $\mathbb{S}^2$ may be considered as a pseudodifferential operator

$$\Delta_{\mathbb{S}^2} f = (\text{Op}^w a) f = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^6} e^{i(v-y)\cdot\xi} a\Big(\frac{v+y}{2}, \xi\Big) f(y)\, dy\, d\xi,$$

whose Weyl symbol is the anisotropic symbol

$$a(v,\xi) = \frac{3}{2} - |v\wedge\xi|^2. \tag{3.50}$$

We now restrict our study to the three-dimensional setting $d = 3$. For

$$\sigma = (\cos\beta\sin\alpha, \sin\beta\sin\alpha, \cos\alpha) \in \mathbb{S}^2,$$

with $\alpha \in [0,\pi]$, $\beta \in [0,2\pi)$, the real spherical harmonics $Y_l^m(\sigma)$ with $l \in \mathbb{N}$, $-l \leq m \leq l$, are defined as $Y_0^0(\sigma) = (4\pi)^{-1/2}$ and for any $l \geq 1$,

$$Y_l^m(\sigma) = \begin{cases} \left(\frac{2l+1}{4\pi}\right)^{1/2} P_l(\cos\alpha) & \text{if } m = 0, \\ \left(\frac{2l+1}{2\pi}\frac{(l-m)!}{(l+m)!}\right)^{1/2} P_l^m(\cos\alpha)\cos m\beta & \text{if } m = 1, ..., l, \\ \left(\frac{2l+1}{2\pi}\frac{(l+m)!}{(l-m)!}\right)^{1/2} P_l^{-m}(\cos\alpha)\sin m\beta & \text{if } m = -l, ..., -1, \end{cases}$$

where P_l stands for the l-th Legendre polynomial and P_l^m the associated Legendre functions of the first kind of order l and degree m. The family $(Y_l^m)_{l\geq 0, -l\leq m\leq l}$ constitutes an orthonormal basis of the space $L^2(\mathbb{S}^2, d\sigma)$ with $d\sigma$ being the surface measure on $\mathbb{S}^2$. We consider for any $n, l \geq 0$, $-l \leq m \leq l$,

$$\varphi_{n,l,m}(v) = 2^{-3/4}\Big(\frac{2n!}{\Gamma(n+l+\frac{3}{2})}\Big)^{1/2}\Big(\frac{|v|}{\sqrt{2}}\Big)^l L_n^{[l+\frac{1}{2}]}\Big(\frac{|v|^2}{2}\Big)e^{-\frac{|v|^2}{4}}Y_l^m\Big(\frac{v}{|v|}\Big), \tag{3.51}$$

where $L_n^{[l+\frac{1}{2}]}$ are the generalized Laguerre polynomials. The family $(\varphi_{n,l,m})_{n,l\geq 0,|m|\leq l}$ is an orthonormal basis of $L^2(\mathbb{R}^3)$ composed by eigenvectors of the harmonic oscillator and the Laplace-Beltrami operator on the unit sphere $\mathbb{S}^2$,

$$\Big(-\Delta_v + \frac{|v|^2}{4} - \frac{3}{2}\Big)\varphi_{n,l,m} = (2n+l)\varphi_{n,l,m}, \quad -\Delta_{\mathbb{S}^2}\varphi_{n,l,m} = l(l+1)\varphi_{n,l,m}. \tag{3.52}$$

The space of collisional invariants may be expressed throughout this basis as

$$\mathcal{N} = \text{Span}\big\{\varphi_{0,0,0}, \varphi_{0,1,-1}, \varphi_{0,1,0}, \varphi_{0,1,1}, \varphi_{1,0,0}\big\}.$$

According to Proposition 3.9, the linearized Landau operator is diagonal in the $L^2(\mathbb{R}^3)$ orthonormal basis $(\varphi_{n,l,m})_{n,l\geq 0,|m|\leq l}$,

$$\mathcal{L}_L\varphi_{n,l,m} = \lambda_L(n,l,m)\varphi_{n,l,m}, \quad n,l \geq 0, \ -l \leq m \leq l, \tag{3.53}$$

where $\lambda_L(0,0,0) = \lambda_L(0,1,0) = \lambda_L(0,1,\pm 1) = \lambda_L(1,0,0) = 0$, $\lambda_L(0,2,m) = 12$, and for $2n+l > 2$,

$$\lambda_L(n,l,m) = 2(2n+l) + l(l+1). \tag{3.54}$$

The linearized non-cutoff Boltzmann operator with Maxwellian molecules (3.12) whose cross section satisfies to the assumptions (3.4), (3.5) and (3.6) with $\gamma = 0$, is also diagonal in the very same orthonormal basis $(\varphi_{n,l,m})_{n,l\geq 0,|m|\leq l}$. In the cutoff case i.e. when

$$b(\cos\theta)\sin\theta \in L^1([0,\pi/2]),$$

it was shown in [**129**] (see also [**19, 28, 47**]) that

$$\mathscr{L}\varphi_{n,l,m} = \lambda_B(n,l,m)\varphi_{n,l,m}, \quad n,l \geq 0, \ -l \leq m \leq l, \tag{3.55}$$

with

$$\begin{aligned} \lambda_B(n,l,m) = 4\pi \int_0^{\frac{\pi}{4}} & b(\cos 2\theta)\sin(2\theta) \\ & \times \big(1 + \delta_{n,0}\delta_{l,0} - P_l(\cos\theta)(\cos\theta)^{2n+l} - P_l(\sin\theta)(\sin\theta)^{2n+l}\big)d\theta, \end{aligned} \tag{3.56}$$

where P_l are the Legendre polynomials. These algebraic relations easily extend to the non-cutoff case and the diagonalization (3.55) holds true as well in this case. The eigenvalues $\lambda_B(n,l,m)$ are all non-negative and satisfy to

$$\lambda_L(n,l,m) = 0 \Leftrightarrow \lambda_B(n,l,m) = 0. \tag{3.57}$$

By using classical results on special functions [**83, 115**], we prove in [**KPS8**] (Theorem 2.2) that the eigenvalues of the linearized non-cutoff Boltzmann operator $\lambda_B(n,l,m)$ have the same growth as the fractional eigenvalues of the linearized Landau operator $\lambda_L(n,m,l)^s$: $\exists c_0>0$, $\forall n,l\geq 0$, $-l\leq m\leq l$,

$$\frac{1}{c_0}\lambda_L(n,l,m)^s\leq\lambda_B(n,l,m)\leq c_0\lambda_L(n,l,m)^s. \tag{3.58}$$

According to (3.54) and (3.56), the eigenvalues $\lambda_L(n,l,m)$ and $\lambda_B(n,l,m)$ only depend on the non-negative parameters $2n+l$, $l(l+1)$. We therefore deduce from these estimates the following result:

Theorem 3.10. ([**KPS8**], Lerner, Morimoto, KPS, Xu) *In the case of Maxwellian molecules $\gamma=0$, there exists*

$$A=a(\mathcal{H},\Delta_{\mathbb{S}^2}):L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3),$$

a positive bounded isomorphism defined by the functional calculus of the two commuting operators $\mathcal{H}$ and $\Delta_{\mathbb{S}^2}$,

$$\exists c>0,\forall f\in L^2(\mathbb{R}^3),\quad c\|f\|_{L^2}^2\leq\big(a(\mathcal{H},\Delta_{\mathbb{S}^2})f,f\big)_{L^2}\leq\frac{1}{c}\|f\|_{L^2}^2,$$

such that

$$\mathscr{L}=a(\mathcal{H},\Delta_{\mathbb{S}^2})\mathscr{L}_L^s.$$

By using that the Hermite functions are Schwartz functions, Proposition 3.9 and (3.50) shows that the Weyl symbol of linearized Landau operator

$$\mathscr{L}_L=l^w(v,D_v),$$

satisfies to

$$l(v,\xi)=2\Big(|\xi|^2+\frac{|v|^2}{4}-\frac{3}{2}\Big)+|v\wedge\xi|^2-\frac{3}{2}\ \operatorname{mod}\ \mathbf{S}^{-\infty}(\mathbb{R}^6). \tag{3.59}$$

According to (3.15), (3.49), (3.50) and (3.58), we obtain the following coercive estimates:

Theorem 3.11. ([**KPS8**], Lerner, Morimoto, KPS, Xu) *In the case of Maxwellian molecules $\gamma=0$, the linearized non-cutoff Boltzmann operator satisfies to the following coercive estimates for all $f\in\mathscr{S}(\mathbb{R}^3)$,*

$$(\mathscr{L}f,f)_{L^2}+\|f\|_{L^2}^2\sim\Big\|\Big(\mathrm{Op}^w\Big(|\xi|^2+\frac{|v|^2}{4}\Big)\Big)^{\frac{s}{2}}f\Big\|_{L^2}^2+\big\|\big(\mathrm{Op}^w(|\xi\wedge v|^2)\big)^{\frac{s}{2}}f\big\|_{L^2}^2$$

and

$$\begin{aligned}\|\mathcal{H}^{\frac{s}{2}}(1-\mathbf{P})f\|_{L^2}^2&+\|(-\Delta_{\mathbb{S}^2})^{\frac{s}{2}}(1-\mathbf{P})f\|_{L^2}^2\\&\lesssim(\mathscr{L}f,f)_{L^2}\lesssim\|\mathcal{H}^{\frac{s}{2}}(1-\mathbf{P})f\|_{L^2}^2+\|(-\Delta_{\mathbb{S}^2})^{\frac{s}{2}}(1-\mathbf{P})f\|_{L^2}^2.\end{aligned}$$

Here the two operators

$$\Big(\mathrm{Op}^w\Big(|\xi|^2+\frac{|v|^2}{4}\Big)\Big)^{\frac{s}{2}}\ \text{and}\ \big(\mathrm{Op}^w(|\xi\wedge v|^2)\big)^{\frac{s}{2}},$$

are defined through functional calculus.

We now consider the general three-dimensional case when the molecules are not necessarily Maxwellian, that is, when the parameter γ in the kinetic factor (3.5) may range over the interval $]-3,+\infty[$. In this case, the linearized non-cutoff Boltzmann operator satisfies to the following weighted coercive estimates:

THEOREM 3.12. ([**KPS8**], Lerner, Morimoto, KPS, Xu) *In the case of general molecules $\gamma \in]-3,+\infty[$, the linearized non-cutoff Boltzmann operator satisfies to the following coercive estimates for all $f \in \mathscr{S}(\mathbb{R}^3)$,*

$$(\mathscr{L}f,f)_{L^2} \sim \Big\| \Big(\mathrm{Op}^w\Big(|\xi|^2+\frac{|v|^2}{4}\Big)\Big)^{\frac{s}{2}} \langle v\rangle^{\frac{\gamma}{2}}(1-\mathbf{P})f\Big\|_{L^2}^2 + \big\|\big(\mathrm{Op}^w(|\xi\wedge v|^2)\big)^{\frac{s}{2}}\langle v\rangle^{\frac{\gamma}{2}}(1-\mathbf{P})f\big\|_{L^2}^2,$$

and

$$\begin{aligned}\|\mathcal{H}^{\frac{s}{2}}\langle v\rangle^{\frac{\gamma}{2}}(1-\mathbf{P})f\|_{L^2}^2 + \|(-\Delta_{\mathbb{S}^2})^{\frac{s}{2}}\langle v\rangle^{\frac{\gamma}{2}}(1-\mathbf{P})f\|_{L^2}^2 \\ \lesssim (\mathscr{L}f,f)_{L^2} \lesssim \|\mathcal{H}^{\frac{s}{2}}\langle v\rangle^{\frac{\gamma}{2}}(1-\mathbf{P})f\|_{L^2}^2 + \|(-\Delta_{\mathbb{S}^2})^{\frac{s}{2}}\langle v\rangle^{\frac{\gamma}{2}}(1-\mathbf{P})f\|_{L^2}^2.\end{aligned}$$

PROOF. These coercive estimates are a direct byproduct of the coercive estimates established in the Maxwellian case (Theorem 3.11) and the link between the Maxwellian and non-Maxwellian cases highlighted in [**10**]. It follows from the equivalence between the norm $|||\cdot|||_0$ in the Maxwellian case and the norm $|||\cdot|||_\gamma$ for general molecules $\gamma \in]-3,+\infty[$ proven in [**10**] (Proposition 2.4):

$$|||f|||_\gamma \sim |||\langle v\rangle^{\frac{\gamma}{2}} f|||_0.$$

□

5. Gelfand-Shilov regularizing properties of the Boltzmann and Landau equations

5.1. The spatially homogeneous non-cutoff Boltzmann equation. We consider the linearized non-cutoff Boltzmann operator with Maxwellian molecules

$$\mathscr{L}f = -\mu_3^{-1/2}Q(\mu_3,\mu_3^{1/2}f) - \mu_3^{-1/2}Q(\mu_3^{1/2}f,\mu_3),$$

acting on the radially symmetric Schwartz space

$$\mathscr{S}_r(\mathbb{R}^3) = \big\{f \in \mathscr{S}(\mathbb{R}^3),\ \forall v \in \mathbb{R}^3, \forall A \in O(3),\ f(v)=f(Av)\big\} = \big\{f(|v|)\big\}_{\substack{f \text{ even} \\ f\in\mathscr{S}(\mathbb{R})}}, \tag{3.60}$$

where $O(3)$ stands for the orthogonal group of $\mathbb{R}^3$. We recall that the case of Maxwellian molecules corresponds to the case when $\gamma=0$ in the kinetic factor (3.5) and that the non-negative cross section $b(\cos\theta)$ is assumed to be supported where $\cos\theta \geq 0$ and to satisfy the assumption (3.6).

When $f \in L^2(\mathbb{R}^3)$ is a radial function, we notice from (3.52) that the following scalar products

$$(f,\varphi_{n,l,m})_{L^2} = 0, \quad n,l \geq 0,\ -l \leq m \leq l,\ (l,m) \neq (0,0), \tag{3.61}$$

are zero, implying that $\mathbb{P}_{2n}f = (f,\varphi_{n,0,0})_{L^2}\varphi_{n,0,0}$. When acting on radial functions, the linearized non-cutoff Boltzmann operator with Maxwellian molecules is therefore given by (Corollary 3.7),

$$\mathscr{L}f = \sum_{n=1}^{+\infty} \lambda_{2n}(f,\varphi_{n,0,0})_{L^2}\varphi_{n,0,0}, \tag{3.62}$$

with

$$\lambda_2 = 0, \qquad \lambda_{2n} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \beta(\theta)(1-(\cos\theta)^{2n} - (\sin\theta)^{2n})d\theta \geq 0, \quad n \geq 2. \tag{3.63}$$

Following the Bobylev's theory [**19**], the Cauchy problem associated to the non-cutoff radially symmetric spatially homogeneous Boltzmann equation with Maxwellian molecules

$$\begin{cases} \partial_t g + \mathscr{L} g = \mu_3^{-1/2} Q(\sqrt{\mu_3} g, \sqrt{\mu_3} g), \\ g|_{t=0} = g_0, \end{cases}$$

may be solved explicitly for any small initial radial L^2-fluctuation around the standard Maxwellian distribution. In [**19**] (p. 215), Bobylev constructs explicit global radial solutions for initial radial L^2-fluctuations

$$f_0 = \mu_3 + \sqrt{\mu_3} g_0, \qquad g_0 = \sum_{n=2}^{+\infty} b_n(0)\varphi_{n,0,0},$$

satisfying

$$\sup_{n\geq 2} \sqrt[n]{|b_n(0)|\sqrt{\frac{\pi^{1/2}\ n!}{2\Gamma(n+\frac{3}{2})}}} < \frac{3}{7}, \tag{3.64}$$

and establishes the exponential return to equilibrium for the density distribution of the particles

$$f = \mu_3 + \sqrt{\mu_3} g,$$

in the $L^\infty(\mathbb{R}^3_v)$-norm

$$\exists C > 0, \forall t \geq 0, \quad \|f(t) - \mu_3\|_{L^\infty} \leq Ce^{-\lambda_4 t}.$$

In [**KPS9**], we do not request the specific structure (3.64) and perform the construction of explicit global radial solutions for any sufficiently small initial radial L^2-fluctuation. The main novelty in [**KPS9**] relates to the property of exponential convergence to zero for the fluctuation which is established in a specific weighted space emphasizing that the non-cutoff radially symmetric spatially homogeneous Boltzmann equation enjoys regularizing properties in the Gelfand-Shilov space $S^{1/2s}_{1/2s}(\mathbb{R}^3)$ for any positive time. We refer the reader to Appendix (Section 3) for the definition and the characterization of the Gelfand-Shilov regularity.

Regarding the smoothing features of the Boltzmann equation, the non-cutoff spatially homogeneous Boltzmann equation is known to enjoy a $\mathscr{S}(\mathbb{R}^d)$-regularizing effect for the weak solutions to the Cauchy problem [**46**]. Regarding the Gevrey regularity, Ukai showed in [**121**] that the Cauchy problem for the Boltzmann equation has a unique local solution in Gevrey classes. Then, Desvillettes, Furioli and Terraneo proved in [**42**] the propagation of Gevrey regularity for solutions of the Boltzmann equation with Maxwellian molecules. For mild singularities $\gamma + 2s < 1$, Morimoto and Ukai proved in [**95**] the $G^{1/2s}$-Gevrey regularity of smooth Maxwellian decay solutions to the Cauchy problem of the spatially homogeneous Boltzmann equation with a modified kinetic factor $\Phi(|v-v_*|) = \langle v - v_*\rangle^\gamma$. This result for mild singularities was recently extended by Zhang and Yin [**131**] for the standard kinetic factor $\Phi(|v-v_*|) = |v-v_*|^\gamma$. In [**96**], Morimoto, Ukai, Xu and Yang have established the property of $G^{1/s}$-Gevrey smoothing effect for the weak solutions to the Cauchy problem associated to the linearized spatially homogeneous Boltzmann equation with Maxwellian molecules when $0 < s < 1$. On the other hand, Lekrine and Xu proved in [**84**] the property of $G^{1/2s'}$-Gevrey smoothing effect for the weak solutions to the Cauchy problem associated to the radially symmetric spatially homogeneous Boltzmann equation with Maxwellian molecules for any $0 < s' < s$, when the singularity is mild $0 < s < 1/2$. This result was then completed by Glangetas and Najeme who established in [**56**] the analytic smoothing effect in the case when $1/2 < s < 1$.

Setting

$$\alpha_{2n,2m} = \sqrt{C_{2n+2m}^{2n}}\Big(\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \beta(\theta)(\sin\theta)^{2n}(\cos\theta)^{2m}d\theta\Big), \quad n \geq 1,\ m \geq 0, \tag{3.65}$$

$$\alpha_{0,2m} = -\Big(\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \beta(\theta)(1-(\cos\theta)^{2m})d\theta\Big), \quad m \geq 1, \quad \alpha_{0,0} = 0, \tag{3.66}$$

where $C_n^k = \frac{n!}{k!(n-k)!}$ stands for the binomials coefficients, we consider the infinite system of differential equations

$$\begin{cases} \partial_t b_0(t) = 0, \\ \forall n \geq 1, \quad \partial_t b_n(t) + \lambda_{2n} b_n(t) = \alpha_{0,2n} b_0(t) b_n(t) \\ \qquad + \displaystyle\sum_{\substack{k+l=n \\ k\geq 1, l\geq 0}} \alpha_{2k,2l}\sqrt{\frac{(2k+2l+1)}{(2k+1)(2l+1)}}\, b_k(t) b_l(t), \end{cases} \tag{3.67}$$

where λ_{2n} stands for the eigenvalue of the linearized radially symmetric Boltzmann operator (3.63). This system is triangular

$$\begin{cases} \forall t \geq 0, \quad b_0(t) = b_0(0), \\ \forall t \geq 0, \quad b_1(t) = b_1(0), \\ \forall n \geq 2, \forall t \geq 0, \quad \partial_t b_n(t) + \lambda_{2n}(1 + b_0(0)) b_n(t) \\ \qquad = \displaystyle\sum_{\substack{k+l=n \\ k\geq 1, l\geq 1}} \alpha_{2k,2l}\sqrt{\frac{(2k+2l+1)}{(2k+1)(2l+1)}}\, b_k(t) b_l(t), \end{cases} \tag{3.68}$$

since the $(n+1)^{\text{th}}$ equation is a linear differential equation for the function b_n with a right-hand-side involving only the functions $(b_k)_{1\leq k\leq n-1}$. This system may therefore be explicitly solved while solving a sequence of linear differential equations. This allows to solve explicitly the non-cutoff radially symmetric spatially homogeneous Boltzmann equation:

THEOREM 3.13. ([**KPS9**], Lerner, Morimoto, KPS, Xu) *Let $0 < \delta < 1$ be a positive constant. There exists a positive constant $\varepsilon_0 > 0$ such that if $g_0 \in \mathcal{N}^{\perp}$ is a radial $L^2(\mathbb{R}^3)$-function satisfying $\|g_0\|_{L^2} \leq \varepsilon_0$, then the Cauchy problem for the fluctuation associated to the non-cutoff spatially homogeneous Boltzmann equation with Maxwellian molecules*

$$\begin{cases} \partial_t g + \mathscr{L} g = \mu_3^{-1/2} Q(\sqrt{\mu_3} g, \sqrt{\mu_3} g), \\ g|_{t=0} = g_0, \end{cases}$$

has a unique global radial solution $g \in L^\infty(\mathbb{R}_t^+, L^2(\mathbb{R}_v^3))$ *given by*

$$g(t) = \sum_{n=0}^{+\infty} b_n(t)\varphi_{n,0,0},$$

where $(\varphi_{n,0,0})_{n\in\mathbb{N}}$ *are the functions defined in* (3.51) *and where the functions* $(b_n(t))_{n\geq 0}$ *are the solutions of the system of differential equations* (3.68) *with initial conditions*

$$\forall n \geq 0, \quad b_n(t)|_{t=0} = (g_0, \varphi_{n,0,0})_{L^2}.$$

Furthermore, this fluctuation around the Maxwellian distribution is exponentially convergent to zero in the following weighted L^2*-space*

$$\forall t \geq 0, \quad \|e^{\frac{t}{2}\mathscr{L}}g(t)\|_{L^2} = \Big(\sum_{n=0}^{+\infty} e^{\lambda_{2n}t}|b_n(t)|^2\Big)^{1/2} \leq e^{-\frac{\lambda_4}{2}(1-\delta)t}\|g_0\|_{L^2} \tag{3.69}$$

and belongs to the Gelfand-Shilov class $S_{1/2s}^{1/2s}(\mathbb{R}^3)$ *for any positive time*

$$\forall t > 0, \quad g(t) \in S_{1/2s}^{1/2s}(\mathbb{R}^3), \tag{3.70}$$

where $0 < s < 1$ *is the parameter appearing in the singularity assumption* (3.6).

This result emphasizes that the non-cutoff radially symmetric spatially homogeneous Boltzmann equation enjoys specific Gelfand-Shilov regularizing properties which depend directly on the value of the parameter $0 < s < 1$ in the singularity assumption (3.6). In particular, this result points out an ultra-analytic smoothing effect for the range of parameter $1/2 < s < 1$. The Gelfand-Shilov smoothing effect

$$\forall t > 0, \quad g(t) \in S_{1/2s}^{1/2s}(\mathbb{R}^3),$$

is a direct consequence of the a priori estimate (3.69) and the spectral asymptotics given in Corollary 3.7. This result is sharp. It can be checked by solving explicitly the triangular system of differential equations (3.68). Thanks to the non-resonance condition satisfied by the eigenvalues

$$\lambda_{2n} \leq \lambda_{2j_1} + \lambda_{2j_2} + ... + \lambda_{2j_k}, \quad j_1, ..., j_k \geq 2, \quad j_1 + j_2 + ... + j_k = n,$$

we notice that when the initial fluctuation is a radial function satisfying $g_0 \in \mathcal{N}^\perp$, i.e. $b_0(0) = b_1(0) = 0$, the non-linear effects do not appear before the component b_4,

$$\forall t \geq 0, \quad b_0(t) = b_1(t) = 0, \tag{3.71}$$

$$\forall t \geq 0, \quad b_2(t) = b_2(0)e^{-\lambda_4 t}, \quad b_3(t) = b_3(0)e^{-\lambda_6 t}, \tag{3.72}$$

$$\begin{aligned}\forall t \geq 0, \quad b_4(t) = \Big[b_4(0) - \frac{3}{5}\sqrt{C_8^4}\frac{b_2(0)^2}{\lambda_8 - 2\lambda_4}\Big(\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \beta(\theta)(\sin\theta)^4(\cos\theta)^4 d\theta\Big)\Big]e^{-\lambda_8 t} \\ + \frac{3}{5}\sqrt{C_8^4}\frac{b_2(0)^2}{\lambda_8 - 2\lambda_4}\Big(\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \beta(\theta)(\sin\theta)^4(\cos\theta)^4 d\theta\Big)e^{-2\lambda_4 t},\end{aligned}$$

and we establish by induction that for any $n \geq 1$, there exist some constants $\gamma_{j_1,j_2,\dots,j_k}$ such that

$$\forall t \geq 0, \quad b_n(t) = \sum_{1\leq k\leq n}\ \sum_{\substack{j_1+j_2+...+j_k=n \\ j_1,...,j_k\geq 2}} \gamma_{j_1,j_2,\dots,j_k} e^{-(\lambda_{2j_1}+\lambda_{2j_2}+...+\lambda_{2j_k})t}. \tag{3.73}$$

The choice of particular initial radial fluctuations allows to check explicitly that the index $1/2s$ for the symmetric Gelfand-Shilov regularity is sharp.

Proof. The Boltzmann collision operator is shown to enjoy noticeable algebraic identities:

(i) $\mu_3^{-1/2}Q(\mu_3^{1/2}\varphi_{0,0,0},\mu_3^{1/2}\varphi_{0,0,0})=0$

(ii) $\mu_3^{-1/2}Q(\mu_3^{1/2}\varphi_{0,0,0},\mu_3^{1/2}\varphi_{m,0,0})=-\Big(\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\beta(\theta)\big(1-(\cos\theta)^{2m}\big)d\theta\Big)\varphi_{m,0,0}$, $m\geq 1$

(iii) $\mu_3^{-1/2}Q(\mu_3^{1/2}\varphi_{n,0,0},\mu_3^{1/2}\varphi_{m,0,0})$
$=\sqrt{\dfrac{2n+2m+1}{(2n+1)(2m+1)}}\sqrt{C_{2n+2m}^{2n}}\Big(\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\beta(\theta)(\sin\theta)^{2n}(\cos\theta)^{2m}d\theta\Big)\varphi_{n+m,0,0}$, $n\geq 1,\ m\geq 0$

By using these algebraic identities, we deduce that the radial function

$$g(t)=\sum_{n=0}^{+\infty}b_n(t)\varphi_{n,0,0}, \tag{3.74}$$

is a solution to the non-cutoff spatially homogeneous Boltzmann equation with Maxwellian molecules

$$\begin{cases}\partial_t g+\mathscr{L}g=\mu_3^{-1/2}Q(\sqrt{\mu_3}g,\sqrt{\mu_3}g),\\ g|_{t=0}=g_0,\end{cases}$$

if and only if the functions $(b_n(t))_{n\geq 0}$ are the solutions to the infinite triangular system of differential equations (3.68) with initial conditions

$$\forall n\geq 0,\quad b_n(t)|_{t=0}=(g_0,\varphi_{n,0,0})_{L^2}.$$

The above algebraic identities are also used to derive sharp trilinear estimates for any given $f,g,h\in\mathscr{S}_r(\mathbb{R}^3)\cap\mathcal{N}^\perp$, $n\geq 2$, $t\geq 0$,

$$|(\mu_3^{-1/2}Q(\mu_3^{1/2}f,\mu_3^{1/2}g),e^{t\mathscr{L}}\mathbf{S}_nh)_{L^2}|\\ \leq C\|e^{\frac{t}{2}\mathscr{L}}\mathbf{S}_{n-2}f\|_{L^2}\|e^{\frac{t}{2}\mathscr{L}}\mathcal{H}^{\frac{s}{2}}\mathbf{S}_{n-2}g\|_{L^2}\|e^{\frac{t}{2}\mathscr{L}}\mathcal{H}^{\frac{s}{2}}\mathbf{S}_nh\|_{L^2},$$

where $\mathscr{L}$ is the linearized non-cutoff Boltzmann operator, $\mathcal{H}=-\Delta_v+\frac{|v|^2}{4}$ the 3-dimensional harmonic oscillator and $\mathbf{S}_n$ the orthogonal projector onto the $n+1$ lowest energy levels

$$\mathbf{S}_nf=\sum_{k=0}^{n}(f,\varphi_{k,0,0})_{L^2}\varphi_{k,0,0},\quad e^{\frac{t}{2}\mathscr{L}}\mathbf{S}_nf=\sum_{k=0}^{n}e^{\frac{1}{2}\lambda_{2k}t}(f,\varphi_{k,0,0})_{L^2}\varphi_{k,0,0}.$$

These trilinear estimates allow to establish the following a priori estimate

$$\frac{d}{dt}\|e^{\frac{t}{2}\mathscr{L}}(\mathbf{S}_Ng)(t)\|_{L^2}^2+\sum_{n=2}^{N}\lambda_{2n}e^{\lambda_{2n}t}|b_n(t)|^2\\ \leq C\|e^{\frac{t}{2}\mathscr{L}}(\mathbf{S}_{N-2}g)(t)\|_{L^2}\sum_{n=2}^{N}\lambda_{2n}e^{\lambda_{2n}t}|b_n(t)|^2, \tag{3.75}$$

implying that

$$\forall N \geq 0, \forall t \geq 0, \quad \|e^{\frac{t}{2}\mathscr{L}}(\mathbf{S}_N g)(t)\|_{L^2} \leq \varepsilon,$$

when $g_0 \in \mathcal{N}^\perp$, $\|g_0\|_{L^2} \leq \varepsilon \ll 1$. For small enough initial L^2-fluctuation around the standard Maxwellian distribution, the function (3.74) is then the unique global radial solution of the non-cutoff radially symmetric spatially homogeneous Boltzmann equation belonging to the space $L^\infty(\mathbb{R}_t^+, L^2(\mathbb{R}_v^3))$. Furthermore, this function is shown to satisfy the following weighted L^2-estimate

$$\forall t \geq 0, \quad \|e^{\frac{t}{2}\mathscr{L}} g(t)\|_{L^2} = \Big(\sum_{n=0}^{+\infty} e^{\lambda_{2n} t}|b_n(t)|^2\Big)^{1/2} \leq e^{-\frac{\lambda_4}{2}(1-\delta)t}\|g_0\|_{L^2},$$

implying its $S_{1/2s}^{1/2s}(\mathbb{R}^3)$ Gelfand-Shilov regularity for any positive time. □

5.2. The spatially homogeneous Landau equation. We consider the spatially homogeneous Landau equation with Maxwellian molecules

$$\begin{cases} \partial_t f = Q_L(f,f), \\ f|_{t=0} = f_0. \end{cases} \tag{3.76}$$

We recall that this corresponds to the case when the parameter $\gamma = 0$ in the definition of the function (3.47) defining the Landau collision operator (3.46). At least formally, it is easily checked that the mass, the momentum and the kinetic energy are conserved quantities by this evolution equation

$$\int_{\mathbb{R}^d} f(t,v)dv = M, \quad \int_{\mathbb{R}^d} f(t,v)vdv = MV, \quad \frac{1}{2}\int_{\mathbb{R}^d} f(t,v)|v|^2dv = E, \tag{3.77}$$

when $t \geq 0$, with $M > 0$, $V \in \mathbb{R}^d$, $E > 0$. The Cauchy problem (3.76) associated to the spatially homogeneous Landau equation with Maxwellian molecules and some quantitative features of the solutions were thoroughly studied by Villani in [**124**]. The Propositions 4 and 6 in [**124**] show that, for each non-negative measurable initial density distribution f_0 having finite mass and finite energy

$$f_0 \geq 0, \quad 0 < \int_{\mathbb{R}^d} f_0(v)dv = M < +\infty, \quad 0 < \frac{1}{2}\int_{\mathbb{R}^d} f_0(v)|v|^2dv = E < +\infty, \tag{3.78}$$

the Cauchy problem (3.76) admits a unique global classical solution $f(t,v)$ defined for all $t \geq 0$. Furthermore, this solution is shown to be a non-negative bounded smooth function

$$f(t) \geq 0, \quad f(t) \in L^\infty(\mathbb{R}_v^d) \cap C^\infty(\mathbb{R}_v^d),$$

for any positive time $t > 0$. As in Section 4, we consider a close-to-equilibrium framework

$$f = \mu_d + \sqrt{\mu_d}g,$$

around the Maxwellian equilibrium distribution and we study the Cauchy problem for the fluctuation

$$\begin{cases} \partial_t g + \mathscr{L}_L g = \mu_d^{-1/2} Q_L(\sqrt{\mu_d}g, \sqrt{\mu_d}g), \\ g|_{t=0} = g_0. \end{cases} \tag{3.79}$$

By elaborating on the solutions constructed by Villani in [**124**], we aim at studying the Gelfand-Shilov regularizing properties of the Cauchy problem (3.79). For the sake of simplicity, we may assume without loss of generality that the density distribution satisfies (3.77) with $V = 0$. Furthermore, by changing the unknown function f to $\tilde{f}$ as

$$f = \frac{M}{\alpha^d}\tilde{f}\Big(\frac{\cdot}{\alpha}\Big), \qquad \alpha = \sqrt{\frac{2E}{Md}}, \tag{3.80}$$

we may reduce the study to the case when

$$\int_{\mathbb{R}^d} f(t,v)dv = 1, \quad \int_{\mathbb{R}^d} f(t,v)v dv = 0, \quad \int_{\mathbb{R}^d} f(t,v)|v|^2 dv = d, \quad t \geq 0. \tag{3.81}$$

Let $f_0 = \mu_d + \sqrt{\mu_d}g_0 \geq 0$, with $g_0 \in L^1(\mathbb{R}_v^d) \cap L^2(\mathbb{R}_v^d)$, be a non-negative initial density distribution having finite mass and finite energy such that

$$\int_{\mathbb{R}^d} f_0(v)dv = 1, \quad \int_{\mathbb{R}^d} f_0(v)v dv = 0, \quad \int_{\mathbb{R}^d} f_0(v)|v|^2 dv = d. \tag{3.82}$$

Such an initial density distribution is rapidly decreasing with a finite temperature tail

$$\frac{1}{2} \leq \frac{1}{T(f_0)} = \sup\Big\{\beta \geq 0 : \int_{\mathbb{R}^d} f_0(v)e^{\beta\frac{|v|^2}{2}}dv < +\infty\Big\},$$

since

$$\int_{\mathbb{R}^d} f_0(v)e^{\frac{|v|^2}{4}}dv = \frac{1}{(2\pi)^{\frac{d}{4}}}\int_{\mathbb{R}^d}(\sqrt{\mu_d(v)} + g_0(v))dv < +\infty, \tag{3.83}$$

when $g_0 \in L^1(\mathbb{R}_v^d)$. The analysis of the evolution of the temperature tail led in [**124**] (Section 6, p. 972-974) shows that

$$\int_{\mathbb{R}^d} f_0(v)e^{\frac{|v|^2}{4}}dv < +\infty \Rightarrow \forall t > 0, \quad \int_{\mathbb{R}^d} f(t,v)e^{\frac{|v|^2}{4}}dv < +\infty.$$

This implies that the fluctuation $f = \mu_d + \sqrt{\mu_d}g \geq 0$, around the Maxwellian equilibrium distribution defined by

$$g(t) = \mu_d^{-1/2}(f(t) - \mu_d) \in L^1(\mathbb{R}_v^d) \cap C^\infty(\mathbb{R}_v^d) \subset \mathscr{S}'(\mathbb{R}_v^d), \quad t > 0, \tag{3.84}$$

belongs to $L^1(\mathbb{R}_v^d)$ and therefore remains a tempered distribution for all $t > 0$. The following result shows that the Cauchy problem (3.79) enjoys the same Gelfand-Shilov regularizing properties as the Cauchy problem defined by the evolution equation associated to the harmonic oscillator:

THEOREM 3.14. ([**KPS11**], Morimoto, KPS, Xu) *Let $f_0 = \mu_d + \sqrt{\mu_d}g_0 \geq 0$, with $g_0 \in L^1(\mathbb{R}_v^d) \cap L^2(\mathbb{R}_v^d)$, be a non-negative measurable function having finite mass and finite energy such that*

$$\int_{\mathbb{R}^d} f_0(v)dv = 1, \quad \int_{\mathbb{R}^d} f_0(v)v dv = 0, \quad \int_{\mathbb{R}^d} f_0(v)|v|^2 dv = d. \tag{3.85}$$

Let $f(t) = \mu_d + \sqrt{\mu_d}g(t)$, with $g(t) \in L^1(\mathbb{R}_v^d) \cap C^\infty(\mathbb{R}_v^d)$ when $t > 0$, be the unique global classical solution of the Cauchy problem associated to the spatially homogeneous Landau equation with Maxwellian molecules

$$\begin{cases} \partial_t f = Q_L(f,f), \\ f|_{t=0} = f_0, \end{cases}$$

constructed by Villani in [**124**]. *Then, there exists a positive constant* $\delta > 0$ *such that*

$$\exists C > 0, \forall t \geq 0,\ \|e^{t\delta\mathcal{H}}g(t)\|_{L^2} = \Big(\sum_{k\geq 0} e^{\delta(2k+d)t}\|\mathbb{P}_k g(t)\|_{L^2}^2\Big)^{1/2} \leq Ce^{d(d-1)t}(\|g_0\|_{L^2}+1),$$

with $\mathcal{H} = -\Delta_v + \frac{|v|^2}{4}$, *where* $\|\cdot\|_{L^2}$ *stands for the* $L^2(\mathbb{R}^d_v)$*-norm and* $\mathbb{P}_k$ *are the orthogonal projections onto the Hermite basis defined in Appendix (Section 2). In particular, this implies that the fluctuation belongs to the Gelfand-Shilov space* $S^{1/2}_{1/2}(\mathbb{R}^d)$ *for any positive time*

$$\forall t > 0, \quad g(t) \in S^{1/2}_{1/2}(\mathbb{R}^d).$$

Remark. The orthogonal projection $\mathbb{P}_k : \mathscr{S}'(\mathbb{R}^d_v) \to \mathscr{S}(\mathbb{R}^d_v)$ is well-defined on tempered distributions since the Hermite functions are Schwartz functions.

This result shows that the Cauchy problem (3.79) enjoys an ultra-analytic regularizing effect in the Gevrey class $G^{1/2}(\mathbb{R}^d)$ both for the fluctuation and its Fourier transform in the velocity variable for any positive time

$$g(t),\ \widehat{g}(t) \in G^{1/2}(\mathbb{R}^d), \quad t > 0.$$

It is consistent with the link between the linearized Boltzmann and Landau operators unveiled by Theorem 3.10 and the result of Gelfand-Shilov regularizing smoothing effect proven for the radially symmetric spatially homogeneous Boltzmann equation (Theorem 3.13). The existence, the uniqueness, the Sobolev regularity and the polynomial decay of the weak solutions to the Cauchy problem (3.76) have been studied by Desvillettes and Villani in [**45**] (Theorem 6) for hard potentials, that is, when the parameter satisfies $0 < \gamma \leq 1$ in the assumption (3.47). Under rather weak assumptions on the initial datum, e.g. $f_0 \in L^1_{2+\delta}$, with $\delta > 0$, they prove that there exists a weak solution to the Cauchy problem such that $f \in C^\infty([t_0, +\infty[, \mathscr{S}(\mathbb{R}^d_v))$, for all $t_0 > 0$, and for all $t_0 > 0$, $s > 0$, $m \in \mathbb{N}$,

$$\sup_{t\geq t_0} \|f(t,\cdot)\|_{H^m_s} < +\infty.$$

The Gevrey regularity $f(t,\cdot) \in G^\sigma$, for any $\sigma > 1$, for all positive time $t > 0$ of the solution to the Cauchy problem (3.76) with an initial datum f_0 with finite mass, energy and entropy satisfying

$$\forall t_0 > 0,\ m \geq 0, \quad \sup_{t\geq t_0} \|f(t,\cdot)\|_{H^m_\gamma} < +\infty,$$

was later established by Chen, Li and Xu in [**30**] for the hard potential case and the Maxwellian molecules case. Under the same assumptions on the solution, this result was later extended to analytic regularity [**32**]:

$$\forall t_0 > 0, \exists c_0, C > 0, \forall t \geq t_0, \quad \|e^{c_0(-\Delta_v)^{1/2}} f(t,\cdot)\|_{L^2} \leq C(t+1),$$

in the hard potential case and the Maxwellian molecules case. Regarding specifically the Maxwellian molecules case $\gamma = 0$, Morimoto and Xu established in [**98**] (Theorem 1.1) the ultra-analyticity

$$\forall\ 0 < t < T, \quad f(t,\cdot) \in G^{1/2}(\mathbb{R}^d),$$

$$\forall\ 0 < T_0 < T, \exists c_0 > 0, \forall\ 0 < t \leq T_0, \quad \|e^{-c_0 t\Delta_v} f(t,\cdot)\|_{L^2} \leq e^{\frac{d}{2}t}\|f_0\|_{L^2},$$

of any positive weak solution $f(t,x) > 0$ to the Cauchy problem (3.76) satisfying
$$f \in L^\infty(]0,T[, L^2(\mathbb{R}^d) \cap L^1_2(\mathbb{R}^d)),$$
with $0 < T \leq +\infty$, with an initial datum satisfying $f_0 \in L^2(\mathbb{R}^d) \cap L^1_2(\mathbb{R}^d)$. The result of Theorem 3.14 allows to specify further the property of ultra-analytic smoothing proven by Morimoto and Xu [**98**] in the close-to-equilibrium framework. This result points out the specific decay of the fluctuation both in the velocity and its dual Fourier variable. As for the Boltzmann equation, the Gelfand-Shilov regularity seems relevant to describe the regularizing properties of the Landau equation in the close-to-equilibrium framework.

6. Comments and perspectives

There are still many open questions on the regularizing properties of the non-cutoff Boltzmann equation. We aim in future works at studying whether this Gelfand-Shilov smoothing effect proven for the radially symmetric spatially homogeneous non-cutoff Boltzmann equation with Maxwellian molecules still holds in the non-radially symmetric case, or more generally for non-Maxwellian molecules. It would be also most interesting to study the possible Gevrey (ultra-analytic) smoothing effect in the position variable for the spatially inhomogeneous non-cutoff Boltzmann equation.

CHAPTER 4

Appendix

1. Wick calculus

We refer the reader to the works [**85, 87, 88**] for comprehensive expositions of the Wick calculus. The main property of the Wick quantization is its property of positivity, i.e., that non-negative Hamiltonians get quantized in non-negative operators

$$a \geq 0 \Rightarrow a^{\mathrm{Wick}} \geq 0.$$

This is not the case for the Weyl, nor the standard quantization[1]. The wave packets transform of a Schwartz function $u \in \mathscr{S}(\mathbb{R}^n)$ is defined as

$$Wu(y,\eta) = (u, \varphi_{y,\eta})_{L^2(\mathbb{R}^n)} = 2^{n/4} \int_{\mathbb{R}^n} u(x) e^{-\pi|x-y|^2} e^{-2i\pi(x-y)\cdot\eta} dx, \quad (y,\eta) \in \mathbb{R}^{2n},$$

with

$$\varphi_{y,\eta}(x) = 2^{n/4} e^{-\pi|x-y|^2} e^{2i\pi(x-y)\cdot\eta}, \quad x \in \mathbb{R}^n.$$

The mapping $u \mapsto Wu$ is continuous from $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}(\mathbb{R}^{2n})$ and isometric from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^{2n})$ (not onto). The following reconstruction formula holds

$$\forall u \in \mathscr{S}(\mathbb{R}^n), \forall x \in \mathbb{R}^n, \quad u(x) = \int_{\mathbb{R}^{2n}} Wu(y,\eta)\varphi_{y,\eta}(x) dy d\eta. \tag{4.1}$$

The Wick quantization of an Hamiltonian a is defined as

$$a^{\mathrm{Wick}} = W^* a W, \tag{4.2}$$

according to the commutative diagram

$$\begin{array}{ccc} L^2(\mathbb{R}^{2n}) & \xrightarrow[\text{(multiplication by } a)]{a} & L^2(\mathbb{R}^{2n}) \\ W \big\uparrow & & \big\downarrow W^* \\ L^2(\mathbb{R}^n) & \xrightarrow[a^{\mathrm{Wick}}]{} & L^2(\mathbb{R}^n) \end{array}$$

where W^* stands for the adjoint of the wave packets transform. The Wick quantization is a positive quantization

$$a \geq 0 \Rightarrow a^{\mathrm{Wick}} \geq 0. \tag{4.3}$$

In particular, real Hamiltonians get quantized by formally self-adjoint operators and $L^\infty(\mathbb{R}^{2n})$ symbols define bounded operators on $L^2(\mathbb{R}^n)$,

$$\|a^{\mathrm{Wick}}\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq \|a\|_{L^\infty(\mathbb{R}^{2n})}. \tag{4.4}$$

[1]An example of a non-negative Hamiltonian getting quantized in the Weyl quantization in an operator failing non-negativity is given in [**85**].

Furthermore, the Wick and Weyl quantizations of a same symbol are linked by the following identities

$$a^{\mathrm{Wick}} = \tilde{a}^w = a^w + r(a)^w, \tag{4.5}$$

with

$$\tilde{a}(X) = \int_{\mathbb{R}^{2n}} a(X+Y)e^{-2\pi|Y|^2}2^n dY, \quad X \in \mathbb{R}^{2n}, \tag{4.6}$$

and

$$r(a)(X) = \int_0^1 \int_{\mathbb{R}^{2n}} (1-\theta)a''(X+\theta Y)Y^2 e^{-2\pi|Y|^2}2^n dY d\theta, \quad X \in \mathbb{R}^{2n}. \tag{4.7}$$

The normalization chosen here for the Weyl quantization is

$$(a^w u)(x) = \int_{\mathbb{R}^{2n}} e^{2i\pi(x-y)\cdot\xi} a\Big(\frac{x+y}{2}, \xi\Big)u(y)dyd\xi. \tag{4.8}$$

Furthermore, we have the following composition formula

$$a^{\mathrm{Wick}}b^{\mathrm{Wick}} = \Big[ab - \frac{1}{4\pi}\nabla a \cdot \nabla b + \frac{1}{4i\pi}\{a,b\}\Big]^{\mathrm{Wick}} + S, \tag{4.9}$$

with $\|S\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq c_n \|a\|_{L^\infty}\gamma_2(b)$, when $a \in L^\infty(\mathbb{R}^{2n})$ and b is a smooth symbol satisfying

$$\gamma_2(b) = \sup_{\substack{X \in \mathbb{R}^{2n}, \\ T \in \mathbb{R}^{2n}, |T|=1}} |b''(X)T^2| < +\infty.$$

The positive constant $c_n > 0$ only depends on the dimension and the notation $\{a,b\}$ denotes the Poisson bracket

$$\{a,b\} = \frac{\partial a}{\partial \xi} \cdot \frac{\partial b}{\partial x} - \frac{\partial a}{\partial x} \cdot \frac{\partial b}{\partial \xi}.$$

2. The harmonic oscillator

The standard Hermite functions $(\phi_n)_{n\in\mathbb{N}}$ are defined for $x \in \mathbb{R}$,

$$\phi_n(x) = \frac{(-1)^n}{\sqrt{2^n n!\sqrt{\pi}}} e^{\frac{x^2}{2}} \frac{d^n}{dx^n}(e^{-x^2}) = \frac{1}{\sqrt{2^n n!\sqrt{\pi}}}\Big(x - \frac{d}{dx}\Big)^n (e^{-\frac{x^2}{2}}) = \frac{a_+^n \phi_0}{\sqrt{n!}},$$

where a_+ is the creation operator

$$a_+ = \frac{1}{\sqrt{2}}\Big(x - \frac{d}{dx}\Big).$$

The family $(\phi_n)_{n\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{R})$. We set for $n \in \mathbb{N}$, $\alpha = (\alpha_j)_{1\leq j\leq d} \in \mathbb{N}^d$, $x \in \mathbb{R}$, $v \in \mathbb{R}^d$,

$$\psi_n(x) = 2^{-1/4}\phi_n(2^{-1/2}x) = \frac{1}{\sqrt{n!}}\Big(\frac{x}{2} - \frac{d}{dx}\Big)^n \psi_0,$$

$$\Psi_\alpha(v) = \prod_{j=1}^d \psi_{\alpha_j}(v_j), \quad \mathcal{E}_k = \mathrm{Span}\{\Psi_\alpha\}_{\alpha\in\mathbb{N}^d, |\alpha|=k},$$

with $|\alpha| = \alpha_1 + \cdots + \alpha_d$. The family $(\Psi_\alpha)_{\alpha\in\mathbb{N}^d}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$ composed by the eigenfunctions of the d-dimensional harmonic oscillator

$$\mathcal{H} = -\Delta_v + \frac{|v|^2}{4} = \sum_{k\geq 0}\Big(\frac{d}{2} + k\Big)\mathbb{P}_k, \quad \mathrm{Id} = \sum_{k\geq 0}\mathbb{P}_k, \tag{4.10}$$

where $\mathbb{P}_k$ is the orthogonal projection onto the vector subspace $\mathcal{E}_k$ whose dimension is $\binom{k+d-1}{d-1}$. The eigenvalue $d/2$ is simple in all dimensions and $\mathcal{E}_0$ is generated by the function

$$\Psi_0(v) = \frac{1}{(2\pi)^{\frac{d}{4}}} e^{-\frac{|v|^2}{4}} = \mu_d^{1/2}(v),$$

where μ_d is the Maxwellian distribution

$$\mu_d(v) = (2\pi)^{-\frac{d}{2}} e^{-\frac{|v|^2}{2}}, \quad v \in \mathbb{R}^d.$$

3. Gelfand-Shilov regularity

We refer the reader to the works [**55, 57, 102, 116**] and the references herein for extensive expositions of the Gelfand-Shilov regularity. The Gelfand-Shilov spaces $S_\nu^\mu(\mathbb{R}^d)$, with $\mu, \nu > 0$, $\mu + \nu \geq 1$, are defined as the spaces of smooth functions $f \in C^\infty(\mathbb{R}^d)$ satisfying to the estimates

$$\exists A, C > 0, \quad |\partial_v^\alpha f(v)| \leq C A^{|\alpha|} (\alpha!)^\mu e^{-\frac{1}{A}|v|^{1/\nu}}, \quad v \in \mathbb{R}^d,\ \alpha \in \mathbb{N}^d,$$

or, equivalently

$$\exists A, C > 0, \quad \sup_{v \in \mathbb{R}^d} |v^\beta \partial_v^\alpha f(v)| \leq C A^{|\alpha|+|\beta|} (\alpha!)^\mu (\beta!)^\nu, \quad \alpha, \beta \in \mathbb{N}^d.$$

These Gelfand-Shilov spaces $S_\nu^\mu(\mathbb{R}^d)$ may also be characterized as the spaces of Schwartz functions $f \in \mathscr{S}(\mathbb{R}^d)$ satisfying to the estimates

$$\exists C > 0, \varepsilon > 0, \quad |f(v)| \leq C e^{-\varepsilon |v|^{1/\nu}}, \quad v \in \mathbb{R}^d, \qquad |\widehat{f}(\xi)| \leq C e^{-\varepsilon |\xi|^{1/\mu}}, \quad \xi \in \mathbb{R}^d.$$

In particular, we notice that Hermite functions belong to the symmetric Gelfand-Shilov space $S_{1/2}^{1/2}(\mathbb{R}^d)$. More generally, the symmetric Gelfand-Shilov spaces $S_\mu^\mu(\mathbb{R}^d)$, with $\mu \geq 1/2$, can be nicely characterized through the decomposition into the Hermite basis $(\Psi_\alpha)_{\alpha \in \mathbb{N}^d}$, see e.g. [**116**] (Proposition 1.2),

$$\begin{aligned} (4.11) \quad f \in S_\mu^\mu(\mathbb{R}^d) &\Leftrightarrow f \in L^2(\mathbb{R}^d),\ \exists t_0 > 0,\ \|e^{t_0 \mathcal{H}^{1/2\mu}} f\|_{L^2} < +\infty \\ &\Leftrightarrow f \in L^2(\mathbb{R}^d),\ \exists t_0 > 0,\ \big\|\big((f, \Psi_\alpha)_{L^2} \exp(t_0 |\alpha|^{\frac{1}{2\mu}})\big)_{\alpha \in \mathbb{N}^d}\big\|_{l^2(\mathbb{N}^d)} < +\infty, \end{aligned}$$

where $(\Psi_\alpha)_{\alpha \in \mathbb{N}^d}$ stands for the Hermite basis defined in the previous section and where

$$\mathcal{H} = -\Delta_v + \frac{|v|^2}{4},$$

is the d-dimensional harmonic oscillator. The Cauchy problem defined by the evolution equation associated to the harmonic oscillator

$$(4.12) \qquad \begin{cases} \partial_t f + \mathcal{H} f = 0, \\ f|_{t=0} = f_0 \in L^2(\mathbb{R}^d), \end{cases}$$

enjoys nice regularizing properties. The smoothing effect for the solutions to this Cauchy problem is naturally described in term of the Gelfand-Shilov regularity. The characterization (4.11) proves that there is a regularizing effect for the solutions to the Cauchy problem (4.12) in the symmetric Gelfand-Shilov space $S_{1/2}^{1/2}(\mathbb{R}^d)$ for any positive time, whereas the smoothing effect for the solutions to the Cauchy problem defined by the evolution equation associated to the fractional harmonic oscillator

$$\begin{cases} \partial_t f + \mathcal{H}^s f = 0, \\ f|_{t=0} = f_0 \in L^2(\mathbb{R}^d), \end{cases} \tag{4.13}$$

with $0 < s < 1$, occurs for any positive time in the symmetric Gelfand-Shilov space $S^{1/2s}_{1/2s}(\mathbb{R}^d)$.

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