Brownian motion on the space of univalent functions via Brownian motion on $\mathrm{Diff}(\mathbb{S}^1)$

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d'après Airault, Fang, Malliavin, Thalmaier etc.

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1. Some probabilistic and algebraic motivations
   - Brownian motion on some quotient spaces
   - Representations of the Virasoro algebra

2. Brownian motion on the diffeomorphism group of the circle
   - Canonical horizontal diffusion
   - Construction via regularization
   - An alternative pointwise approach

3. Brownian motion on the space of univalent functions
   - Beurling-Ahlfors extension
   - Stochastic conformal welding
   - Some properties of the resulting process

Notations and preliminaries :

a drop of complex analysis

The space of Jordan curves

Let consider

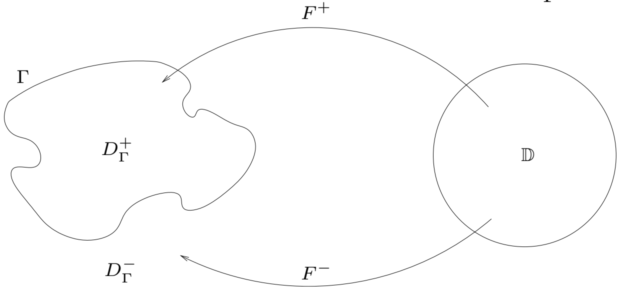
$$\begin{aligned}\mathcal{J} &:= \{\Gamma \subset \mathbb{C},\ \Gamma \text{ is a Jordan curve}\},\\ \mathcal{J}^\infty &:= \{\Gamma \subset \mathbb{C},\ \Gamma \text{ is a } C^\infty \text{ Jordan curve}\}.\end{aligned}$$

Facts :

- $\Gamma \in \mathcal{J} \longleftrightarrow \exists\, \phi : \mathbb{S}^1 \to \mathbb{C}$ continuous and injective such that $\phi(\mathbb{S}^1) = \Gamma$;
- Let $h \in \mathrm{Homeo}(\mathbb{S}^1)$, then ϕ and $\phi \circ h$ are parametrizations of the same Jordan curve Γ.

Riemann mapping theorem

- Γ splits the complex plane into two domains D_Γ^+ and D_Γ^- :



- Let $\mathbb{D} := \{z \in \mathbb{C},\ |z| < 1\}$, the Riemann mapping theorem ensures that :

$$\exists\ F^+ : \mathbb{D} \to D_\Gamma^+ \text{ biholomorphic,}$$
$$\exists\ F^- : \mathbb{D} \to D_\Gamma^- \text{ biholomorphic.}$$

Riemann mapping theorem

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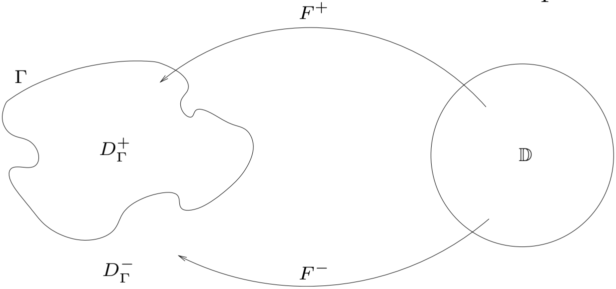


- Let $\mathbb{D} := \{z \in \mathbb{C},\ |z| < 1\}$, the Riemann mapping theorem ensures that :

$$\exists\ F^+ : \mathbb{D} \to D_\Gamma^+ \text{ biholomorphic, unique mod } \mathrm{SU}(1,1),$$
$$\exists\ F^- : \mathbb{D} \to D_\Gamma^- \text{ biholomorphic, unique mod } \mathrm{SU}(1,1).$$

Restrictions to $\mathbb{S}^1$ of homographic transformations

$\mathrm{SU}(1,1)$:= Poincaré group of automorphisms of the disk

$\simeq$ restrictions to $\mathbb{S}^1$ of homographic transformations

$$z \mapsto \frac{az+b}{\bar{b}z+\bar{a}}, \quad |a|^2 - |b|^2 = 1.$$

Holomorphic parametrizations

- By a theorem of Caratheodory, the maps $F^{\pm}$ extend to homeomorphisms :

$$F^{+} : \overline{\mathbb{D}} \to \overline{D_{\Gamma}^{+}}, \qquad F^{-} : \overline{\mathbb{D}} \to \overline{D_{\Gamma}^{-}} \;;$$

- in particular, $F^{\pm}_{|\mathbb{S}^1}$ define (canonical) parametrizations of Γ ;
- we have

$$g_{\Gamma} := (F^{-})^{-1} \circ F^{+}_{|\mathbb{S}^1} \in \mathrm{Homeo}(\mathbb{S}^1) \quad \text{(orientation preserving).}$$

Notations and preliminaries :

a touch of algebra

The Lie group $\mathrm{Diff}(\mathbb{S}^1)$ and its Lie algebra

$\mathrm{Diff}(\mathbb{S}^1)$:= the group of $C^\infty,$ orientation preserving diffeomorphisms of the circle

$\mathfrak{diff}(\mathbb{S}^1)$:= Lie algebra of right invariant vector fields on $\mathrm{Diff}(\mathbb{S}^1)$

$\simeq$ C^∞ functions on $\mathbb{S}^1$ via the identification
$u \in C^\infty(\mathbb{S}^1, \mathbb{R}) \longleftrightarrow$ vector field $u\frac{d}{d\theta}$
Lie bracket given by $[u,v]_{\mathfrak{diff}(\mathbb{S}^1)} := u\dot{v} - \dot{u}v$

$$\mathfrak{diff}_0(\mathbb{S}^1) := \left\{ u \in \mathfrak{diff}(\mathbb{S}^1),\ \frac{1}{2\pi}\int_{\mathbb{S}^1} u(\theta)d\theta = 0 \right\}$$

Central extensions of $\mathrm{Diff}(\mathbb{S}^1)$ and Virasoro algebra

The central extensions of $\mathrm{Diff}(\mathbb{S}^1)$, that is

$$1 \;\to\; A \;\to\; E\,? \;\to\; \mathrm{Diff}(\mathbb{S}^1) \;\to\; 1, \quad A \subset Z(E)$$

or equivalently

$$0 \;\to\; \mathfrak{a} \;\to\; \mathfrak{e}\,? \;\to\; \mathfrak{diff}(\mathbb{S}^1) \;\to\; 0,$$

have been classified by Gelfand-Fuchs.

Central extensions of $\mathrm{Diff}(\mathbb{S}^1)$ and Virasoro algebra

They are of the form

$$\mathcal{V}_{c,h} = \mathbb{R} \oplus \mathfrak{diff}(\mathbb{S}^1),$$

and are associated to a fundamental cocyle on $\mathfrak{diff}(\mathbb{S}^1)$:

$$\omega_{c,h}(f,g) := \int_{\mathbb{S}^1} \left[\left(h - \frac{c}{12} \right) f' - \frac{c}{12} f''' \right] g \, d\theta,$$

where $c, h > 0$, via

$$[\alpha\kappa + f,\ \beta\kappa + g]_{\mathcal{V}_{c,h}} := \omega_{c,h}(f,g)\kappa + [f,g]_{\mathfrak{diff}(\mathbb{S}^1)}.$$

Define Brownian motions on some natural quotient spaces of $\mathrm{Diff}(\mathbb{S}^1)$

Brownian motion on the space of Jordan curves

Theorem (Beurling-Ahlfors-Letho, $\sim$ 1970, conformal welding)

The application

$$\mathcal{J}^\infty \to \mathrm{Diff}(\mathbb{S}^1), \quad \Gamma \mapsto g_\Gamma = (F^-)^{-1} \circ F^+_{|\mathbb{S}^1}$$

is surjective and induces a canonical isomorphism :

$$\mathcal{J}^\infty \longrightarrow {}_{\mathrm{SU}(1,1)}\backslash \mathrm{Diff}(\mathbb{S}^1) / {}_{\mathrm{SU}(1,1)}.$$

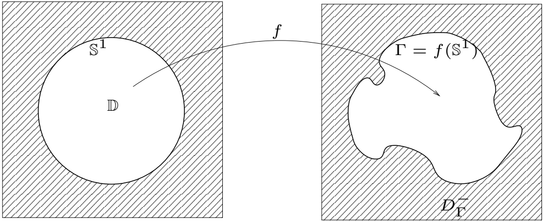
Idea : to construct a Brownian motion on $\mathcal{J}^\infty$, a first step consists in defining a Brownian motion on $\mathrm{Diff}(\mathbb{S}^1)$ and pray that the construction passes to the quotient !

The space of univalent functions

In the same spirit, consider

$$\mathcal{U}^\infty := \{f \in C^\infty(\overline{\mathbb{D}}, \mathbb{C}),\ f \text{ univalent s.t. } f(0) = 0,\ f'(0) = 1\}.$$

To $f \in \mathcal{U}^\infty$, one can associate $\Gamma = f(\mathbb{S}^1) \in \mathcal{J}^\infty$:



Riemann mapping theorem again

The Riemann mapping theorem provides a biholomorphic mapping h_f such that

$$h_f : \mathbb{C} \backslash \overline{\mathbb{D}} \to D_\Gamma^-, \quad h_f(\infty) = \infty.$$

It is unique up to a rotation of $\mathbb{C} \backslash \overline{\mathbb{D}}$, i.e. up to an element of $\mathbb{S}^1$ and extends to the boundary :

$$h_f : \mathbb{C} \backslash \mathbb{D} \to \overline{D_\Gamma^-}, \quad h_f(\infty) = \infty.$$

Using this construction, we thus have an application

$$\mathcal{U}^\infty \to \mathrm{Diff}(\mathbb{S}^1), \quad f \mapsto g_f := f^{-1} \circ h_{f_{|\mathbb{S}^1}}.$$

The space $\mathcal{U}^\infty$ as a quotient of $\mathrm{Diff}(\mathbb{S}^1)$

Theorem (Kirillov, 1982)

The application

$$\mathcal{U}^\infty \to \mathrm{Diff}(\mathbb{S}^1), \quad f \mapsto g_f = f^{-1} \circ h_{f_{|\mathbb{S}^1}}$$

induces a canonical isomorphism :

$$\mathcal{U}^\infty \longrightarrow \mathrm{Diff}(\mathbb{S}^1)/_{\mathbb{S}^1}.$$

Idea : as before, the construction of a Brownian motion on $\mathcal{U}^\infty$ appears closely related to the construction of a Brownian motion on $\mathrm{Diff}(\mathbb{S}^1)$...

Unitarizing measures and representations of the Virasoro algebra

Facts :

- The theory of Segal and Bargmann shows that the infinite dimensional Heisenberg group $\mathcal{H}$ has a representation :

$$\mathcal{H} \to \mathrm{End}\left(\mathbb{L}^2_{hol}(H,\nu)\right), \quad u \mapsto \rho(u),$$

  where H is an Hilbert space and ν a Gaussian measure ;
- a similar Gaussian realization was proved by Frenkel in the case of Loop groups.

Question :

Does there exists a space $\mathcal{M}$, a measure μ, and a representation of the Virasoro algebra of the following form ?

$$\mathcal{V}_{c,h} \to \mathrm{End}\left(\mathbb{L}^2_{hol}(\mathcal{M},\mu)\right), \quad u \mapsto \rho(u).$$

- Heuristics : $\mathcal{M} := \mathrm{Diff}(\mathbb{S}^1)/\mathrm{SU}(1,1)$ is a good candidate. It carries a canonical Kählerian structure, associated to a Kähler potential K such that $\partial\bar{\partial}K = \omega_{c,h}$;
- heuristics again : the measure μ should look like :

$$\mu = c_0 \exp\left(-K\right) d\mathrm{vol}.$$

- Idea : realize μ as an invariant measure for a Brownian motion $+$ drift on $\mathcal{M}$, with infinitesimal generator :

$$\mathcal{G} := \frac{1}{2}\Delta - \nabla K \nabla.$$

Up to technical difficulties, this method works !

Canonical horizontal diffusion

Horizontal diffusion on the tangent space

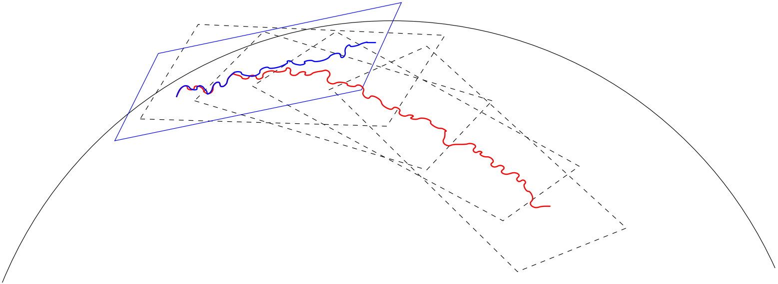
Given a differentiable structure $\mathcal{M}$, the canonical way to define a brownian motion on $\mathcal{M}$ is to do a stochastic development of a diffusion living on the tangent space $T\mathcal{M}$ to $\mathcal{M}$, that is :

1. first define a brownian motion on the tangent space $T\mathcal{M}$;
2. then "roll it without slipping" from $T\mathcal{M}$ to $\mathcal{M}$.

The notion of stochastic development

——— diffusion on the tangent space $T\mathcal{M}$

——— diffusion on the underlying manifold $\mathcal{M}$



Stochastic development and metric structure

Remarks :

1. the notion stochastic development, that is “roll without slipping”, implies a pre-existing metric structure on the manifold $\mathcal{M}$, i.e. on $T\mathcal{M}$;
2. the resulting process on $\mathcal{M}$ will inherit from the invariance properties of the metric choosen on $T\mathcal{M}$.

The way forward

To define a Brownian motion on $\mathrm{Diff}(\mathbb{S}^1)$, we thus have to :

1. choose a metric structure on $\mathrm{Diff}(\mathbb{S}^1)$;
2. construct a Brownian motion on $\mathfrak{diff}(\mathbb{S}^1)$;
3. roll it without slipping on $\mathrm{Diff}(\mathbb{S}^1)$ via the exponential mapping.

How to choose a metric on $\mathrm{Diff}(\mathbb{S}^1)$?

... if you have in mind to construct a Brownian motion on the space of smooth Jordan curves...

How to choose a metric on $\mathrm{Diff}(\mathbb{S}^1)$?

Theorem (Airault, Malliavin, Thalmaier)

There exists, up to a multiplicative constant, a unique Riemannian metric on

$$\mathrm{SU}(1,1)\backslash^{\mathrm{Diff}(\mathbb{S}^1)}/_{\mathrm{SU}(1,1)}$$

which is invariant under the left and right action of $\mathrm{SU}(1,1)$.

How to choose a metric on $\mathrm{Diff}(\mathbb{S}^1)$?

... if you have in mind to construct a Brownian motion on the space of univalent functions...

How to choose a metric on $\mathrm{Diff}(\mathbb{S}^1)$?

Theorem

There exists a canonical Kähler metric on $\mathcal{U}^\infty \simeq \mathrm{Diff}(\mathbb{S}^1)/\mathbb{S}^1$, i.e. on $\mathfrak{diff}_0(\mathbb{S}^1)$. It is associated to the fundamental cocycle $\omega_{c,h}$ defining the Virasoro algebra :

$$||u||^2 := \omega_{c,h}(u, Ju).$$

Here $\mathfrak{diff}_0(\mathbb{S}^1) \simeq \{u \in C^\infty(\mathbb{S}^1), \int_{\mathbb{S}^1} u d\theta = 0\}$, and

$$\begin{aligned} u(\theta) &= \sum_{k=1}^{+\infty} \left(a_k \cos(k\theta) + b_k \sin(k\theta)\right), \\ Ju(\theta) &:= \sum_{k=1}^{+\infty} \left(-a_k \sin(k\theta) + b_k \cos(k\theta)\right), \\ \omega_{c,h}(u, Ju) &= \sum_{k=1}^{+\infty} \alpha_k^2 \left(a_k^2 + b_k^2\right), \quad \alpha_k^2 := \left(hk + \frac{c}{12}(k^3 - k)\right). \end{aligned}$$

The algebra $\mathfrak{diff}(\mathbb{S}^1)$ as a Sobolev space

- The metric $||.||$ is invariant under the adjoint action of $\mathbb{S}^1$;
- the sequence α_k grows like $k^{3/2}$, thus

$$\left(\mathfrak{diff}_0(\mathbb{S}^1), ||.||\right) \simeq H^{3/2}(\mathbb{S}^1) ;$$

- an orthonormal system for $\left(\mathfrak{diff}_0(\mathbb{S}^1), ||.||\right)$ is, for $k \geq 1$:

$$e_{2k-1}(\theta) := \frac{\cos(k\theta)}{\alpha_k}, \quad e_{2k}(\theta) := \frac{\sin(k\theta)}{\alpha_k}.$$

Brownian motion on $\mathfrak{diff}(\mathbb{S}^1)$

Definition

The Brownian motion $(u_t)_{t\geq 0}$ on $\mathfrak{diff}(\mathbb{S}^1)$ (with $H^{3/2}$ structure) is the solution of the following Stratonovitch SDE

$$du_t(\theta) = \sum_{k\geq 1}\left(e_{2k-1}(\theta)\circ dX_t^k + e_{2k}(\theta)\circ dY_t^k\right),$$

where $X^k, Y^k, k\geq 1$ are independant, real valued, standard Brownian motions.

Remark : almost surely, the above series converges uniformly on $[0,T]\times\mathbb{S}^1$.

Stochastic development via exponential map

- The stochastic development of the diffusion $(u_t)_{t\geq 0}$ on $\mathfrak{diff}(\mathbb{S}^1)$ to a Brownian motion g_t on $\mathrm{Diff}(\mathbb{S}^1)$ writes formally :

$$(\star) \quad dg_t = (\circ du_t)\, g_t, \qquad g_0 = \mathrm{Id},$$

in other words

$$dg_t = \sum_{k\geq 1} \left(e_{2k-1}(g_t) \circ dX_t^k + e_{2k}(g_t) \circ dY_t^k \right), \quad g_0 = \mathrm{Id}.$$

- Problem : the classical Kunita's theory of stochastic flow works with a regularity $H^{3/2+\varepsilon}$ for any $\varepsilon > 0$, but not in the critical case $H^{3/2}$.

Malliavin's approach :

construction via regularization

Regularized Brownian motion on $\mathfrak{diff}(\mathbb{S}^1)$

Malliavin's approach of the problem is to regularize the horizontal diffusion, i.e. consider the following SDE for $0 < r < 1$:

$$(\star)_r \begin{cases} du_t^r(\theta) = \sum_{k\geq 1} r^k \left(e_{2k-1}(\theta)\circ dX_t^k + e_{2k}(\theta)\circ dY_t^k\right), \\ dg_t^r = (\circ du_t^r)\, g_t^r, \qquad g_0^r = \mathrm{Id}. \end{cases}$$

Theorem (Airault, Malliavin, Thalmaier)

For any $0 < r < 1$, the equation $(\star)_r$ admits a unique solution $t \mapsto g_t^r \in \mathrm{Diff}(\mathbb{S}^1)$. The limit $g_t(\theta) := \lim_{r\to 1} g_t^r(\theta)$ exists uniformly in θ and defines a solution of $(\star)$.

The limit $g_t \in \mathrm{Homeo}(\mathbb{S}^1)$ only!

Alternative approach :

finite dimensional approximation

The pointwise approach by S. Fang

Fang's approach of the problem is to consider the following approximating SDE's, for $n \geq 1$:

$$(\star)_n \begin{cases} du_t^n(\theta) = \sum_{k=1}^{n} \left(e_{2k-1}(\theta) \circ dX_t^k + e_{2k}(\theta) \circ dY_t^k\right), \\ dg_t^n = (\circ du_t^n)\, g_t^n, \qquad g_0^n = \mathrm{Id}. \end{cases}$$

Theorem (Fang)

For any $n \geq 1$, the equation $(\star)_n$ admits a unique solution $t \mapsto g_t^n \in \mathrm{Diff}(\mathbb{S}^1)$. For θ given, the limit $g_n(\theta) := \lim_{n\to+\infty} g_t^n(\theta)$ exists uniformly in $[0, T]$ and defines a solution of equation $(\star)$.

The pointwise approach by S. Fang

Theorem (Fang)

There exists a version of g_t such that, almost surely, $g_t \in \mathrm{Homeo}(\mathbb{S}^1)$ for all t. Moreover, there exists $c_0 > 0$ such that

$$|g_t(\theta) - g_t(\theta')| \leq C_t |\theta - \theta'|^{e^{-c_0 t}}.$$

In other words, the mappings g_t are δ_t-Hölderian homeomorphisms with δ_t going to zero when $t \to +\infty$.

Morality : Brownian motion on $\mathrm{Diff}(\mathbb{S}^1)$ with its $H^{3/2}$ metric structure must be realized in the bigger space $\mathrm{Homeo}(\mathbb{S}^1)$.

1. Some probabilistic and algebraic motivations
2. Brownian motion on the diffeomorphism group of the circle
3. Brownian motion on the space of univalent functions

Brownian motion on the space of univalent functions

What a wonderful world...

We have defined a Brownian motion g_t on the group of diffeomorphisms of the circle.

To construct a Brownian motion ϕ_t on $\mathcal{U}^\infty$, the space of univalent functions, we would like to factorize g_t thanks to the notion of conformal welding :

$$g_t = (\phi_t)^{-1} \circ h_t.$$

The classical theory of conformal welding is well developed for diffeomorphisms of the circle that have a quasi-conformal extension to the unit disk.

What a wonderful world... or not

The class of diffeomorphisms preserving the point at infinity and admitting a quasi-conformal extension to the half-plane is caracterized by the quasi-symmetry property :

$$\sup_{\theta,\theta'\in\mathbb{S}^1} \frac{h(\theta+\theta')-h(\theta')}{h(\theta)-h(\theta-\theta')} < \infty.$$

Theorem (Airault, Malliavin, Thalmaier)

Almost surely, the Brownian motion g_t on $\mathrm{Diff}(\mathbb{S}^1)$ does not satisfy the above quasi-symmetry property :

$$\limsup_{h\to 0} \frac{1}{\sqrt{\log^- |h|}} \left(\sup_{t\in[0,1],\ \theta\in\mathbb{S}^1} \log \left| \frac{g_t(\theta+h)-g_t(\theta)}{g_t(\theta)-g_t(\theta-h)} \right| \right) = \log 2.$$

From $BM(\mathrm{Diff}(\mathbb{S}^1))$ to $BM(\mathcal{U}^\infty)$

The problem : classical theory of conformal welding cannot be applied directly here...

The solution :

1. extend the stochastic flow g_t to a stochastic flow of diffeomorphisms in the unit disk $\mathbb{D}$;
2. use conformal welding “inside” the disk, i.e. on a disk of radius $0 < \rho < 1$;
3. pray and let ρ goes to 1...

Beurling-Ahlfors extension

Beurling-Ahlfors extension

- Smooth vector fields on the circle are of the form $u(\theta)d/d\theta$ where $u \in C^\infty(\mathbb{S}^1) \simeq C^\infty_{2\pi}(\mathbb{R})$.
- Given $u \in C^\infty(\mathbb{S}^1)$, Beurling-Ahlfors extension provides a vector field U on $\mathbb{H} := \{\zeta = x + iy \in \mathbb{C},\ y > 0\}$ via :

$$U(\zeta) = U(x+iy) := \int u(x-sy)K(s)ds - 6i \int u(x-sy)sK(s)ds,$$

where

$$K(s) := (1 - |s|)\, 1_{[-1,1]}(s).$$

Beurling-Ahlfors extension

In Fourier series, if $u(x) = \sum_{n=-\infty}^{+\infty} c_n e^{inx}$, then

$$U(\zeta) = U(x+iy) = \sum_{n=-\infty}^{+\infty} c_n \left(\widehat{K}(ny) + 6\widehat{K}'(ny)\right) e^{inx},$$

where

$$\widehat{K}(\xi) = \left(\frac{\sin(\xi/2)}{\xi/2}\right)^2 .$$

Beurling-Ahlfors extension

Consider the holomorphic chart $\zeta \mapsto z = \exp(i\zeta)$, denote $\log^-(a) = \max\{0, -\log(a)\}$, and define

$$\begin{aligned}\widetilde{U}(z) &:= izU(\zeta) \\ &:= iz\left(\sum_{n=-\infty}^{+\infty} c_n \left(\widehat{K}(n\log^-(|z|)) + 6\widehat{K}'(n\log^-(|z|))\right) e^{inx}\right).\end{aligned}$$

Proposition

Given $u \in \mathbb{L}^2(\mathbb{S}^1)$, the vector field $\widetilde{U}$ vanishes at the origin $z = 0$ an is C^1 in the unit disk $\mathbb{D}$.

From the circle to the disk

We now apply the preceding machinery to the stochastic flow $(u_t)_{t\geq 0}$ on $\mathfrak{diff}(\mathbb{S}^1)$. The complex version of u_t simply writes :

$$u_t(\theta) := \sum_{n\in\mathbb{Z}\backslash\{0\}} e_n(\theta) X_t^n,$$

where

$$\begin{cases} e_n(\theta) := \dfrac{e^{in\theta}}{\alpha_{|n|}}, \text{ with } \alpha_{|n|}^2 = \left(h|n| + \dfrac{c}{12}(|n|^3 - |n|)\right), \\ \\ X_t^n,\ n \neq 0, \text{ are independant Brownian motions.} \end{cases}$$

From the circle to the disk

- We thus obtain a flow $\widetilde{U}_t$ of C^1 vector fields on the unit disk $\mathbb{D}$, vanishing at zero :

$$\widetilde{U}_t(z) := iz \left(\sum_{n \in \mathbb{Z} \setminus \{0\}} \left(\widehat{K}(n \log^-(|z|)) + 6\widehat{K}'(n \log^-(|z|)) \right) e_n(x) X_t^n \right).$$

- At this stage, it is possible (rhymes with technical) to control the covariance of the resulting process, i.e. to control the expectations $\mathbb{E}[\widetilde{U}_t(z)\widetilde{U}_t(z')]$, $\mathbb{E}[\bar{\partial}\widetilde{U}_t(z)\bar{\partial}\widetilde{U}_t(z')]$...

Stochastic development again

It is then possible to integrate the development equation :

$$(\widetilde{\star}) \quad d\widetilde{\Psi}_t = \left(\circ d\widetilde{U}_t\right)\widetilde{\Psi}_t, \qquad \widetilde{\Psi}_0 = \mathrm{Id}.$$

Theorem (Airault, Malliavin, Thalmaier)

The equation $(\widetilde{\star})$ defines a unique stochastic flow $\widetilde{\Psi}_t$ of C^1, orientation preserving, diffeomorphisms of the unit disk $\mathbb{D}$. Moreover,

$$\lim_{\rho \to 1} \widetilde{\Psi}_t\left(\rho e^{i\theta}\right) = g_t(\theta) \text{ uniformly in } \theta,$$

where g_t is the solution of $(\star)$.

Stochastic conformal welding

Beltrami equation in a small disk

For $0 < \rho < 1$, let $\mathbb{D}_\rho := \rho\mathbb{D}$ and

$$\nu_t^\rho(z) := \begin{cases} \dfrac{\bar\partial\widetilde{\Psi}_t}{\partial\widetilde{\Psi}_t}(z) & \text{if } \ z \in \overline{\mathbb{D}_\rho}, \\ 0 & \text{otherwise.} \end{cases}$$

Let F_t^ρ be a solution of the following Beltrami equation, defined on the whole complex place $\mathbb{C}$:

$$\frac{\bar\partial F_t^\rho}{\partial F_t^\rho}(z) = \nu_t^\rho(z).$$

We normalize the solution s.t. $\partial_z F_t^\rho(z) - 1 \in \mathbb{L}^p$, $F_t^\rho(0) = 0$.

Stochastic conformal welding

Theorem (Airault, Malliavin, Thalmaier)

Let $\widetilde{\Psi}_t$ the solution of equation $(\widetilde{\star})$, i.e. the extension of g_t in the unit disk. Define

$$\begin{aligned} f_t^\rho(z) &:= F_t^\rho \circ (\widetilde{\Psi}_t)^{-1}(z), \quad && z \in \widetilde{\Psi}_t(\mathbb{D}_\rho), \\ g_t^\rho(z) &:= F_t^\rho(z), && z \notin \widetilde{\Psi}_t(\mathbb{D}_\rho). \end{aligned}$$

Then

$$\begin{aligned} &f_t^\rho \text{ is holomorphic and univalent on } \widetilde{\Psi}_t(\mathbb{D}_\rho), \\ &g_t^\rho \text{ is holomorphic and univalent on } \left(\widetilde{\Psi}_t(\mathbb{D}_\rho)\right)^c, \end{aligned}$$

and

$$(f_t^\rho)^{-1} \circ g_t^\rho(z) = \widetilde{\Psi}_t(z), \quad z \in \partial\mathbb{D}_\rho.$$

Towards Brownian motion on $\mathcal{U}^\infty$

We can now let ρ go to 1...

Theorem (Airault, Malliavin, Thalmaier)

For each $0 < r < 1$, the following limit

$$\phi_t(z) := \lim_{\rho \to 1} f_t^\rho$$

exists uniformly in $z \in \mathbb{D}_r$ and it defines an univalent function ϕ_t in the unit disk $\mathbb{D}$.

A factorization of g_t or almost

Consider the following extra assumption :

$$(\mathcal{H}) \quad \phi_t \text{ is continuous and injective on } \overline{\mathbb{D}}.$$

Theorem (Airault, Malliavin, Thalmaier)

Suppose that the function ϕ_t satisfies $(\mathcal{H})$, then there exists a function h_t univalent outside the unit disk $\mathbb{D}$ such that :

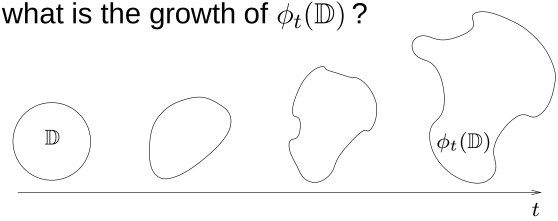
$$(\phi_t)^{-1} \circ h_t \left(e^{i\theta}\right) = g_t \left(e^{i\theta}\right),$$

where g_t is the solution of $(\star)$, i.e. the Brownian motion on $\mathrm{Diff}(\mathbb{S}^1)$.

Some properties of the resulting process

Area of the random domain

Question : what is the growth of $\phi_t(\mathbb{D})$?



Theorem (Airault, Malliavin, Thalmaier)

Let $\mathcal{A}_t^\rho := \mathrm{area}(F_t^\rho(\mathbb{D}_\rho))$. Then, there exist constants c_1, c_2, c_3, independent of $\rho < 1$ such that

$$\mathbb{P}\left(\sup_{t\in[0,T]} \log(\mathcal{A}_t^\rho) - c_1 T > c_2 + R\right) \leq \exp\left(-c_3 \frac{R^2}{T}\right).$$

Area of the random domain

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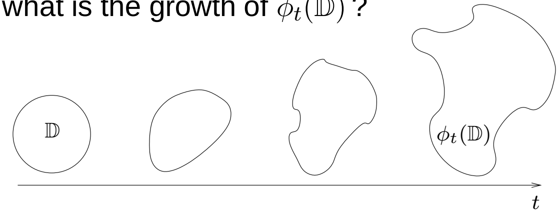

Theorem (Airault, Malliavin, Thalmaier)

Let $\mathcal{A}_t = \text{area}(\phi_t(\mathbb{D}))$. Then, there exist constants c_1, c_2, c_3 s.t.

$$\mathbb{P}\left(\sup_{t\in[0,T]} \log(\mathcal{A}_t) - c_1 T > c_2 + R\right) \leq \exp\left(-c_3 \frac{R^2}{T}\right).$$

A diffusion on Jordan curves

Theorem (Airault, Malliavin, Thalmaier)

Let ϕ_t be the stochastic flow of univalent functions defined above. Then $t \mapsto \phi_t(\mathbb{S}^1)$ defines a Markov process with values in $\mathcal{J}$, the space of Jordan curves.