Lecture on the construction of a

Brownian motion on the space of univalent functions via Brownian motion on Diff($\mathbb{S}^1$)

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Contents of the lecture

1 Notations and preliminaries

1.1 A drop of complex analysis

1.1.1 The space of Jordan curves

Let consider

$$\mathcal{J} := \{\Gamma \subset \mathbb{C},\ \Gamma \text{ is a Jordan curve}\},$$

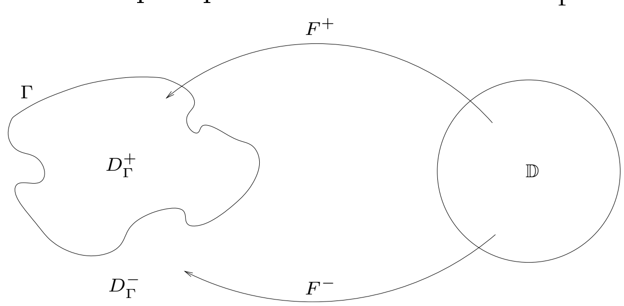
$$\mathcal{J}^\infty := \{\Gamma \subset \mathbb{C},\ \Gamma \text{ is a } C^\infty \text{ Jordan curve}\}.$$

Facts :

- $\Gamma \in \mathcal{J} \longleftrightarrow \exists\, \phi : \mathbb{S}^1 \to \mathbb{C}$ continuous and injective s.t. $\phi(\mathbb{S}^1) = \Gamma$;
- Let $h \in \mathrm{Homeo}(\mathbb{S}^1)$, then ϕ and $\phi \circ h$ are parametrizations of the same Jordan curve Γ.

1.1.2 Riemann mapping theorem

- Γ splits the complex plane into two domains D_Γ^+ and D_Γ^- :



- Let $\mathbb{D} := \{z \in \mathbb{C},\ |z| < 1\}$, the Riemann mapping theorem ensures that :

$$\exists\, F^+ : \mathbb{D} \to D_\Gamma^+ \text{ biholomorphic, unique mod } \mathrm{SU}(1,1),$$
$$\exists\, F^- : \mathbb{D} \to D_\Gamma^- \text{ biholomorphic, unique mod } \mathrm{SU}(1,1),$$

where

$$\mathrm{SU}(1,1) := \text{Poincaré group of automorphisms of the disk}$$

$$\simeq \text{restrictions to } \mathbb{S}^1 \text{ of homographic transformations } z \mapsto \frac{az+b}{\bar{b}z+\bar{a}},\quad |a|^2 - |b|^2 = 1.$$

- By a theorem of Caratheodory, the maps $F^{\pm}$ extend to homeomorphisms :
$$F^{+}:\overline{\mathbb{D}}\to\overline{D_{\Gamma}^{+}},\qquad F^{-}:\overline{\mathbb{D}}\to\overline{D_{\Gamma}^{-}}\ ;$$
- in particular, $F^{\pm}_{|\mathbb{S}^1}$ define (canonical) parametrizations of Γ ;
- we have
$$g_\Gamma := (F^-)^{-1}\circ F^+_{|\mathbb{S}^1}\in \mathrm{Homeo}(\mathbb{S}^1)\quad \text{(orientation preserving)}.$$

1.2 A touch of algebra

1.2.1 The Lie group Diff($\mathbb{S}^1$) and its Lie algebra

$\mathrm{Diff}(\mathbb{S}^1)$:= the group of C^∞, orientation preserving diffeomorphisms of the circle

$\mathfrak{diff}(\mathbb{S}^1)$:= Lie algebra of right invariant vector fields on $\mathrm{Diff}(\mathbb{S}^1)$

$\simeq$ C^∞ functions on $\mathbb{S}^1$ via the identification
$u\in C^\infty(\mathbb{S}^1,\mathbb{R})\longleftrightarrow$ vector field $u\frac{d}{d\theta}$
Lie bracket given by $[u,v]_{\mathfrak{diff}(\mathbb{S}^1)} := u\dot v-\dot u v$

$$\mathfrak{diff}_0(\mathbb{S}^1) := \left\{u\in\mathfrak{diff}(\mathbb{S}^1),\ \frac{1}{2\pi}\int_{\mathbb{S}^1}u(\theta)d\theta=0\right\}$$

1.2.2 Central extensions of Diff($\mathbb{S}^1$) and Virasoro algebra

The central extensions of $\mathrm{Diff}(\mathbb{S}^1)$, that is

$$1\ \to\ A\ \to\ E\,?\ \to\ \mathrm{Diff}(\mathbb{S}^1)\ \to\ 1,\quad A\subset Z(E)$$

or equivalently

$$0\ \to\ \mathfrak{a}\ \to\ \mathfrak{e}\,?\ \to\ \mathfrak{diff}(\mathbb{S}^1)\ \to\ 0,$$

have been classified by Gelfand-Fuchs. They are of the form

$$\mathcal{V}_{c,h}=\mathbb{R}\oplus\mathfrak{diff}(\mathbb{S}^1),$$

and are associated to a fundamental cocyle on $\mathfrak{diff}(\mathbb{S}^1)$:

$$\omega_{c,h}(f,g) := \int_{\mathbb{S}^1}\left[\left(h-\frac{c}{12}\right)f'-\frac{c}{12}f'''\right]g\,d\theta,$$

where $c,h>0$, via

$$[\alpha\kappa+f,\ \beta\kappa+g]_{\mathcal{V}_{c,h}} := \omega(f,g)\kappa+[f,g]_{\mathfrak{diff}(\mathbb{S}^1)}.$$

2 Some probabilistic and algebraic motivations for the introduction of Brownian motion on Diff($\mathbb{S}^1$)

2.1 Brownian motion on some quotient spaces

2.1.1 Brownian motion on the space of Jordan curves

Theorem 1 (Beurling-Ahlfors-Letho, $\sim$ 1970, conformal welding)**.** The application

$$\mathcal{J}^\infty \to \mathrm{Diff}(\mathbb{S}^1), \quad \Gamma \mapsto g_\Gamma = (F^-)^{-1} \circ F^+_{|\mathbb{S}^1}$$

is surjective and induces a canonical isomorphism :

$$\mathcal{J}^\infty \longrightarrow {}_{\mathrm{SU}(1,1)}\backslash \mathrm{Diff}(\mathbb{S}^1) /_{\mathrm{SU}(1,1)}.$$

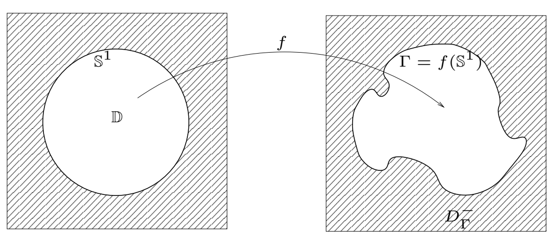
Idea : to construct a Brownian motion on $\mathcal{J}^\infty$, a first step consists in defining a Brownian motion on Diff($\mathbb{S}^1$) and pray that the construction passes to the quotient !

2.1.2 The space of univalent functions

In the same spirit, consider

$$\mathcal{U}^\infty := \{f \in C^\infty(\overline{\mathbb{D}}, \mathbb{C}),\ f \text{ univalent s.t. } f(0) = 0,\ f'(0) = 1\}.$$

To $f \in \mathcal{U}^\infty$, one can associate $\Gamma = f(\mathbb{S}^1) \in \mathcal{J}^\infty$:



The Riemann mapping theorem provides a biholomorphic mapping h_f such that

$$h_f : \mathbb{C}\backslash\overline{\mathbb{D}} \to D_\Gamma^-, \quad h_f(\infty) = \infty.$$

It is unique up to a rotation of $\mathbb{C}\backslash\overline{\mathbb{D}}$, i.e. up to an element of $\mathbb{S}^1$ and extends to the boundary :

$$h_f : \mathbb{C}\backslash\mathbb{D} \to \overline{D_\Gamma^-}, \quad h_f(\infty) = \infty.$$

Using this construction, we thus have an application

$$\mathcal{U}^\infty \to \mathrm{Diff}(\mathbb{S}^1), \quad f \mapsto g_f := f^{-1} \circ h_{f|\mathbb{S}^1}.$$

Theorem 2 (Kirillov, 1982)**.** The application

$$\mathcal{U}^\infty \to \mathrm{Diff}(\mathbb{S}^1), \quad f \mapsto g_f = f^{-1} \circ h_{f|\mathbb{S}^1}$$

induces a canonical isomorphism :

$$\mathcal{U}^\infty \longrightarrow \mathrm{Diff}(\mathbb{S}^1)/_{\mathbb{S}^1}.$$

Idea : as before, the construction of a Brownian motion on $\mathcal{U}^\infty$ appears closely related to the construction of a Brownian motion on $\mathrm{Diff}(\mathbb{S}^1)$...

2.2 Representations of the Virasoro algebra

Facts :

- The theory of Segal and Bargmann shows that the infinite dimensional Heisenberg group $\mathcal{H}$ has a representation :

  $$\mathcal{H} \to \mathrm{End}\left(\mathbb{L}^2_{hol}(H,\nu)\right), \quad u \mapsto \rho(u),$$

  where H is an Hilbert space and ν a Gaussian measure ;

- a similar Gaussian realization was proved by Frenkel in the case of Loop groups.

Question :

Does there exists a space $\mathcal{M}$, a measure μ, and a representation of the Virasoro algebra of the following form ?

$$\mathcal{V}_{c,h} \to \mathrm{End}\left(\mathbb{L}^2_{hol}(\mathcal{M},\mu)\right), \quad u \mapsto \rho(u).$$

- Heuristics : $\mathcal{M} := \mathrm{Diff}(\mathbb{S}^1)/\mathrm{SU}(1,1)$ is a good candidate. It carries a canonical Kählerian structure, associated to a Kähler potential K such that $\partial\bar{\partial}K = \omega_{c,h}$;

- heuristics again : the measure μ should look like :

  $$\mu = c_0 \exp\left(-K\right) d\mathrm{vol}.$$

- Idea : realize μ as an invariant measure for a Brownian motion + drift on $\mathcal{M}$, with infinitesimal generator :

  $$\mathcal{G} := \frac{1}{2}\Delta - \nabla K \nabla.$$

  Up to technical difficulties, this method works !

3 Brownian motion on the diffeomorphism group of the circle

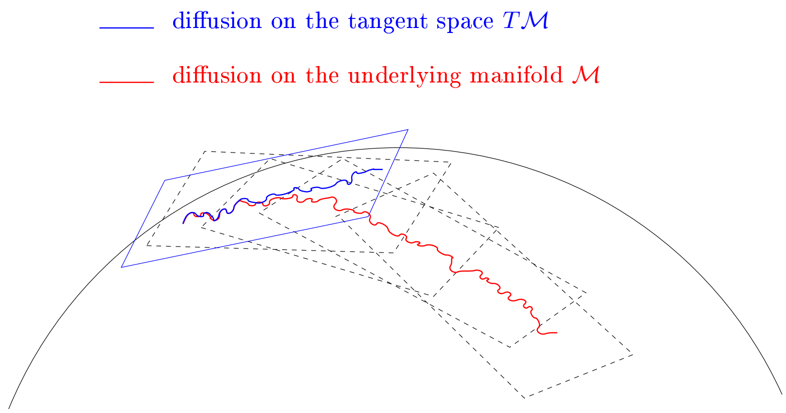
3.1 Canonical horizontal diffusion

3.1.1 The notion of stochastic development

Given a differentiable structure $\mathcal{M}$, the canonical way to define a brownian motion on $\mathcal{M}$ is to do a stochastic development of a diffusion living on the tangent space $T\mathcal{M}$ to $\mathcal{M}$, that is :

(i) first define a brownian motion on the tangent space $T\mathcal{M}$;

(ii) then "roll it without slipping" from $T\mathcal{M}$ to $\mathcal{M}$.



Remarks :

(i) the notion stochastic development, i.e. "roll without slipping", implies a pre-existing metric structure on the manifold $\mathcal{M}$, i.e. on $T\mathcal{M}$;

(ii) the resulting process on $\mathcal{M}$ will inherit from the invariance properties of the metric choosen on $T\mathcal{M}$.

To define a Brownian motion on $\mathrm{Diff}(\mathbb{S}^1)$, we thus have to :

(i) choose a metric structure on $\mathrm{Diff}(\mathbb{S}^1)$;

(ii) construct a Brownian motion on $\mathfrak{diff}(\mathbb{S}^1)$;

(iii) roll it without slipping on $\mathrm{Diff}(\mathbb{S}^1)$ via the exponential mapping.

3.1.2 How to choose a metric on Diff($\mathbb{S}^1$) ?

Theorem 3 (Airault, Malliavin, Thalmaier)**.** There exists, up to a multiplicative constant, a unique Riemannian metric on

$$\mathrm{SU}(1,1)\backslash\mathrm{Diff}(\mathbb{S}^1)/\mathrm{SU}(1,1)$$

which is invariant under the left and right action of $\mathrm{SU}(1,1)$.

Theorem 4. There exists a canonical Kähler metric on $\mathcal{U}^\infty \simeq \mathrm{Diff}(\mathbb{S}^1)/\mathbb{S}^1$, i.e. on $\mathfrak{diff}_0(\mathbb{S}^1)$. It is associated to the fundamental cocycle $\omega_{c,h}$ defining the Virasoro algebra :

$$||u||^2 := \omega_{c,h}(u, Ju).$$

Here $\mathfrak{diff}_0(\mathbb{S}^1) \simeq \{u \in C^\infty(\mathbb{S}^1), \int_{\mathbb{S}^1} u d\theta = 0\}$, and

$$\begin{aligned} u(\theta) &= \textstyle\sum_{k=1}^{+\infty} \left(a_k \cos(k\theta) + b_k \sin(k\theta)\right), \\ Ju(\theta) &:= \textstyle\sum_{k=1}^{+\infty} \left(-a_k \sin(k\theta) + b_k \cos(k\theta)\right), \\ \omega_{c,h}(u, Ju) &= \textstyle\sum_{k=1}^{+\infty} \alpha_k^2 \left(a_k^2 + b_k^2\right), \quad \alpha_k^2 := \left(hk + \frac{c}{12}(k^3 - k)\right). \end{aligned}$$

- The metric $||.||$ is invariant under the adjoint action of $\mathbb{S}^1$;
- the sequence α_k grows like $k^{3/2}$, thus

$$\left(\mathfrak{diff}_0(\mathbb{S}^1), ||.||\right) \simeq H^{3/2}(\mathbb{S}^1) \ ;$$

- an orthonormal system for $(\mathfrak{diff}_0(\mathbb{S}^1), ||.||)$ is, for $k \geq 1$:

$$e_{2k-1}(\theta) := \frac{\cos(k\theta)}{\alpha_k}, \quad e_{2k}(\theta) := \frac{\sin(k\theta)}{\alpha_k}.$$

3.1.3 Brownian motion on $\mathfrak{diff}(\mathbb{S}^1)$

Definition I.1 — The Brownian motion on $\mathfrak{diff}(\mathbb{S}^1)$ (with $H^{3/2}$ structure) is the solution of the following Stratonovitch SDE

$$dZ_t(\theta) = \sum_{k\geq 1} \left(e_{2k-1}(\theta) \circ dX_t^k + e_{2k}(\theta) \circ dY_t^k\right),$$

where $X^k, Y^k, k \geq 1$ are independant, real valued, standard Brownian motions.

Remark : almost surely, the above series converges uniformly on $[0, T] \times \mathbb{S}^1$.

- The stochastic development of the diffusion Z_t on $\mathfrak{diff}(\mathbb{S}^1)$ to a Brownian motion g_t on $\mathrm{Diff}(\mathbb{S}^1)$ writes formally :

$$(\star) \quad dg_t = (\circ dZ_t)\, g_t, \qquad g_0 = \mathrm{Id},$$

  in other words

$$dg_t = \sum_{k\geq 1} \left(e_{2k-1}(g_t) \circ dX_t^k + e_{2k}(g_t) \circ dY_t^k \right), \quad g_0 = \mathrm{Id}.$$

- Problem : the classical Kunita's theory of stochastic flow works with a regularity $H^{3/2+\varepsilon}$ for any $\varepsilon > 0$, but not in the critical case $H^{3/2}$.

3.2 Construction via regularization

3.2.1 Regularized Brownian motion on $\mathfrak{diff}(\mathbb{S}^1)$

Malliavin's approach of the problem is to regularize the horizontal diffusion, i.e. consider the following SDE for $0 < r < 1$:

$$(\star)_r \begin{cases} dZ_t^r(\theta) = \sum_{k\geq 1} r^k \left(e_{2k-1}(\theta) \circ dX_t^k + e_{2k}(\theta) \circ dY_t^k \right), \\ dg_t^r = (\circ dZ_t^r)\, g_t^r, \qquad g_0^r = \mathrm{Id}. \end{cases}$$

Theorem 5 (Airault, Malliavin, Thalmaier)**.** For any $0 < r < 1$, the equation $(\star)_r$ admits a unique solution $t \mapsto g_t^r \in \mathrm{Diff}(\mathbb{S}^1)$. The limit $g_t(\theta) := \lim_{r\to 1} g_t^r(\theta)$ exists uniformly in θ and defines a solution of $(\star)$.

The limit $g_t \in \mathrm{Homeo}(\mathbb{S}^1)$ only !

3.3 An alternative pointwise approach

3.3.1 The pointwise approach by S. Fang

Fang's approach of the problem is to consider the following approximating SDE's, for $n \geq 1$:

$$(\star)_n \begin{cases} dZ_t^n(\theta) = \sum_{k=1}^{n} \left(e_{2k-1}(\theta) \circ dX_t^k + e_{2k}(\theta) \circ dY_t^k \right), \\ dg_t^n = (\circ dZ_t^n)\, g_t^n, \qquad g_0^n = \mathrm{Id}. \end{cases}$$

Theorem 6 (Fang)**.** For any $n \geq 1$, the equation $(\star)_n$ admits a unique solution $t \mapsto g_t^n \in \mathrm{Diff}(\mathbb{S}^1)$. For θ given, the limit $g_n(\theta) := \lim_{n\to+\infty} g_t^n(\theta)$ exists uniformly in $[0, T]$ and defines a solution of equation $(\star)$.

Theorem 7 (Fang)**.** There exists a version of g_t such that, almost surely, $g_t \in \mathrm{Homeo}(\mathbb{S}^1)$ for all t. Moreover, there exists $c_0 > 0$ such that

$$|g_t(\theta) - g_t(\theta')| \leq C_t |\theta - \theta'|^{e^{-c_0 t}}.$$

In other words, the mappings g_t are δ_t-Hölderian homeomorphisms with δ_t going to zero when $t \to +\infty$.

Morality : Brownian motion on $\mathrm{Diff}(\mathbb{S}^1)$ with its $H^{3/2}$ metric structure must be realized in the bigger space $\mathrm{Homeo}(\mathbb{S}^1)$.

4 Brownian motion on the space of univalent functions

Our goal : construct a Brownian motion on $\mathcal{U}^\infty$, the space of univalent functions, starting from the Brownian motion on $\mathrm{Diff}(\mathbb{S}^1)$.

The two main steps :

(i) starting from the Brownian motion on $\mathfrak{diff}(\mathbb{S}^1)$, use the Beurling-Ahlfors extension to construct a stochastic flow of diffeomorphisms of the unit disk ;

(ii) use stochastic conformal welding to get a Brownian motion on the space of univalent functions.

The classical theory of conformal welding is well developed for diffeomorhisms of the circle that have a quasi-conformal extension to the unit disk. The class of diffeomorphisms preserving the point at infinity and admitting a quasi-conformal extension to the half-plane is caracterized by the quasi-symmetry property :

$$\sup_{\theta,\theta' \in \mathbb{S}^1} \frac{h(\theta+\theta') - h(\theta')}{h(\theta) - h(\theta-\theta')} < \infty.$$

Theorem 8 (Airault, Malliavin, Thalmaier)**.** Almost surely, the Brownian motion g_t on $\mathrm{Diff}(\mathbb{S}^1)$ does not satisfy the above quasi-symmetry property.

4.1 Beurling-Ahlfors extension

4.1.1 Beurling-Ahlfors extension

- Smooth vector fields on the circle are of the form $u(\theta)d/d\theta$ where $u \in C^\infty(\mathbb{S}^1) \simeq C^\infty_{2\pi}(\mathbb{R})$.

- Given $u \in C^\infty(\mathbb{S}^1)$, Beurling-Ahlfors extension provides a vector field U on $\mathbb{H} := \{\zeta = x + iy \in \mathbb{C},\ y > 0\}$ via :

$$U(\zeta) = U(x+iy) := \int u(x-sy)K(s)ds - 6i\int u(x-sy)sK(s)ds,$$

where

$$K(s) := (1-|s|)\,1_{[-1,1]}(s).$$

In Fourier series, if $u(x) = \sum_{n=-\infty}^{+\infty} c_n e^{inx}$, then

$$U(\zeta) = U(x+iy) = \sum_{n=-\infty}^{+\infty} c_n\left(\widehat{K}(ny) + 6\widehat{K}'(ny)\right)e^{inx},$$

where

$$\widehat{K}(\xi) = \left(\frac{\sin(\xi/2)}{\xi/2}\right)^2.$$

Consider the chart $\zeta \mapsto z = \exp(i\zeta)$, denote $\log^-(a) := \max\{0, -\log(a)\}$, and define

$$\begin{aligned}\widetilde{U}(z) &:= izU(\zeta)\\ &:= iz\left(\sum_{n=-\infty}^{+\infty} c_n\left(\widehat{K}(n\log^-(|z|)) + 6\widehat{K}'(n\log^-(|z|))\right)e^{inx}\right).\end{aligned}$$

Proposition 1. Given $u \in \mathbb{L}^2(\mathbb{S}^1)$, the vector field $\widetilde{U}$ vanishes at the origin $z = 0$ an is C^1 in the unit disk $\mathbb{D}$.

4.1.2 From the circle to the disk

We now apply the preceding machinery to the stochastic flow $(u_t)_{t\geq 0}$ on $\mathfrak{Diff}(\mathbb{S}^1)$. The complex version of u_t simply writes :

$$u_t(\theta) := \sum_{n\in\mathbb{Z}\setminus\{0\}} e_n(\theta)X_t^n,$$

where

$$\begin{cases} e_n(\theta) := \dfrac{e^{in\theta}}{\alpha_{|n|}}, \ \text{ with } \ \alpha_{|n|}^2 = \left(h|n| + \dfrac{c}{12}(|n|^3 - |n|)\right), \\ \\ X_t^n, \ n \neq 0, \ \text{ are independant Brownian motions.} \end{cases}$$

- We thus obtain a flow $\widetilde{U}_t$ of C^1 vector fields on the unit disk $\mathbb{D}$, vanishing at zero :

$$\widetilde{U}_t(z) := iz \left(\sum_{n\in\mathbb{Z}\setminus\{0\}} \left(\widehat{K}(n\log^-(|z|)) + 6\widehat{K}'(n\log^-(|z|))\right) e_n(x) X_t^n \right).$$

- At this stage, it is possible (rhymes with technical) to control the covariance of the resulting process, i.e. to control the expectations $\mathbb{E}[\widetilde{U}_t(z)\widetilde{U}_t(z')]$, $\mathbb{E}[\bar\partial\widetilde{U}_t(z)\bar\partial\widetilde{U}_t(z')]$...

4.1.3 Stochastic development again

It is then possible to integrate the development equation :

$$(\widetilde{\star}) \quad d\widetilde{\Psi}_t = \left(\circ d\widetilde{U}_t\right)\widetilde{\Psi}_t, \qquad \widetilde{\Psi}_0 = \text{Id}.$$

Theorem 9 (Airault, Malliavin, Thalmaier)**.** The equation $(\widetilde{\star})$ defines a unique stochastic flow $\widetilde{\Psi}_t$ of C^1, orientation preserving, diffeomorphisms of the unit disk $\mathbb{D}$. Moreover,

$$\lim_{\rho\to 1} \widetilde{\Psi}_t\left(\rho e^{i\theta}\right) = g_t(\theta) \ \text{ uniformly in } \ \theta,$$

where g_t is the solution of $(\star)$.

The resulting process $\widetilde{\Psi}_t$ effectively extends the Brownian motion on $\text{Diff}(\mathbb{S}^1)$ to the unit disk $\mathbb{D}$.

4.2 Stochastic conformal welding

The idea here is to factorize the flow $\widetilde{\Psi}_t$ via conformal welding to obtain a diffusion taking values in the space of univalent functions. However, in reason of theorem 8, the stochastic conformal welding can not be performed in a brutal way. The good idea is to stay away from the boundary of $\mathbb{D}$, and perform the conformal welding strictly inside the unit disk :

(i) factorize the diffusion $\widetilde{\Psi}_t$ in a disk of radius $\rho < 1$;

(ii) investigate the limit of each factor when ρ goes to 1.

4.2.1 Stochastic Beltrami equation

For $0 < \rho < 1$, let $\mathbb{D}_\rho := \rho\mathbb{D}$ and

$$\nu_t^\rho(z) := \begin{cases} \dfrac{\bar\partial \Psi_t}{\partial \Psi_t}(z) & \text{if } z \in \overline{\mathbb{D}_\rho}, \\ 0 & \text{otherwise.} \end{cases}$$

Now let F_t^ρ be a solution of the following Beltrami equation, defined on the whole complex place $\mathbb{C}$:

$$\frac{\bar\partial F_t^\rho}{\partial F_t^\rho}(z) = \nu_t^\rho(z).$$

We normalize the solution by the conditions

$$\partial_z F_t^\rho(z) - 1 \in \mathbb{L}^p, \quad F_t^\rho(0) = 0.$$

Theorem 10. Let $\widetilde{\Psi}_t$ the solution of equation $(\widetilde{\star})$, i.e. the extension of g_t in the unit disk. Define

$$f_t^\rho(z) := F_t^\rho \circ (\widetilde{\Psi}_t)^{-1}(z), \quad z \in \widetilde{\Psi}_t(\mathbb{D}_\rho),$$

$$g_t^\rho(z) := F_t^\rho(z), \qquad z \notin \widetilde{\Psi}_t(\mathbb{D}_\rho).$$

Then

$$f_t^\rho \text{ is holomorphic and univalent on } \widetilde{\Psi}_t(\mathbb{D}_\rho),$$
$$g_t^\rho \text{ is holomorphic and univalent on } \left(\widetilde{\Psi}_t(\mathbb{D}_\rho)\right)^c,$$

and

$$(f_t^\rho)^{-1} \circ g_t^\rho(z) = \widetilde{\Psi}_t(z), \quad z \in \partial\mathbb{D}_\rho.$$

4.2.2 Brownian motion on $\mathcal{U}^\infty$

As announced, we now look at the limit of f_t^ρ when ρ goes to 1.

Theorem 11. For each $0 < r < 1$, the limit

$$\phi_t(z) := \lim_{\rho \to 1} f_t^\rho$$

exists uniformly in $z \in \mathbb{D}_r$ and defines a univalent function ϕ_t in $\mathbb{D}$.

It is known that if a sequence of univalent functions converges on compact subsets of $\mathbb{D}$ to a non constant function ϕ, then ϕ is univalent in $\mathbb{D}$. On the contrary, univalence on the boundary needs an extra assumption :

$$(\mathcal{H}) \quad \phi \text{ is continuous and injective on } \overline{\mathbb{D}}.$$

Theorem 12. Suppose that the function ϕ_t satisfies $(\mathcal{H})$, then there exists a function h_t univalent outside the unit disk $\mathbb{D}$ such that :

$$(\phi_t)^{-1} \circ h_t\left(e^{i\theta}\right) = g_t\left(e^{i\theta}\right),$$

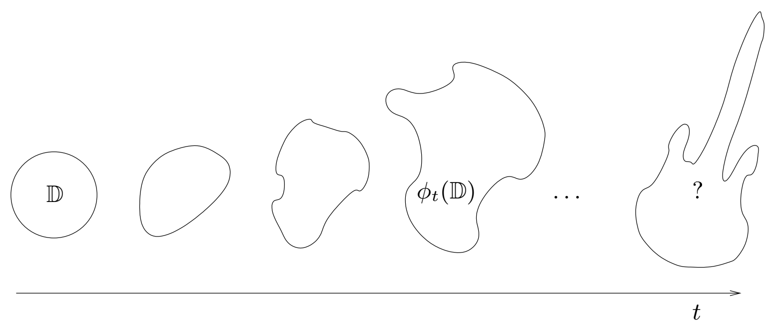
where g_t is the solution of $(\star)$, i.e. the Brownian motion on $\mathrm{Diff}(\mathbb{S}^1)$.

In other words, the Brownian motion on $\mathrm{Diff}(\mathbb{S}^1)$ can effectively be factorized to get a Brownian motion on $\mathcal{U}^\infty$, the space of univalent functions.

4.3 Some properties of the resulting process

4.3.1 Area of the random domain

Is it possible to obtain quantitative informations on the shape of the image of the unit disk by the mapping ϕ_t ?



Theorem 13. Let $\mathcal{A}_t^\rho$ be the area of $F_t^\rho(\mathbb{D}_\rho)$. Then, there exist constants c_1, c_2, c_3, independent of $\rho < 1$ such that

$$\mathbb{P}\left(\sup_{t\in[0,T]} \log(\mathcal{A}_t^\rho) - c_1 T > c_2 + R\right) \leq \exp\left(-c_3 \frac{R^2}{T}\right)$$

Letting ρ go to one, we thus get :

Theorem 14. Let $\mathcal{A}_t$ be the area of $\phi_t(\mathbb{D})$. Then, there exist constants c_1, c_2, c_3 such that

$$\mathbb{P}\left(\sup_{t\in[0,T]} \log(\mathcal{A}_t) - c_1 T > c_2 + R\right) \leq \exp\left(-c_3 \frac{R^2}{T}\right)$$

4.3.2 Back to the Brownian motion on Jordan curves

Theorem 15. Let ϕ_t be the stochastic flow of univalent functions defined in theorem 11. Then $t \mapsto \phi_t(\mathbb{S}^1)$ defines a Markov process with values in $\mathcal{J}$, the space of Jordan curves.

References

[AM01] Hélène Airault and Paul Malliavin. Unitarizing probability measures for representations of Virasoro algebra. *J. Math. Pures Appl. (9)*, 80(6):627–667, 2001.

[AMT02] Hélène Airault, Paul Malliavin, and Anton Thalmaier. Support of Virasoro unitarizing measures. *C. R. Math. Acad. Sci. Paris*, 335(7):621–626, 2002.

[AMT04] Hélène Airault, Paul Malliavin, and Anton Thalmaier. Canonical Brownian motion on the space of univalent functions and resolution of Beltrami equations by a continuity method along stochastic flows. *J. Math. Pures Appl. (9)*, 83(8):955–1018, 2004.

[BR87] M. J. Bowick and S. G. Rajeev. String theory as the Kähler geometry of loop space. *Phys. Rev. Lett.*, 58(6):535–538, 1987.

[Fan02] Shizan Fang. Canonical Brownian motion on the diffeomorphism group of the circle. *J. Funct. Anal.*, 196(1):162–179, 2002.

[Mal99] Paul Malliavin. The canonic diffusion above the diffeomorphism group of the circle. *C. R. Acad. Sci. Paris Sér. I Math.*, 329(4):325–329, 1999.

[Mal05] Paul Malliavin. Heat measures and unitarizing measures for Berezinian representations on the space of univalent functions in the unit disk. In *Perspectives in analysis*, volume 27 of *Math. Phys. Stud.*, pages 253–268. Springer, Berlin, 2005.