Brownian Motion and lorentzian manifolds
The case of Robertson-Walker space-times

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Dudley’s diffusion
Let $\mathbb{R}^{1,d} := \{\xi = (\xi^0, \xi^i) \in \mathbb{R} \times \mathbb{R}^d\}$ denote the Minkowski space-time of special relativity, equipped with the pseudo-metric:

$$\langle \xi, \xi \rangle := |\xi^0|^2 - \sum_{i=1}^{d} |\xi^i|^2,$$

and

$$\mathbb{H}^d := \{\xi \in \mathbb{R}^{1,d} | \xi^0 > 0 \text{ and } \langle \xi, \xi \rangle = 1\}.$$

If $(e_0, e_1, \ldots, e_d)$ is the canonical basis in $\mathbb{R}^{1,d}$, $(e_j^*)$ its dual basis, the matrices $E_j = e_0 \otimes e_j^* + e_j \otimes e_0^*$ generate the hyperbolic rotations.
Let $\dot{\xi}_s$ be an hyperbolic brownian motion on $\mathbb{H}^d$ and

$$\xi_s := \xi_0 + \int_0^s \dot{\xi}_u du.$$ 

Then $(\xi_s, \dot{\xi}_s)$ is a diffusion on $\mathbb{R}^{1,d} \times \mathbb{H}^d$.

Its law is invariant under the action of the Lorentz group.
If \( \xi_s = (\xi^0_s, \vec{\xi}_s) \) and \( s(t) \) is defined as \( \xi^0_s(t) = t \), then the euclidian trajectory

\[
Z_t := \vec{\xi}_s(t)
\]

satisfies

\[
\left| \frac{dZ(t)}{dt} \right| < 1.
\]

The velocity of \( Z(t) \) has norm \( < 1 \) (the velocity of the light).
The relativistic diffusion
of Franchi and Le Jan
The general framework

\[ G(\mathcal{M}) \]  pseudo-orthonormal frame bundle with first element in \( T^1\mathcal{M} \)

\[ \pi_1 \]  positive part of the unitary tangent bundle

\[ T^1\mathcal{M} \]  oriented lorentzian manifold of dimension \( d + 1 \), equipped with the Levi-Civita connexion
The infinitesimal generator of the diffusion

Let $V_j$ be the canonical vertical vector field associated to the matrix $E_j$ and $H_0$ the first horizontal vector field. We define

$$\mathcal{L} := H_0 + \frac{\sigma^2}{2} V,$$

where $V := \sum_{j=1}^{d} V_j^2$.

If $\mathcal{L}_0$ is the infinitesimal generator of the geodesic flow on $T^1 M$ and $\Delta_V$ the vertical laplacian, for all $F \in C^2(T^1 M)$, one has on $G(M)$:

$$(\mathcal{L}_0 F) \circ \pi_1 = H_0(F \circ \pi_1), \quad (\Delta_V F) \circ \pi_1 = V(F \circ \pi_1).$$

On $T^1 M$, the operator $\mathcal{L}$ induce the operator:

$$G := \mathcal{L}_0 + \frac{\sigma^2}{2} \Delta_V.$$
Theorem (Franchi and Le Jan)

- **The stochastic differential equation (stratonovich form)**

\[
(*) \quad d\Psi_s = H_0(\Psi_s) \, ds + \sigma \sum_{j=1}^{d} V_j(\Psi_s) \circ dw^j_s
\]

defines a diffusion \((\xi_s, \dot{\xi}_s) := \pi_1(\Psi_s)\) on \(T^1 \mathcal{M}\), whose infinitesimal generator is \(G = \mathcal{L}_0 + \frac{\sigma^2}{2} \Delta \mathcal{V}\).

- If \(\xi(s) : T_{\xi_s} \mathcal{M} \to T_{\xi_0} \mathcal{M}\) denote the inverse parallel transport along the curves \(C^1 (\xi_s', 0 \leq s' \leq s)\), then \(\zeta_s := \xi(s) \dot{\xi}_s\) is an hyperbolic brownian motion on \(T_{\xi_0} \mathcal{M}\).
Robertson-Walker space-times
They are the lorentzian manifolds $\mathcal{M}$ of the type:

$$\mathcal{M} = I \times M,$$

where $I$ is an interval in $\mathbb{R}$ and $M = S^3$, $\mathbb{R}^3$, or $\mathbb{H}^3$, equipped with pseudo-metric:

$$ds^2 = dt^2 - \alpha^2(t) d\ell^2.$$  

(1)

where $d\ell^2$ is the standard riemannian metric on $M$. 

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Examples of scale factors $\alpha$

![Graph showing examples of scale factors $\alpha$]
Consider the following coordinates system $\xi^\mu = (t, r, \theta)$ on $\mathcal{M}$:

\[
\begin{align*}
t & \in ]0, T[, \quad \text{with} \quad 0 < T \leq +\infty, \\
r & \in [0, 1] \quad \text{if} \quad k = 1 \quad \text{and} \quad r \in \mathbb{R}^+ \quad \text{if} \quad k = -1 \quad \text{or} \quad k = 0,
\end{align*}
\]

\[
\theta = \begin{pmatrix} \sin(\phi) \cos(\psi) \\ \sin(\phi) \sin(\psi) \\ \cos(\phi) \end{pmatrix} \in \mathbb{S}^2, \quad \phi \in [0, \pi], \quad \psi \in \mathbb{R}/2\pi\mathbb{Z}.
\]

In this chart, the pseudo-metric (1) writes:

\[
ds^2 = dt^2 - \alpha^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 d\phi^2 + r^2 \sin^2(\phi) d\psi^2 \right). \tag{2}
\]
Integration of geodesics

Lemma

Let \((\xi_s, \dot{\xi}_s) \in T^1 M\) be a time-like geodesic, parametrized by its arc-length \(s\). Then, the functions

\[ a := \alpha(t_s) \sqrt{\dot{t}_s^2 - 1} \quad \text{and} \quad \vec{b} := \alpha^2(t_s) r_s^2 \theta_s \wedge \dot{\theta}_s \]

are constant along \((\xi_s, \dot{\xi}_s)\).

Lemma

Let \((\xi_u, \dot{\xi}_u) \in T^1 M\) be a light-like (or null) geodesic. Then, the functions

\[ a := \alpha(t_u) \dot{t}_u \quad \text{and} \quad \vec{b} := \alpha^2(t_u) r_u^2 \theta_u \wedge \dot{\theta}_u \]

are constant along \((\xi_u, \dot{\xi}_u)\).
Light-like geodesics

Let \((\xi_u, \dot{\xi}_u)\) be a light-like geodesic. As the function

\[ a := \alpha(t_u) \dot{t}_u \]

is constant, by integrating, one gets:

\[ \int_{t_0}^{t_u} \alpha(v) dv = a \times u. \]
Let \((\xi_u, \dot{\xi}_u)\) be a light-like geodesic. We define \(\rho := |\vec{b}|/a\) and \(\varepsilon := \text{sign}(\dot{r}_0)\); then \(r_u\) is given by:

\[
\begin{align*}
\sqrt{r_u^2 - \rho^2} &= \sqrt{r_0^2 - \rho^2} + \varepsilon \int_{t_0}^{t_u} \frac{dv}{\alpha(v)}, \\
\arcsin \left( \sqrt{\frac{r_u^2 - \rho^2}{1 - \rho^2}} \right) &= \arcsin \left( \sqrt{\frac{r_0^2 - \rho^2}{1 - \rho^2}} \right) + \varepsilon \int_{t_0}^{t_u} \frac{dv}{\alpha(v)}, \\
\text{argsh} \left( \sqrt{\frac{r_u^2 - \rho^2}{1 + \rho^2}} \right) &= \text{argsh} \left( \sqrt{\frac{r_0^2 - \rho^2}{1 + \rho^2}} \right) + \varepsilon \int_{t_0}^{t_u} \frac{dv}{\alpha(v)}. 
\end{align*}
\]
Let \((\xi_u, \dot{\xi}_u)\) be a light-like geodesic, then \(\theta_u \perp \overrightarrow{v}\) where

\[
\overrightarrow{v} := \frac{\theta_0 \wedge \dot{\theta}_0}{|\theta_0 \wedge \dot{\theta}_0|}.
\]

In the frame \(\mathcal{R} := (\overrightarrow{u} := \theta_0, \overrightarrow{v}, \overrightarrow{w} := \overrightarrow{u} \wedge \overrightarrow{v})\), one has

\[
\begin{align*}
\phi_u & \equiv \phi_0 = \pi/2, \\
\psi_u - \psi_0 &= \text{sign}(\dot{\psi}_0) \times F \left( \int_{t_0}^{t_u} \frac{dv}{\alpha(v)} \right).
\end{align*}
\]
Asymptotic behavior of light-like geodesics

Let \((\xi_u, \dot{\xi}_u)\) be a light-like geodesic; define the time \(\tau := \sup\{u, t_u \leq T\}\).

**Proposition**

- If \(\int^T dv/\alpha(v) < +\infty\), then, when \(u\) goes to \(\tau\), the functions \(r_u\) and \(\psi_u\) converge to \(r_\infty\) and \(\psi_\infty\) (finite).

- If \(\int^T dv/\alpha(v) = +\infty\), then, when \(u\) goes to \(\tau\), \(r_u\) goes to infinity when \(k = 0\) or \(-1\), and goes to 1 when \(k = 1\). Moreover, \(\psi_u\) converge when \(k = 0\) or \(-1\), and tends to \(\pm\infty\) when \(k = 1\).
The relativistic diffusion
in Robertson-Walker space-times
The SDE’s satisfied by \((\xi_s, \dot{\xi}_s)\)

If \(\Psi_s\) is a solution of (*) and \((\xi_s, \dot{\xi}_s) = \pi_1(\Psi_s) \in T^1M\), then:

\[
\begin{align*}
dt_s &= t_s ds, \quad dr_s = \dot{r}_s ds, \quad d\phi_s = \dot{\phi}_s ds, \quad d\psi_s = \dot{\psi}_s ds \\
\dot{t}_s &= \left[ -\alpha(t_s) \alpha'(t_s) \left( \frac{\dot{r}_s^2}{1 - kr_s^2} + r_s^2 |\dot{\theta}_s|^2 \right) + \frac{3\sigma^2}{2} t_s \right] ds + dM^t_s, \\
\dot{r}_s &= \left[ -2 \frac{\alpha'(t_s) t_s}{\alpha(t_s)} \dot{r}_s - \frac{kr_s \dot{r}_s^2}{1 - kr_s^2} + r_s (1 - kr_s^2) |\dot{\theta}_s|^2 + \frac{3\sigma^2}{2} \dot{r}_s \right] ds + dM^r_s, \\
\dot{\phi}_s &= \left[ -2 \left( \frac{\alpha'(t_s) t_s}{\alpha(t_s)} + \frac{\dot{r}_s}{r_s} \right) \dot{\phi}_s + \sin(\phi_s) \cos(\phi_s) \dot{\psi}_s^2 + \frac{3\sigma^2}{2} \dot{\phi}_s \right] ds + dM^\phi_s, \\
\dot{\psi}_s &= \left[ -2 \left( \frac{\alpha'(t_s) t_s}{\alpha(t_s)} + \frac{\dot{r}_s}{r_s} \right) \dot{\psi}_s - 2 \cot(\phi_s) \dot{\phi}_s \dot{\psi}_s + \frac{3\sigma^2}{2} \dot{\psi}_s \right] ds + dM^\psi_s.
\end{align*}
\]
Two natural sub-diffusions

The 2d-diffusion \((t_s, \dot{t}_s)\) satisfies:

\[
dt_s = \dot{t}_s ds,
\]

\[
d\dot{t}_s = \left[ -H(t_s) \left( \dot{t}_s^2 - 1 \right) + \frac{3\sigma^2}{2} \dot{t}_s \right] ds + dM_s^t,
\]

where \(H = \alpha' / \alpha\) is the Hubble function.
The “radial” diffusion \((t_s, r_s, a_s, b_s, c_s)\) with \(a_s^2 = b_s^2/r_s^2 + c_s^2\):

\[
\begin{align*}
    dt_s &= \sqrt{1 + a_s^2/\alpha^2(t_s)} \, ds, \\
    db_s &= \frac{3\sigma^2}{2} b_s \, ds + \frac{\sigma^2 \alpha^2(t_s) r_s^2}{2 b_s} \, ds + dM_s^b, \\
    dr_s &= \frac{\sqrt{1 - k r_s^2}}{\alpha^2(t_s)} c_s \, ds, \\
    dc_s &= \frac{3\sigma^2}{2} c_s \, ds + \frac{b_s^2 \sqrt{1 - k r_s^2}}{\alpha^2(t_s) r_s^3} \, ds + dM_s^c, \\
    da_s &= \frac{3\sigma^2}{2} a_s \, ds + \sigma^2 \frac{\alpha^2(t_s)}{a_s} \, ds + dM_s^a,
\end{align*}
\]

where \(a_s := \alpha(t_s) \sqrt{\dot{i}_s^2 - 1},\) \(b_s := \alpha^2(t_s) r_s^2 |\dot{\theta}_s|,\) \(c_s := \frac{\alpha^2(t_s) \dot{r}_s}{\sqrt{1 - k r_s^2}}.\)
Proposition

Suppose that $t \mapsto H(t)$ is decreasing and non-negative on $\mathbb{R}^+$. Define $H_\infty = \lim_{t \to \infty} H(t)$.

- If $H_\infty > 0$, then the process $t_s$ is recurrent in $1 + \infty$.
- If $H_\infty = 0$, then $t_s$ is transient, and if we define

$$Z_s := \alpha(t_s) t_s \times \left( \int_{t_0}^{t_s} \alpha(u) du \right)^{-1},$$

then

$$0 < \liminf_{s \to +\infty} Z_s < \limsup_{s \to +\infty} Z_s < +\infty \quad a.s.$$

$$Z_s \overset{d}{\to} \sigma^2 / 2 \times \Gamma(2).$$
Let \((\xi_s, \dot{\xi}_s)\) be the diffusion of Franchi and Le Jan and as before 
\(\tau := \sup\{s, \ t_s \leq T\}\).

**Proposition**

- If \(\int_T dv/\alpha(v) < +\infty\), then, when \(s\) goes to \(\tau\), the processes \(r_s\) and \(\theta_s\) converge a.s., to some finite random variables \(r_\infty \in \mathbb{R}^*_+\) and \(\theta_\infty \in \mathbb{S}^2\).

- If \(\int_T dv/\alpha(v) = +\infty\), then, when \(s\) goes to \(\tau\), \(r_s\) goes to plus infinity a.s. when \(k = 0\) ou \(-1\), and it goes to 1 when \(k = 1\). Moreover the point \(\theta_s\) converge a.s. when \(k = 0\) or \(-1\), and has an oscillatory behavior in \(\mathbb{S}^2\) when \(k = 1\).
Thanks for your attention...