Brownian Motion and lorentzian manifolds

The case of Robertson-Walker space-times

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Dudley’s diffusion

Minkowski space-time and hyperbolic space

Let $\mathbb{R}^{1,d} := \{\xi = (\xi^0, \xi^i) \in \mathbb{R} \times \mathbb{R}^d\}$ denote the Minkowski space-time of special relativity, equipped with the pseudo-metric :

$$\langle \xi, \xi \rangle := |\xi^0|^2 - \sum_{i=1}^{d} |\xi^i|^2,$$

and

$$\mathbb{H}^d := \{\xi \in \mathbb{R}^{1,d} \,|\, \xi^0 > 0 \text{ and } \langle \xi, \xi \rangle = 1\}.$$

If $(e_0, e_1, \ldots, e_d)$ is the canonical basis in $\mathbb{R}^{1,d}$, (e_j^*) its dual basis, the matrices $E_j = e_0 \otimes e_j^* + e_j \otimes e_0^*$ generate the hyperbolic rotations.

Dudley's relativistic diffusion

Let $\dot{\xi}_s$ be an hyperbolic brownian motion on $\mathbb{H}^d$ and

$$\xi_s := \xi_0 + \int_0^s \dot{\xi}_u du.$$

Then $(\xi_s, \dot{\xi}_s)$ is a diffusion on $\mathbb{R}^{1,d} \times \mathbb{H}^d$.

Its law is invariant under the action of the Lorentz group.

Dudley's relativistic diffusion

If $\xi_s = (\xi_s^0, \overrightarrow{\xi_s})$ and $s(t)$ is defined as $\xi_{s(t)}^0 = t$, then the euclidian trajectory

$$Z_t := \overrightarrow{\xi_{s(t)}}$$

satisfies

$$\left| \frac{dZ(t)}{dt} \right| < 1.$$

The velocity of $Z(t)$ has norm < 1 (the velocity of the light).

The relativistic diffusion of Franchi and Le Jan

The general framework

$G(\mathcal{M})$ — pseudo-orthonormal frame bundle with first element in $T^1\mathcal{M}$

$\downarrow \pi_1$

$T^1\mathcal{M}$ — positive part of the unitary tangent bundle

$\downarrow$

$\mathcal{M}$ — oriented lorentzian manifold of dimension $d+1$, equipped with the Levi-Civita connexion

The infinitesimal generator of the diffusion

Let V_j be the canonical vertical vector field associated to the matrix E_j and H_0 the first horizontal vector field. We define

$$\mathcal{L} := H_0 + \frac{\sigma^2}{2}\mathcal{V}, \quad \text{where} \quad \mathcal{V} := \sum_{j=1}^{d} V_j^2.$$

If $\mathcal{L}_0$ is the infinitesimal generator of the geodesic flow on $T^1\mathcal{M}$ and $\Delta_{\mathcal{V}}$ the vertical laplacian, for all $F \in C^2(T^1\mathcal{M})$, one has on $G(\mathcal{M})$:

$$(\mathcal{L}_0 F)\circ\pi_1 = H_0(F\circ\pi_1)\,, \quad (\Delta_{\mathcal{V}} F)\circ\pi_1 = \mathcal{V}(F\circ\pi_1)\,.$$

On $T^1\mathcal{M}$, the operator $\mathcal{L}$ induce the operator :

$$\mathcal{G} := \mathcal{L}_0 + \frac{\sigma^2}{2}\Delta_{\mathcal{V}}.$$

Theorem (Franchi and Le Jan)

• *The stochastic differential equation (stratonovich form)*

$$(*) \qquad d\Psi_s = H_0(\Psi_s)\, ds + \sigma \sum_{j=1}^{d} V_j(\Psi_s) \circ dw_s^j$$

defines a diffusion $(\xi_s, \dot{\xi}_s) := \pi_1(\Psi_s)$ *on* $T^1\mathcal{M}$*, whose infinitesimal generator is* $\mathcal{G} = \mathcal{L}_0 + \frac{\sigma^2}{2}\Delta_{\mathcal{V}}$.

• *If* $\overleftarrow{\xi}(s) : T_{\xi_s}\mathcal{M} \to T_{\xi_0}\mathcal{M}$ *denote the inverse parallel transport along the curves* C^1 $(\xi_{s'} \,|\, 0 \leq s' \leq s)$*, then* $\zeta_s := \overleftarrow{\xi}(s)\,\dot{\xi}_s$ *is an hyperbolic brownian motion on* $T_{\xi_0}\mathcal{M}$ *.*

Robertson-Walker space-times

Robertson-Walker space-times

They are the lorentzian manifolds $\mathcal{M}$ of the type :

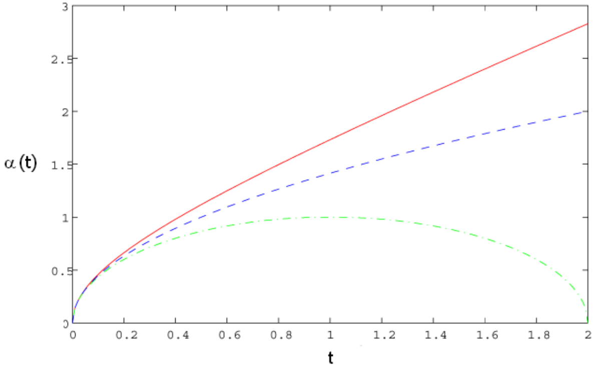
$\mathcal{M} = I \times M,$ where I is an interval in $\mathbb{R}$ and $M = \mathbb{S}^3, \mathbb{R}^3$, or $\mathbb{H}^3$,

equipped with pseudo-metric :

$$ds^2 = dt^2 - \alpha^2(t) d\ell^2. \tag{1}$$

where $d\ell^2$ is the standard riemannian metric on M.

Examples of scale factors α



Consider the following coordinates system $\xi^\mu = (t,\ r,\ \theta)$ on $\mathcal{M}$:

$$\begin{cases} t \in]0, T[, \quad \text{with} \quad 0 < T \leq +\infty, \\ \\ r \in [0,1] \quad \text{if} \quad k = 1 \quad \text{and} \quad r \in \mathbb{R}^+ \quad \text{if} \quad k = -1 \text{ ou } k = 0, \\ \\ \theta = \begin{pmatrix} \sin(\phi)\cos(\psi) \\ \sin(\phi)\sin(\psi) \\ \cos(\phi) \end{pmatrix} \in \mathbb{S}^2, \ \phi \in [0, \pi], \quad \psi \in \mathbb{R}/2\pi\mathbb{Z}. \end{cases}$$

In this chart, the pseudo-metric (1) writes :

$$ds^2 = dt^2 - \alpha^2(t)\left(\frac{dr^2}{1 - kr^2} + r^2 d\phi^2 + r^2 \sin^2(\phi) d\psi^2\right). \qquad (2)$$

Integration of geodesics

Lemma

Let $(\xi_s, \dot{\xi}_s) \in T^1\mathcal{M}$ be a time-like geodesic, parametrized by its arc-length s. Then, the functions

$$a := \alpha(t_s)\sqrt{\dot{t}_s^2 - 1} \text{ and } \vec{b} := \alpha^2(t_s) r_s^2\, \theta_s \wedge \dot{\theta}_s$$

are constant along $(\xi_s, \dot{\xi}_s)$.

Lemma

Let $(\xi_u, \dot{\xi}_u) \in T^1\mathcal{M}$ be a light-like (or null) geodesic. Then, the functions

$$a := \alpha(t_u)\dot{t}_u \text{ and } \vec{b} := \alpha^2(t_u) r_u^2\, \theta_u \wedge \dot{\theta}_u$$

are constant along $(\xi_u, \dot{\xi}_u)$.

Light-like geodesics

Let $(\xi_u, \dot{\xi}_u)$ be a light-like geodesic. As the function

$$a := \alpha(t_u)\dot{t}_u$$

is constant, by integrating, one gets :

$$\int_{t_0}^{t_u} \alpha(v)dv = a \times u.$$

Light-like geodesics

Let $(\xi_u, \dot{\xi}_u)$ be a light-like geodesic. We define $\rho := |\vec{b}|/a$ and $\varepsilon := sign(\dot{r}_0)$; then r_u is given by :

$$\begin{cases} \sqrt{r_u^2 - \rho^2} = \sqrt{r_0^2 - \rho^2} + \varepsilon \displaystyle\int_{t_0}^{t_u} \frac{dv}{\alpha(v)}, \\ \arcsin\left(\sqrt{\dfrac{r_u^2 - \rho^2}{1 - \rho^2}}\right) = \arcsin\left(\sqrt{\dfrac{r_0^2 - \rho^2}{1 - \rho^2}}\right) + \varepsilon \displaystyle\int_{t_0}^{t_u} \frac{dv}{\alpha(v)}, \\ \operatorname{argsh}\left(\sqrt{\dfrac{r_u^2 - \rho^2}{1 + \rho^2}}\right) = \operatorname{argsh}\left(\sqrt{\dfrac{r_0^2 - \rho^2}{1 + \rho^2}}\right) + \varepsilon \displaystyle\int_{t_0}^{t_u} \frac{dv}{\alpha(v)}. \end{cases}$$

Light-like geodesics

Let $(\xi_u, \dot{\xi}_u)$ be a light-like geodesic, then $\theta_u \perp \overrightarrow{v}$ where

$$\overrightarrow{v} := \frac{\theta_0 \wedge \dot{\theta}_0}{|\theta_0 \wedge \dot{\theta}_0|}.$$

In the frame $\mathcal{R} := (\overrightarrow{u} := \theta_0, \overrightarrow{v}, \overrightarrow{w} := \overrightarrow{u} \wedge \overrightarrow{v})$, one has

$$\begin{cases} \phi_u \equiv \phi_0 = \pi/2, \\ \\ \psi_u - \psi_0 = sign(\dot{\psi}_0) \times F\left(\int_{t_0}^{t_u} \frac{dv}{\alpha(v)} \right). \end{cases}$$

Asymptotic behavior of light-like geodesics

Let $(\xi_u, \dot{\xi}_u)$ be a light-like geodesic ; define the time $\tau := \sup\{u,\ t_u \leq T\}$.

Proposition

• *If* $\int^T dv/\alpha(v) < +\infty$, *then, when* u *goes to* τ, *the functions* r_u *and* ψ_u *converge to* r_∞ *and* ψ_∞ *(finite).*

• *If* $\int^T dv/\alpha(v) = +\infty$, *then, when* u *goes to* τ , r_u *goes to infinity when* $k = 0$ *or* -1, *and goes to* 1 *when* $k = 1$. *Moreover,* ψ_u *converge when* $k = 0$ *or* -1, *and tends to* $\pm\infty$ *when* $k = 1$.

The relativistic diffusion

in Robertson-Walker space-times

The SDE's satisfied by $(\xi_s, \dot{\xi}_s)$

If Ψ_s is a solution of $(*)$ and $(\xi_s, \dot{\xi}_s) = \pi_1(\Psi_s) \in T^1\mathcal{M}$, then :

$$
\left\{
\begin{aligned}
& dt_s = \dot{t}_s ds, \quad dr_s = \dot{r}_s ds, \quad d\phi_s = \dot{\phi}_s ds, \quad d\psi_s = \dot{\psi}_s ds \\
& d\dot{t}_s = \left[-\alpha(t_s)\, \alpha'(t_s) \left(\frac{\dot{r}_s^2}{1 - kr_s^2} + r_s^2 |\dot{\theta}_s|^2 \right) + \frac{3\sigma^2}{2} \dot{t}_s \right] ds + dM_s^{\dot{t}}, \\
& d\dot{r}_s = \left[-2 \frac{\alpha'(t_s)\dot{t}_s}{\alpha(t_s)} \dot{r}_s - \frac{k r_s \dot{r}_s^2}{1 - kr_s^2} + r_s(1 - kr_s^2)|\dot{\theta}_s|^2 + \frac{3\sigma^2}{2} \dot{r}_s \right] ds + dM_s^{\dot{r}}, \\
& d\dot{\phi}_s = \left[-2 \left(\frac{\alpha'(t_s)\dot{t}_s}{\alpha(t_s)} + \frac{\dot{r}_s}{r_s} \right) \dot{\phi}_s + \sin(\phi_s)\cos(\phi_s) \dot{\psi}_s^{\,2} + \frac{3\sigma^2}{2} \dot{\phi}_s \right] ds + dM_s^{\dot{\phi}}, \\
& d\dot{\psi}_s = \left[-2 \left(\frac{\alpha'(t_s)\dot{t}_s}{\alpha(t_s)} + \frac{\dot{r}_s}{r_s} \right) \dot{\psi}_s - 2\cot(\phi_s)\dot{\phi}_s\dot{\psi}_s + \frac{3\sigma^2}{2} \dot{\psi}_s \right] ds + dM_s^{\dot{\psi}}.
\end{aligned}
\right.
$$

Two natural sub-diffusions

The 2d-diffusion $(t_s, \dot{t}_s)$ satisfies :

$$dt_s = \dot{t}_s ds,$$

$$d\dot{t}_s = \left[-H(t_s) \left(\dot{t}_s^2 - 1 \right) + \frac{3\sigma^2}{2} \dot{t}_s \right] ds + dM_s^{\dot{t}},$$

where $H = \alpha' / \alpha$ is the Hubble function.

Two natural sub-diffusions

The "radial" diffusion $(t_s, r_s, a_s, b_s, c_s)$ with $a_s^2 = b_s^2/r_s^2 + c_s^2$:

$$\begin{cases} dt_s = \sqrt{1 + a_s^2/\alpha^2(t_s)}ds, \quad db_s = \dfrac{3\sigma^2}{2} b_s ds + \dfrac{\sigma^2 \alpha^2(t_s) r_s^2}{2b_s} ds + dM_s^b, \\ dr_s = \dfrac{\sqrt{1 - kr_s^2}}{\alpha^2(t_s)} c_s\, ds, \quad dc_s = \dfrac{3\sigma^2}{2} c_s ds + \dfrac{b_s^2 \sqrt{1 - kr_s^2}}{\alpha^2(t_s) r_s^3} ds + dM_s^c, \\ da_s = \dfrac{3\sigma^2}{2} a_s ds + \sigma^2 \dfrac{\alpha^2(t_s)}{a_s} + dM_s^a, \end{cases}$$

where $\quad a_s := \alpha(t_s)\sqrt{\dot{t}_s^2 - 1}, \quad b_s := \alpha^2(t_s) r_s^2 |\dot{\theta}_s|, \quad c_s := \dfrac{\alpha^2(t_s)\dot{r}_s}{\sqrt{1 - kr_s^2}}$.

Asymptotic behavior of the 2d-diffusion

Proposition

Suppose that $t \mapsto H(t)$ *is decreasing and non-negative on* $\mathbb{R}^+$*.*
Define $H_\infty = \lim_{+\infty} H(t)$*.*

- *If* $H_\infty > 0$*, then the process* $\dot{t}_s$ *is recurrent in* $]1 + \infty[$*.*
- *If* $H_\infty = 0$*, then* $\dot{t}_s$ *is transient, and if we define*

$$Z_s := \alpha(t_s)\dot{t}_s \times \left(\int_{t_0}^{t_s} \alpha(u)du \right)^{-1}, \text{ then}$$

$$0 < \liminf_{s \to +\infty} Z_s < \limsup_{s \to +\infty} Z_s < +\infty \ \text{ a.s.}$$

$$Z_s \xrightarrow{d} \sigma^2/2 \times \Gamma(2).$$

Asymptotic behavior of the diffusion

Let $(\xi_s, \dot{\xi}_s)$ be the diffusion of Franchi and Le Jan and as before $\tau := \sup\{s,\ t_s \leq T\}$.

Proposition

• *If* $\int^T dv/\alpha(v) < +\infty$*, then, when* s *goes to* τ*, the processes* r_s *and* θ_s *converge a.s., to some finite random variables* $r_\infty \in \mathbb{R}_+^*$ *and* $\theta_\infty \in \mathbb{S}^2$*.*

• *If* $\int^T dv/\alpha(v) = +\infty$*, then, when* s *goes to* τ*,* r_s *goes to plus infinity a.s. when* $k = 0$ *ou* -1*, and it goes to* 1 *when* $k = 1$*. Moreover the point* θ_s *converge a.s. when* $k = 0$ *or* -1*, and has an oscillatory behavior in* $\mathbb{S}^2$ *when* $k = 1$*.*

Thanks for your attention...