EXAM / HOMEWORK

to give back before December 13th

Exercice 1. *Wright–Fisher diffusion*
Let $(B_t)_{t\geq 0}$ be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We consider the Wright–Fisher diffusion, that is the stochastic process $(X_t)_{t\geq 0}$ with values in $[0,1]$, starting from $X_0 = x \in [0,1]$ and solution of the SDE

$$dX_t = \sqrt{X_t(1-X_t)}dB_t.$$

We set $\tau := \inf\{t \geq 0, X_t \in \{0,1\}\}$ and denote by L is the infinitesimal generator of $(X_t)_{t\geq 0}$.

1. Show that $x \mapsto u(x) := \mathbb{E}_x[\tau]$ is the solution of the Dirichlet problem $Lu = -1$ and $u(0) = u(1) = 0$. Deduce the value of $u(x)$ for $x \in]0,1[$.
2. Show that τ is finite almost surely.
3. What can you say about X_t for $t \geq \tau$?

Exercice 2. *Brownian Motion on rotationally invariant manifolds*
We consider a manifold M of dimension $d \geq 3$ with a global coordinates system

$$M \sim (0,+\infty) \times \mathbb{S}^{d-1}, \;\; i.e. \;\; \forall x \in M, \; x = (r,\theta), \; r \in (0,+\infty), \theta \in \mathbb{S}^{d-1}.$$

If we endow M with a metric of the form $g_M = dr^2 + f^2(r)d\theta^2$, with f a C^2 function such that $f > 0$ on $(0,+\infty)$, the Laplace operator reads

$$\Delta_M = \partial_r^2 + (d-1)\frac{f'(r)}{f(r)}\partial_r + \frac{1}{f^2(r)}\Delta_{\mathbb{S}^{d-1}},$$

where $\Delta_{\mathbb{S}^{d-1}}$ is the classical Laplace operator on the sphere. Let us now define the Brownian motion (X_t) on (M, g_M) as the continuous Markov process with generator $\frac{1}{2}\Delta_M$. If we write $X_t = (r_t, \theta_t)$ in the global chart, then there exists a real Brownian motion $(B_t)_{t\geq 0}$ and an independent Brownian motion (W_t) on $\mathbb{S}^{d-1}$ such that

$$dr_t = \frac{d-1}{2}\frac{f'(r_t)}{f(r_t)}dt + dB_t, \qquad \theta_t = W(\tau_t), \quad \text{with} \quad \tau_t := \int_0^t \frac{1}{f(r_s)^2}ds.$$

We denote by ζ the maximal lifetime (i.e. the time of explosion) of the process (X_t) and by $\tau_t^{-1} := \inf\{t > 0, \tau_s \geq t\}$ the general inverse of the clock τ_t.

1. Explicit some necessary and sufficient conditions on f for (r_t) to be recurrent / transient.
2. Explicit some necessary and sufficient on f for the almost sure finiteness of ζ.
3. Show that θ_t converges almost surely as $t \to \zeta$ iff $\tau_\zeta < +\infty$ almost surely.
4. We set and $\rho_t := r_{\tau_t^{-1}}$. Show that ρ_t has lifetime is τ_ζ and satisfies the SDE

$$d\rho_t = \frac{d-1}{2}f'(\rho_t)f(\rho_t)dt + f(\rho_t)dB_t.$$

5. Deduce some necessary and sufficient conditions on f for the almost sure convergence of θ_t as t goes to ζ.
6. Discuss the almost sure asymptotics of the Brownian motion in the hyperbolic space $\mathbb{H}^d$ which, in the above setting, corresponds to the case where $f(r) = \sinh(r)$.