Contrôle continu # 1

1 hour, no document allowed.

Exercice 1 *Gaussian conditional expectation*
Let ${}^t(X, Y, Z)$ be a centered Gaussian vector with covariance matrix

$$\Gamma := \begin{pmatrix} 4 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

1. Compute $\mathbb{E}[X^3|X^2]$.
2. What is the density of (Y, Z) ? Its Fourier transform ?
3. Without calculation, compute $\mathbb{E}[Y|Z]$ and $\mathbb{E}[(Y - \mathbb{E}[Y|Z])^2]$.
4. Compute $\mathbb{E}[X|(Y, Z)]$.

Exercice 2 *Conditionning and finite filtration*
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space where $\Omega = [0, 1]$, $\mathcal{F}$ is the Borelian sigma field on Ω and $\mathbb{P}$ is the uniform probability. Let us fix $K > 1$ an integer and $\alpha > 0$ a real number. For $n \in \mathbb{N}$, let us define

$$\mathcal{F}_n := \sigma\left(\left] \frac{j}{K^n}, \frac{j+1}{K^n} \right], \quad 0 \leq j \leq K^n - 1 \right).$$

Last, consider the random variables $(X_n)_{n \geq 0}$ such that

$$X_n := \begin{cases} \alpha^n & \text{if } 0 \leq \omega \leq K^{-n}, \\ 0 & \text{otherwise.} \end{cases}$$

1. Show that $(\mathcal{F}_n)$ is a filtration of $(\Omega, \mathcal{F}, \mathbb{P})$.
2. Recall that if $Z : \Omega \to \mathbb{R}$ is $\mathcal{F}_n$-measurable then

   $$Z(\omega) = \sum_{j=0}^{K^n - 1} \beta_j \mathbb{1}_{\left] \frac{j}{K^n}, \frac{j+1}{K^n} \right]}(\omega), \quad \text{with } \beta_j \in \mathbb{R} \text{ for } 0 \leq j \leq K^n - 1.$$

   Show that $\mathbb{E}[X_{n+1}|\mathcal{F}_n] = \frac{\alpha^{n+1}}{K} \mathbb{1}_{X_n > 0}$.
3. For which values of α the sequel (X_n) is a martingale with respect to $(\mathcal{F}_n)$? A submartingale ? A supermartingale ?
4. Determine the almost sure limit of X_n when n goes to infinity.
5. Does the converge holds in $\mathbb{L}^1$? In $\mathbb{L}^p$ for $p > 1$?