Brownian motion, moving metrics and entropy formulas

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Workshop
Stochastic Differential Geometry
Centre Henri Lebesgue
Rennes

May 30, 2013

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I. $g(t)$-Brownian motion

Riemannian metrics evolving as functions of time

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Riemannian metrics evolving as functions of time

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- Through the work of Perelman, deformation of Riemannian metrics $g(t)$ under certain evolution equations gained a lot of attention.
- This approach provides analytic and geometric tools for studying topological properties of manifolds.

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- The scalar curvature $\mathrm{R} := \mathrm{trace}\,\mathrm{Ric}$ satisfies the reaction-diffusion equation

$$\frac{\partial}{\partial t}\mathrm{R} = \Delta \mathrm{R} + 2|\mathrm{Ric}|^2.$$

Ricci solitons

Ricci solitons

A complete Riemannian manifold (M, g) is said to be a gradient Ricci soliton if there exists $f \in C^\infty(M;\mathbb{R})$ such that

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for some $\rho \in \mathbb{R}$. The function f is called a potential function of the Ricci soliton.

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Note that if $f = \text{const}$, then (M, g) is Einstein.

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- If (M, g) is Einstein with
$$\mathrm{Ric} = \rho\, g,$$
then
$$g(t) := (1 - 2\rho t)\, g$$
solves the Ricci flow equation.
- Likewise, if (M, g, f) is a gradient Ricci soliton with
$$\mathrm{Ric} + \mathrm{Hess}(f) = \rho\, g,$$
then
$$g(t) := (1 - 2\rho t)\, \varphi_t^* g$$
solves the Ricci flow equation. Here φ_t is the 1-parameter family of diffeomorphisms generated by $\nabla f/(1 - 2\rho t)$.

Depending on the sign $\pm$ in

$$\frac{\partial}{\partial t} g(t) = \pm 2 \,\mathrm{Ric}_{g(t)}, \quad g(0) = g_0$$

we talk about backward/forward Ricci flow.

Brownian motion with respect to a time varying metric

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- We call X shortly a g(t)-Brownian motion on M.

Evolution equation for densities

Let $X_t(x)$ be a $g(t)$-Brownian motion with starting point x.

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- Consider the smooth density

$$p \in C^\infty(]0,\infty[\times M \times M)$$

defined by

$$\mathbb{P}\{X_t(x) \in dy\} = p(t,x,y)\,\mathrm{vol}_t(dy), \quad t > 0\,,$$

where $\mathrm{vol}_t(dy)$ is the Riemannian volume on $(M, g(t))$.

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- We have for $p_t := p(t,x,\cdot)$

$$\begin{cases} \dfrac{d}{dt} p_t + \dfrac{1}{2}\left(\mathrm{trace}\,\dot{g}(t)\right)\, p_t = \Delta_{g(t)} p_t \\ \lim\limits_{t\downarrow 0}\ p_t(y)\,\mathrm{vol}_t(dy) \to \delta_x \ \text{ in law as } t \downarrow 0. \end{cases}$$

Consider the special case of the Ricci flow

$$\frac{d}{dt}g(t) = -2\,\mathrm{Ric}_{g(t)} \quad \text{(forward Ricci flow)}, \quad \text{resp.,}$$
$$\frac{d}{dt}g(t) = 2\,\mathrm{Ric}_{g(t)} \quad \text{(backward Ricci flow)}.$$

For $y \in M$, denote by $\mathrm{R}(t,y) := \operatorname{trace}\mathrm{Ric}_{g(t)}(y)$ the scalar curvature at the point y for the metric $g(t)$.

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Corollary

For the forward Ricci flow, we have:

$$\frac{d}{dt}p(t, x, \cdot) = \Delta_{g(t)}p(t, x, \cdot) + \mathrm{R}(t, \cdot)\,p(t, x, \cdot).$$

For the backward Ricci flow, we have:

$$\frac{d}{dt}p(t, x, \cdot) = \Delta_{g(t)}p(t, x, \cdot) - \mathrm{R}(t, \cdot)\,p(t, x, \cdot).$$

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- Consider positive solutions u of the following heat equation:

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or of the conjugate heat equation under Ricci flow

$$\begin{cases} \dfrac{\partial}{\partial t}u + \Delta_{g(t)}u \pm \mathrm{R}(t,\cdot)u = 0 \\ \dfrac{\partial}{\partial t}g_t = \pm \mathrm{Ric}_{g(t)}, \end{cases}$$

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- It is well-known how to use Stochatic Analysis to establish estimates of $\frac{|\nabla u|}{u}(x, T)$ as well as Harnack type inequalities for u.

Perelman's $\mathcal{F}$-functional

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Let M be a smooth compact manifold without boundary and let $\mathcal{M}$ be the set of Riemannian metrics on M.
Consider

$$\mathcal{F}\colon \mathcal{M} \times C^\infty(M) \to \mathbb{R},$$
$$\mathcal{F}(g,f) := \int_M (R + |\nabla f|^2)\, e^{-f}\, d\mathrm{vol}$$

where $R = \mathrm{trace}\,\mathrm{Ric}$ denotes the scalar curvature of (M,g).

Gradient flow to Perelman's $\mathcal{F}$-functional

The gradient flow of $\mathcal{F}$ on $\mathcal{M} \times C^\infty(M)$, under the constraint that

$$e^{-f}\, d\mathrm{vol} \equiv \text{const} \quad \text{(static measure)},$$

is given by the modified Ricci flow

$$\text{(MRF)} \qquad \begin{cases} \dfrac{\partial}{\partial t} g = -2\,(\mathrm{Ric} + \mathrm{Hess} f), \\ \dfrac{\partial}{\partial t} f = -\Delta f - R. \end{cases}$$

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(MRF)
$$\begin{cases} \dfrac{\partial}{\partial t} g = -2\,(\mathrm{Ric} + \mathrm{Hess} f), \\ \dfrac{\partial}{\partial t} f = -\Delta f - R. \end{cases}$$

If g and f evolve according to MRF, then

$$\frac{d}{dt}\mathcal{F}(g,f) = 2\int_M |\mathrm{Ric}_g + \mathrm{Hess}_g f|_g^2\; e^{-f}\, d\mathrm{vol}_g.$$

MRF modulo time dependent diffeomorphisms

Modulo diffeomorphisms the evolution of the metric is Ricci flow. More precisely, let ϕ_t be the flow generated by the (time-dependent) vector field ∇f, and let

$$\hat{g}(t) := \phi_t^* g(t), \quad \hat{f}(t) := \phi_t^* f(t) \equiv f(t) \circ \phi_t.$$

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Then, by a straightforward calculation,

$$\begin{cases} \dfrac{\partial}{\partial t}\hat{g} = -2\,\mathrm{Ric}_{\hat{g}}, \\ \dfrac{\partial}{\partial t}\hat{f} = -\hat{\Delta}\hat{f} - \hat{R} + |\hat{\nabla}\hat{f}|^2_{\hat{g}}, \end{cases}$$

where $\hat{R}$ and $\hat{\Delta}$ are taken with respect to the metric $\hat{g}(t)$.

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where $\hat{R}$ and $\hat{\Delta}$ are taken with respect to the metric $\hat{g}(t)$. Perelman's $\mathcal{F}$-functional is invariant under diffeomorphisms, hence

$$\mathcal{F}(g(t), f(t)) = \mathcal{F}(\hat{g}(t), \hat{f}(t)).$$

Conclusion Let g and f evolve according to

$$\begin{cases} \dfrac{\partial}{\partial t} g = -2\,\mathrm{Ric}, \\ \dfrac{\partial}{\partial t} f = -\Delta f - R + |\nabla f|^2, \end{cases}$$

then

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In particular, $\mathcal{F}(g(t), f(t))$ is non-decreasing in time and the monotonicity is strict unless

$$\mathrm{Ric} + \mathrm{Hess} f = 0 \quad \text{(steady Ricci soliton)}.$$

Proposition (Ricci flow under conjugate backward heat equation)
Let

$$u := e^{-f}.$$

Using $\Delta \log u = \frac{\Delta u}{u} - |\nabla \log u|^2$, then g and u evolve according to

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For $\mathcal{F}(g,u) = \int_M (R + |\nabla \log u|^2)\, u\, d\mathrm{vol}_g$ we have

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The integral $\int_M u(t,y)\,\mathrm{vol}_{g(t)}(dy)$ stays constant under the flow.

Theorem (Boltzmann-Shannon entropy) Let

$$\mu_t(dy) := u(t, y)\,\mathrm{vol}_{g(t)}(dy).$$

be the measure on M with density $u(t, \cdot)$ with respect to the volume measure to $g(t)$ as reference measure.

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Let $\mathcal{E}(t)$ be the Boltzmann-Shannon entropy of μ_t,

$$\mathcal{E}(t) = \int_M (u \log u)(t, y) \operatorname{vol}_{g(t)}(dy).$$

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$$\mathcal{E}(t) = \int_M (u\log u)(t,y)\,\mathrm{vol}_{g(t)}(dy).$$

Then the first two derivatives of $\mathcal{E}(t)$ are given by

$$\mathcal{E}'(t) = \int_M (R + |\nabla \log u|^2)\,u\,d\mathrm{vol}_g \equiv \mathcal{F}(g,u),$$
$$\mathcal{E}''(t) = 2\int_M |\mathrm{Ric} - \mathrm{Hess}\log u|^2\,u\,d\mathrm{vol}_g.$$

Backward Ricci flow under backward heat equation

Now M not necessarily compact. Consider

$$\begin{cases} \dfrac{\partial}{\partial t} g = 2\,\mathrm{Ric}, \\ \dfrac{\partial}{\partial t} u + \Delta u = 0. \end{cases}$$

III. A probabilistic version: first examples

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Let $X_t(x)$ be a $g(t)$-Brownian motion on M starting from x, and let $p(t,x,y)$ be the density of $X_t(x)$ with respect to $\mathrm{vol}_{g(t)}$. Instead of $\mathrm{vol}_{g(t)}$ take the heat kernel measure

$$m_t(dy) := p(t,x,y)\,\mathrm{vol}_{g(t)}(dy) = \mathbb{P}\left\{X_t(x) \in dy\right\}$$

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as reference measure. Note that the integral

$$\int_M u(t,y)\,m_t(dy) = \int_M u(t,y)\,p(t,x,y)\,\mathrm{vol}_{g(t)}(dy) = \mathbb{E}[u(t, X_t(x))]$$

stays constant under the flow, taking into account that $u(t, X_t(x))$ is a martingale.

Theorem

The entropy of $\mu_t := u(t,\cdot)\, dm_t \equiv u(t, X_t(x))\, d\mathbb{P}$ *calculates as*

$$\begin{aligned}\mathcal{E}(t) &= \int_M (u \log u)(t,y)\, p(t,x,y)\, \mathrm{vol}_{g(t)}(dy) \\ &= \mathbb{E}[(u \log u)(t, X_t(x))].\end{aligned}$$

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The first two derivatives of $\mathcal{E}(t)$ *are given by*

$$\mathcal{E}'(t) = \mathbb{E}\left[\frac{|\nabla u|^2}{u}(t, X_t(x))\right] \equiv \mathbb{E}\left[\left(|\nabla \log u|^2 u\right)(t, X_t(x))\right]$$

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Now the Wiener measure $\mathbb{P}$ takes the role as reference measure, and $\mathcal{E}(t)$ is the entropy of the measure $\mu_t = u(t, X_t(x))\, d\mathbb{P}$.

Application to ancient solutions of the heat equation

Hongxin Guo, Robert Philipowski, ATh. (2013)

- With the substitution $\tau := -t$ we get

$$\frac{\partial u}{\partial \tau} = \Delta_{g(-\tau)}, \quad \frac{\partial g}{\partial \tau} = -2\mathrm{Ric}, \quad \tau \le 0.$$

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- Let

$$\theta := \lim_{t\to\infty} \mathcal{E}'(t) \in [0, +\infty].$$

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Example Consider $u(t, y) = e^{y-t}$ on $\mathbb{R}$ with the standard metric. Then

$$\mathcal{E}(t) = t \quad \text{and} \quad \theta = 1.$$

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- Then u is constant if and only if $\theta = 0$.
- If the entropy $\mathcal{E}(t)$ grows sublinearly, i.e.

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  then $\theta = 0$ and hence u is constant.

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  then $\theta = 0$ and hence u is constant.
- If $2\mathrm{Ric} - \frac{\partial g}{\partial t}$ is strictly positive everywhere, then nonconstant solutions to the backward heat equation cannot have linear entropy.

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$$\mathbb{E}\left[\exp\left(-\int_0^t R(s, X_s(x))\,ds\right)\, u(t, X_t(x))\right] = u(0,x) \quad \text{indep. of } t.$$

IV. Ricci flow under conjugate backward heat equation

Consider

$$\begin{cases} \dfrac{\partial}{\partial t} g = -2\,\mathrm{Ric}, \\ \dfrac{\partial}{\partial t} u + \Delta u = Ru. \end{cases}$$

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$$\mathbb{E}\left[\exp\left(-\int_0^t R(s, X_s(x))\,ds\right)\, u(t, X_t(x))\right] = u(0, x) \quad \text{indep. of } t.$$

Take

$$\mathbb{P}_t(x) := \exp\left(-\int_0^t R(s, X_s(x))\,ds\right) d\mathbb{P}$$

as reference measure and to consider the entropy of the measure

$$\mu_t(x) := u(t, X_t(x))\, d\mathbb{P}_t(x).$$

The entropy of

$$\mu_t(x) := u(t, X_t(x))\, d\mathbb{P}_t(x)$$

reads as

$$\mathcal{E}(t) = \mathbb{E}_{t,x}\big[(u \log u)(t, X_t(x))\big].$$

where $\mathbb{E}_{t,x} := \mathbb{E}_{\mathbb{P}_t(x)}$ denotes expectation w/r to $\mathbb{P}_t(x)$.
The derivative of $\mathcal{E}(t)$ is given by

$$\mathcal{E}'(t) = \mathbb{E}_{t,x}\left[\left((R + \left|\nabla \log u\right|^2)u\right)(t, X_t(x))\right].$$

Theorem

Consider the following entropy functional

$$\mathrm{Ent}(g,u,t) := \mathbb{E}_{t,x}\big[(u\log u)(t,X_t(x))\big] - 2\int_0^t \mathbb{E}_{s,x}\big[\Delta u(s,X_s(x))\big]\,ds.$$

Then

$$\frac{d}{dt}\,\mathrm{Ent}(g,u,t) = \mathbb{E}_{t,x}\left[\left(\frac{|\nabla u|^2}{u} - 2\Delta u + Ru\right)(t,X_t(x))\right],$$

$$\frac{d^2}{dt^2}\,\mathrm{Ent}(g,u,t) = 2\,\mathbb{E}_{t,x}\left[\left(\left|\mathrm{Ric} - \mathrm{Hess}\log u\right|^2 u\right)(t,X_t(x))\right].$$

In terms of the martingale

$$\mathbb{M}_t := u(t, X_t(x)) \exp\left(-\int_0^t R(s, X_s(x))\, ds\right)$$

(mild conditions on u and R are required to make this local martingale a true martingale) and the measure $\mathbb{P}^*$ defined via

$$\mathbb{P}^*|\mathscr{F}_t := (\mathbb{M}_t \cdot \mathbb{P})|\mathscr{F}_t \equiv (\mathbb{M}_T \cdot \mathbb{P})|\mathscr{F}_t, \quad (t \le T),$$

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the above derivative formulas for $\mathrm{Ent}(g, u, t)$ write intrinsically as

$$\frac{d}{dt}\,\mathrm{Ent}(g, u, t) = \mathbb{E}^*\left[\left(\frac{|\nabla u|^2}{u^2} - 2\frac{\Delta u}{u} + R\right)(t, X_t(x))\right],$$

$$\frac{d^2}{dt^2}\,\mathrm{Ent}(g, u, t) = 2\,\mathbb{E}^*\left[\left(\left|\mathrm{Ric} - \mathrm{Hess}\log u\right|^2\right)(t, X_t(x))\right].$$

V. Relative entropies – Perelman's $\mathcal{W}$-functional

Let M again be a compact manifold. To study shrinking solitons, Perelman introduced the so-called $\mathcal{W}$-functional.

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Instead of the $\mathcal{F}$-functional one considers

$$\mathcal{W}\colon\ \mathcal{M}\times C^\infty(M)\times\mathbb{R}_+^*\to\mathbb{R},$$

$$\mathcal{W}(g,f,\tau):=\int_M\Big[\tau\left(R+|\nabla f|^2\right)+f-n\Big]\,\frac{e^{-f}}{(4\pi\tau)^{n/2}}\,d\mathrm{vol}_g$$

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$$\mathcal{W}(g,f,\tau) := \int_M \left[\tau\,(R+|\nabla f|^2)+f-n\right]\,\frac{e^{-f}}{(4\pi\tau)^{n/2}}\,d\mathrm{vol}_g$$

One studies the gradient flow of $\mathcal{W}(g,f,\tau)$. This leads to evolutions $g(t)$, $f(t)$ and $\tau(t)$ where τ is then a strictly positive smooth function $\tau(t)$.

Theorem (Perelman 2002)

Let $f(t)$, $u(t)$ and $\tau(t)$ develop according to

$$\begin{cases} \dfrac{\partial}{\partial t} g = -2\,\mathrm{Ric}, \\ \dfrac{\partial}{\partial t} f = -\Delta f - R + |\nabla f|^2 + \dfrac{n}{2\tau}, \\ \dfrac{\partial}{\partial t} \tau = -1. \end{cases}$$

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Then

$$\frac{d}{dt}\mathcal{W}(g, f, \tau) = 2\tau \int_M \left|\mathrm{Ric} + \mathrm{Hess}\, f - \frac{g}{2\tau}\right|^2 \frac{e^{-f}}{(4\pi\tau)^{n/2}}\, d\mathrm{vol}_g.$$

In particular, $\mathcal{W}(g, f, \tau)$ is non-decreasing in time and monotonicity is strict unless (M, g) satisfies

$$\mathrm{Ric} + \mathrm{Hess}\, f = \frac{g}{2\tau} \quad \text{(shrinking Ricci soliton).}$$

Let

$$u := \frac{e^{-f}}{(4\pi\tau)^{n/2}} \quad \text{or} \quad f = -\left(\log u + \frac{n}{2}\log(4\pi\tau)\right).$$

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Let

$$\mathcal{W}(g,u,\tau) := \int_M \left[\tau\left(R + |\nabla \log u|^2\right) - \log u - \frac{n}{2}\log(4\pi\tau) - n\right] u \, d\mathrm{vol}_g \,.$$

Then

$$\frac{d}{dt}\mathcal{W}(g,u,\tau) = 2\tau \int_M \left|\mathrm{Ric} - \mathrm{Hess}\log u - \frac{g}{2\tau}\right|^2 u\, d\mathrm{vol}_g.$$

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In particular, $\mathcal{W}(g,u,\tau)$ is non-decreasing in time and monotonicity is strict unless

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Entropy of the Gaussian measure on $\mathbb{R}^n$

Let

$$d\mu_t(y) = (4\pi t)^{-n/2}\, e^{-|y|^2/4t}\, dy =: \gamma_t(y)\, dy$$

be the standard Gaussian measure on $\mathbb{R}^n$. The Boltzmann-Shannon entropy of μ_τ is given as

$$\mathcal{E}_0(t) := \int_{\mathbb{R}^n} (\gamma_\tau \log \gamma_\tau)(y)\, dy = -\frac{n}{2}\big[1 + \log(4\pi\tau)\big].$$

Theorem (Relative entropy)

Let $g(t)$, $u(t)$ and $\tau(t)$ evolve according to Eq. (1). We normalize u such that

$$\int_M u(t)\, d\mathrm{vol}_{g(t)} \equiv 1.$$

Let

$$H(g,u,t) := \mathcal{E}(t) - \mathcal{E}_0(t) \equiv \int_M u \log u \, d\mathrm{vol}_g + \frac{n}{2}\big[1 + \log(4\pi\tau)\big].$$

Then

$$\frac{d}{dt} H(g,u,t) = \int_M \left[R + |\nabla \log u|^2 - \frac{n}{2\tau}\right] u \, d\mathrm{vol}_g$$

and

$$\frac{d}{dt} \tau\, H(g,u,t) = \mathcal{W}(g,u,\tau).$$

Remark Compare with Lei Ni's entropy formula for positive solutions of the heat equation on a Riemannian static manifold.

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Lei Ni (2004) Let $u > 0$ be a positive solution of the heat equation

$$\left(\frac{\partial}{\partial t} - \Delta\right) u = 0$$

on a compact static Riemannian manifold (M, g). Let

$$H(u,t) := \int_M u \log u \, d\mathrm{vol} + \frac{n}{2}\big[1 + \log(4\pi t)\big]$$

be the difference between the Boltzmann entropy of the measure $u(x)\,\mathrm{vol}(dx)$ on M (normalized to be a probability measure) and the Boltzmann entropy of the standard Gaussian measure $\mu(dy)$ on $\mathbb{R}^n$.

Then

$$\frac{d}{dt}H(u,t) = \int_M \left(\Delta \log u + \frac{n}{2t}\right) u \, d\mathrm{vol}.$$

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Then, by the differential Harnack inequality,

$$|\nabla \log u|^2 - \frac{\Delta u}{u} \leq \frac{n}{2t},$$

equivalently

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In this case $H(u,t)$ non-decreasing as function of t.

VI. Probabilistic version of Perelman's $\mathcal{W}$-functional

From now on, M no longer necessarily compact; mild integrability have to be imposed at certain places to make terms integrable or to get true martingales out of local martingales.

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From now on, M no longer necessarily compact; mild integrability have to be imposed at certain places to make terms integrable or to get true martingales out of local martingales.

We retake the Perelman's situation, i.e. $g(t)$, $u(t)$ and $\tau(t)$ evolve according to

$$\begin{cases} \dfrac{\partial}{\partial t} g = -2\,\mathrm{Ric}, \\ \dfrac{\partial}{\partial t} u + \Delta u = Ru, \\ \dfrac{\partial}{\partial t} \tau = -1. \end{cases} \tag{2}$$

For simplicity take $\tau(t) = T - t$.

Consider again on M the entropy functional

$$\mathrm{Ent}(g,u,t) := \mathbb{E}_{t,x}\big[(u\log u)(t,X_t(x))\big] \\ - 2\int_0^t \mathbb{E}_{s,x}\big[\Delta u(s,X_s(x))\big]\,ds,$$

and the corresponding expression on $\mathbb{R}^n$,

$$\mathrm{Ent}_0(t) = \mathbb{E}\big[(\gamma_{\tau(t)}\log\gamma_{\tau(t)})(B_t)\big] - 2\int_0^t \mathbb{E}\left[\Delta\gamma_{\tau(s)}(B_s)\right]ds,$$

where γ_t is the standard Gaussian kernel and B_t standard Brownian motion on $\mathbb{R}^n$ starting at 0.

A straightforward manipulation shows (with $\tau(t) = T - t$)

$$\begin{aligned}
\mathrm{Ent}_0(t) &= \int_{\mathbb{R}^n} \gamma_\tau(y) \log \gamma_\tau(y)\, dy - 2\, t\Delta\gamma_T(0) \\
&= -\frac{1}{2}\,\frac{n}{(4\pi T)^{n/2}}\left(\frac{t}{T} + \log(4\pi\tau)\right) + \frac{1}{2}\,\frac{n}{(4\pi T)^{n/2}}\,\frac{t}{T} \\
&= -\frac{1}{2}\,\frac{n}{(4\pi T)^{n/2}}\,\log(4\pi\tau).
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Normalize u such that

$$\mathbb{E}_{t,x}\big[u(t, X_t(x))\big] \equiv \frac{1}{(4\pi T)^{n/2}},$$

and consider

$$\mathbb{H}(g, u, t) := \mathrm{Ent}(g, u, t) - \mathrm{Ent}_0(t).$$

Theorem (Relative entropy; probabilistic version)

Let $g(t)$, $u(t)$ and $\tau(t)$ solve Eq. (2) with u normalized as above. Let

$$\mathbb{H}(t) \equiv \mathbb{H}(g,u,t) := \mathrm{Ent}(g,u,t) - \mathrm{Ent}_0(t) \quad \textit{and}$$

$$\mathbb{W}(\tau) \equiv \mathbb{W}(g,u,\tau) := \mathbb{E}^*\left[\left(\tau\left(R - 2\,\frac{\Delta u}{u} + |\nabla \log u|^2\right) - \log u - \frac{n}{2}\log(4\pi\tau) - \frac{n}{2}\right)(t, X_t(x))\right].$$

Then

$$\frac{d}{dt}\mathbb{H}(t) = \mathbb{E}^*\left[\left(|\nabla \log u|^2 - 2\,\frac{\Delta u}{u} + R - \frac{n}{2\tau}\right)(t, X_t(x))\right],$$

$$\frac{d}{dt}\mathbb{W}(\tau) = 2\tau\,\mathbb{E}^*\left[\left|\mathrm{Ric} - \mathrm{Hess}\log u - \frac{g}{2\tau}\right|^2(t, X_t(x))\right].$$

The right-hand-side of $\frac{d}{dt}\mathbb{H}(g, u, t)$ is non-positive due to the Harnack inequality for solutions of the conjugate heat equation under Ricci flow: if $R \geq 0$ then

$$|\nabla \log u|^2 - 2\,\frac{\Delta u}{u} + R - \frac{n}{2\tau} \leq 0,$$

equivalently

$$2\Delta \log u + |\nabla \log u|^2 - R + \frac{n}{2\tau} \geq 0.$$

VII. Applications

No breather theorems for non-compact manifolds

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- A general problem is to rule out non-trivial breathers, e.g. no steady or expanding breather theorems, like
every steady breather is Ricci-flat,
every expanding breather is a gradient soliton, etc
- The above formulas are appropriate to deal with non-compact manifolds.

The case of a surface $(\dim M = 2)$

On a surface of positive curvature things simplify a lot:

Instead of

$$\begin{cases} \dfrac{\partial}{\partial t} g = -2\,\mathrm{Ric}, \\ \dfrac{\partial}{\partial t} u + \Delta u = Ru \end{cases}$$

we may use

$$\begin{cases} \dfrac{\partial}{\partial t} g = -R\, g, \\ \left(\dfrac{\partial}{\partial t} - \Delta - R\right) R = 0. \end{cases}$$

Hence R itself can take over the role of u and then there is no conjugate heat equation involved anymore in the definition of the surface entropy.

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Hence after time reversal we have (without restrictions $T = 0$)

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and in addition

$$\left(\frac{\partial}{\partial t} R + \Delta + R\right) R = 0.$$

Note that

$$R(t, X_t)\, \exp\left(\int_0^t R(s, X_s)\, ds\right), \quad t \geq 0,$$

is a positive local martingale, and hence a supermartingale; in particular, it has an almost sure limit, as $t \to \infty$.

Change the measure as above

$$\frac{d\mathbb{P}^*|\mathscr{F}_t}{d\mathbb{P}|\mathscr{F}_t} := R(t, X_t)\, \exp\left(\int_0^t R(s, X_s)\, ds\right),$$

and define the surface entropy under the backward Ricci flow as

$$\mathcal{E}(t) = \mathbb{E}^*\left[\log R(t, X_t) - \int_0^t \frac{\Delta R}{R}(s, X_s)\, ds\right], \quad t \geq 0.$$

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$$\mathcal{E}(t) = \mathbb{E}^*\left[\log R(t, X_t) - \int_0^t \frac{\Delta R}{R}(s, X_s)\,ds\right], \quad t \geq 0.$$

The following derivative formulas hold:

$$\mathcal{E}'(t) = -\mathbb{E}^*\left[(R + \Delta \log R)\,(t, X_t)\right],$$
$$\mathcal{E}''(t) = \ 2\mathbb{E}^*\left[\left|\nabla\nabla \log R + \frac{1}{2} R g\right|^2 (t, X_t)\right].$$