Realizations of the Quantized Hamiltonian via Finite Dimensional Approximations to Wiener Measure

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Outline

I) Canonical quantization in $\mathbb{R}^d$ and (not) in M.

II) Motivating piecewise geodesic spaces for approximation

III) The metrics of piecewise geodesic spaces

IV) Results

V) Some notes on proof methods and further discussions

Canonical quantization in $\mathbb{R}^d$ and (not) in M.

$\mathbb{R}^d$ Quantum Mechanics

Classical Hamiltonian: With a potential $V : \mathbb{R}^d \to \mathbb{R}$,

$$\begin{aligned} H(q,p) &= \frac{1}{2} p \cdot p + V(q) \\ &= \text{K.E.} + \text{P.E.} \\ &= \frac{1}{2}|\dot{q}|^2 + V(q) \end{aligned}$$

with $p = \dot{q}$ the momentum.

Goal: Solve Schrödinger's Equation:

$$-\sqrt{-1}\frac{\partial}{\partial t}\psi(t,x) = \hat{H}\psi(t,x) \iff \psi(t,x) = e^{\frac{t}{i}\hat{H}}\psi_0(x)$$

where $\hat{H}$ is the *quantized* Hamiltonian.

$\mathbb{R}^d$ Canonical Quantization

The rule: Observables Become Operators.

$$q \rightsquigarrow \hat{q} = M_q \quad \text{and} \quad p \rightsquigarrow \hat{p} = -\sqrt{-1}\frac{\partial}{\partial q}$$

Therefore,

$$\hat{H} = \frac{1}{2}\hat{p} \cdot \hat{p} + V(\hat{q}) = -\frac{1}{2}\Delta + M_V(q)$$

M non-Canonical Quantization

Note: (M, g) is a d-dimension Riemannian manifold.

Curved Classical Hamiltonian: $V : M \to \mathbb{R}$ a potential,

$$\begin{aligned} H(q,p) &= \frac{1}{2} g^{ij}(q) p_i p_j + V(q) && (P1) \\ &= \frac{1}{2} \frac{1}{\sqrt{g}} p_i \sqrt{g} g^{ij}(q) p_j + V(q) && (P2) \end{aligned}$$

So, $H \rightsquigarrow \hat{H} =$??:

$$\begin{aligned} (P1) &\rightsquigarrow -\frac{1}{2} g^{ij}(q) \frac{\partial^2}{\partial q_i \partial q_j} + M_V(q) \\ (P2) &\rightsquigarrow -\frac{1}{2} \Delta_M + M_V(q) \\ (P3) &\rightsquigarrow \cdots \end{aligned}$$

Motivating piecewise geodesic spaces for approximation

Wiener Measure

- $\mathcal{P} = \{0 = s_0 < s_1 < \cdots < s_n = t\}$
- $\Delta_i s = s_i - s_{i-1}$
- $p_t(x, y) = (2\pi t)^{-d/2} e^{-\frac{|x-y|^2}{2t}} =$ heat kernel at time t

Theorem (Norbert Wiener)

There is a unique probability measure ν_t *on*

$$W_t(\mathbb{R}^d) := \{\sigma \in C([0, t] \to \mathbb{R}^d) : \sigma(0) = 0\}$$

such that

$$\int_{W_t(\mathbb{R}^d)} f(\sigma) d\nu_t(\sigma) = \int_{(\mathbb{R}^d)^n} F(x_1, \ldots, x_n) \prod_{i=1}^{n} p_{\Delta_i s}(x_i, x_{i-1}) dx_1 \cdots dx_n$$

whenever $f : W_t(\mathbb{R}^d) \to \mathbb{R}$ *is of the form* $f(\sigma) = F(\sigma(s_1), \ldots, \sigma(s_n))$.

The Heat Equation - Path Integral

Corollary

For $f : \mathbb{R}^d \to \mathbb{R}$ continuous,

$$e^{\frac{t}{2}\Delta} f(0) = \int_{W_t(\mathbb{R}^d)} f(\sigma(t)) d\nu_t(\sigma).$$

Proof.

From the previous Theorem,

$$\begin{aligned} \int_{W_t(\mathbb{R}^d)} f(\sigma(t)) d\nu_t(\sigma) &= \int_{\mathbb{R}^d} f(x) p_t(x, o) dx \\ &= e^{\frac{t}{2}\Delta} f(0). \end{aligned}$$

□

Potential Goal

- Add a potential $V \in C(\mathbb{R}^d \to \mathbb{R}_+)$
- Consider the Schrödinger operator $\hat{H} = -\frac{1}{2}\Delta + V$.

Goal: Describe $e^{-t\hat{H}}f(0)$ by a path integral.

Theorem (Trotter Product Formula)

If A and B are $N \times N$ matrices, then

$$e^{A+B} = \lim_{n\to\infty} \left(e^{\frac{A}{n}} e^{\frac{B}{n}} \right)^n$$

Approach to a Solution

I) Apply Trotter's formula with $A = \frac{t}{2}\Delta$ and $B = -tV$.

$$e^{(t\Delta/2 - tV)} = \lim_{n\to\infty} \left[e^{\frac{t\Delta}{2n}} e^{-\frac{tV}{n}} \right]^n$$

II) Use the fact that $e^{\frac{t\Delta}{2}} g(0) = \int_{\mathbb{R}^d} g(x) p_t(x,0) dx$.

$$\left[e^{\frac{t\Delta}{2n}} e^{-\frac{tV}{n}} \right]^n f(0) =$$

$$\int_{(\mathbb{R}^d)^n} \left[\prod_{i=1}^n p_{\Delta_i s}(x_{i-1}, x_i) e^{-tV(x_i)/n} \right] f(x_n) \prod_{i=1}^n dx_i$$

$$= \left(\frac{1}{2\pi t} \right)^{\frac{dn}{2}} \int_{(\mathbb{R}^d)^n} e^{-\sum_{i=1}^n \left[\frac{1}{2} \left(\frac{|x_i - x_{i-1}|}{\Delta_i s} \right)^2 + V(x_i) \right] \Delta_i s} f(x_n) \prod_{i=1}^n dx_i$$

$H_{\mathcal{P},t}(\mathbb{R}^d)$

Recall: $\mathcal{P} = \{0 = s_0 < s_1 < \cdots < s_n = t\}$

Definition

$H_{\mathcal{P},t}(\mathbb{R}^d) = \{$*continuous piecewise linear maps* $\sigma : [0,t] \to \mathbb{R}^d\}$.

Note: $H_{\mathcal{P},t}(\mathbb{R}^d)$ " $\to$ " $W_t(\mathbb{R}^d)$ as $\mathcal{P} \to 0$.

How We Think of $H_{\mathcal{P}}(\mathbb{R}^d)$

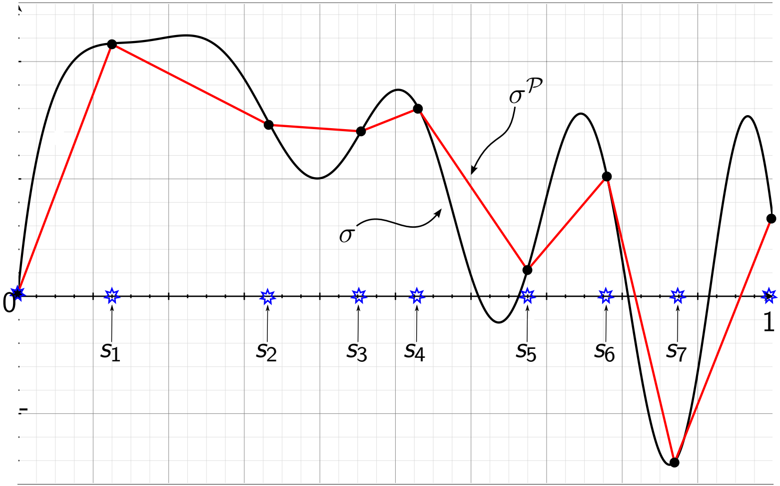

FIGURE: $\sigma^{\mathcal{P}} \in H_{\mathcal{P},1}(\mathbb{R}^d)$ approximating $\sigma \in W(\mathbb{R}^d)$.

From Whence the Path Integral Came

I. $H_{\mathcal{P}}(\mathbb{R}^d) \simeq (\mathbb{R}^d)^n$

II. **Take** $\sigma^{\mathcal{P}} \in H_{\mathcal{P}}(\mathbb{R}^d)$:

$$\Longrightarrow (\sigma^{\mathcal{P}})'(s) = \frac{\sigma^{\mathcal{P}}(s_i) - \sigma^{\mathcal{P}}(s_{i-1})}{\Delta_i s} \quad \text{for } s \in (s_{i-1}, s_i)$$

$$\Longrightarrow \int_0^1 |(\sigma^{\mathcal{P}})'(r)|^2 dr = \sum_{i=1}^{n} \left(\frac{|\sigma^{\mathcal{P}}(s_i) - \sigma^{\mathcal{P}}(s_{i-1})|}{\Delta_i s} \right)^2 \Delta_i s$$

$$\simeq \sum_{i=1}^{n} \left(\frac{|x_i - x_{i-1}|}{\Delta_i s} \right)^2 \Delta_i s$$

From Whence the Path Integral Came

Lemma

Define the measure $\nu_{\mathcal{P}}$ on $H_{\mathcal{P}}(\mathbb{R}^d)$ by

$$\nu_{\mathcal{P}}(\sigma) = \left(\frac{1}{2\pi t}\right)^{\frac{dn}{2}} e^{-\frac{1}{2}\int_0^t |\sigma'(s)|^2 ds} dm_{\mathcal{P}},$$

where $m_{\mathcal{P}}$ is the Lebesgue measure on $H_{\mathcal{P}}(\mathbb{R}^d)$. Then,

$$\left(\frac{1}{2\pi t}\right)^{\frac{dn}{2}} \int_{(\mathbb{R}^d)^n} e^{-\sum_{i=1}^n \frac{1}{2}\left(\frac{|x_i - x_{i-1}|}{\Delta_i s}\right)^2 \Delta_i s} f(x_n) \prod_{i=1}^n dx_i = \int_{H_{\mathcal{P},t}(\mathbb{R}^d)} f(\sigma(t)) d\nu_{\mathcal{P}}(\sigma).$$

The Point of the Comparison

To understand the limit as $n \to \infty$ of:

$$\left(\frac{1}{2\pi t}\right)^{\frac{dn}{2}} \int_{(\mathbb{R}^d)^n} e^{-\sum_{i=1}^{n} \frac{1}{2}\left(\frac{|x_i - x_{i-1}|}{\Delta_i s}\right)^2 \Delta_i s} f(x_n) \prod_{i=1}^{n} dx_i$$

we can instead attempt to understand the limit as mesh$(\mathcal{P}) \to 0$ of,

$$\int_{H_{\mathcal{P}}(\mathbb{R}^d)} f(\sigma(t)) d\nu_{\mathcal{P}}(\sigma)$$

A Flat Example of the Main Theorem

Theorem

Let $\Gamma_{\mathcal{P}}$ be any inner product on $H_{\mathcal{P},t}(\mathbb{R}^d)$. If $f : W(\mathbb{R}^d) \to \mathbb{R}$ is bounded and continuous then,

$$\lim_{|\mathcal{P}|\to 0} \int_{H_{\mathcal{P},t}(\mathbb{R}^d)} f(\sigma) d\nu_{\Gamma_{\mathcal{P}}}(\sigma) = \int_{W_t(\mathbb{R}^d)} f(\sigma) d\nu_t(\sigma).$$

Here $d\nu_{\Gamma_{\mathcal{P}}}$ is a probability measure on $H_{\mathcal{P}}(\mathbb{R}^d)$ defined by,

$$d\nu_{\Gamma_{\mathcal{P}}} = \frac{1}{Z_t} e^{-\frac{1}{2}\int_0^t |\sigma'(s)|^2 ds} d\,\mathrm{Vol}_{\Gamma_{\mathcal{P}}}.$$

What's Special Here?

- If we normalize $\nu_{\Gamma_{\mathcal{P}}}$ to be a probability measure, then it will be independent of choice of inner product $\Gamma_{\mathcal{P}}$.
- For a more general Riemannian metric, this is not true!

Back to H

If:

$$\lim_{|\mathcal{P}|\to 0}\int_{H_{\mathcal{P},t}(\mathbb{R}^d)} f(\sigma)d\nu_{\Gamma_{\mathcal{P}}}(\sigma) = \int_{W_t(\mathbb{R}^d)} f(\sigma)\rho(\sigma)d\nu_t(\sigma).$$

Then:

$$e^{-t\hat{H}}f(o) \quad = \quad \int_{W_t(\mathbb{R}^d)} f(\sigma)d\nu_t(\sigma).$$

The metrics of piecewise geodesic spaces

Going Backwards?

Background: (M, g, o) is a d-dimensional Riemannian manifold with metric g and fixed point $o \in M$.

Goal: Use the flat-case derivation of the path integral representation of $e^{-t\hat{H}}$ to derive an analogous expression in the manifold case. Then, use this expression to understand $\hat{H}$ on $L^2(M)$.

Necessary Background

- $W_t(M) = \{\sigma \in C([0,t] \to M) : \sigma(0) = o\}$
- $H_{\mathcal{P},t}(M) = \{\sigma \in W_t(M) : \sigma \text{ is piecewise geodesic (w.r.t. } \mathcal{P})\}$
- $\mathcal{P} = \{0 = s_0 < s_1 < \cdots < s_n = t\}$

Theorem

There is a unique probability measure ν_t on $W_t(M)$ such that

$$\int_{W_t(M)} f(\sigma) d\nu_t(\sigma) = \int_{M^n} F(x_1, ..., x_n) \prod_{i=1}^{n} p_{\Delta_i s}(x_i, x_{i-1}) dx_1 \cdots dx_n$$

whenever $f : W_t(M) \to \mathbb{R}$ is of the form $f(\sigma) = F(\sigma(s_1), ..., \sigma(s_n))$.

Main Type Theorem

- For metric $\Gamma_{\mathcal{P}}$ on $H_{\mathcal{P},t}(M)$ define

$$d\nu_{\Gamma_{\mathcal{P}}}(\sigma) \;=\; \frac{1}{Z_{\Gamma_{\mathcal{P}}}} e^{-\frac{1}{2}\int_0^1 |\sigma'(s)|^2 ds} d\,\mathrm{Vol}_{\Gamma_{\mathcal{P}}}(\sigma)$$

Theorem (Version Beta)

If $f : W_t(M) \to \mathbb{R}$ is bounded and continuous, then there exists bounded measurable map $\rho : W_t(M) \to [0, \infty)$ such that,

$$\lim_{|\mathcal{P}|\to 0} \int_{H_{\mathcal{P},t}(M)} f(\sigma) d\nu_{\Gamma_{\mathcal{P}}}(\sigma) = \int_{W_t(M)} f(\sigma)\rho(\sigma) d\nu_t(\sigma).$$

- $\implies e^{-t\hat{H}} f(o) = \int_{W_t(M)} f(\sigma)\rho(\sigma) d\nu_t(\sigma)$

Notational Ease

From here on, we set $t = 1$.

- $W(M) := W_1(M)$
- $H_{\mathcal{P}}(M) := H_{\mathcal{P},1}(M)$
- $\nu := \nu_1$

A Note on $X \in T_\sigma H_{\mathcal{P}}(M)$

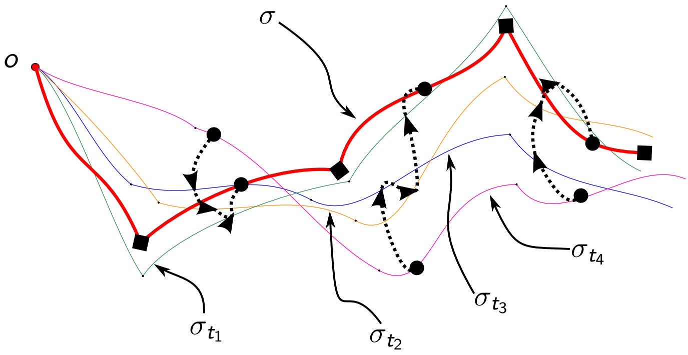

Figure: Family of curves passing through σ.

A Note on $X \in T_\sigma H_{\mathcal{P}}(M)$

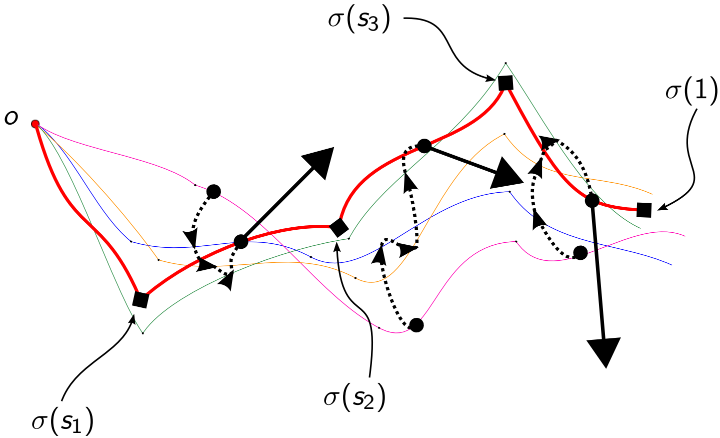

Figure: $X \in T_\sigma H_{\mathcal{P}}(M) \iff X$ a piecewise Jacobi field along σ.

Metrics on $H_{\mathcal{P}}(M)$

Definition

If $X, Y \in T_\sigma H_{\mathcal{P}}(M)$ then define

$$S^1_{\mathcal{P}}(X, Y) := \sum_{i=1}^{n} g(X'(s_{i-1}+), Y'(s_{i-1}+))\Delta_i s$$

$$S_{\mathcal{P}}(X, Y) := \sum_{i=1}^{n} g(X(s_i), Y(s_i))\Delta_i s$$

$$G^1_{\mathcal{P}}(X, Y) := \int_0^1 g(X'(s), Y'(s))ds$$

$$G_{\mathcal{P}}(X, Y) := \int_0^1 g(X(s), Y(s))ds$$

- $\Delta_i s := s_i - s_{i-1}$
- X' (resp. Y') is the covariant derivative of X (resp. Y) with respect to the Levi-Civita connection

Results

A probabilistically natural result of Andersson and Driver

- $S^1_{\mathcal{P}}(X, Y) = \sum_{i=1}^{n} g(X'(s_{i-1}+), Y'(s_{i-1}+))\Delta_i s$
- $d\nu_{S^1_{\mathcal{P}}}(\sigma) = \frac{1}{Z_{S^1_{\mathcal{P}}}} e^{-\frac{1}{2}\int_0^1 |\sigma'(s)|^2 ds}\, \mathrm{dVol}_{S^1_{\mathcal{P}}}$

Theorem (From Andersson and Driver in 1999 (JFA))

Suppose (M, g, o) is a pointed d-dimensional Riemannian manifold in which the curvature and its derivative are bounded. If $f : W(M) \to \mathbb{R}$ is bounded and continuous then,

$$\lim_{|\mathcal{P}|\to 0} \int_{H_{\mathcal{P}}(M)} f(\sigma) d\nu_{S^1_{\mathcal{P}}}(\sigma) = \int_{W(M)} f(\sigma) d\nu(\sigma).$$

How?

In the paper Andersson and Driver showed that

$$\nu_{S^1_{\mathcal{P}}} = \mathsf{Law}_{\nu}(b^{\mathcal{P}})$$

- $b^{\mathcal{P}}$ is a piecewise approximation to Brownian motion

Results II

- $S_{\mathcal{P}}(X,Y) = \sum_{i=1}^{n} g(X(s_i), Y(s_i))\Delta_i s$

Theorem (Andersson & Driver, 1999)

Suppose (M, g, o) is a pointed d-dimensional Riemannian manifold in which the curvature and its derivative are bounded. If $f : W(M) \to \mathbb{R}$ is bounded and continuous then,

$$\lim_{|\mathcal{P}|\to 0} \int_{H_{\mathcal{P}}(M)} f(\sigma) d\nu_{S_{\mathcal{P}}}(\sigma) = \int_{W(M)} f(\sigma) e^{-\frac{1}{6}\int_0^1 \mathrm{Scal}(\sigma(s))ds} d\nu(\sigma).$$

Results III

- $G^1_{\mathcal{P}}(X,Y) := \int_0^1 g(X'(s), Y'(s))ds$

Theorem (Lim, 2007)

Suppose (M, g, o) is a pointed d-dimensional Riemannian manifold in which the curvature and its derivative are bounded. Also suppose that the sectional curvature of M is bounded such that $0 \le SC < \frac{1}{17d}$. If $f : W(M) \to \mathbb{R}$ is bounded and continuous then,

$$\lim_{|\mathcal{P}|\to 0} \int_{H_{\mathcal{P}}(M)} f(\sigma) d\nu_{G^1_{\mathcal{P}}}(\sigma) = \int_{W(M)} f(\sigma)\rho(\sigma) d\nu(\sigma),$$

where

$$\rho(\sigma) := e^{-\frac{1}{6}\int_0^1 \mathrm{Scal}(\sigma(s))ds} \sqrt{\det(I + K_\sigma)},$$

and K_σ is an infinite dimensional trace class operator.

Result IV

- $G_{\mathcal{P}}(X, Y) = \int_0^1 g(X(s), Y(s))ds$

Theorem (TL, 2012)

Suppose (M, g, o) is a pointed d-dimensional Riemannian manifold in which the curvature and its derivative are bounded. Also suppose that the sectional curvature of M is non-positive. If $f : W(M) \to \mathbb{R}$ is bounded and continuous then,

$$\lim_{|\mathcal{P}|\to 0} \int_{H_{\mathcal{P}}(M)} f(\sigma) d\nu_{G^1_{\mathcal{P}}}(\sigma) = \int_{W(M)} f(\sigma) e^{-\tau_G \int_0^1 \mathrm{Scal}(\sigma(s))ds} d\nu(\sigma),$$

where $\tau_G = \frac{2+\sqrt{3}}{20\sqrt{3}}$.

Understanding H

Theorem (Curved Feynmann-Kac Formula)

For a non-negative and continuous map $S : M \to \mathbb{R}$ and a bounded and continuous map $f : M \to \mathbb{R}$ then

$$e^{-tK} f(o) = \int_{W_t(M)} f(\sigma(t)) e^{-\int_0^t S(\sigma(s))ds} d\nu_t(\sigma),$$

implies $K = -\frac{1}{2}\Delta + S$.

$\therefore$ Feynmann-Kac says:

- $H_{S^1_{\mathcal{P}}} = -\frac{1}{2}\Delta + V$
- $H_{S_{\mathcal{P}}} = -\frac{1}{2}\Delta + V + \frac{1}{6}\,\mathrm{Scal}$
- $H_{G^1_{\mathcal{P}}} = \;\; ??$
- $H_{G_{\mathcal{P}}} = -\frac{1}{2}\Delta + V + \frac{2+\sqrt{3}}{20\sqrt{3}}\,\mathrm{Scal}$

Other Possible Realizations

$\hat{H} = \frac{1}{2}\Delta + V + \frac{1}{\kappa}$ Scal

- $\kappa = 6$ Cheng 72.
- $\kappa = 12$, De Witt 1957, Um 73, Atsuchi & Maeda 85, and Darling 85. Geometric Quantization.
- $\kappa = 8$ Marinov 1980 and De Witt 1992.
- Inahama (2005) Osaka J. Math.
- Semi-group proofs and extensions of Andersson & Driver;
  - Butko (2006)
  - O. G. Smolyanov, Weizsäcker, Wittich, Potential Anal. 26 (2007).
  - Bär and Frank Pfäffle, Crelle 2008.
- Fine and Sawin CMP (2008) – supersymmetic version.

Some notes on proof methods and further discussions

Proof Outline

Recall: $d\nu_\Gamma(\sigma) = \frac{1}{Z_\Gamma} e^{-\frac{1}{2}\int_0^1 |\sigma'(s)|^2 ds}\, \mathrm{dVol}_\Gamma.$

Define: $\rho_{\mathcal{P}} = \frac{d\nu_{G_{\mathcal{P}}}}{d\nu_{S^1_{\mathcal{P}}}}.$

1) Get a uniform integrability type estimate

$$\sup_{\mathcal{P}} \int_{H_{\mathcal{P}}(M)} |\rho_{\mathcal{P}}|^p d\nu_{S_{\mathcal{P}}} < \infty.$$

2) Find a limit

$$\lim_{|\mathcal{P}|\to 0} \rho_{\mathcal{P}}(\sigma^{\mathcal{P}}) = \exp\left\{ -\frac{2+\sqrt{3}}{20\sqrt{3}} \int_0^1 \mathrm{Scal}(\sigma(s))ds \right\}$$

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Why Derive?

If we have a theorem like,

Theorem (Meta 1)

There exists some $\rho : W(M) \to \mathbb{R}$ such that $\rho_{\mathcal{P}} \to \rho$.

Then,

Theorem (Meta 2)

Given $\nu_{G_{\mathcal{P}}}$ as above, $d\nu_{G_{\mathcal{P}}} \to \rho d\nu$.

Meta 2 Proof.

Andersson and Driver proved that $d\nu_{S^1_{\mathcal{P}}} \to d\nu$. Therefore,

$$d\nu_{G_{\mathcal{P}}} = \rho_{\mathcal{P}} d\nu_{S^1_{\mathcal{P}}} \to \rho d\nu.$$

□

Finding $\rho_{\mathcal{P}}$

Recall: $d\nu_\Gamma(\sigma) = \frac{1}{Z_\Gamma} e^{-\frac{1}{2}\int_0^1 |\sigma'(s)|^2 ds}\, \mathrm{dVol}_\Gamma$.

Given a frame $\{f_{i,a} : 1 \leq i \leq n, 1 \leq a \leq d\}$ on $H_{\mathcal{P}}(M)$,

$$\rho_{\mathcal{P}} = \frac{d\nu_{G_{\mathcal{P}}}}{d\nu_{S^1_{\mathcal{P}}}} = \frac{Z_{S^1_{\mathcal{P}}}}{Z_{G_{\mathcal{P}}}} \sqrt{\frac{\det(G_{\mathcal{P}}(f_{i,a}, f_{j,c}))}{\det(S^1_{\mathcal{P}}(f_{i,a}, f_{j,c}))}}$$

Note: This equation is true for any choice of $\{f_{i,a}\}$.

The Basis $\{f_{i,a}\}$

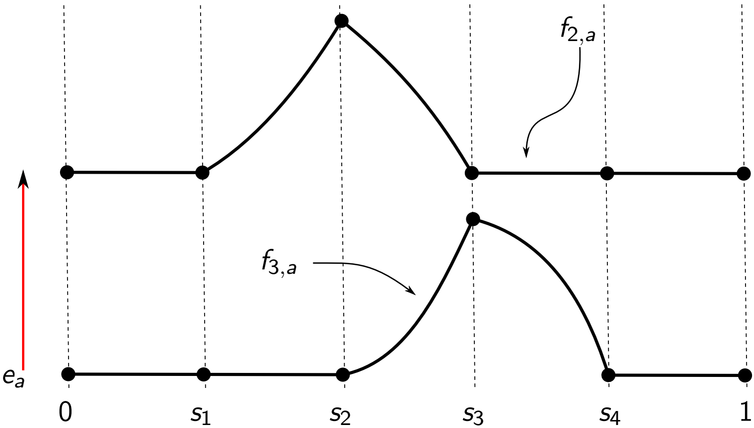

Figure: Two of the basis vectors from $\{f_{i,a}\}$

Appearance of $\rho_{\mathcal{P}}$

- From here on $\mathcal{P}$ is evenly spaced: $s_i = \frac{i}{n}$.

Proposition

If $\#(\mathcal{P}) = n+1$, then there exist $n \times n$ tri-diagonal block matrices with $d \times d$ blocks, $\mathcal{L}_{\mathcal{P}}$ and $\mathcal{R}_{\mathcal{P}}$, such that

$$\rho_{\mathcal{P}} = \sqrt{\frac{\det(\mathcal{L}_{\mathcal{P}} + \mathcal{R}_{\mathcal{P}})}{\det(\mathcal{L}_{\mathcal{P}})}}.$$

Moreover, $\mathcal{L}_{\mathcal{P}}$ is a constant matrix and $\mathcal{R}_{\mathcal{P}}$ depends on $\sigma \in H_{\mathcal{P}}(M)$ as well as the curvature of M.

The Matrix $\mathcal{L}_{\mathcal{P}}$

$$\mathcal{L}_{\mathcal{P}} = \frac{1}{6n^3} \begin{pmatrix} 4\mathcal{I} & \mathcal{I} & 0 & 0 & 0 \\ \mathcal{I} & 4\mathcal{I} & \mathcal{I} & 0 & 0 \\ 0 & \mathcal{I} & \ddots & \mathcal{I} & 0 \\ 0 & 0 & \mathcal{I} & 4\mathcal{I} & \mathcal{I} \\ 0 & 0 & 0 & \mathcal{I} & 2\mathcal{I} \end{pmatrix}$$

The Matrix $\mathcal{R}_{\mathcal{P}}$

$$\mathcal{R}_{\mathcal{P}} = \frac{1}{360n^5}\begin{pmatrix} D_1 & M_2 & 0 & 0 & 0 \\ M_2 & D_2 & M_3 & 0 & 0 \\ 0 & M_3 & \ddots & M_4 & 0 \\ 0 & 0 & M_{n-1} & D_{n-1} & M_n \\ 0 & 0 & 0 & M_n & D_n \end{pmatrix} + O\left(\text{small}\right)$$

$$D_i = 64A_i - 16A_{i+1}$$
$$M_i = 13A_i - 7A_{i+1}$$

where for $\mathbf{x} \in \mathbb{R}^d$,

- $A_i(\sigma)\mathbf{x} = //_{s_{i-1}}(\sigma)^{-1}\mathbf{R}\left(\sigma'(s_{i-1}+), //_{s_{i-1}}(\sigma)\mathbf{x}\right)\sigma'(s_{i-1}+)$
- $//_s(\sigma) : T_oM \simeq \mathbb{R}^d \to T_{\sigma(s)}M$ is parallel translation along σ.

From $H_{\mathcal{P}}(M)$ to $H_{\mathcal{P}}(\mathbb{R}^d)$

Definition

The map $b^{\mathcal{P}} : H_{\mathcal{P}}(M) \to H_{\mathcal{P}}(\mathbb{R}^d)$ *defined by*

$$b^{\mathcal{P}}(\sigma)(s) = b_s^{\mathcal{P}}(\sigma) = \int_0^s //_r(\sigma)^{-1}\sigma'(r)dr$$

will be called the rolling map.

Friendly Translation:

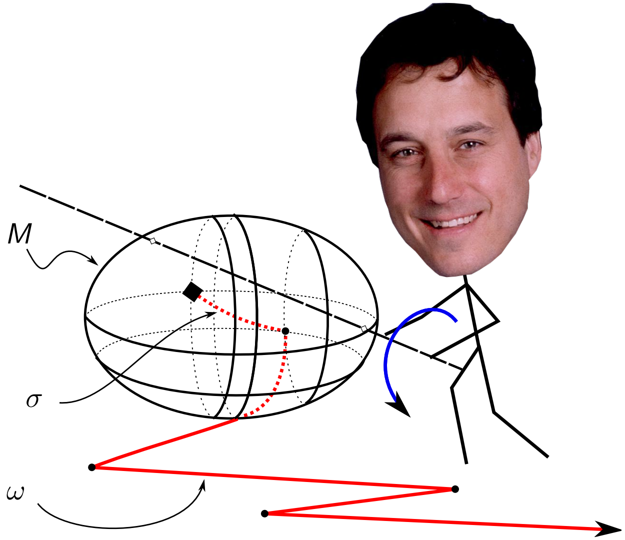

Figure: $\omega = b^{\mathcal{P}}(\sigma) \in \mathbb{R}^d$ "rolls" onto $\sigma \in M$

Some Brownian Motion

Notation

- $(B_s)_{s\in[0,1]}$ *is an* $\mathbb{R}^d$*-valued Brownian motion*
- $\Delta_i B = B_{s_i} - B_{s_{i-1}}$
- $\Delta_i b^{\mathcal{P}} = b^{\mathcal{P}}_{s_i} - b^{\mathcal{P}}_{s_{i-1}}$

Proposition

The distribution of $(\Delta_1 b^{\mathcal{P}}, ..., \Delta_n b^{\mathcal{P}})$ *with respect to the measure* $\nu_{S^1_{\mathcal{P}}}$ *is the same as* $(\Delta_1 B, ..., \Delta_n B)$.

$\rho_{\mathcal{P}}$ Size

Proposition

There exists positive numbers $\zeta, p_{i,j} \in \mathbb{R}$ *such that* $\sum_j p_{i,j} = 1$, $\sum_i p_{i,j} < 3$ *and,*

$$|\rho_{\mathcal{P}}| \le \prod_{i=1}^{n} \left(\sum_{j=1}^{n} p_{i,j} \cosh(\zeta \|\Delta_i b^{\mathcal{P}}\|) \right)^d .$$

Lemma

Given any $\zeta > 0$, $p \in \mathbb{N}$, *and* $p_{i,j}$ *as above,*

$$\limsup_{n\to\infty} \mathbb{E}\left[\prod_{i=1}^{n} \left(\sum_{j=i}^{n} p_{i,j} \cosh(\zeta \|\Delta_j B\|) \right)^p \right] < \infty$$

Uniform Integrability Type Estimate

Corollary

$$\limsup_{\mathcal{P}} \int_{H_{\mathcal{P}}(M)} |\rho_{\mathcal{P}}|^p d\nu_{S^1_{\mathcal{P}}} < \infty.$$

Proof.

$$\int_{H_{\mathcal{P}}(M)} |\rho_{\mathcal{P}}|^p d\nu_{S^1_{\mathcal{P}}} \leq \int_{H_{\mathcal{P}}(M)} \prod_{i=1}^{n} \left(\sum_{j=1}^{n} p_{i,j} \cosh(\zeta \|\Delta_i b^{\mathcal{P}}\|) \right)^{dp} d\nu_{S^1_{\mathcal{P}}}$$

$$= \mathbb{E}\left[\prod_{i=1}^{n} \left(\sum_{j=i}^{n} p_{i,j} \cosh(\zeta \|\Delta_j B\|) \right)^{dp} \right]$$

□

$\rho_{\mathcal{P}}$ Rewritten

$$\rho_{\mathcal{P}} = \sqrt{\frac{\det(\mathcal{L}_{\mathcal{P}} + \mathcal{R}_{\mathcal{P}})}{\det(\mathcal{L}_{\mathcal{P}})}} = \sqrt{\det(\mathcal{I} + \mathcal{L}_{\mathcal{P}}^{-1/2}\mathcal{R}_{\mathcal{P}}\mathcal{L}_{\mathcal{P}}^{-1/2})}$$

Proposition

If U is a square matrix with $|U| << 1$,

$$\det(\mathcal{I} + U) \approx \exp\{\mathrm{tr}(U)\}.$$

Corollary

$$\textit{For large } n,\ \rho_{\mathcal{P}} \approx \sqrt{\exp\{\mathrm{tr}(\mathcal{L}_{\mathcal{P}}^{-1/2}\mathcal{R}_{\mathcal{P}}\mathcal{L}_{\mathcal{P}}^{-1/2})\}}.$$

Scal's Cameo and Limiting

Proposition

$$\mathrm{tr}(\mathcal{L}_{\mathcal{P}}^{-1/2}\mathcal{R}_{\mathcal{P}}\mathcal{L}_{\mathcal{P}}^{-1/2})(\sigma^{\mathcal{P}}) \sim -2\tau_G \sum_{i=1}^{n} \mathrm{Scal}(\sigma^{\mathcal{P}}(s_{i-1}))\Delta_i s$$
$$\to -2\tau_G \int_0^1 \mathrm{Scal}(\sigma(s))ds.$$

Corollary

$$\rho_{\mathcal{P}}(\sigma^{\mathcal{P}}) \to e^{-\tau_G \int_0^1 \mathrm{Scal}(\sigma(s))ds}$$

Proof.

$$\rho_{\mathcal{P}}(\sigma^{\mathcal{P}}) \approx \sqrt{\exp\{\mathrm{tr}(\mathcal{L}_{\mathcal{P}}^{-1/2}\mathcal{R}_{\mathcal{P}}\mathcal{L}_{\mathcal{P}}^{-1/2})(\sigma^{\mathcal{P}})\}} \to \sqrt{e^{-2\tau_G \int_0^1 \mathrm{Scal}(\sigma(s))ds}}.$$

□

Thank You!

Fin

Encore II

Recall: $A_i(\sigma)\mathbf{x} = //_{s_{i-1}}(\sigma)^{-1}\mathbf{R}\left(\sigma'(s_{i-1}+), //_{s_{i-1}}(\sigma)\mathbf{x}\right)\sigma'(s_{i-1}+)$

A rough outline of how Scal arises:

1. $\mathrm{tr}(\mathbf{R}) = \mathrm{Ric} \implies \mathrm{tr}(A_i) \sim \langle \mathrm{Ric}\,\Delta_i b^{\mathcal{P}}, \Delta_i b^{\mathcal{P}} \rangle$
2. $\mathrm{tr}(\mathrm{Ric}) = \mathrm{Scal}$

Proposition

Let $R : \mathbb{R}^d \to \mathbb{R}^d$ be a linear map. Then,

$$\mathbb{E}\left[\langle R\Delta_i B, \Delta_i B\rangle\right] = \mathrm{tr}(R)\Delta_i s.$$

Proof is just applying Ito's formula to the map $f(x) = \langle Rx, x\rangle$ and neglect the martingale term.