Stochastic analysis and rough paths

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Programme

1. Background on Lyons' rough path theory: rough differential equations; flows and applications in stochastic analysis.
2. Sub-problem: Tail estimates for the tangent process for Gaussian-driven RDEs.
3. Ilustration of its use in Gaussian Hörmander theory.

Rough paths

$$dy_t = V(y_t)\,dx_t,\ \ y(0) = y_0$$

$(x_t)_{t\in[0,T]} = \left(x_t^1, ..., x_t^d\right)_{t\in[0,T]}$ typically continuous but rough (finite $p-$var, $p > 2$). Sufficiently smooth vector fields $(V_1, ..., V_d)$ on $\mathbb{R}^e$.

$$\textbf{Signature}:\ S(x)_{s,t} = \sum_{i=0}^{\infty} \int_{s<t_1<....<t_i<t} dx_{t_1} \otimes ... \otimes dx_{t_i} \in T\left(\mathbb{R}^d\right).$$

- Algebraic properties (K.T. Chen): Invariant under reparameterisation; algebra homomorphism.
- Projection onto $\left(\mathbb{R}^d\right)^{\otimes 2}$:

$$\text{Antisymm}\left(\int_{s<t_1<t_2<t} dx_{t_1} \otimes dx_{t_2}\right) = \left(\text{Area}^{i,j}_{s,t}\right).$$

Rough paths

Definition

$p \in (2,3)$ a rough path is a path $\mathbf{x} = (x, \mathbb{X}) \in \mathbb{R}^d \oplus \left(\mathbb{R}^d\right)^{\otimes 2}$ satisfying the algebraic and analytic constraints

$$\begin{aligned} \mathbb{X}_{s,u} - \mathbb{X}_{s,t} - \mathbb{X}_{t,u} &= x_{s,t} \otimes x_{t,u} \quad \text{"multiplicative"} \\ \sup_{s<t\in[0,T]} \frac{|x_{s,t}|}{|t-s|^{1/p}} &< \infty, \quad \sup_{s<t\in[0,T]} \frac{|\mathbb{X}_{s,t}|}{|t-s|^{2/p}} < \infty. \end{aligned}$$

Weakly geometric rough paths are those for which $\text{Symm}(\mathbb{X}_{s,t}) = \frac{1}{2} x_{s,t} \otimes x_{s,t}$.

Controlled rough paths, integration theory, RDEs

- (Gubinelli, Hairer, ...case: $p \in (2,3)$) A pair of paths (y, y') is controlled by $\mathbf{X}$ if
$$y_{s,t} = y'_s x_{s,t} + R^y_{s,t},$$
with $\left|R^y_{s,t}\right| \lesssim |t-s|^{2/p}$.
- We can then integrate y against $\mathbf{X}$:
$$\int_s^t y_u d\mathbf{x}_u = \lim_{|D|\to 0} \sum_{t_i \in D} \left(y_{t_i} x_{t_i,t_{i+1}} + y'_{t_i} \mathbb{X}_{t_i,t_{i+1}} \right)$$

Theorem (Universal Limit Theorem, Lyons)

If $\mathbf{x} \in WG\Omega_p\left(\mathbb{R}^d\right)$ and $V = (V_1, .., V_d) \in Lip^\gamma$ for $\gamma > p$,
$$dy = V(y)\, d\mathbf{x}$$
has a unique solution, and $\mathbf{x} \mapsto y$ is continuous in the rough path topology.

Historical results on smoothness of the density

Setup: $(X_t)_{t\in[0,T]} = (X_t^1, ..., X_t^d)_{t\in[0,T]}$ zero-mean continuous Gaussian process, i.i.d. components, covariance function $R(s,t) = E\left[X_s^1 X_t^1\right]$. C_b^∞ vector fields $(V_1, ..., V_d)$

$$dY_t = V(Y_t)\, d\mathbf{X}_t + V_0(Y_t)\, dt, \ \ Y(0) = y_0.$$

When does Y_t have a smooth density?

- Hörmander (1967), Malliavin (1978): if $X = B$, Brownian motion and Hörmander's condition.
- Nualart & Sausserau (2005) if X fBm, $H > 1/2$ and ellipticity.
- Baudoin & Hairer (2006) if X fBm, $H > 1/2$ and (H).
- C., Friz & Victoir (2009) X is a non-degnerate Gaussian process (includes fBm, $H > 1/4$) and (E);
- C., Friz (2010) X is a non-degenerate Gaussian process (includes fBm, $H > 1/4$) and (H);
- Driscoll (2011) smoothness of the density for the Lévy area fBm $H > 1/3$.
- Hairer & Pillai (2011): if X fBm, $H > 1/3$ and (H).

From classical theory we need to check that: (1) $Y_t \equiv U^{\mathbf{X}}_{t\leftarrow 0}(y_0) \in D^{\infty}(\mathbb{R}^e)$ and (2) $\det C_T^{-1} \in \cap_{p>1} L^p$. Key ingredients:

1. High order Malliavin differentiability & moments.

$$DY_T : h \mapsto \sum_i \int_0^T J^{\mathbf{X}}_{T\leftarrow t}(y_0)\, V_i(Y_t)\, dh^i_t.$$

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    1. Norris-type lemma (à la Hairer & Pillai (2011)): if $Z_t = Z_0 + \int_0^t A_s dX_s$ and X is $\theta-$*Hölder rough*, then

$$|A|_{\infty;[0,t]} \leq CL_\theta(X)^{-q} |Z|^r_{\infty;[0,t]}.$$

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   2. "Base case" for the induction in the proof of Hörmander's Theorem.

The Malliavin covariance matrix

Lemma (C., Friz, Victoir (2009))

We have

$$v^T C_T v = \sum_{i=1}^{d} \int_{[0,T]\times[0,T]} \left\langle v, J^{\mathbf{X}(\cdot)}_{T\leftarrow s} V_i\left(Y_s\right)\right\rangle \left\langle v, J^{\mathbf{X}(\cdot)}_{T\leftarrow s'} V_i\left(Y_{s'}\right)\right\rangle dR\left(s, s'\right).$$

The objective is to prove that

$$P\left(\inf_{|v|=1} v^T C_T v < \epsilon\right) \text{ is } O\left(\epsilon^p\right) \text{ as } \epsilon \downarrow 0 \text{ for all } p > 1.$$

Idea: Let $f \in C^\gamma\left([0, T], \mathbb{R}\right)$, prove a bound of the form

$$|f|_\infty \leq C\,|f|_\gamma^a \left(\int_{[0,T]^2} f_s f_t \, dR\left(s,t\right)\right)^b \quad \text{some } a + 2b = 1,$$

then use Norris iteratively together with the identity

$$J^{\mathbf{X}}_{0\leftarrow T}\left(y_0\right) W\left(Y_t\right) = W\left(y_0\right) + \sum_i \int J^{\mathbf{X}}_{0\leftarrow s}\left(y_0\right)\left[W, V_i\right]\left(Y_s\right) dX_s$$

Regularity of flows

A solution to a RDE

$$dy = V(y)\, d\mathbf{x},\ y(0) = y_0$$

induces a flow

$$y_0 \mapsto U^{\mathbf{x}}_{t\leftarrow 0}(y_0).$$

Lemma (Lyons, Friz-Victoir,...)

If $V = \left(V^1, \ldots V^d\right)$ are Lip-γ then $U^{\mathbf{x}}_{t\leftarrow 0}(\cdot)$ is differentiable and the derivative

$$J^{\mathbf{x}}_{t\leftarrow 0}(y_0) : a \mapsto \frac{d}{d\epsilon} U^{\mathbf{x}}_{t\leftarrow 0}(y_0 + \epsilon a)\bigg|_{\epsilon=0} \quad \text{"Jacobian of the flow"}$$

and solves the RDE (note: with linear-growth vector fields)

$$dJ^{\mathbf{x}}_{t\leftarrow 0}(y_0) = \sum_{i=1}^{d} DV_i(y_t)\, J^{\mathbf{x}}_{t\leftarrow 0}(y_0)\, d\mathbf{x},\ \ J^{\mathbf{x}}_{0\leftarrow 0}(y_0) = I$$

Problem I: Jacobian growth estimates

Theorem (Non-explosion estimate I, Friz-Victoir 2010)

Let $\mathbf{x} \in WG\Omega_p\left(\mathbb{R}^d\right)$ *then*

$$\left|J^{\mathbf{x}}_{\cdot\leftarrow 0}\left(y_0\right)\right|_{p-var;[0,T]} \leq C \exp\left(C\,||\mathbf{x}||^p_{p\text{-}var;[0,T]}\right).$$

Problem

If $\mathbf{x} = \mathbf{B}\left(\omega\right)$*, the Brownian rough path, then RHS is not in* L^1*, but we know the LHS has moments of all order (BDG etc.)!*

Jacobian growth estimates II

Definition

Let $\omega_{\mathbf{x}}(s,t) = ||\mathbf{x}||^p_{p-var;[s,t]}$. The **accumulated α-local p-variation** is the functional

$$M_{\alpha,p,[0,T]}(\mathbf{x}) := \sup_{D=(t_i)} \sum_{\omega(t_i,t_{i+1})\leq\alpha} \omega_{\mathbf{x}}(t_i,t_{i+1})$$

Theorem (Non-explosion estimate II)

Let $\mathbf{x} \in WG\Omega_p(\mathbb{R}^d)$ *then*

$$\left|J^{\mathbf{x}}_{\cdot\leftarrow 0}(y_0)\right|_{p-var;[0,T]} \leq C\exp\left(CM_{\alpha,p,[0,T]}(\mathbf{x})\right)$$

The accumulated local p-variation

Definition

For $\alpha > 0$, $p \geq 1$ we call

$$M_{\alpha,I,p}(\mathbf{x}) = \sup_{\substack{D=(t_i),\\ \omega(t_i,t_{i+1})\leq\alpha}} \sum_{i:t_i\in D} \omega_{\mathbf{x}}(t_i, t_{i+1})$$

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2. increasing and continuous for α in $(0, \infty)$;
3. $M_{\alpha,I,p}(\mathbf{x}) \leq ||\mathbf{x}||^p_{p-var;I}$ (controls are super-additive);
4. If $\alpha \geq ||\mathbf{x}||^p_{p-var;I}$ then $M_{\alpha,I,p}(\mathbf{x}) = ||\mathbf{x}||^p_{p-var;I}$.

The accumulated local p-variation ctd.

1. $M_{\alpha,I,p}(\mathbf{x})$ not so easy to study directly.
2. Let $I=[0,T]$ and define a "greedy" sequence of stopping times $\tau_0(\alpha)=0$

$$\tau_{i+1}(\alpha)=\inf\left\{t\geq\tau_i(\alpha):\omega_{\mathbf{x}}(\tau_i(\alpha),t)\geq\alpha\right\}\wedge T.$$

Lemma

Let $N_{\alpha,I,p}(\mathbf{x})=\sup\left\{n\in\mathbb{N}:\tau_i(\alpha)<T\right\}$ *then*

$$M_{\alpha,I,p}(\mathbf{x})\leq 2\alpha\left(N_{\alpha,I,p}(\mathbf{x})+1\right).$$

Gaussian rough paths

Let $\mathcal{W} = C\left([0,T], \mathbb{R}^d\right) \rightsquigarrow$ abstract Wiener space $(\mathcal{W}, \mathcal{H}, \mu)$;

- $(X_t(\omega))_{t\in[0,T]} = (\omega(t))_{t\in[0,T]}$ canonical process with law μ;
- $(X_t)_{t\in[0,T]} = \left(X_t^1, ..., X_t^d\right)_{t\in[0,T]}$ zero mean Gaussian process, i.i.d. components, covariance function $R(s,t) = E\left[X_s^1 X_t^1\right]$.

Key assumption : (Friz & Victoir) $R \in C^{\rho\text{-var}}\left([0,T]^2\right)$ in 2D sense

$$V_\rho\left(R; [0,T]^2\right) := \sup_D \left(\sum_{i,j} \left|\mathbb{E}\left[\left(X_{t_{i+1}} - X_{t_i}\right)\left(X_{t_{j+1}} - X_{t_j}\right)\right]\right|^\rho\right)^{1/\rho} < \infty$$

- Examples: BM with $\rho_{BM} = 1$, fBM with $\rho_{fBM} = 1/(2H)$ ...

Theorem

Under the key assumption X lifts to a natural geometric p-rough path $\mathbf{X}$.

Embedding of the CM-space

Theorem (Friz, Victoir 2009)

If $R \in C^{\rho\text{-var}}\left([0,T]^2\right)$ *then* $\mathcal{H} \hookrightarrow C^{\rho\text{-var}}\left([0,T],\mathbb{R}^d\right)$:

$$|h|_{\rho-var;[0,T]} \leq |h|_{\mathcal{H}} \sqrt{V_\rho\left(R;[0,T]^2\right)}$$

Remark: Recall for BM $\rho = 1$ so this embedding is sharp ($\mathcal{H}^{\mathrm{BM}}$ is identified with $W_0^{1,2}$).

Translated rough paths

If x and h have Young-complementrary variation regularity, say p and q such that $1/p + 1/q > 1$ then can define $(T_h \mathbf{x})^1 = \mathbf{x}^1 + h$,

$$(T_h \mathbf{x})^2 = \mathbf{x}^2 + \int h \otimes d\mathbf{x}^1 + \int \mathbf{x}^1 \otimes dh + \int h \otimes dh, \ldots$$

Theorem

There exists measurable $A \subseteq \mathcal{W}$ such that (i) $\mu(A) = 1$, (ii) $\forall \omega \in A$

$$T_h \mathbf{X}(\omega) \equiv \mathbf{X}(\omega + h) \quad \forall h \in \mathcal{H}$$

Lemma

$$||T_h \mathbf{x}||^p_{p-var} \leq C \left[||\mathbf{x}||^p_{p-var} + |h|^q_{q-var} \right]$$

Gaussian isoperimetric inequality

Let $\mathcal{K}$ denote the unit ball in $\mathcal{H}$. For any $A \subseteq \mathcal{W}$ the Minkowski sum:

$$A + r\mathcal{K} := \{x + ry : x \in A, y \in \mathcal{K}\}.$$

Theorem (Borell)

Let A be a Borel subset of $\mathcal{W}$ such that $\mu(A) \geq \Phi(a)$ for some real number a. Then, for every $r \geq 0$

$$\mu_*(A + r\mathcal{K}) \geq \Phi(a + r),$$

where μ_* is the inner measure of μ and Φ denotes the standard normal cumulative distribution function.

Theorem (C, Litterer, Lyons)

There exists a subset $E \subset \mathcal{W}$, $\mu(E) = 1$, *such that for all* $\omega \in E$, $h \in \mathcal{H}$ *and* $\alpha > 0$ *we have*

$$||\mathbf{X}(\omega - h)||_{p-var;I} \leq \alpha \Rightarrow |h|_{\rho-var;I} \geq c\alpha N_{\bar{\alpha}^{1/p},I,p}(\mathbf{X}(\omega))^{1/\rho}.$$

Corollary (C., Litterer, Lyons - the tai of $N_{\alpha,I,p}(\mathbf{X}(\omega))$)

$$\mu(\{\omega : N_{\alpha,I,p}(\mathbf{X}(\omega)) > n\}) \leq c \exp\left(-cn^{2/\rho}\right)$$

Proof.

Step 1: CM-embedding gives $|h|_{\rho-var;I} \lesssim |h|_{\mathcal{H}}$. Estimate gives

$$\{\omega : N_{\bar{\alpha},I,p}(\mathbf{X}(\omega)) > n\} \cap E \subseteq \mathcal{W} \backslash (A_\alpha + r_n \mathcal{K}), \quad (1)$$

where $A_\alpha = \left\{\omega : ||\mathbf{X}(\omega)||_{p-var;I} \leq \alpha\right\}$ and $r_n := cn^{1/\rho}$.

Step 2: Borell's inequality with (1)

$$\mu(\{\omega : N_{\bar{\alpha},I,p}(\mathbf{X}(\omega)) > n\}) \leq 1 - \Phi(a_\alpha + r_n) \leq \exp\left[-cr_n^2\right].$$

□

Remarks: (i) For BM ($\rho = 1$) we have that $\mu\left(N_{\alpha,I,p}\left(\mathbf{X}\left(\omega\right)\right) > n\right) \lesssim \exp\left(-cn^2\right)$ hence

$$\log\left|J_{\cdot\leftarrow 0}^{\mathbf{X}(\omega)}\left(y_0\right)\right|_{p-var;I} \lesssim A + N_{\alpha,I,p}\left(\mathbf{X}\left(\omega\right)\right)$$

has a Gaussian tail. This is sharp (elementary Stratonovich calculus)
(ii) In the general Gaussian setting $\mu\left(N_{\alpha,I,p}\left(\mathbf{X}\left(\omega\right)\right) > n\right) \lesssim \exp\left(-cn^{2/\rho}\right)$ (recall: $\rho < 3/2$).

Corollary

$\left|J_{\cdot\leftarrow 0}^{\mathbf{X}(\omega)}\left(y_0\right)\right|_{p-var;I}$ *has finite moments of all order.*

Problem 2: Back to the Malliavin covariance matrix

If $R = R^{BM}$ we have (Baudoin & Hairer):

$$|f|_{\infty;[0,T]} \leq 2 \max \left(T^{-1/2} |f|_{L^2[0,T]}, |f|_{L^2[0,T]}^{2\gamma/(2\gamma+1)} |f|_{\gamma-H\ddot{o}l;[0,T]}^{1/(2\gamma+1)} \right).$$

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- Fix a finite partition D of $[0,T]$. Look for a lower bound on the quadratic form:
$$f(D)^T Qf(D).$$
- Let mesh$(D) \to 0$.
- Formulate as a constrained quadratic optimisation problem:
$$\min x^T Qx \text{ subject to } x_1, ..., x_k \geq b > 0.$$

Solution to the optimisation problem

Let $(Z_1, ..., Z_k, Z_{k+1}, ..., Z_n) \sim N(0, Q)$, $(Z_1, ..., Z_k) \sim N(0, Q_{11})$ and $(Z_1, ..., Z_k) \sim N(0, Q_{22})$ and write:

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{pmatrix},$$

Then

$$(Z_1, ..., Z_k) \,|\, (Z_{k+1}, ..., Z_n) \sim N(\bar{\mu}, \bar{Q})$$

where $\bar{\mu} = Q_{12} Q_{22}^{-1} (Z_{k+1}, ..., Z_n)^T$ and $\bar{Q} = Q_{12} Q_{22}^{-1} Q_{12}^T$. Therefore

$$\begin{aligned} & E\left[E\left[(x_1 Z_1 + ... + x_n Z_n)^2 \,|Z_{k+1}, ..., Z_n\right]\right] \\ = \; & \sum_{i,j=1}^{k} x_i \bar{Q}_{i,j} x_j + E\left[\left(\sum_{i=1}^{k} x_i \bar{\mu}_i + \sum_{i=k+1}^{n} x_i Z_i\right)^2\right], \end{aligned}$$

implies $\inf_{x_1 \geq b, ..., x_k \geq b} x^T Q x = \inf_{x_1 \geq b, ..., x_k \geq b} \sum_{i,j=1}^{k} x_i \bar{Q}_{i,j} x_j$.

Back to the non-degeneracy inequality

Theorem

If X is non degenerate then $\forall [s,t] \subseteq [0,T]$

$$\int_{[0,T]^2} f_s f_t dR(s,t) \geq \inf_{u \in [s,t]} |f(u)|^2 \, var\left(X_{s,t} | \mathcal{F}_{0,s} \vee \mathcal{F}_{t,T}\right) \tag{2}$$

provided

$$cov\left(X_{s,t}, X_{u,v} | \mathcal{F}_{0,s} \vee \mathcal{F}_{t,T}\right) \geq 0, \quad \forall [u,v] \subseteq [s,t] \subseteq [0,T]. \tag{3}$$

Main theorem – Hörmander for Gaussian RDEs

The interpolation inequality can be further quantified if we assume the *non-determinancy-type* condition:

$$\exists\alpha > 0 \text{ s.t. } \inf_{0\leq s<t\leq T} \frac{var\left(X_{s,t}|\mathcal{F}_{0,s}\vee\mathcal{F}_{t,T}\right)}{(t-s)^{\alpha}} > 0. \quad (4)$$

Theorem

$X_t = \left(X_t^1, ..., X_t^d\right)$ continuous Gaussian process with i.i.d. components. One (any) component satisfies:

1. *R has finite 2D ρ-var, X lifts to a geometric p-rough path $\mathbf{X}$ some $p > 2\rho$ and $\mathcal{H} \hookrightarrow C^{q-var}$ for some $1/p + 1/q > 1$;*
2. *$cov\left(X_{s,t}, X_{u,v}|\mathcal{F}_{0,s}\vee\mathcal{F}_{t,S}\right) \geq 0$, $\forall [u,v] \subseteq [s,t] \subseteq [0,S] \subseteq [0,T]$;*
3. *Non-determinism-type condition (4).*

If $dY_t = V(Y_t)\,d\mathbf{X}_t + V_0(Y_t)\,dt$, $Y_0 = y_0$ then Y_t has a smooth density for $t > 0$ under Hörmander's condition at y_0.

Checking the conditions

1. Non-determinancy can be checked for fBm $H < 1/2$, OU-process Brownian bridge
2. A matrix Q is diagonally dominant if
$$Q_{ii} \geq \sum_{i \neq j} |Q_{i,j}| .$$
Diagonal dominance is preserved under Schur complementation! For processes with negatively correlated increments we need only show that $E\left[X_{0,T}X_{s,t}\right] \geq 0$ for all $s < t$.
3. Geometrical interpretation: $cov\left(X, Y|\mathcal{G}\right) \geq 0$ iff $\cos\theta \geq 0$ where θ is the angle between the projections $P^{\perp}X$ and $P^{\perp}Y$ onto $L^2\left(\Omega, \mathcal{G}, P\right)^{\perp}$.

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