Quasi-invariant Measures on Path Space

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Transformation of measure under the flow of a vector field

Let E be a vector space (or a manifold), equipped with a finite Borel measure γ and let Z be a vector field on E.

Consider the corresponding *flow* on E, i.e. the maps $\rho_s : x \mapsto x^s$ where

$$\frac{dx^s}{ds} = Z(x^s)$$

$$x^0 = x$$

We want to study the transformation of γ under ρ_s and establish conditions under which the measures $\gamma_s \equiv \gamma \circ \rho_s^{-1}$ are mutually absolutely continuous (say γ is *quasi-invariant* under the flow of Z).

Example (Cameron-Martin Theorem)

Let E be the Wiener space (space of continuous paths starting at 0, equipped with the Wiener measure γ).

Let Z be a deterministic path of finite energy:

$$\int_0^1 \dot{Z}_s^2 ds < \infty.$$

(Cameron-Martin path)

Since h is constant we have $\rho_s(w) = w + sZ$.

The measures γ_s and γ are equivalent and

$$\frac{d\gamma_s}{d\gamma}(w) = \exp\left(s\int_0^1 \dot{Z}_t dw - \frac{s^2}{2}\int_0^1 \dot{Z}_t^2 dt\right).$$

Definition. We say that Z is *admissible* if there exists an L^1 function $DivZ$ such that

$$\int_E D_Z \Phi d\gamma = \int_E \Phi DivZ d\gamma$$

for all test functions Φ on E.

The two properties - quasi-invariance and admissibilty - are obviously related.

Supppose quasi-invariance holds and denote

$$\frac{d\gamma_s}{d\gamma} = X_s.$$

Then we have

$$\int_E \Phi \circ \rho_s d\gamma = \int_E \Phi d\gamma_s = \int_E \Phi X_s d\gamma$$

Differentiating wrt s and setting $s = 0$ gives

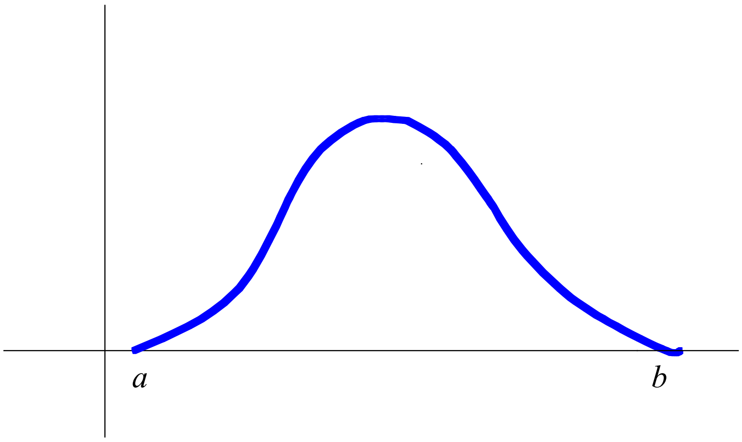
$$\int_E D_Z \Phi d\gamma = \int_E \Phi \frac{dX_s}{ds} \Big/_{s=0} d\gamma$$

i.e. Z is admissible and

$$DivZ = \frac{dX_s}{ds} \Big/_{s=0}.$$

Does the converse hold?

No. Let γ be a measure on $\mathbf{R}^n$ with a compactly supported C^1 density F and let Z be a (non-zero) constant vector.



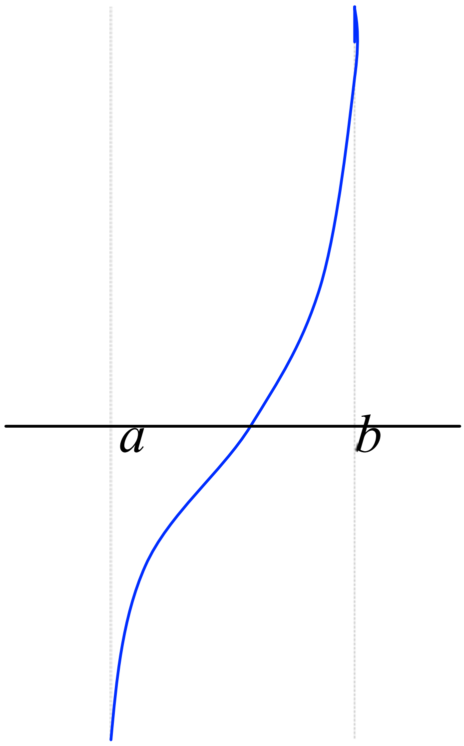
Then Z is admissible.

$$\int_{\mathbf{R}^n} D_Z \Phi \, d\gamma = \int_{\mathbf{R}^n} D_Z \Phi F \, dx$$

$$-\int_{\mathbf{R}^n} \Phi D_Z F \, dx = -\int_{\mathbf{R}^n} \Phi \frac{D_Z F}{F} \, d\gamma$$

Quasi-invariance under the flow $(x \mapsto x + sZ)$ would imply that γ and $\gamma(.+sZ)$ have the same null-sets for all s. This is obviously not the case.

Note that $DivZ = -\frac{D_Z F}{F}$ blows up at the boundary of the support of F.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Under what conditions does admissibility imply quasi-invariance?

In order to address this question we need to develop the relationship between the two properties further.

Suppose Z is admissible and also the family of measures $\gamma_s \equiv \gamma_{\rho_s}$ are alsolutely continuous wrt γ. Denote as before $\frac{d\gamma_s}{d\gamma} = X_s$.

For test functions Φ

$$\int_E \Phi \circ \rho_s d\gamma = \int_E \Phi d\gamma_s = \int_E \Phi X_s d\gamma \qquad (1)$$

Replacing Φ by $\Phi \circ \rho_s^{-1}$ we have

$$\int_E \Phi d\gamma = \int_E \Phi \circ \rho_s^{-1} X_s d\gamma.$$

Thus the RHS is independent of s. Differentiating wrt s gives

$$\int_E \Big\{ D\Phi(\rho_s^{-1}(x)) \frac{d}{ds} \rho_s^{-1}(x) X_s$$

$$+ \Phi \circ \rho_s^{-1} X_s' \Big\} d\gamma = 0.$$

This relation can be rewritten in the form

.

$$\int_E \Big\{ - D(\Phi \circ \rho_s^{-1})(x)(ZX_s) + \Phi \circ \rho_s^{-1} X_s' \Big\} d\gamma$$

$$= 0.$$

Using the defining property of divergence we have

$$\int_E \Phi \circ \rho_s^{-1} \Big\{ X_s' - Div(ZX_s) \Big\} d\gamma = 0. \qquad (2)$$

Since (2) holds for all test functions Φ, we conclude that X_s is the (unique) solution to the IVP

$$X_s' = Div(ZX_s) \qquad (3)$$

$$X_0 = 1.$$

We can show that

$$X_s(x) = \exp \int_0^s DivZ(x^{-u}) du$$

satisfies (3).

Working the previous argument backwards (under suitable conditions), we arrive at (1) and prove the following result.

Theorem 1. *Suppose Z is admissible and there exists a set $B \subset E$ with $\gamma_s(B) = 1$ for all s such that $DivZ$ is defined and differentiable along Z on B. Suppose further that the function $u \mapsto DivZ(x^{-u})$ is continuous for $x \in B$. Then γ is quasi-invariant under Z and*

$$\frac{d\gamma_s}{d\gamma} = \exp \int_0^s (DivZ)(x^{-u})du.$$

Note: The continuity hypothesis precludes the singular behavior seen in the previous example.

Example. Let (E, γ) be the Wiener space and Z be a (deterministic) Cameron-Martin path. Then

$$w^s = w + sZ.$$

It can be shown that

$$DivZ = \int_0^1 \dot{Z} dw.$$

Applying Thm 1 we see that γ is quasi-invariant under Z and

$$\frac{d\gamma_s}{d\gamma} = \exp \int_0^s (DivZ)(w^{-u}) du$$

$$= \exp \int_0^s \Big(\int_0^1 \dot{Z} d(w - uZ) \Big) du$$

$$= \exp \Big(s \int_0^1 \dot{Z} dw - \frac{s^2}{2} \int_0^1 \dot{Z}^2 dt \Big).$$

(C-M Theorem)

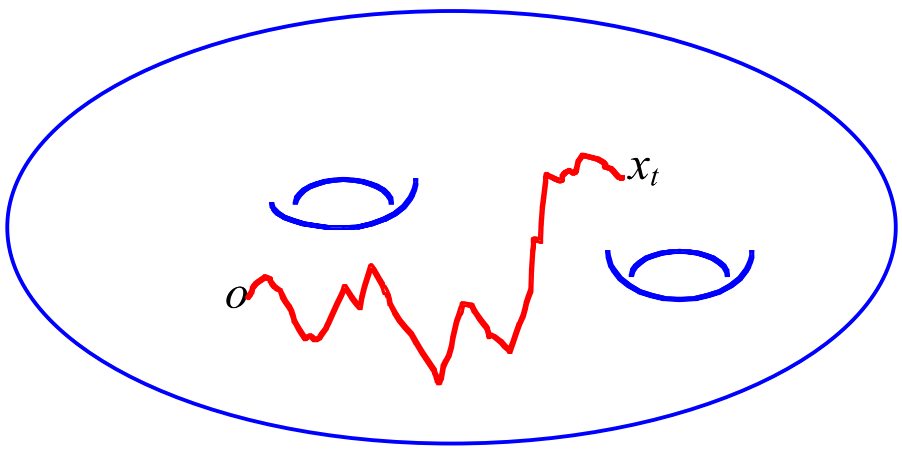
Application to measures induced by stochastic differential equations

Let M denote a closed compact manifold and $X_0, \ldots, X_n$ smooth vector fields on M. Let o be a fixed point in M.

Consider the Stratonovich SDE

$$dx_t = \sum_{i=1}^{n} X_i(x_t) \circ dw_i, \ t \in [0, T]$$

with initial point $x_0 = o$.



We consider two measures associated with this equation

(1) The law γ_T of x_T, a measure on M (the *endpoint* problem).

(2) The law γ of x, a measure on $C_o(M)$, the space of paths $\{\sigma : [0, T] \mapsto M / \sigma(0) = o\}$ (the *pathspace* problem).

The objective is to construct admissible vector fields and to establish quasi-invariance under the corresponding flows.

The endpoint problem

Let γ_T denote the law of x_T where x is the solution of the SDE

$$dx_t = \sum_{i=1}^{n} X_i(x_t) \circ dw_i,\ t \in [0,T].$$

Deote by μ denote the Wiener measure on $C_0(\mathbf{R}^n)$. Note that γ_T is the image of μ under the map

$$g_T : (w_1, \ldots, w_n) \mapsto x_T$$

Suppose the vector fields X_i satisfy the Hormander's condition (HC):

$X_1, \ldots, X_n$ together with the collection of iterated Lie brackets $[X_i, X_j], [[X_i, X_j], X_k], \ldots$ span TM at every point.

Then we can show that every C^1 vector field Z on M is admissible wrt the measure γ_T:

Let Φ be a test function on M. Consider the integral

$$\int_M D_Z \Phi d\gamma_T = \int_{C_0(\mathbf{R}^n)} D_Z \Phi(g_T(w)) d\mu \qquad (4)$$

We now perform integration by parts on the Wiener space in the style of the Malliavin calculus.

This involves constructing a lift of Z to Wiener space via the map g_T, i.e. a function η satisfying $Dg_T(w)\eta = Z$ (possible under HC).

THe RHS of (4) may now be expressed in the form

$$\int_{C_0(\mathbf{R}^n)} D_\eta(\Phi \circ g_T)(w)) d\mu = \int_{C_0(\mathbf{R}^n)} \Phi(x_T) Div(\eta) d\mu$$

where Div denotes the divergence operator in Wiener space:

$$Div(\eta) = (\eta, w) - Trace_H D\eta(w).$$

Writing this as an integral over the original measure space (M, γ_T), we have

$$\int_M D_Z \Phi d\gamma_T = \int_M \Phi(x) E[Div(\eta)/x_T = x] d\gamma_T$$

We conclude that Z is admissible with divergence

$$DivZ(x) = E[Div(\eta)/x_T = x].$$

This implies that Z has a density (in the Euclidean case), a result that is related to the hypoellipticity of the generator of x.

However, it is seems very difficult to establish the regularity of $DivZ$ and hence to verify the continuity condition in Theorem 1.

In fact, quasi-invariance in the Euclidean case implies *everywhere positivity* of the density of γ_T and this is known not to hold in certain cases.

Example in $\mathbf{R}$

$$dx = x_t \circ dw$$

$$x_0 = 1$$

Then $x_T = e^{w_T}$, so $x_t > 0$.

In this case, we have

$$X \equiv Div(1)(x) = \frac{1}{xT}\Big(\ln x + T\Big),\ x > 0$$

which blows up as $x \to 0^+$.

We can compute the density F of x_T from X via

$$F(x) = C \exp\left(-\int X dx\right)$$

$$= \frac{1}{\sqrt{2\pi}xT} \exp \frac{-(\ln x)^2}{2T}$$

The pathspace problem

The space E under study here is the set of paths $\{\sigma : [0, T] \mapsto M \text{ with } \sigma(0) = o\}$.

The measure on E is the law of x.

The tangent space $T_x \equiv \{V : [0, T] \mapsto TM / \, V(0) = 0, V_t \in T_{x_t}M\}$.

Assume the vector fields $X_1, \ldots, X_n$ span TM at each point of M.

We construct admissible vector fields on the pathspace in the form

$$Z_t = \sum_{i=1}^{n} X_i(x_t) h_i(t)$$

where the h_i are suitably chosen real-valued processes.

We introduce a Riemannian structure $[g_{jk}]$ on M, by defining

$$g^{jk} = a_{ij}a_{ik}$$

where

$$X_i = a_{ir}\frac{\partial}{\partial x_r}$$

is a local representation of X_i.

Note that here, and from this point on, we are using the summation convention.

A class of admissible vector fields on $C_0(M)$

Let ∇ denote the Levi-Civita covariant derivative. Define a set of 1-forms on M

$$\omega^{jk} = < \nabla_{X_j} X_k, \cdot > - < \nabla . X_j, X_k >$$

and functions

$$B^{jk} = \frac{1}{2}\Big(< L_{ji} X_i, X_k > - < L_{ij} X_k, X_i > -$$

$$< \nabla_{X_j} X_k, \nabla_{X_i} X_i > + < \nabla_{X_l} X_i, X_k > < \nabla_{X_j} X_l, X_i > \Big)$$

where L_{ij} is the (Hessian) operator on vector fields

$$H(X_i, X_j) = \nabla_{X_i} \nabla_{X_j} - \nabla_{\nabla_{X_i} X_j}.$$

In the case of a gradient system terms defining B^{jk} other than the first two vanish.

Theorem 2. *Let r be an adapted C-M path in $\mathbf{R}^n$ and define h by the following linear system of SDE's*

$$dh_t^i = \omega^{ji}(\circ dx_t)h_t^j + \Big(B^{ji}(x_t)h_t^j + \dot{r}_t^i\Big)dt$$

$$h_0^i = 0.$$

Define $Z_t \equiv X_i(x_t)h_t^i$. Then there is the integration-by-parts formula

$$E[D_Z\Phi(x)] = E[\Phi(x)R] = E[\Phi(x)E[R/x]]$$

where

$$R = \int_0^T \left(\dot{r}_t^i + \frac{1}{2} < Ric(Z_t), X_i(x_t) > \right)dw_i$$

By choosing r appropriately, we can ensure that R is x-*measurable*. In this case we have

$$DivZ = R.$$

For example, let V be a vector field on M and α is a real-valued continuous x-measurable process and define

$$\dot{r}_t^i =< X_i(x_t), V(x_t) > \alpha(t)$$

Then

$$R = \int_0^T \left(\dot{r}_t^i + \frac{1}{2} < Ric(Z_t), X_i(x_t) > \right) dw_i$$

$$= \int_0^T \left\langle V(x_t)\alpha_t + \frac{1}{2} Ric(Z_t), X_i(x_t) dw_i \right\rangle$$

and converting

$$dx_t = X_i(x_t) \circ dw$$

to Ito form, we have

$$X_i(x_t) dw_i = dx_t - \frac{1}{2} \nabla_{X_i} X_i(x_t) dt.$$

Theorem 3. *There exists a solution in* $C_o(M)$ *to the flow equation*

$$\frac{dx^s}{ds} = Z(x^s).$$

$$x^0 = x.$$

The processes x^s *are semi-martingales of the form*

$$dx^s_t = X_i(s,t) \circ dw_i(t) + X_0(s,t)dt$$

where $X_0(s,\cdot), \ldots, X_n(s,\cdot)$ *are adapted process in* TM *such that* $s \mapsto X_j(s,\cdot)$ *are continuous into the space* $L^2[0,T]$.

In particular

$$DivZ(x^{-u}) =$$

$$\int_0^T \left\langle V(x_t^{-u})\alpha_t^{-u} + \frac{1}{2}Ric(Z(x^{-u})_t), X_i(-u,t)dw_i(t) \right\rangle.$$

We can verify continuity in u. Hence we obtain

Theorem 4. *The measure γ is quasi-invariant under the flow of Z and*

$$\frac{d\gamma_s}{d\gamma} = \exp \int_0^s (DivZ)(x^{-u})du.$$