Curvature bounds and heat kernels methods in subriemannian geometry

Fabrice Baudoin

Based on joint works with N. Garofalo, M. Bonnefont, B. Kim, I. Munive, J. Wang

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$$\Gamma(f,g) = \frac{1}{2}\left(\Delta(fg) - f\Delta g - g\Delta f\right) = \langle \nabla f, \nabla g \rangle$$

and

$$\Gamma_2(f,g) = \frac{1}{2}\left(\Delta\Gamma(f,g) - \Gamma(f,\Delta g) - \Gamma(\Delta f, g)\right).$$

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We have $\mathbf{Ric} \geq \rho$ *and* $\mathbf{dim}\mathbb{M} \leq n$ *if and only if for every smooth* f*,*

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This leads to the notion of intrinsic curvature-dimension bounds for diffusion operators. To be satisfied, the curvature-dimension inequality requires some form of ellipticity.

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A basic question is:

Are there curvature dimension bounds for such structures ?

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The triple $(\mathbb{M}, \theta, g)$ is called a Riemannian contact manifold. The Riemannian geometry of $(\mathbb{M}, \theta, g)$ is confined to the background.

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This operator is not elliptic but locally subelliptic of order $1/2$. There is one missing direction.

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These are the model spaces of the K-contact geometry. A contact triple $(\mathbb{M}, \theta, g)$ is K-contact if the Reeb vector field acts by isometry on the horizontal bundle. These geometries are the simplest contact geometries.

Sub-Riemannian geometry

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Can we throw away the geometry and only work with intrinsic curvatures of Dirichlet forms (Bakry-Ledoux approach to Riemannian geometry) ?

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The main idea is to introduce the vertical intrinsic curvature of the Dirichlet form:

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The main idea is to introduce the vertical intrinsic curvature of the Dirichlet form:

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where T is the Reeb vector field. We stress that the direction T is canonical and given by the geometry.

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Theorem (B. , Garofalo 2011)

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Theorem (B. , Garofalo 2011)

Let $\mathbb{M}$ be a $2n+1$ dimensional K-contact manifold. We have $\mathbf{Ric}_\nabla \geq \rho_1$ *if and only if for every* $\nu > 0$,

$$\Gamma_2(f) + \nu \Gamma_2^T(f) \geq \frac{1}{2n}(Lf)^2 + \left(\rho_1 - \frac{1}{\nu}\right)\Gamma(f) + \frac{n}{2}(Tf)^2.$$

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The previous theorem opens the door to the use of the powerful heat kernel methods.

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In the context of contact manifolds, the commutation is equivalent to the fact that the manifold is K-contact (B., J. Wang 2013).

Generalized curvature dimension inequality

Definition (B., Garofalo 2009)

We say that L satisfies the generalized-curvature inequality $CD(\rho_1, \rho_2, \kappa, d)$ if for every $\nu > 0$,

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$CD(\rho_1, \rho_2, \kappa, d)$ is the linearization of

$$\Gamma_2(f) + 2\sqrt{\kappa\Gamma(f)\Gamma_2^Z(f)} \geq \frac{1}{d}(Lf)^2 + \rho_1\Gamma(f) + \rho_2\Gamma^Z(f).$$

Examples

We have the following general class of examples:

- Let $\mathbb{M}$ be a n-dimensional complete Riemannian manifold wiose Ricci curvature is bounded from below by ρ. The Laplacian of $\mathbb{M}$ satisfies the curvature dimension inequality $\mathrm{CD}(\rho, 0, 0, d)$ with $\Gamma^Z = 0$.

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- K-contact manifolds with lower bounds on the Tanno-Ricci tensor.
- Two-step nilpotent Lie groups.
- Bundles over Riemannian manifolds.
- Riemannian submersions
- Infinite dimensional examples (B.-Gordina-Melcher, 2012)

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- Boundedness of Riesz transforms (B., Garofalo 2011);
- Improved Sobolev inequalities and isoperimetric estimates (B., Kim 2012)
- Quasi-invariance results in infinite dimension (B., Gordina, Melcher 2012).

Subelliptic Myers theorem

If the parameter ρ_1 is positive, then it is possible to prove sharp Gaussian upper bounds for the heat kernel that lead to (almost) sharp Sobolev-type inequalities.

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If the inequality $CD(\rho_1, \rho_2, \kappa, d)$ holds for some constants $\rho_1 > 0, \rho_2 > 0, \kappa > 0$, then the metric space $(\mathbb{M}, d)$ is compact in the metric topology and we have

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Conjecture: If L satisfies the generalized-curvature inequality $CD(\rho_1, \rho_2, \kappa, d)$, with $\rho_1 \geq 0$, then $(\mathbb{M}, d)$ satisfies $MCP\left(0, d\left(1 + \frac{3\kappa}{4\rho_2}\right)\right)$.

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The conjecture is true if L is the sub-Laplacian on a 3 -dimensional K-contact manifold (Agrachev-Lee 2011)

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