# Time discretization and simulation of quadratic BSDEs

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The aim is to study the time discretization of the (decoupled) forward backward system

$$
\begin{aligned}
X_t &= x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \quad 0 \leqslant t \leqslant T, \\
Y_t &= g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)dr - \int_t^T Z_s dW_s, \quad 0 \leqslant t \leqslant T,
\end{aligned}
$$

when $f$ has a quadratic growth with respect to $z$.

- Simplification (for this talk) :

$$X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \quad 0 \leqslant t \leqslant T \quad (1.1)$$

$$Y_t = g(X_T) + \int_t^T f(Z_s)dr - \int_t^T Z_s dW_s, \quad 0 \leqslant t \leqslant T \quad (1.2)$$

- Standard assumption : $f$ is assumed to be locally Lipschitz

$$\left|f(z) - f(z')\right| \leqslant K\left|z - z'\right| \left(1 + |z| + \left|z'\right|\right).$$

| | | $\sigma$ bounded | | $\sigma$ unbounded |
|---|---|---|---|---|
| | | $\sigma(t)$ | $\sigma(t,x)$ | $\sigma(t,x)$ |
| $g$ bounded | $g$ Lipschitz | | | |
| | $g$ not Lipschitz | | | |
| $g$ unbounded | $g$ Lipschitz | | | |
| | $g$ locally Lipschitz | | | |
| | $g$ not locally Lipschitz | | | |

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See [R. 2011], [E. Gobet - P. Turkedjiev Preprint]

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Stays true for super-quadratic BSDEs

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Proved by P. Briand

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Good speed of convergence for the truncated BSDE. See [R. 2012]. Stays true for super-quadratic BSDEs.

| | | $\sigma$ bounded | | $\sigma$ unbounded |
|---|---|---|---|---|
| | | $\sigma(t)$ | $\sigma(t,x)$ | $\sigma(t,x)$ |
| $g$ bounded | $g$ Lipschitz | $\lvert Z\rvert \leqslant C$ | | Aim of the talk |
| | $g$ not Lipschitz | $\lvert Z_t\rvert \leqslant \frac{C}{(T-t)^\alpha}$ | | |
| $g$ unbounded | $g$ Lipschitz | $\lvert Z\rvert \leqslant C$ | $\lvert Z\rvert \leqslant C$ | |
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# Existence and uniqueness

Thanks to [Kobylanski 2000] we have :

- Since $g$ is bounded there exists a solution $(Y, Z)$ such that $Y$ is bounded.
- Since $f$ is locally Lipschitz we have a uniqueness result among bounded solutions.
- $\left(\int_0^t Z_s dW_s\right)_{t\in[0,T]}$ is a BMO martingale :

$$\|Z * W\|^2_{BMO} = \sup_{0\leqslant\tau\leqslant T \text{ stopping time}} \mathbb{E}_\tau\left[\int_\tau^T |Z_s|^2 ds\right] < +\infty.$$

# The linearization trick

Let us consider two solutions $(Y^1, Z^1)$, $(Y^2, Z^2)$ for two terminal conditions $g_1$, $g_2$ and two generators $f_1$, $f_2$. We denote

$$\delta Y := Y^1 - Y^2, \quad \delta Z := Z^1 - Z^2, \quad \delta g := g_1 - g_2, \quad \delta f := f_1 - f_2.$$

We have

$$\delta Y_t = \delta g(X_T) + \int_t^T \delta f(Z_s^1) ds - \int_t^T \delta Z_s \left(dW_s - \gamma_s ds\right),$$

with

$$\gamma_s = \delta Z_s \frac{f_2(Z_s^1) - f_2(Z_s^2)}{|\delta Z_s|^2}.$$

# Why BMO martingales are nice ?

Let us denote $\mathcal{E}(\gamma)$ the Doléans-Dade exponential associated to the martingale $(\int_0^t \gamma_s dW_s)_t$. Since $|\gamma_s| \leqslant C(1 + |Z_s^1| + |Z_s^2|)$, we have

$$\|\gamma * W\|_{BMO}^2 \leqslant C(1 + \left\|Z^1 * W\right\|_{BMO}^2 + \left\|Z^2 * W\right\|_{BMO}^2) < +\infty.$$

- $\mathcal{E}(\gamma)$ is a martingale, we are allowed to apply Girsanov theorem.
- $\mathcal{E}(\gamma) \in L^p$ with $p > 1$ that depends only on $\|\gamma * W\|_{BMO}$.

# Comparison and stability

- Comparison :

$$\delta Y_t = \mathbb{E}_t^{\mathbb{Q}}\left[\delta g(X_T) + \int_t^T \delta f(Z_s^1)ds\right].$$

- Stability :

$$|\delta Y_t|^q \leqslant C\mathbb{E}_t\left[|\delta g(X_T)|^q + \left|\int_t^T \delta f(Z_s^1)ds\right|^q\right].$$

# Time discretization scheme

Let us consider a grid $0 = t_0 < t_1 < ... < t_n = T$ with $h_i = t_{i+1} - t_i$ and $h = \max_i h_i$. $(X_i^n)_i$ discrete approximation of $X$ with "good" convergence properties. $(Y_i^n, Z_i^n)_i$ solution of the scheme

$$\begin{cases} Y_n^n = g(X_n^n) \\ Y_i^n = \mathbb{E}_{t_i}[Y_{i+1}^n + h_i f(Z_i^n)] \\ Z_i^n = \mathbb{E}_{t_i}[Y_{i+1}^n H_i] \end{cases}$$

with $(H_i)_i$ such that

- $H_i$ is $\mathcal{F}_{t_{i+1}}$ measurable, independant with $\mathcal{F}_{t_i}$,
- $\mathbb{E}_{t_i}[H_i] = 0$,
- $0 < \lambda \leqslant h_i \mathbb{E}[|H_i|^2] \leqslant \Lambda$.

Example : $H_i = \frac{W_{t_{i+1}} - W_{t_i}}{h_i} = \frac{\Delta W_i}{h_i}$.

# Linearization of time discretization schemes

Let us consider two discretized solutions $(Y^1, Z^1)$, $(Y^2, Z^2)$ for two terminal conditions $g_1$, $g_2$ and two generators $f_1$, $f_2$.

$$\delta Y := Y^1 - Y^2, \quad \delta Z := Z^1 - Z^2, \quad \delta g := g_1 - g_2, \quad \delta f := f_1 - f_2.$$

We have

$$\begin{aligned} \delta Y_i &= \mathbb{E}_{t_i}[\delta Y_{i+1} + h_i(f_1(Z_i^1) - f_2(Z_i^1)) + h_i(f_2(Z_i^1) - f_2(Z_i^2))] \\ &= \mathbb{E}_{t_i}[\delta Y_{i+1} + h_i \delta f(Z_i^1) + h_i \gamma_i \delta Z_i], \end{aligned}$$

with

$$\gamma_i = \delta Z_i \frac{f_2(Z_i^1) - f_2(Z_i^2)}{|\delta Z_i|^2}.$$

Since, $\delta Z_i = \mathbb{E}_{t_i}[H_i \delta Y_{i+1}]$, we have

$$\begin{aligned} \delta Y_i &= \mathbb{E}_{t_i}[(1 + h_i \gamma_i H_i)(\delta Y_{i+1} + h_i \delta f(Z_i^1))] \\ &= \mathbb{E}_{t_i}\left[\prod_{j=i}^{n-1}(1 + h_j \gamma_j H_j)\left(\delta g(X_n^n) + \sum_{k=i}^{n-1} h_k \delta f(Z_k^1)\right)\right]. \end{aligned}$$

# New assumptions for comparison and stability

- $E_t = \prod_{t_j \leqslant t}(1 + h_j \gamma_j H_j)$ is the Doléans-Dade exponential of the martingale $M_t := \sum_{t_i \leqslant t} h_i \gamma_i H_i$.
- To have $E_t \geqslant 0$, we need to have $(\gamma_i)_i$ and $(H_i)_i$ bounded.
- We take $H_i = \frac{\rho_R(\Delta W_i)}{h_i}$ with $R$ well chosen.
- For $\gamma_i$ we need to truncate $f$.

# Truncation of the initial BSDE

Let us denote $(Y^N, Z^N)$ the solution of the BSDE

$$Y_t^N = g(X_T) + \int_t^T f(\rho_N(Z_s^N))ds - \int_t^T Z_s^N dW_s,$$

and $(Y^{N,n}, Z^{N,n})$ the solution of the scheme

$$\begin{cases} Y_n^{N,n} = g(X_n^n) \\ Y_i^{N,n} = \mathbb{E}_{t_i}[Y_{i+1}^{N,n} + h_i f(\rho_N(Z_i^{N,n}))] \\ Z_i^{N,n} = \mathbb{E}_{t_i}[Y_{i+1}^{N,n} H_i] \end{cases}$$

# Error due to the truncation

**[P. Imkeller - G. dos Reis 2010], [A. R. 2012]**

For all $q > 0$, there exists $C_q > 0$ such that

$$\mathbb{E}\left[\sup_{0\leqslant t\leqslant T}\left|Y_t - Y_t^N\right|^2\right] + \mathbb{E}\left[\int_0^T \left|Z_s - Z_s^N\right|^2 ds\right] \leqslant \frac{C_q}{N^q}.$$

# Application of the comparison result

By taking $R$ and $N$ such that

$$E_t = \prod_{t_j \leqslant t} (1 + h_j \gamma_j H_j) > 0$$

we obtain a comparison theorem.

**Corollary**

$\left| Y^{N,n} \right| \leqslant C$ with $C$ that does not depend on $n$, $N$, $R$.

# Stability

We will study the error between $(Y^N, Z^N)$ and $(Y^{N,n}, Z^{N,n})$ by using our stability result on schemes. We need to write the initial BSDE as a perturbed time discretization scheme.

$$\begin{cases} Y^N_{t_n} & = & g(X_T) \\ Y^N_{t_i} & = & \mathbb{E}_{t_i}[Y^N_{t_{i+1}} + \int_{t_i}^{t_{i+1}} f(\rho_N(Z^N_s))ds] \\ & = & \mathbb{E}_{t_i}[Y^N_{t_{i+1}} + h_i\left(f(\rho_N(\bar{Z}^N_{t_i})) + \zeta_i\right)] \\ \bar{Z}^N_{t_i} & = & \mathbb{E}_{t_i}[Y^N_{t_{i+1}} H_i] \end{cases}$$

with

$$\zeta_i := \frac{1}{h_i}\mathbb{E}_{t_i}\left[\int_{t_i}^{t_{i+1}} \left(f(\rho_N(Z^N_s)) - f(\rho_N(\bar{Z}^N_{t_i}))\right) ds\right].$$

# Stability

If we apply the linearization trick to $(Y^N, Z^N)$ and $(Y^{N,n}, Z^{N,n})$ we obtain

$$Y^N_{t_i} - Y^{N,n}_i = \mathbb{E}_{t_i}\left[\prod_{j=i}^{n-1}(1 + h_j\gamma_j H_j)\left(g(X_T) - g(X^n_n) + \sum_{k=i}^{n-1} h_k\zeta_k\right)\right].$$

## Proposition

$M_t := \sum_{t_i \leqslant t} h_i\gamma_i H_i$ is a BMO martingale. Moreover, $\|M\|_{BMO}$ is bounded by a constant that does not depend $n$, $N$ and $R$. Finally, there exists $q > 1$ independent of $N$, $n$ and $R$ such that

$$\left|Y^N_{t_i} - Y^{N,n}_i\right|^q \leqslant \mathbb{E}_{t_i}\left[|g(X_T) - g(X^n_n)|^q + \left|\sum_{k=i}^{n-1} h_k\zeta_k\right|^q\right].$$

# An explicit speed of convergence

**Theorem**

- $h_i = T/n = h$,
- $H_i = \frac{\rho_R(\Delta W_i)}{h}$ with $R = \sqrt{2h}\log n$,
- $N = n^{1/4}$.

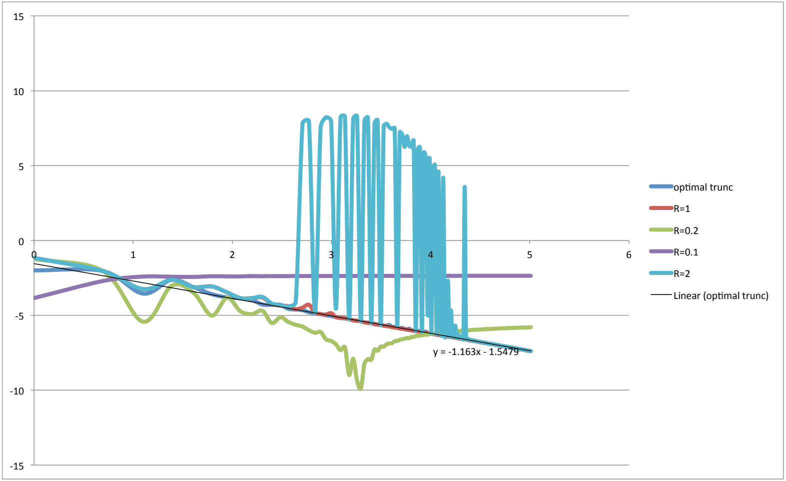
Then, for all $\eta > 0$ we have

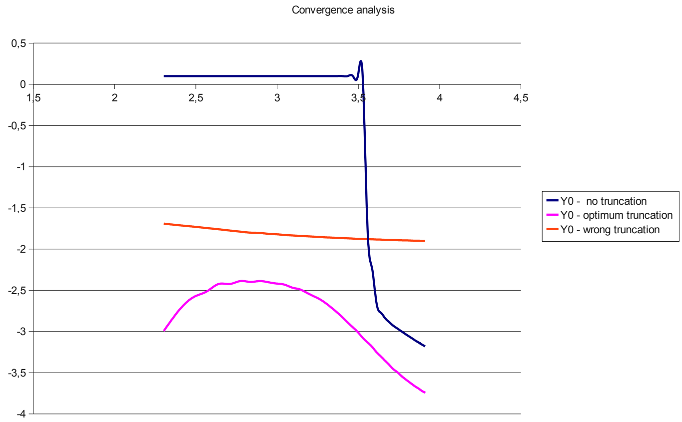
$$\mathbb{E}\left[\sup_{0\leqslant i\leqslant n}\left|Y_{t_i} - Y_i^{N,n}\right|^2\right] + \mathbb{E}\left[\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\left|Z_s - Z_i^{N,n}\right|^2 ds\right] \leqslant C_\eta h^{1-\eta}.$$

# Example

- $X$ is a geometric Brownian motion without drift (dimension 1),
- $g(x) = \sin^2(x)$,
- $f(z) = az^2$ with $a = 5$ or $a = 6$,
- $n$ from 10 to 50,
- conditional expectation approximated by tree method or quantification method.

We know the real solution.


15
10
5
0
-5
-10
-15
0
1
2
3
4
5
6
y = -1.163x - 1.5479
optimal trunc
R=1
R=0.2
R=0.1
R=2
Linear (optimal trunc)


Convergence analysis
0,5
0
-0,5
-1
-1,5
-2
-2,5
-3
-3,5
-4
1,5
2
2,5
3
3,5
4
4,5
Y0 - no truncation
Y0 - optimum truncation
Y0 - wrong truncation