BSDEs with weak terminal condition: theory and application to Finance

Anthony Réveillac

Joint work with Bruno Bouchard and Romuald Elie

Université Paris Dauphine-CEREMADE

May, 23 2013 - Rennes

Motivation

Financial market :

- $W := (W_t)_{t\in[0,T]}$ a p-dim. Brownian motion defined on $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$
- Riskless asset $S^0 := (S^0_t)_{t\in[0,T]}$,

$$dS^0_t = S^0_t r\, dt$$

- Risky assets $S := (S_t)_{t\in[0,T]}$,

$$dS^i_t = S^i_t \left(\alpha^i_t\, dt + \sigma^i_t dW_t\right), \quad i = 1, \ldots, d, \ (d \leq p).$$

Motivation

Investment strategy $(r = 0)$: $(x, (Z_t)_t)$ with associated wealth process $(X_t^{x,Z})_t$ defined as

$$X_t^{x,Z} := x + \sum_{i=1}^{d} \int_0^t Z_u^i \frac{dS_u^i}{S_u^i} = x + \sum_{i=1}^{d} \int_0^t Z_u^i (\sigma_u^i dW_u^i + \alpha_u^i du), \quad t \in [0, T]$$

Motivation

Investment strategy $(r = 0)$: $(x, (Z_t)_t)$ with associated wealth process $(X_t^{x,Z})_t$ defined as

$$X_t^{x,Z} := x + \sum_{i=1}^{d} \int_0^t Z_u^i \frac{dS_u^i}{S_u^i} = x + \sum_{i=1}^{d} \int_0^t Z_u^i (\sigma_u^i dW_u^i + \alpha_u^i du), \quad t \in [0, T]$$

$\hookrightarrow$ "Super-replication" (Incomplete markets as energy markets (Ex. : weather derivatives))

$$\forall \xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}), \quad \exists (x, Z), \quad X_T^{x,Z} \geq \xi, \ \mathbb{P} - a.s..$$

$$\text{Price of } \xi := Y_0(\xi) := \inf\{x > 0, \ \exists Z, \ X_T^{x,Z} \geq \xi\}.$$

Motivation

Investment strategy $(r = 0)$: $(x, (Z_t)_t)$ with associated wealth process $(X_t^{x,Z})_t$ defined as

$$X_t^{x,Z} := x + \sum_{i=1}^{d} \int_0^t Z_u^i \frac{dS_u^i}{S_u^i} = x + \sum_{i=1}^{d} \int_0^t Z_u^i(\sigma_u^i dW_u^i + \alpha_u^i du), \quad t \in [0, T]$$

$\hookrightarrow$ "Super-replication" (Incomplete markets as energy markets (Ex. : weather derivatives))

$$\forall \xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}), \quad \exists (x, Z), \quad X_T^{x,Z} \geq \xi, \ \mathbb{P} - a.s..$$

$$\text{Price of } \xi := Y_0(\xi) := \inf\{x > 0, \ \exists Z, \ X_T^{x,Z} \geq \xi\}.$$

$$Y_t^Z := x - \int_0^t g(s, Z_s)ds + \int_0^t Z_s dW_s, \quad t \in [0, T], \quad Y_T^Z \geq \xi, \ \mathbb{P} - a.s..$$

Motivation

Investment strategy $(r=0)$: $(x,(Z_t)_t)$ with associated wealth process $(X_t^{x,Z})_t$ defined as

$$X_t^{x,Z} := x + \sum_{i=1}^d \int_0^t Z_u^i \frac{dS_u^i}{S_u^i} = x + \sum_{i=1}^d \int_0^t Z_u^i(\sigma_u^i dW_u^i + \alpha_u^i du), \quad t \in [0,T]$$

$\hookrightarrow$ "Super-replication" (Incomplete markets as energy markets (Ex. : weather derivatives))

$$\forall \xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}), \quad \exists (x,Z), \quad X_T^{x,Z} \geq \xi,\ \mathbb{P}-a.s..$$

$$\text{Price of } \xi := Y_0(\xi) := \inf\{x>0,\ \exists Z,\ X_T^{x,Z} \geq \xi\}.$$

$$Y_t^Z := x - \int_0^t g(s,Z_s)ds + \int_0^t Z_s dW_s, \quad t \in [0,T], \quad Y_T^Z \geq \xi,\ \mathbb{P}-a.s..$$

super-replication price is too high !

Motivation

In practice one replaces the too stringent condition

$$X_T^{x,Z} \geq \xi, \ \mathbb{P} - p.s..$$

by a condition of the form

$$\mathbb{E}[\ell(X_T^{x,Z} - \xi)] \geq m, \quad m \in (0,1].$$

Motivation

In practice one replaces the too stringent condition

$$X_T^{x,Z} \geq \xi, \ \mathbb{P} - p.s..$$

by a condition of the form

$$\mathbb{E}[\ell(X_T^{x,Z} - \xi)] \geq m, \quad m \in (0,1].$$

Ex. : $\ell(x) := \mathbf{1}_{[0,+\infty[}$ $\Rightarrow$ "quantile-hedging" (Föllmer-Leukert)

Motivation

In practice one replaces the too stringent condition

$$X_T^{x,Z} \geq \xi, \ \mathbb{P} - p.s..$$

by a condition of the form

$$\mathbb{E}[\ell(X_T^{x,Z} - \xi)] \geq m, \quad m \in (0,1].$$

Ex. : $\ell(x) := \mathbf{1}_{[0,+\infty[} \Rightarrow$ "quantile-hedging" (Föllmer-Leukert)

Price of $\xi := Y_0(\xi) := \inf\{x > 0, \ \exists Z, \ \mathbb{E}[\ell(X_T^{x,Z} - \xi)] \geq m\}$.

Motivation

In practice one replaces the too stringent condition

$$X_T^{x,Z} \geq \xi, \ \mathbb{P} - p.s..$$

by a condition of the form

$$\mathbb{E}[\ell(X_T^{x,Z} - \xi)] \geq m, \quad m \in (0,1].$$

Ex. : $\ell(x) := \mathbf{1}_{[0,+\infty[} \Rightarrow$ "quantile-hedging" (Föllmer-Leukert)

$$\text{Price of } \xi := Y_0(\xi) := \inf\{x > 0, \ \exists Z, \ \mathbb{E}[\ell(X_T^{x,Z} - \xi)] \geq m\}.$$

$$Y_t^Z := x - \int_0^t g(s, Z_s)ds + \int_0^t Z_s dW_s, \quad t \in [0,T], \quad \mathbb{E}[\ell(Y_T^Z - \xi)] \geq m$$

Motivation

In practice one replaces the too stringent condition

$$X_T^{x,Z} \geq \xi, \ \mathbb{P} - p.s..$$

by a condition of the form

$$\mathbb{E}[\ell(X_T^{x,Z} - \xi)] \geq m, \quad m \in (0,1].$$

<u>Ex.</u> : $\ell(x) := \mathbf{1}_{[0,+\infty[} \Rightarrow$ "quantile-hedging" (Föllmer-Leukert)

Price of $\xi := Y_0(\xi) := \inf\{x > 0, \ \exists Z, \ \mathbb{E}[\ell(X_T^{x,Z} - \xi)] \geq m\}$.

$$Y_t^Z := x - \int_0^t g(s, Z_s)ds + \int_0^t Z_s dW_s, \quad t \in [0,T], \quad \mathbb{E}[\ell(Y_T^Z - \xi)] \geq m$$

Stochastic target problems with controlled loss (Soner, Touzi ; Bouchard, Elie, Touzi)

BSDEs with weak terminal condition

Definition (Super-solution of a BSDE with weak terminal condition)

A pair of predictable processes (Y, Z) (in $\mathbb{S}^2 \times \mathbb{H}^2$) is a super-solution to BSDE(g,ψ,m) if

$$Y_t \geq Y_T + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad (1)$$

$$\mathbb{E}[\psi(Y_T)] \geq m. \quad (2)$$

Definition (g-expectation)

A pair of predictable processes (Y, Z) (in $\mathbb{S}^2 \times \mathbb{H}^2$) is a solution to BSDE($g$,$\zeta$) if

$$\mathcal{E}_t^g[\zeta] := Y_t = \zeta + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s.$$

BSDEs with weak terminal condition

Proposition

(Y, Z) *is a super-solution of* (1)-(2) *iff.* (Y, Z) *enjoys* (1) *and there exists a predictable process* $\tilde{\alpha}$ *such that* $Y_t \geq \mathcal{E}_t^g[\psi^{-1}(M_T^{m,\tilde{\alpha}})]$, $M_T^{m,\tilde{\alpha}} := m + \int_0^T \tilde{\alpha}_s dW_s$.

BSDEs with weak terminal condition

Proposition

(Y, Z) is a super-solution of (1)-(2) iff. (Y, Z) enjoys (1) and there exists a predictable process $\tilde{\alpha}$ such that $Y_t \geq \mathcal{E}^g_t[\psi^{-1}(M^{m,\tilde{\alpha}}_T)]$, $M^{m,\tilde{\alpha}}_T := m + \int_0^T \tilde{\alpha}_s dW_s$.

Idea of the proof : $\Rightarrow$?

$$\mathbb{E}[\psi(Y_T)] \geq m \Rightarrow \psi(Y_T) = \rho + \int_0^T \tilde{\alpha}_t dW_t, \quad \rho \geq m$$

$$\psi(Y_T) \geq m + \int_0^T \tilde{\alpha}_t dW_t =: M^{m,\tilde{\alpha}}_T$$

BSDEs with weak terminal condition

Proposition

(Y, Z) is a super-solution of (1)-(2) iff. (Y, Z) enjoys (1) and there exists a predictable process $\tilde{\alpha}$ such that $Y_t \geq \mathcal{E}_t^g[\psi^{-1}(M_T^{m,\tilde{\alpha}})]$, $M_T^{m,\tilde{\alpha}} := m + \int_0^T \tilde{\alpha}_s dW_s$.

Idea of the proof : $\Rightarrow$?

$$\mathbb{E}[\psi(Y_T)] \geq m \Rightarrow \psi(Y_T) = \rho + \int_0^T \tilde{\alpha}_t dW_t, \quad \rho \geq m$$

$$\psi(Y_T) \geq m + \int_0^T \tilde{\alpha}_t dW_t =: M_T^{m,\tilde{\alpha}}$$

$$Y_T \geq (\psi^{-1} \circ \psi)(Y_T) \geq \psi^{-1}(M_T^{m,\tilde{\alpha}})$$

Comparaison for BSDEs $\Rightarrow$: $Y_t \geq \mathcal{E}_t^g[\psi^{-1}(M_T^{m,\tilde{\alpha}})]$.

$\square$

BSDEs with weak terminal condition

$$
\begin{aligned}
Y_t &\geq Y_T + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, && (1)\\
\mathbb{E}[\psi(Y_T)] &\geq m, && (2)
\end{aligned}
$$

$$\Gamma(0,m) := \{ Y_0,\ (Y,Z) \text{ super-solution of } (1)-(2)\}$$

$$\mathcal{Y}_0(m) := \inf_{\tilde{\alpha}} \ \mathcal{E}_0^g[\psi^{-1}(M_T^{m,\tilde{\alpha}})]$$

$$M_T^{m,\tilde{\alpha}} := m + \int_0^T \tilde{\alpha}_s dW_s$$

Proposition

$$\inf \Gamma(0,m) = \mathcal{Y}_0(m)$$

BSDEs with weak terminal condition : minimal initial value

$$\mathcal{Y}^{\alpha}_{\cdot} := \operatorname{essinf}_{\tilde\alpha}\ \mathcal{E}^{g}_{\cdot}[\psi^{-1}(M_T^{M_{\cdot}^{m,\alpha},\tilde\alpha})], \quad \mathcal{Y}^{\alpha}_0 = \mathcal{Y}_0$$

Theorem

For every α, $\mathcal{Y}^{\alpha}$ is a làdlàg g-sub-martingale and :

(i) *(DPP) $\mathcal{Y}^{\alpha}_{\tau_1} = \operatorname{essinf}_{\tilde\alpha}\mathcal{E}^{g}_{\tau_1,\tau_2}[\mathcal{Y}^{\tilde\alpha}_{\tau_2}]$, for every $\tau_1 \le \tau_2$ s.t..*

(ii) *if ψ^{-1} is continuous, there exists $(\mathcal{Z}^{\alpha}, \mathcal{K}^{\alpha})$ such that*

$$\mathcal{Y}^{\alpha}_t = \psi^{-1}(M_T^{m,\alpha}) + \int_t^T g(s, \mathcal{Y}^{\alpha}_s, \mathcal{Z}^{\alpha}_s)ds - \int_t^T \mathcal{Z}^{\alpha}_s dW_s + \mathcal{K}^{\alpha}_t - \mathcal{K}^{\alpha}_T, \quad t \in [0,T].$$

with the minimality condition

$$\mathcal{K}^{\alpha}_{\tau_1} = \operatorname{essinf}_{\tilde\alpha}\mathbb{E}\left[\mathcal{K}^{\tilde\alpha}_{\tau_2}|\mathcal{F}_{\tau_1}\right], \ \forall\, \tau_1 \le \tau_2,\ s.t.,$$

and satisfying the "independence on the future property"

$$(\mathcal{Y}^{\alpha}, \mathcal{Z}^{\alpha}, \mathcal{K}^{\alpha})\mathbf{1}_{[0,\tau]} = (\mathcal{Y}^{\tilde\alpha}, \mathcal{Z}^{\tilde\alpha}, \mathcal{K}^{\tilde\alpha})\mathbf{1}_{[0,\tau]}, \ \forall\, \tau \ s.t., \ \forall \tilde\alpha, \ with\ \tilde\alpha\mathbf{1}_{[0,\tau]} = \alpha\mathbf{1}_{[0,\tau]}.$$

BSDEs with weak terminal condition : other results

- $\exists\,?\ \hat{\alpha}$, which provides the minimal value at any time t, *i.e.*

$$\mathcal{Y}_t^{\hat{\alpha}} = \mathcal{E}_t^g[\psi^{-1}(M_T^{\hat{\alpha}})](= \mathrm{essinf}_{\tilde{\alpha}}\mathcal{E}_t^g[\psi^{-1}(M_T^{(M_t^{(m,\hat{\alpha})},\tilde{\alpha})})])$$

BSDEs with weak terminal condition : other results

- $\exists$? $\hat{\alpha}$, which provides the minimal value at any time t, *i.e.*

$$\mathcal{Y}_t^{\hat{\alpha}} = \mathcal{E}_t^g[\psi^{-1}(M_T^{\hat{\alpha}})](= \text{essinf}_{\tilde{\alpha}}\mathcal{E}_t^g[\psi^{-1}(M_T^{(M_t^{(m,\hat{\alpha})},\tilde{\alpha})})])$$

$\hookrightarrow$ g and ψ^{-1} convex $\Rightarrow$ Yes

Proposition

If ψ is deterministic and if $\hat{\psi^{-1}}$ the convex envelope of ψ^{-1} is continuous then

$$\mathcal{Y}_{T-}^{\alpha} = \hat{\psi^{-1}}(M_T^{\alpha}), \quad and \quad \mathcal{Y}_t^{\alpha} = essinf_{\tilde{\alpha}}\mathcal{E}_t^g[\hat{\psi^{-1}}(M_T^{\tilde{\alpha}})].$$

- Continuity and convexity of the map $m \mapsto \mathcal{Y}_t^{\alpha}(m), \quad \forall t < T$.

BSDEs with weak terminal condition : other results

- $\exists\,?\ \hat{\alpha}$, which provides the minimal value at any time t, *i.e.*

$$\mathcal{Y}_t^{\hat{\alpha}} = \mathcal{E}_t^g[\psi^{-1}(M_T^{\hat{\alpha}})](= \text{essinf}_{\tilde{\alpha}} \mathcal{E}_t^g[\psi^{-1}(M_T^{(M_t^{(m,\hat{\alpha})},\tilde{\alpha})})])$$

$\hookrightarrow$ g and ψ^{-1} convex $\Rightarrow$ Yes

Proposition

If ψ is deterministic and if $\hat{\psi^{-1}}$ the convex envelope of ψ^{-1} is continuous then

$$\mathcal{Y}_{T-}^{\alpha} = \hat{\psi^{-1}}(M_T^{\alpha}), \quad \text{and} \quad \mathcal{Y}_t^{\alpha} = \text{essinf}_{\tilde{\alpha}} \mathcal{E}_t^g[\hat{\psi^{-1}}(M_T^{\tilde{\alpha}})].$$

- Continuity and convexity of the map $m \mapsto \mathcal{Y}_t^{\alpha}(m), \quad \forall t < T$.
- Dual characterization of the minimization problem.

BSDEs with weak terminal condition : other results

- $\exists$? $\hat{\alpha}$, which provides the minimal value at any time t, *i.e.*

$$\mathcal{Y}_t^{\hat{\alpha}} = \mathcal{E}_t^g[\psi^{-1}(M_T^{\hat{\alpha}})](= \text{essinf}_{\tilde{\alpha}}\mathcal{E}_t^g[\psi^{-1}(M_T^{(M_t^{(m,\hat{\alpha})},\tilde{\alpha})})])$$

$\hookrightarrow$ g and ψ^{-1} convex $\Rightarrow$ Yes

Proposition

If ψ is deterministic and if $\hat{\psi^{-1}}$ the convex envelope of ψ^{-1} is continuous then

$$\mathcal{Y}_{T-}^{\alpha} = \hat{\psi^{-1}}(M_T^{\alpha}), \quad \text{and} \quad \mathcal{Y}_t^{\alpha} = \text{essinf}_{\tilde{\alpha}}\mathcal{E}_t^g[\hat{\psi^{-1}}(M_T^{\tilde{\alpha}})].$$

- Continuity and convexity of the map $m \mapsto \mathcal{Y}_t^{\alpha}(m), \quad \forall t < T$.
- Dual characterization of the minimization problem.
- Recover the PDE of Bouchard, Elie and Touzi in the Markovian framework.

BSDEs with weak terminal condition : other results

- $\exists\,?\ \hat{\alpha}$, which provides the minimal value at any time t, *i.e.*

$$\mathcal{Y}_t^{\hat{\alpha}} = \mathcal{E}_t^g[\psi^{-1}(M_T^{\hat{\alpha}})](= \text{essinf}_{\tilde{\alpha}} \mathcal{E}_t^g[\psi^{-1}(M_T^{(M_t^{(m,\hat{\alpha})},\tilde{\alpha})})])$$

$\hookrightarrow$ g and ψ^{-1} convex $\Rightarrow$ Yes

Proposition

If ψ is deterministic and if $\hat{\psi^{-1}}$ the convex envelope of ψ^{-1} is continuous then

$$\mathcal{Y}_{T-}^{\alpha} = \hat{\psi^{-1}}(M_T^{\alpha}), \quad \text{and} \quad \mathcal{Y}_t^{\alpha} = \text{essinf}_{\tilde{\alpha}} \mathcal{E}_t^g[\hat{\psi^{-1}}(M_T^{\tilde{\alpha}})].$$

- Continuity and convexity of the map $m \mapsto \mathcal{Y}_t^{\alpha}(m), \quad \forall t < T$.
- Dual characterization of the minimization problem.
- Recover the PDE of Bouchard, Elie and Touzi in the Markovian framework.
- Link with 2nd. order BSDEs (Cheridito, Soner, Touzi, Victoir, Zhang...).

Feynman-Kac formula for semi-linear PDEs (Pardoux-Peng)

$$\begin{cases} \partial_t v(t,x) + b(t,x)\cdot Dv(t,x) + \frac{1}{2} Tr.[\sigma\sigma^T(t,x) D^2 v(t,x)] = f(t,\cdot,v,\sigma^T\cdot Dv) \\ v(T,\cdot) = h(\cdot). \end{cases}$$

$$"\Leftrightarrow"$$

$$\begin{cases} dX_s^{t,x} = b(s,X_s^{t,x})ds + \sigma(s,X_s^{t,x})dW_s; \quad X_t^{t,x} = x. \\ dY_s^{t,x} = f(t,X_s^{t,x},Y_s^{t,x},Z_s^{t,x})ds - Z_s^{t,x}dW_s; \quad Y_T^{t,x} = h(X_T^{t,x}). \end{cases}$$

Feynman-Kac formula for semi-linear PDEs (Pardoux-Peng)

$$\begin{cases} \partial_t v(t,x) + b(t,x)\cdot Dv(t,x) + \frac{1}{2} Tr.[\sigma\sigma^T(t,x) D^2 v(t,x)] = f(t,\cdot,v,\sigma^T\cdot Dv) \\ v(T,\cdot) = h(\cdot). \end{cases}$$

$$" \Leftrightarrow "$$

$$\begin{cases} dX_s^{t,x} = b(s,X_s^{t,x})ds + \sigma(s,X_s^{t,x})dW_s; \quad X_t^{t,x} = x. \\ dY_s^{t,x} = f(t,X_s^{t,x},Y_s^{t,x},Z_s^{t,x})ds - Z_s^{t,x}dW_s; \quad Y_T^{t,x} = h(X_T^{t,x}). \end{cases}$$

$$v(t,x) = Y_t^{t,x}, \quad (v \in \mathcal{C}^{1,2})$$

Feynman-Kac formula for fully non-linear PDEs

How to provide a probabilistic representation of fully non-linear PDEs of the form

$$\begin{cases} -\partial_t v(t,x) - H(\cdot, v, Dv, D^2 v) = 0 \\ v(T,\cdot) = h(\cdot), \end{cases}$$

with $H(x,r,p,\gamma) := \sup_{a\geq 0}\{\frac{1}{2}a\gamma - F(x,r,p,a)\}$?

Feynman-Kac formula for fully non-linear PDEs

How to provide a probabilistic representation of fully non-linear PDEs of the form

$$\begin{cases} -\partial_t v(t,x) - H(\cdot, v, Dv, D^2 v) = 0 \\ v(T,\cdot) = h(\cdot), \end{cases}$$

with $H(x,r,p,\gamma) := \sup_{a\geq 0}\{\frac{1}{2}a\gamma - F(x,r,p,a)\}$?

Intuition from PDEs : $v(t,x) := \sup_a v^a(t,x)$,

$$\begin{cases} \partial_t v^a(t,x) + \frac{1}{2}aD^2 v^a(t,x) - F(\cdot, v^a, Dv^a, a) = 0 \\ v^a(T,\cdot) = g(\cdot). \end{cases}$$

Feynman-Kac formula for fully non-linear PDEs

How to provide a probabilistic representation of fully non-linear PDEs of the form

$$\begin{cases} -\partial_t v(t,x) - H(\cdot, v, Dv, D^2 v) = 0 \\ v(T,\cdot) = h(\cdot), \end{cases}$$

with $H(x,r,p,\gamma) := \sup_{a\geq 0}\{\frac{1}{2}a\gamma - F(x,r,p,a)\}$?

Intuition from PDEs : $v(t,x) := \sup_a v^a(t,x)$,

$$\begin{cases} \partial_t v^a(t,x) + \frac{1}{2}aD^2 v^a(t,x) - F(\cdot, v^a, Dv^a, a) = 0 \\ v^a(T,\cdot) = g(\cdot). \end{cases}$$

$$Y_t^a = h(X_T^a) + \int_t^T F(X_s^a, Y_s^a, Z_s^a, a_s)ds - \int_t^T Z_s^a a_s^{1/2} dW_s; \quad dX_t^a = a_t^{1/2} dW_t.$$

Feynman-Kac formula for fully non-linear PDEs

How to provide a probabilistic representation of fully non-linear PDEs of the form

$$\begin{cases} -\partial_t v(t,x) - H(\cdot, v, Dv, D^2 v) = 0 \\ v(T,\cdot) = h(\cdot), \end{cases}$$

with $H(x,r,p,\gamma) := \sup_{a \geq 0}\{\frac{1}{2}a\gamma - F(x,r,p,a)\}$?

Intuition from PDEs : $v(t,x) := \sup_a v^a(t,x)$,

$$\begin{cases} \partial_t v^a(t,x) + \frac{1}{2}aD^2 v^a(t,x) - F(\cdot, v^a, Dv^a, a) = 0 \\ v^a(T,\cdot) = g(\cdot). \end{cases}$$

$$Y_t^a = h(X_T^a) + \int_t^T F(X_s^a, Y_s^a, Z_s^a, a_s)ds - \int_t^T Z_s^a a_s^{1/2} dW_s; \quad dX_t^a = a_t^{1/2} dW_t.$$

$$Y_t := \sup_{a \geq 0} Y_t^a$$

EDSR du 2nd. ordre

$$Y_t^a = h(X_T^a) + \int_t^T F(X_s^a, Y_s^a, Z_s^a, a_s)ds - \int_t^T Z_s^a a_s^{1/2} dW_s; \quad dX_t^a = a_t^{1/2} dW_t.$$

EDSR du 2nd. ordre

$$Y_t^a = h(X_T^a) + \int_t^T F(X_s^a, Y_s^a, Z_s^a, a_s)ds - \int_t^T Z_s^a a_s^{1/2} dW_s; \quad dX_t^a = a_t^{1/2} dW_t.$$

$$Y_t^a = h(B_T) + \int_t^T F(B_s, Y_s^a, Z_s^a, a_s)ds - \int_t^T Z_s^a dB_s; \quad dB_t = a_t^{1/2} dW_t, \text{ sous } \mathbb{P}^a$$

Definition

A pair of predictable processes (Y, Z) is a solution to 2EDSR(h,g) if

$$Y_t \geq h(B_T) + \int_t^T \hat{F}(B_s, Y_s, Z_s)ds - \int_t^T Z_s dB_s, \quad \mathbb{P}^a - p.s., \forall a.$$

Lien entre $\mathcal{Y}$ et les EDSR du 2nd. ordre

Formally if g does not depend on Z, we have :

$$Y_t^a = \mathcal{E}_t^g[\psi^{-1}(M_T^a)] = \psi^{-1}(M_T^a) - \int_t^T Z_s^a dW_s + \int_t^T g(s, Y_s^a) ds.$$

Setting

$$\tilde{Y}^a := -Y^a,\ \tilde{Z}^a := -\frac{Z^a}{a}, \quad h(\cdot) := -\psi^{-1}(m + \cdot), \quad f(t, y) := -g(t, -y),$$

and $B^a := \int_0^\cdot a_s dW_s$

$$\tilde{Y}_t^a = h(B_T^a) - \int_t^T \tilde{Z}_s^a dB_s^a + \int_t^T f(s, \tilde{Y}_s^a) ds,$$

we have

$$-\mathcal{Y}_0 = \sup_{a \geq 0} \tilde{Y}_0^a = \sup_{a > 0} \tilde{Y}_0^a = Y_0^{2ndOrder}.$$