# $G$-Sobolev Spase, BSDE & PPDE

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with

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# Introduction

- Backward stochastic differential equations (BSDE) driven by a $G$-Brownian motion $(B_t)_{t\geq 0}$ in the following form:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)d\langle B\rangle_s - \int_t^T Z_s dB_s - (K_T - K_t).$$

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- Under a Lipschitz condition of $f$ and $g$ in $Y$ and $Z$.
- The existence and uniqueness of the solution $(Y, Z, K)$ is proved, where $K$ is a decreasing $G$-martingale.

$G$-martingale $M$ is of the form

$$
\begin{aligned}
M_t &= M_0 + \bar{M}_t + K_t, \\
\bar{M}_t &:= \int_0^t z_s B_s, \\
K_t &:= \int_0^t \eta_s \langle B \rangle_s - \int_0^t 2G(\eta_s) ds.
\end{aligned}
$$

# BSDE driven by G-Brownian motion

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- Definition of $G$-BM: given a linear space of functions of paths:

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\begin{aligned}
L_{ip}(\Omega_T) \quad := \quad & \{\varphi(\omega(t_1), \cdots, \omega(t_n) : t_1, \cdots, t_n \in [0, T], & (1) \\
& \varphi \in C_{l.Lip}(\mathbb{R}^n),\ n \in \mathbb{N}\}, & (2)
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- $\mathbb{E}_t^G[\xi] := u_k(t, \omega(t); \omega(t_1), \cdots, \omega(t_{k-1}))$ for each $t \in [t_{k-1}, t_k)$, $k = 1, \cdots, n$.

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- "$G$-Sobolev space" is implicitly "given"!

# Space of cylinder functions $C^\infty(\Omega_T)$

**Definition**

- A function $\xi : \Omega_T \to \mathbb{R}$ is called a cylinder function of paths on $[0, T]$ if
$$\xi(\omega_T) = \varphi(\omega(t_1), \cdots, \omega(t_n)), \omega_T \in \Omega_T,$$
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- $C^\infty(\Omega_T)$: collection of all cylinder functions of paths on $[0, T]$.

# Step processes $M^0(0, T)$

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A function $\eta(\omega_t) : [0, T] \times \Omega_T \to \mathbb{R}$ is called a step process if $\exists$ a time partition $\{t_i\}_{i=0}^n$, $0 = t_0 < t_1 < \cdots < t_n = T$, such that for each $t \in (t_k, t_{k+1}]$

$$\eta(\omega_t) = \varphi_k(\omega(t_1), \cdots, \omega(t_k)), \quad k = 0, 1, \cdots, n-1.$$

$\varphi_k(\omega(t_1), \cdots, \omega(t_k)) \in C^\infty(\Omega_T)$ is bounded.
We denote by $M^0(0, T)$ the collection of all step processes.

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- $M^2_G(0,T)$: the completion of $M^0(0,T)$ under $\|\cdot\|_{M^2_G}$.
- In linear case: $(\Omega_T, \mathcal{F}, \{\mathcal{F}_t\}, P)$: $\|\cdot\|_{L^2_P}$, $L^2_P(\Omega_T)$, $\|\cdot\|_{M^2_P}$ and $M^2_P(0,T)$.

# Space $C^\infty(0, T)$ of cylinder path process

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- all derivatives of $u_k$ have at most polynomial growth.
- $\mathcal{C}^\infty(0, T) :=$ the collection of all cylinder path processes.

# An easy Proposition:

**Proposition.**

Let $\eta, \zeta$ be step processes. Then

$$u(\omega_t) := \int_0^t \eta(\omega_s) ds + \int_0^t \zeta(\omega_s) dB_s$$

belongs to $\mathcal{C}^\infty(0, T)$. □

# Time derivatives of $u \in \mathcal{C}^\infty(0, T)$

For $t \in [t_k, t_{k+1})$, $n \in \mathbb{N}$, we denote

$$D_t^{(n)} u(\omega_t) := \partial_{t+}^{(n)} u_k(t, x; x_1, \cdots, x_k)|_{x=\omega(t), x_1=\omega(t_1), \cdots, x_k=\omega(t_k)}.$$

We denote $D_t = D_t^{(1)}$ for simplicity.

- For $t \in (t_k, t_{k+1}]$, we denote

$$D_x u(\omega_t) := \partial_x u_k(t, x; x_1, \cdots, x_k)|_{x=\omega(t), x_1=\omega(t_1), \cdots, x_k=\omega(t_k)}, \quad (3)$$

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- Let $\mathbb{D}$ be the Malliavin derivative operator. For $u \in \mathcal{C}^\infty(0,T)$, we have

$$\mathbb{D}_t u_t(\omega) = D_x u(\omega_t).$$

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- An advantage of our this formulation: is that we do not need to define of our derivatives on a space of right continuous paths with left limit.

# Wiener probability space $(\Omega, \mathcal{F}, P)$

- For the case $G(A) = \frac{1}{2}\mathrm{tr}(A)$, $G$-expectation$= E_P$ of the exp. in Wiener Prob. space $(\Omega, \mathcal{F}, P)$ and the $G$-BM $B$ becomes the Wiener process.

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**Proposition.**

For each given $u \in \mathcal{C}^\infty(0, \infty)$ we have

$$u(\omega_t) = u(\omega_0) + \int_0^t \mathcal{A}u(\omega_s)ds + \int_0^t D_x u(\omega_s)dB_s,$$

where

$$\mathcal{A}u(\omega_s) := D_s u(\omega_s) + \frac{1}{2}\Delta_x u(\omega_s).$$

□

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- $S_P^2(0, T)$: the completion of $\mathcal{C}^\infty(0, T)$ under the norm $\|\cdot\|_{S_P^2}$.

# Space $S_P^2(0, T)$

## Definition

- For $u \in \mathcal{C}^\infty(0, T)$, set

$$\|u\|_{S_P^2}^2 = E_P[\sup_{s\in[0,T]} |u_s|^2].$$

- $S_P^2(0, T)$: the completion of $\mathcal{C}^\infty(0, T)$ under the norm $\|\cdot\|_{S_P^2}$.
- For $u \in \mathcal{C}^\infty(0, T)$, set

$$\|u\|_{W_P^1}^2 = E_P[\sup_{s\in[0,T]} |u_s|^2 + \int_0^T (|\mathcal{A}u_s|^2 + |D_x u_s|^2) ds].$$

# $\|\cdot\|_{W_P^1(0,T)}$-norm

**Proposition.**

The norm $\|\cdot\|_{W_P^1(0,T)}$ is closable in the space $S_P^2(0,T)$: Let $u^n \in \mathcal{C}^\infty(0,T)$ be a Cauchy sequence w.r.t. the norm $\|\cdot\|_{W_P^1}$. If $\|u^n\|_{S_P^2} \to 0$, we have $\|u^n\|_{W_P^1} \to 0$. □

# $\|\cdot\|_{W^1_P(0,T)}$-norm

**Proposition.**

The norm $\|\cdot\|_{W^1_P(0,T)}$ is closable in the space $S^2_P(0,T)$: Let $u^n \in \mathcal{C}^\infty(0,T)$ be a Cauchy sequence w.r.t. the norm $\|\cdot\|_{W^1_P}$. If $\|u^n\|_{S^2_P} \to 0$, we have $\|u^n\|_{W^1_P} \to 0$. □

**Proof.**

The proposition follows directly from the uniqueness of the decomposition for Itô processes. □

# Space $W_P^1(0, T)$

Denote by $W_P^1(0, T)$ the closure of $\mathcal{C}^\infty(0, T)$ w.r.t. the norm $\|\cdot\|_{W_P^1}$ in $S_P^2(0, T)$. Now the operators $\mathcal{A}$, $D_x$ can be continuously extended to the space $W_P^1(0, T)$.

# A new point of view for Itô processes:

## Proposition.

Assume $u \in S_P^2(0, T)$. Then the following two conditions are equivalent:

□

# A new point of view for Itô processes:

**Proposition.**

Assume $u \in S_P^2(0,T)$. Then the following two conditions are equivalent:

- (i) $u \in W_P^1(0,T)$;

□

# A new point of view for Itô processes:

## Proposition.

Assume $u \in S_P^2(0, T)$. Then the following two conditions are equivalent:

- (i) $u \in W_P^1(0, T)$;
- (ii) $u(\omega_t) = u(\omega_0) + \int_0^t \eta(\omega_s)ds + \int_0^t v(\omega_s)dB_s$ with $\eta, v \in M_P^2(0, T)$.
  Moreover, we have

$$\mathcal{A}u(\omega_t) = \eta(\omega_t),\ D_x u(\omega_t) = v(\omega_t).$$

□

**Proof.**

(i) $\Longrightarrow$ (ii) is obvious. Let's prove (ii) $\Longrightarrow$ (i). Choose step processes $\eta^n, v^n$ such that $\|\eta^n - \eta\|_{M_P^2} \to 0$ and $\|v^n - v\|_{M_P^2} \to 0$. Set

$$u^n(\omega_t) := u(\omega_0) + \int_0^t \eta^n(\omega_s)ds + \int_0^t v^n(\omega_s)dB_s.$$

Clearly $u^n$ belongs to $\mathcal{C}^\infty(0, T)$ by Proposition 12. By Proposition 16 and the uniqueness of the decomposition for Itô processes, we have

$$\mathcal{A}u^n(\omega_t) = \eta^n(\omega_t), \; D_x u^n(\omega_t) = v^n(\omega_t).$$

So $u$ belongs to $W_P^1(0, T)$ with

$$\mathcal{A}u(\omega_t) = \eta(\omega_t), \; D_x u(\omega_t) = v(\omega_t).$$

□

**Proposition.**

- For each given $u \in \mathcal{C}^\infty(0,\infty)$ we have

$$
\begin{aligned}
&u(\omega_t)\\
=&u(\omega_0)+\int_0^t D_s u(\omega_s)ds+\int_0^t D_x u(\omega_s)dB_s+\frac{1}{2}\int_0^t D_x^2 u(\omega_s)d\langle B\rangle_s\\
&= u(\omega_0)+\int_0^t \mathcal{A}_G u(\omega_s)ds+\int_0^t D_x u(\omega_s)dB_s+K_t,
\end{aligned}
$$

□

## Proposition.

- For each given $u \in \mathcal{C}^{\infty}(0,\infty)$ we have

$$
\begin{aligned}
& u(\omega_t) \\
= & u(\omega_0) + \int_0^t D_s u(\omega_s) ds + \int_0^t D_x u(\omega_s) dB_s + \frac{1}{2}\int_0^t D_x^2 u(\omega_s) d\langle B\rangle_s \\
& = u(\omega_0) + \int_0^t \mathcal{A}_G u(\omega_s) ds + \int_0^t D_x u(\omega_s) dB_s + K_t,
\end{aligned}
$$

- where

$$
\mathcal{A}_G u(\omega_s) := D_s u(\omega_s) + G(D_x^2 u(\omega_s)),
$$

□

**Proposition.**

- For each given $u \in \mathcal{C}^\infty(0,\infty)$ we have

$$
\begin{aligned}
& u(\omega_t) \\
=& u(\omega_0) + \int_0^t D_s u(\omega_s) ds + \int_0^t D_x u(\omega_s) dB_s + \frac{1}{2}\int_0^t D_x^2 u(\omega_s) d\langle B\rangle_s \\
&= u(\omega_0) + \int_0^t \mathcal{A}_G u(\omega_s) ds + \int_0^t D_x u(\omega_s) dB_s + K_t,
\end{aligned}
$$

- where

$$\mathcal{A}_G u(\omega_s) := D_s u(\omega_s) + G(D_x^2 u(\omega_s)),$$

- $$K_t := \frac{1}{2}\int_0^t D_x^2 u(\omega_s) d\langle B\rangle_s - \int_0^t G(D_x^2 u(\omega_s)) ds$$

  $K$ is a non-increasing $G$-martingale.

□

## Definition

- 1) For $u \in \mathcal{C}^\infty(0, T)$, we set

$$\|u\|_{S_G^2}^2 = \mathbb{E}^G[\sup_{s\in[0,T]} |u_s|^2].$$

# Space $S_G^2(0, T)$

## Definition

- 1) For $u \in \mathcal{C}^\infty(0, T)$, we set

$$\|u\|^2_{S^2_G} = \mathbb{E}^G[\sup_{s\in[0,T]} |u_s|^2].$$

- We denote by $S_G^2(0, T)$ the completion of $u \in \mathcal{C}^\infty(0, T)$ w.r.t. the norm $\|\cdot\|_{S^2_G}$.

# Space $S_G^2(0,T)$

## Definition

- 1) For $u \in \mathcal{C}^\infty(0,T)$, we set

$$\|u\|_{S_G^2}^2 = \mathbb{E}^G[\sup_{s\in[0,T]} |u_s|^2].$$

- We denote by $S_G^2(0,T)$ the completion of $u \in \mathcal{C}^\infty(0,T)$ w.r.t. the norm $\|\cdot\|_{S_G^2}$.
- 2) For $u \in \mathcal{C}^\infty(0,T)$, we set

$$\|u\|_{W_G^2}^2 = \mathbb{E}^G[\sup_{s\in[0,T]} |u_s|^2 + \int_0^T (|D_s u_s|^2 + |D_x u_s|^2 + |D_x^2 u_s|^2) ds].$$

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- Actually solved by Song (2012)
- see Peng, Song and Zhang (2012) for the multi-dimensional case.
- We only consider the 1-dim $G$-BM and assume $\bar{\sigma}^2 := \mathbb{E}^G[B_1^2] > \underline{\sigma}^2 := -\mathbb{E}^G[-B_1^2]$

**Lemma**

*If*

$$u(\omega_t) = \int_0^t \zeta(\omega_s)ds + \int_0^t v(\omega_s)dB_s + \frac{1}{2}\int_0^t w(\omega_s)d\langle B\rangle_s = 0$$

*with* $\zeta, v, w \in M_G^2(0,T)$, *we have* $\zeta = v = w = 0$.

**Proof.**

By the uniqueness of the decomposition for continuous semimartingales we have $v = 0$ and $\int_0^t \zeta(\omega_s)ds + \frac{1}{2}\int_0^t w(\omega_s)d\langle B\rangle_s = 0$. By Corollary 3.5 in Song (2012) we conclude that $\zeta = w = 0$. □

**Proposition.**

The norm $\|\cdot\|_{W_G^2}$ is closable in the space $S_G^2(0,T)$: Let $u^n \in \mathcal{C}^\infty(0,T)$ be a Cauchy sequence w.r.t. the norm $\|\cdot\|_{W_G^2}$. If $\|u^n\|_{S_G^2} \to 0$, we have $\|u^n\|_{W_G^2} \to 0$. □

**Proposition.**

The norm $\|\cdot\|_{W_G^2}$ is closable in the space $S_G^2(0,T)$: Let $u^n \in \mathcal{C}^\infty(0,T)$ be a Cauchy sequence w.r.t. the norm $\|\cdot\|_{W_G^2}$. If $\|u^n\|_{S_G^2} \to 0$, we have $\|u^n\|_{W_G^2} \to 0$. □

**Proof.**

The proposition follows directly from the uniqueness of the decomposition for $G$-Itô processes. □

# Spase $W_G^2(0, T)$

Denote by $W_G^2(0, T)$ the closure of $\mathcal{C}^\infty(0, T)$ w.r.t. the norm $\|\cdot\|_{W_G^2}$ in $S_G^2(0, T)$. Now the differential operators $D_t$, $D_x$ and $D_x^2$ defined on $\mathcal{C}^\infty(0, T)$ can be continuously extended to the space $W_G^2(0, T)$.

**Proposition.**

Assume $u \in S_G^2(0,T)$. Then the following two conditions are equivalent:
(i) $u \in W_G^2(0,T)$;
(ii) $u$ is of the form:

$$u(\omega_t) = u(\omega_0) + \int_0^t \zeta(\omega_s)ds + \int_0^t v(\omega_s)dB_s + \frac{1}{2}\int_0^t w(\omega_s)d\langle B\rangle_s,$$

$\zeta, v, w \in M_G^2(0,T)$

Moreover, we have

$$D_t u(\omega_t) = \zeta(\omega_t),\ D_x u(\omega_t) = v(\omega_t),\ D_x^2 u(\omega_t) = w(\omega_t).$$

□

**Remark.**

Compared to $W_P^2(0,T)$ case, here the derivatives $D_t u$, $D_x u$, $D_x^2 u$ can be distinguished clearly. □

: to find a pair of processes $(Y, Z) \in S_P^2(0, T) \times M_P^2(0, T)$ s. t.

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s, \quad (6)$$

$g$: a given function and $\xi$ is a given $\mathcal{F}_T$=measurable.
The backward SDE (6) is equivalent to the path dependent PDE: to find $u \in W_P^1(0, T)$ such that

$$\mathcal{A}u(\omega_t) + f(t, u(\omega_t), D_x u(\omega_t)) = 0, \quad t \in [0, T), \quad (7)$$

$$u(\omega_T)) = \xi(\omega_T). \quad (8)$$

**Assumption 1**. $f(t, Y_t, Z_t) \in M_P^2(0, T)$ for any $(Y, Z) \in S_P^2(0, T) \times M_P^2(0, T)$.

**Theorem**

*Let $(Y, Z)$ be a solution to the backward SDE (6). Then we have $u(\omega_t) := Y_t(\omega) \in W_P^1(0, T)$ with $D_x u(\omega_t) = Z_t(\omega)$.*
*Moreover, given $u(\omega_t) \in W_P^1(0, T)$, the following (i) and (ii) are equivalent:*
*(i) $(u, D_x u)$ is a solution to the backward SDE (6);*
*(ii) $u$ is a solution to the path dependent PDE (7-8).*

Proof.

(i) $\Longrightarrow$ (ii). Assume that $(Y,Z)$ is a solution to the backward SDE. We know that $u(\omega_t) := Y_t(\omega) \in W_P^1(0,T)$ with $D_x u(\omega_t) = Z_t(\omega)$ and

$$\mathcal{A}u(\omega_t) + f(t, u(\omega_t), D_x u(\omega_t)) = 0.$$

(ii) $\Longrightarrow$ (i). Assume that $u(\omega_t) \in W_P^1(0,T)$ is a solution to the path dependent PDE (7-8). By Proposition **??** we have

$$\begin{aligned} u(\omega_t) &= u(\omega_0) + \int_0^t \mathcal{A}u(\omega_s)ds + \int_0^t D_x u(\omega_s)dB_s \\ &= u(\omega_0) - \int_0^t f(s, u(\omega_s), D_x u(\omega_s))ds + \int_0^t D_x u(\omega_s)dB_s \\ &= \xi(\omega_T) + \int_t^T f(s, u(\omega_s), D_x u(\omega_s))ds - \int_t^T D_x u(\omega_s)dB_s. \end{aligned}$$

□

**Remark.**

The advantage of our formulation is the path dependent PDE can be a system of PDEs, namely $u(\omega_t)$ can be $\mathbb{R}^m$-valued, or even $H$-valued for a Hilbert space $H$. □

**Remark.**

Let us consider Markovian situations:
$\xi = \varphi(B_T)$, $f(\omega_t, y, z) = h(t, B_t(\omega), y, z)$ for deterministic and continuous functions $\varphi(x)$ and $h(t, x, y, z)$ satisfying Lipschitz conditions in $(y, z)$ and polynomial growth condition in $x$. Assume that $(Y, Z)$ is the solution to the backward SDE (6) with the coefficients $(\varphi(B_T),$ $h(t, B_t, y, z))$. By the classical arguments in the BSDE theory, we know that $Y$ is Markovian, i.e., there exists a deterministic function $u(t, x)$ such that $Y_t = u(t, B_t)$. Assuming $u(t, x)$ is smooth, we have

$$\mathcal{A}u(\omega_t) = \partial_t u(t, B_t) + \frac{1}{2}\Delta_x u(t, B_t), \ D_x u(\omega_t) = Du(t, B_t).$$

By Theorem 7, we have

$$\partial_t u(t, B_t) + \frac{1}{2}\Delta_x u(t, B_t) + h(t, B_t, u(t, B_t), Du(t, B_t)) = 0.$$

Equivalently

Let us consider backward SDEs driven by $G$-Brownian motion in the following from: to find $Y \in S_G^2(0,T)$, $Z, \eta \in M_G^2(0,T)$ such that

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, \eta_s) ds - \int_t^T Z_s dB_s - (K_T - K_t), \tag{9}$$

where $K_t = \frac{1}{2}\int_0^t \eta_s d\langle B\rangle_s - \int_0^t G(\eta_s) ds$, $f$ is a given function and $\xi$ is a given random variable.

The related path dependent PDEs: to find $u \in W_G^2(0,T)$ such that

$$D_t u + G(D_x^2 u) + f(t, u, D_x u, D_x^2 u) = 0, \quad t \in [0, T), \tag{10}$$

$$u(\omega_T)) = \xi(\omega_T). \tag{11}$$

**Assumption 2.**
$f(t, Y_t, Z_t, \eta_t) \in M_G^2(0,T)$ for any
$(Y, Z, \eta) \in S_G^2(0,T) \times M_G^2(0,T) \times M_G^2(0,T)$.

**Theorem**

*Let $(Y, Z, \eta)$ be a solution to the backward SDE (9). Then we have $u(\omega_t) := Y_t(\omega) \in W_G^2(0,T)$ with $D_x u(\omega_t) = Z_t(\omega)$ and $D_x^2 u(\omega_t) = \eta_t(\omega)$.*
*Moreover, for $u(\omega_t) \in W_G^2(0,T)$, the following conditions are equivalent:*
*(i) $(u, D_x u, D_x^2 u)$ is a solution to the backward SDE (9);*
*(ii) $u$ is a solution to the path dependent PDE (10-11).*

$W^1_G(0, T)$ in $G$-expectation space

Let us see how to define the $G$-Sobolev space $W^1_G(0, T)$.

For $u, v \in \mathcal{C}^\infty(0, T)$, set

$$\tilde{d}^2_{W^1_G}(u, v) = \mathbb{E}^G[\sup_{s\in[0,T]} |u_s - v_s|^2 + \int_0^T (|\mathcal{A}_G u_s - \mathcal{A}_G v_s|^2 + |D_x(u_s - v_s)|^2) d$$

However, we don't know whether the metric $\tilde{d}_{W^1_G}$ is closable in the space $S^2_G(0, T)$, which is equivalent to the uniqueness of the following decomposition:

$\mathcal{Q}$ : Assume $\int_0^t \eta_s ds + K_t = L_t$, where $\eta \in M^2_G(0, T)$, $K_t, L_t$ are non-increasing $G$-martingales with $K_T, L_T \in L^2_G(\Omega_T)$. Does it hold that $\int_0^t \eta_s ds = 0$ and $K_t = L_t$?

Now we modify the metric $\tilde{d}_{W_G^1}$ slightly. For $u, v \in \mathcal{C}^\infty(0, T)$, set

$$d^2_{W_G^1}(u, v) = \mathbb{E}^G[\sup_{s\in[0,T]} |u_s - v_s|^2 + \int_0^T |D_x(u_s - v_s)|^2 ds] + \int_0^T \mathbb{E}^G[|\mathcal{A}_G u_s -$$

For a step process $\eta$, set $\|\eta\|^2_{\tilde{M}_G^2} = \int_0^T \mathbb{E}^G[|\eta_s|^2]ds$. Denote by $\tilde{M}_G^2(0, T)$ the completion of the collection of step processes with respect to the norm $\|\cdot\|_{\tilde{M}_G^2}$.

In order to define the first-order $G$-Sobolev space, we first recall the structure of $G$-martingales. S.P., Y.Song & J.Zhang proved that for any $\xi \in C^{\infty}(\Omega_T)$, the $G$-martingale $X_t = \mathbb{E}_t^G[\xi]$ has the following representation:

$$X_t = \mathbb{E}^G[\xi] + \int_0^t Z_s dB_s + \frac{1}{2}\int_0^t \eta_s d\langle B\rangle_s - \int_0^t G(\eta_s)ds \qquad (12)$$

for some $Z, \eta \in M_G^2(0,T)$ and conjectured that for any $\xi \in L_G^2(\Omega_T)$ the representation (12) holds. Besides, **[?]** showed that for any $\eta \in M_G^2(0,T)$,

$$K_t := \frac{1}{2}\int_0^t \eta_s d\langle B\rangle_s - \int_0^t G(\eta_s)ds$$

is a non-increasing $G$-martingale.

For $p \geq 1$ and $\xi \in C^{\infty}(\Omega_T)$, set $\|\xi\|^p_{\mathbb{L}^p_G} = \mathbb{E}^G[\sup_{t\in[0,T]} |\mathbb{E}^G_t[\xi]|^p]$. Denote by $\mathbb{L}^p_G(\Omega_T)$ the closure of $C^{\infty}(\Omega_T)$ with respect to the norm $\|\cdot\|_{\mathbb{L}^p_G}$ in $L^p_G(\Omega_T)$. [?] showed that for any $\xi \in \mathbb{L}^2_G(\Omega_T)$ the $G$-martingale $X_t := \mathbb{E}^G_t[\xi]$ has the following decomposition:

$$X_t = \mathbb{E}^G[\xi] + \int_0^t Z_s dB_s + K_t, \tag{13}$$

where $K_t$ is a non-increasing $G$-martingale.

Song [?] showed that $\mathbb{L}_G^p(\Omega_T) \supset L_G^q(\Omega_T)$ for any $1 \leq p < q$. Moreover, [?] proved that the decomposition (13) holds for any $\xi \in L_G^p(\Omega_T)$ with $p > 1$. Independently, [?] showed that $\mathbb{L}_G^2(\Omega_T) \supset L_G^q(\Omega_T)$ for any $q > 2$.

Below we present a property of non-increasing $G$-martingales.

**Lemma**

*Assume $\int_0^t \eta_s ds + K_t = L_t$, where $\eta \in \tilde{M}_G^2(0,T)$, $K_t, L_t$ are non-increasing $G$-martingales with $K_T, L_T \in L_G^2(\Omega_T)$. Then we have $\int_0^t \eta_s ds = 0$ and $K_t = L_t$.*

**Proof.**

Let $\zeta \in \tilde{M}_G^2(0,T)$. We claim that $A = 0$ if $A_t := \int_0^t \zeta_s ds$ is a $G$-martingale. In fact $A_t$ must be a non-increasing $G$-martingale by $G$-martingale decomposition theorem. For $n \in \mathbb{N}$, set $h = T/n$ and

$$\hat{\zeta}_t^n = \sum_{k=0}^{n-1} 1_{(kh,(k+1)h]}(t) \frac{1}{h} \int_{kh}^{(k+1)h} \zeta_s ds,$$
$$\check{\zeta}_t^n = \sum_{k=1}^{n-1} 1_{(kh,(k+1)h]}(t) \frac{1}{h} \int_{(k-1)h}^{kh} \zeta_s ds.$$

For $t \in (kh,(k+1)h]$, we have $\mathbb{E}^G[\hat{\zeta}_t^n - \check{\zeta}_t^n] = \frac{1}{h}\mathbb{E}^G[-(A_{kh} - A_{(k-1)h})]$.

$$0 \leftarrow \{T \int_0^T \mathbb{E}^G[|\hat{\zeta}_t^n - \check{\zeta}_t^n|^2]dt\}^{1/2} \geq \int_0^T \mathbb{E}^G[\hat{\zeta}_t^n - \check{\zeta}_t^n]dt$$
$$= \sum^{n-1} \mathbb{E}^G[-(A_{kh} - A_{(k-1)h})]$$

By the uniqueness of the decomposition we obtain the following result.

**Proposition.**

The metric $d_{W_G^1}$ is closable in the space $S_G^2(0,T)$: Let $u^n, v^n \in \mathcal{C}^\infty(0,T)$ be two Cauchy sequences w.r.t. the metric $d_{W_G^1}$. If $\|u^n - v^n\|_{S_G^2} \to 0$, we have $d_{W_G^1}(u^n, v^n) \to 0$. □

# Operators $\mathcal{A}_G$ and $D_x$

$W_G^1(0,T) :=$ the closure of $\mathcal{C}^\infty(0,T)$ w.r.t. the metric $d_{W_G^1}$ in $S_G^2(0,T)$.
$\mathcal{A}_G, D_x$ is continuously extended to $W_G^1(0,T)$.

**Proposition.**

Assume $u \in S_G^2(0,T)$. Then the following two conditions are equivalent:
(i) $u \in W_G^1(0,T)$;
(ii) $u(\omega_t) = u(\omega_0) + \int_0^t \eta(\omega_s)ds + \int_0^t v(\omega_s)dB_s + K_t$ with $\eta \in \tilde{M}_G^2(0,T), v \in M_G^2(0,T)$ $K_T \in L_G^2(\Omega_T)$ and $K_t$ a non-increasing $G$-martingale.
Moreover, we have $\mathcal{A}_G u = \eta$ and $D_x u = v$. □

# Fully nonlinear path dependent PDEs

Let's consider a special case of backward SDEs driven by $G$-Brownian motion studied in [HJPS2012] (see the Appendix): to find $(Y, Z, K)$ with $Y \in S_G^2(0,T)$, $Z \in M_G^2(0,T)$, $K_T \in L_G^2(\Omega_T)$ and $K_t$ a non-increasing $G$-martingale satisfying

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s - (K_T - K_t). \qquad (14)$$

## The related path dependent PDEs:to find $u \in W^1_G(0,T)$, s.t.

$$\mathcal{A}_G u(\omega_t) + f(t, u(\omega_t), D_x u(\omega_t)) = 0, \quad t \in [0,T), \qquad \text{(PPDE)}$$

$$u(\omega_T)) = \xi(\omega_T). \qquad \text{(TC)}$$

**Assumption 3.** $f(t, Y_t, Z_t) \in \tilde{M}^2_G(0,T)$ for any $(Y,Z) \in S^2_G(0,T) \times M^2_G(0,T)$.

**Theorem**

*Let $(Y, Z)$ be a solution to the backward SDE (14). Then we have $u(\omega_t) := Y_t(\omega) \in W_G^1(0, T)$ with $D_x u(\omega_t) = Z_t(\omega)$.*
*Moreover, for $u(\omega_t) \in W_G^1(0, T)$, the following (i) and (ii) are equivalent:*
*(i) $(u, D_x u)$ is a solution to the backward SDE;*
*(ii) $u$ is a solution to the path dependent PDE.*

**Corollary**

*$u \in S_G^2(0,T)$ is a $G$-martingale if and only if $u \in W_G^1(0,T)$ and $\mathcal{A}_G u(\omega_t) = 0$.*

**Proof.**

By the $G$-martingale decomposition theorem, $u \in S_G^2(0,T)$ is a $G$-martingale if and only if $u$ is a solution of backward SDE (14) with $f = 0$. □

# Backward SDEs driven by $G$-BM

In [HJPS2012] the authors studied the backward stochastic differential equations driven by a $G$-Brownian motion $(B_t)_{t\geq 0}$ in the following form:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s - (K_T - K_t). \qquad (15)$$

where $K$ is a non-increasing $G$-martingale.

Solution $(Y, Z, K)$ for equation (15) in the $G$-framework under the following assumption: there exists some $\beta > 1$ such that

(H1) for any $y, z$, $f(\cdot, \cdot, y, z) \in M_G^\beta(0, T)$;

(H2) $|f(\omega_t, y, z) - f(\omega_t, y', z')| \leq L(|y - y'| + |z - z'|)$ for some $L > 0$.

Denote by $\mathfrak{S}_G^{\alpha}(0,T)$ the collection of processes $(Y,Z,K)$ such that $Y\in S_G^{\alpha}(0,T)$, $Z\in H_G^{\alpha}(0,T)$, $K$ is a decreasing $G$-martingale with $K_0=0$ and $K_T\in L_G^{\alpha}(\Omega_T)$.

**Definition**

Let $\xi\in L_G^{\beta}(\Omega_T)$ and $f$ satisfy (H1) and (H2) for some $\beta>1$. A triplet of processes $(Y,Z,K)$ is called a solution of equation (15) if for some $1<\alpha\leq\beta$ the following properties hold:

(a) $(Y,Z,K)\in\mathfrak{S}_G^{\alpha}(0,T)$;

(b) $Y_t=\xi+\int_t^T f(s,Y_s,Z_s)ds-\int_t^T Z_s dB_s-(K_T-K_t)$.

The main result in [HJPS2012] is the following theorem:

**Theorem**

*Assume that $\xi \in L_G^{\beta}(\Omega_T)$ and $f$ satisfies (H1) and (H2) for some $\beta > 1$. Then equation (15) has a unique solution $(Y, Z, K)$. Moreover, for any $1 < \alpha < \beta$ we have $Y \in S_G^{\alpha}(0, T)$, $Z \in H_G^{\alpha}(0, T)$ and $K_T \in L_G^{\alpha}(\Omega_T)$.*

Merci