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Perspectives in Analysis and Probability
WS3 Backward Stochastic Differential Equations

Robust utility maximization from terminal wealth and consumption

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We consider a filtered probability space $(\Omega, \mathbb{G}, \mathbb{P})$.

• $\mathcal{Q}$ is the space of all probability measures $\mathbb{Q}$ on $(\Omega, \mathcal{G}_T)$ with $\mathbb{Q} \ll \mathbb{P}$ and $d\mathbb{Q}|_{\mathcal{G}_T} = Z_T^{\mathbb{Q}} dP|_{\mathcal{G}_T}$ and $\mathbb{E}^{\mathbb{Q}}\left[\ln Z_T^{\mathbb{Q}}\right] < \infty$.

• $\mathcal{Q}^e \subset \mathcal{Q}$ is the sub-space of all probability measures equivalent to $\mathbb{P}$.

Note that the reference probability measure $\mathbb{P}$ belongs to $\mathcal{Q}^e$.

• The robust utility maximization problem $\mathcal{P}(U, \bar{U}_T)$ is to find the infimum of $\Gamma(\mathbb{Q})$ over the set $\mathcal{Q}$ where

$$\begin{aligned}\Gamma(\mathbb{Q}) &:= \mathbb{E}^{\mathbb{Q}}\left[\int_0^T S_s^\delta U_s ds + S_T^\delta \bar{U}_T\right] + \mathbb{E}^{\mathbb{Q}}\left[\int_0^T \delta_s S_s^\delta \ln Z_s^{\mathbb{Q}} ds + S_T^\delta \ln Z_T^{\mathbb{Q}}\right] \\ &=: \mathbb{E}^{\mathbb{Q}}\left[\mathcal{U}_{0,T}^\delta\right] + \mathbb{E}^{\mathbb{Q}}\left[\mathcal{R}_{0,T}^\delta(\mathbb{Q})\right]\end{aligned}$$

• $S_t^\delta := e^{-\int_0^t \delta_s ds}$ where δ is a non-negative adapted process is a discounting process

• U is a given process (the cost process) and $\bar{U}_T$ a given random variable (the terminal target).

References:

Bordigoni G. : Stochastic control and BSDEs in a robust utility maximization problem with an entropic penality term. *PhD Thesis of Politecnico di Milano* (2005). (continuous case)

Bordigoni, G., Matoussi, A., Schweizer, M. : A Stochastic control approach to a robust utility maximization problem. *F. E. Benth et al. (eds.), Stochastic Analysis and Applications. Proceedings of the Second Abel Symposium, Oslo, 2005*, *Springer*, 125-151 (2007).(continuous case)

Gundel, A. : Robust utility maximization for complete and incomplete market models. (Case $\delta = 0$)

We make the following assumptions:

1. For each $i = 1, \ldots, d$, N^i is a counting process and there exists a positive adapted process λ^i, such that the process M^i with

$$M_t^i := N_t^i - \int_0^t \lambda_s^i ds$$

is a **martingale**. We assume that the processes N^i have no common jumps.

2. Any discontinuous martingale admits a representation of the form $\sum_{i=1}^d \vartheta_t^i dM_t^i$ where ϑ^i are predictable processes.

We make the following further assumptions:

i) the discount rate δ is a non-negative bounded process, more precisely there exists a positive constant c such that for any $t \geq 0$, $0 \leq \delta_t \leq c$, a.s.

ii) the cost process U belongs to $D_1^{\exp}$ and the terminal target $\bar{U}_T$ is in $L^{\exp}$.

iii) the process $\Lambda_t^i := \int_0^t \lambda_s^i ds$ is uniformly bounded, i.e., $\Lambda_T^i \leq C$, $a.s.$.

- $D_1^{\exp}$ is the space of progressively measurable processes X such that

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left(\gamma \int_0^T |X_s| ds\right)\right] < \infty, \quad \forall \gamma > 0.$$

- $L^{\exp}$ is the space of all $\mathcal{G}_T$-measurable r.vs ζ with $\mathbb{E}^{\mathbb{P}}\left[\exp\left(\gamma|\zeta|\right)\right] < \infty, \quad \forall \gamma > 0.$

Under the above Assumptions, there exists a unique $\mathbb{Q}^*$ which minimizes $\Gamma(\mathbb{Q})$ over all $\mathbb{Q} \in \mathcal{Q}$:

$$\Gamma(\mathbb{Q}^*) = \inf_{\mathbb{Q} \in \mathcal{Q}} \Gamma(\mathbb{Q})$$

Furthermore, $\mathbb{Q}^*$ is equivalent to $\mathbb{P}$, i.e., $\mathbb{Q}^* \in \mathcal{Q}^e$.

A triple of processes $(Y, M^{Y,c}, y)$ such that

Y is a $\mathbb{P}$-semimartingale,

$M^{Y,c}$ is a locally square-integrable continuous local $\mathbb{P}$-martingale null at 0

$y = (y^1, \cdots, y^d)$ an $\mathbb{R}^d$-valued predictable locally bounded process,

is called solution of quadratic-exponential BSDEs, if it satisfies:

$$(*) \begin{cases} dY_t = \Big[\sum_{i=1}^{d} g(y_t^i)\lambda_t^i - U_t + \delta_t Y_t\Big]dt + \frac{1}{2}d\langle M^{Y,c}\rangle_t + dM_t^{Y,c} + \sum_{i=1}^{d} y_t^i dM_t^i \\ Y_T = \bar{U}_T \end{cases}$$

where g is the convex function $g(x) = e^{-x} + x - 1$.

Note that Y is a special $\mathbb{P}$-semimartingale.

We consider solutions of the BSDE (*) in some specific spaces :
$(Y, M^{Y,c}, y) \in D_0^{\exp} \times \mathcal{M}^c_{0,loc}(\mathbb{P}) \times \mathcal{L}^2(\lambda, \mathbb{P}) =: \mathbb{B}$ where

- $D_0^{\exp}$ is the space of progressively measurable processes Y with

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left(\gamma \operatorname{ess\,sup}_{0\le t\le T}|Y_t|\right)\right] < \infty, \quad \forall \gamma > 0.$$

- $\mathcal{M}^c_{0,loc}(\mathbb{P})$ is the space of continuous local martingale, null at 0
- $\mathcal{L}^2(\lambda, \mathbb{P})$ is the space of $\mathbb{R}^d$-valued predictable processes y such that

$$\sum_{i=1}^{d} \mathbb{E}^{\mathbb{P}}\Big[\int_0^T (y_s^i)^2 \lambda_s^i ds\Big] < \infty.$$

(*) $dY_t = (\sum_{i=1}^d g(y_t^i)\lambda_t^i - U_t + \delta_t Y_t)\, dt + \frac{1}{2} d\langle M^{Y,c}\rangle_t + dM_t^{Y,c} + \sum_{i=1}^d y_t^i dM_t^i$

Let $(Y, M^{Y,c}, y) \in \mathbb{B}$ be a solution of the BSDE (*). Then, Y satisfies the following recursion equality: for any stopping time τ valued in $[t, T]$,

$$Y_t = -\ln \mathbb{E}^{\mathbb{P}}\left[\exp\left(-Y_\tau + \int_t^\tau (\delta_s Y_s - U_s)ds\right)\Big|\mathcal{G}_t\right].$$

Moreover the BSDE (*) admits at most one solution which belongs to $\mathbb{B}$.

(*) $dY_t = (\sum_{i=1}^d g(y_t^i)\lambda_t^i - U_t + \delta_t Y_t)\, dt + \frac{1}{2} d\langle M^{Y,c}\rangle_t + dM_t^{Y,c} + \sum_{i=1}^d y_t^i dM_t^i$

Recursive equation:

Assume that $(Y, M^{Y,c}, y)$ is a solution of (*), and define

$$X_t = Y_t - Y_0 - \int_0^t (\delta_s Y_s - U_s) ds$$

and $Z_t = e^{-X_t}$.

Itô's formula leads to $dZ_t = Z_{t-} \left[-dM_t^{Y,c} + \sum_{i=1}^d \left(e^{-y_t^i} - 1 \right) dM_t^i \right]$.

Hence, Z is a non-negative local martingale. Assuming that Z is a martingale, one obtains, for $t < \tau < T$:

$$e^{-Y_t} = \mathbb{E}^{\mathbb{P}} \left[\exp \left(-Y_\tau + \int_t^\tau (\delta_s Y_s - U_s) ds \right) \Big| \mathcal{G}_t \right].$$

Using a localizing sequence we obtain the result.

(*) $dY_t = (\sum_{i=1}^d g(y_t^i)\lambda_t^i - U_t + \delta_t Y_t)\, dt + \frac{1}{2} d\langle M^{Y,c} \rangle_t + dM_t^{Y,c} + \sum_{i=1}^d y_t^i dM_t^i$

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Uniqueness of the solution of the BSDE (*):

Assume that $(Y, M^{Y,c}, y)$ and $(\bar{Y}, M^{\bar{Y},c}, \bar{y})$ are two solutions of (*). Suppose that, for some $t \in [0, T]$, the set $A = \{Y_t > \bar{Y}_t\} \in \mathcal{G}_t$ satisfies $\mathbb{P}(A) > 0$ and define $\tau = \inf\{s \geq t | \bar{Y}_s \geq Y_s\}$, so that $\bar{Y}_\tau \geq Y_\tau$. Since $Y_T = \bar{Y}_T$, one has $\tau \leq T$, and:

$$\int_t^\tau (\delta_s Y_s - U_s) ds - Y_\tau > \int_t^\tau (\delta_s \bar{Y}_s - U_s) ds - \bar{Y}_\tau \text{ on } A,$$

then from the recursion relation, it follows that

$$\exp\left(-Y_t\right) = \mathbb{E}^{\mathbb{P}}\left[\exp\left(\int_t^\tau \delta_s Y_s - U_s) ds - Y_t\right) \Big| \mathcal{G}_t\right] > \exp\left(-\bar{Y}_t\right) \text{ on } A$$

which implies that $Y_t < \bar{Y}_t$ on A in contradiction with the definition of A; therefore Y and $\bar{Y}$ are indistinguishable. Therefore $M^{Y,c} = M^{Y',c}$ and $y = y'$.

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Theorem: There exists a unique triple
$(Y, M^{Y,c}, y) \in D_0^{\exp} \times \mathcal{M}_0^p(\mathbb{P}) \times \mathcal{L}^2(\lambda, \mathbb{P}) = \mathbb{B}(p)$ (for any $p \geq 1$) solution of (*).

- $\mathcal{M}_0^p(\mathbb{P})$ is the space of continuous martingales M with $M_0 = 0$ and

$$\mathbb{E}^{\mathbb{P}}\Big[\sup_{0\leq t\leq T} |M_t|^p\Big] < \infty.$$

The optimal measure $\mathbb{Q}^*$ solution of $\min \Gamma(\mathbb{Q})$ admits the Radon-Nikodym density $Z^{\mathbb{Q}^*} = \mathcal{E}(L)$ w.r.t. $\mathbb{P}$ where

$$dL_t = -dM_t^{Y,c} + \sum_{i=1}^{d} \left(e^{-y_t^i} - 1\right) dM_t^i, \quad L_0 = 0.$$

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Proof: We divide the proof in three steps.

• We prove that the value process V of the control problem is a $\mathbb{P}$-special semimartingale, i.e. $V = V_0 + M^V + A^V$ with $M^V = M^{V,c} + \sum_{i=1}^d v^i dM^i$.

• We prove that $(V, M^{V,c}, v)$ is solution of the BSDE (*).

• We show that $(V, M^{V,c}, v) \in \mathbb{B}(p)$.

(*) $dY_t = (\sum_{i=1}^d g(y_t^i)\lambda_t^i - U_t + \delta_t Y_t)\,dt + \frac{1}{2}d\langle M^{Y,c}\rangle_t + dM_t^{Y,c} + \sum_{i=1}^d y_t^i dM_t^i$

Step 1: We embed the minimization of $\Gamma(\mathbb{Q})$ in a stochastic control problem and we use the martingale optimality principle.

We define the minimal conditional cost

$$J(\tau, \mathbb{Q}) := \mathbb{Q}\,\mathrm{essinf}_{\mathbb{Q}' \in \mathcal{D}(\mathbb{Q},\tau)} \Gamma(\tau, \mathbb{Q}')$$

with

$$\begin{aligned} \Gamma(\tau, \mathbb{Q}') &:= \mathbb{E}_{\mathbb{Q}}\left[\mathcal{U}^{\delta}_{0,T} + \mathcal{R}^{\delta}_{0,T}(\mathbb{Q}') \,|\, \mathcal{G}_\tau\right] \\ \mathcal{D}(\mathbb{Q}, \tau) &= \{Z^{\mathbb{Q}'} \,|\, \mathbb{Q}' \in \mathcal{Q}_f \text{ and } \mathbb{Q}' = \mathbb{Q} \text{ on } \mathcal{G}_\tau\} \end{aligned}$$

we define

$$V(\tau, \mathbb{Q}) := \mathbb{Q}\mathrm{essinf}_{\mathbb{Q}' \in \mathcal{D}(\mathbb{Q},\tau)} \mathbb{E}_{\mathbb{Q}'}\left[\mathcal{U}^{\delta}_{\tau,T} + \mathcal{R}^{\delta}_{\tau,T}(\mathbb{Q}') \,|\, \mathcal{G}_\tau\right]$$

which is the **value** of the control problem started at time τ and assuming one has used the model $\mathbb{Q}$ up to time τ.

Then, we prove that there exist processes V and $J^{\mathbb{Q}}$ such that
$V(\tau) = V_\tau, J(\tau, \mathbb{Q}) = J^{\mathbb{Q}}_\tau$ and

$$J^{\mathbb{Q}}_t = S^\delta_t V_t + \int_0^t S^\delta_s U_s ds + \int_0^t \delta_s S^\delta_s \ln Z^{\mathbb{Q}}_s ds + S^\delta_t \ln Z^{\mathbb{Q}}_t.$$

. As $\mathbb{P} \in \mathcal{Q}^e_f$, and $J^{\mathbb{P}}$ a submartingale, we obtain from $J^{\mathbb{P}} = S^\delta V + \int S^\delta_s U_s ds$ that V is a $\mathbb{P}$-special semimartingale, $V = V_0 + M^V + A^V$.

Since S^δ is uniformly bounded from below and $J^{\mathbb{P}}$ is a $\mathbb{P}$-submartingale, M^V is a true $\mathbb{P}$-martingale and

$$dM^V_t = dM^{V,c}_t + \sum_{i=1}^d v^i_t dM^i_t$$

where $M^{V,c}$ is a continuous $\mathbb{P}$-martingale.

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. As $\mathbb{P} \in \mathcal{Q}^e_f$, and $J^{\mathbb{P}}$ a submartingale, we obtain from $J^{\mathbb{P}} = S^\delta V + \int S^\delta_s U_s ds$ that V is a $\mathbb{P}$-special semimartingale, $V = V_0 + M^V + A^V$.

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where $M^{V,c}$ is a continuous $\mathbb{P}$-martingale.

Step 2: We now prove that $(V, M^{V,c}, v)$ is solution of the BSDE (*).
For $\mathbb{Q} \in \mathcal{Q}_f^e$, we denote by $L^{\mathbb{Q}}$ the stochastic logarithm of $Z^{\mathbb{Q}}$, i.e., the $\mathbb{P}$-local martingale such that $dZ_t^{\mathbb{Q}} = Z_{t-}^{\mathbb{Q}} dL_t^{\mathbb{Q}}$.

From assumption, the local martingale $L^{\mathbb{Q}}$ admits the decomposition

$$dL_t^{\mathbb{Q}} = dL_t^{\mathbb{Q},c} + \sum_{i=1}^{d} \ell_t^i dM_t^i$$

One has

$$d\ln Z_t^{\mathbb{Q}} = dL_t^{\mathbb{Q},c} - \frac{1}{2} d\langle L^{\mathbb{Q},c}\rangle_t + \sum_{i=1}^{d} \ln(1+\ell_t^i) dM_t^i + \sum_{i=1}^{d} (\ln(1+\ell_t^i) - \ell_t^i)\lambda_t^i dt.$$

Using integration by parts formula,

$$\begin{aligned} dJ_t^{\mathbb{Q}} &= S_t^{\delta}\Big[(-\delta_t V_t + U_t)dt + dM_t^{V,c} + dA_t^V + dL_t^{\mathbb{Q},c} - \frac{1}{2}d\langle L^{\mathbb{Q},c}\rangle_t \\ &\quad + \sum_{i=1}^{d}(v_t^i + \ln(1+\ell_t^i))dM_t^i + \sum_{i=1}^{d}(\ln(1+\ell_t^i) - \ell_t^i)\lambda_t^i dt\Big] \end{aligned}$$

From Girsanov's theorem, the processes $(\widetilde{M}_t^i)_{t\geq 0}$ and $(\widetilde{M}_t^c)_{t\geq 0}$ defined as $d\widetilde{M}_t^i = dM_t^i - \ell_t^i\lambda_t^i dt$, and $d\widetilde{M}_t^c = d(M_t^{V,c} + L_t^{\mathbb{Q},c}) - d\langle M^{V,c} + L^{\mathbb{Q},c}, L^{\mathbb{Q},c}\rangle_t$ are $\mathbb{Q}$-local martingales, and:

$$\begin{aligned} dJ_t^{\mathbb{Q}} &= S_t^{\delta}\Big[(-\delta_t V_t + U_t)dt + d\widetilde{M}_t^c + dA_t^V + d\langle M^{V,c} + L^{\mathbb{Q},c}, L^{\mathbb{Q},c}\rangle_t - \frac{1}{2}d\langle L^{\mathbb{Q},c}\rangle_t \\ &\quad + \sum_{i=1}^{d}(v_t^i + \ln(1+\ell_t^i))d\widetilde{M}_t^i + \sum_{i=1}^{d}\left(\ell_t^i(v_t^i - 1) + (1+\ell_t^i)\ln(1+\ell_t^i)\right)\lambda_t^i dt\Big]. \end{aligned}$$

$dJ_t^{\mathbb{Q}} = d(S_t^{\delta}V_t) + S_t^{\delta}U_t dt + \delta_t S_t^{\delta}\ln Z_t^{\mathbb{Q}}dt + d(S_t^{\delta}\ln Z_t^{\mathbb{Q}})$.

In order that the process $J^{\mathbb{Q}}$ is a $\mathbb{Q}$-submartingale for each $\mathbb{Q} \in \mathcal{Q}^e$, we impose that its finite variation part is a non-decreasing process.

$$
\begin{aligned}
A_t^V &= -\operatorname{ess\,inf}_{\mathcal{Q}^e} \int_0^t (U_s - \delta_s V_s) ds + \langle M^{V,c} + L^{\mathbb{Q},c}, L^{\mathbb{Q},c} \rangle_t - \frac{1}{2} \langle L^{\mathbb{Q},c} \rangle_t \\
&\quad + \sum_{i=1}^d \int_0^t \left(\ell_s^i (v_s^i - 1) + (1 + \ell_s^i) \ln(1 + \ell_s^i) \right) \lambda_s^i ds.
\end{aligned}
$$

To find the essinf, we divide in two parts, the continuous part and the discontinuous part; hence we have two optimization problems:

$$\begin{aligned} A_t^V &= \int_0^t (\delta_s V_s - U_s) ds - \operatorname*{ess\,inf}_{\mathcal{Q}^e} \{ \langle M^{V,c}, L^{\mathbb{Q},c} \rangle_t + \frac{1}{2} \langle L^{\mathbb{Q},c} \rangle_t \} \\ &- \operatorname*{ess\,inf}_{\mathcal{Q}^e} \sum_{i=1}^d \int_0^t \left(\ell_s^i (v_s^i - 1) + (1 + \ell_s^i) \ln(1 + \ell_s^i) \right) \lambda_s^i ds. \end{aligned}$$

The first infimum is obtained for $L^{\mathbb{Q},c} = -M^{V,c}$ and

$$-\operatorname*{ess\,inf}_{\mathcal{Q}^e} \{ \langle M^{V,c}, L^{\mathbb{Q},c} \rangle + \frac{1}{2} \langle L^{\mathbb{Q},c} \rangle \} = \frac{1}{2} \langle M^{V,c} \rangle$$

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The second part of the optimisation problem reduces to find the optimal ℓ^i, solution of

$$\operatorname{ess\,inf}\left(\ell_s^i(v_s^i - 1) + (1 + \ell_s^i)\ln(1 + \ell_s^i)\right)$$

The solution is $\ell_s^{*,i} = e^{-v_s^i} - 1$, which leads to

$$-\operatorname{ess\,inf}\left(\ell_s^i(v_s^i - 1) + (1 + \ell_s^i)\ln(1 + \ell_s^i)\right) = e^{-v_s^i} + v_s^i - 1 = g(v_s^i)$$

where $g(x) = e^{-x} + x - 1$.

Therefore,

$$A_t^V = \int_0^t (\delta_s V_s - U_s)ds + \frac{1}{2}\langle M^{V,c}\rangle_t + \int_0^t \sum_{i=1}^d g(v_s^i)\lambda_s^i ds\,.$$

It follows that $(V, M^{V,c}, v)$ is a solution of

$$dV_t = \left(\sum_{i=1}^d g(v_t^i)\lambda_t^i - U_t + \delta_t V_t\right)dt + \frac{1}{2}d\langle M^{V,c}\rangle_t + dM_t^{V,c} + \sum_{i=1}^d v_t^i dM_t^i,\ V_T = \bar{U}_T$$

(*) $dY_t = [\sum_{i=1}^d g(y_t^i)\lambda_t^i - U_t + \delta_t Y_t]dt + \frac{1}{2}d\langle M^{Y,c}\rangle_t + dM_t^{Y,c} + \sum_{i=1}^d y_t^i dM_t^i$

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Therefore,

$$A^V_t = \int_0^t (\delta_s V_s - U_s)ds + \frac{1}{2}\langle M^{V,c}\rangle_t + \int_0^t \sum_{i=1}^d g(v^i_s)\lambda^i_s ds\,.$$

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where $g(x)=e^{-x}+x-1$.

Therefore,

$$A^V_t=\int_0^t(\delta_sV_s-U_s)ds+\frac{1}{2}\langle M^{V,c}\rangle_t+\int_0^t\sum_{i=1}^d g(v^i_s)\lambda^i_s ds\,.$$

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Furthermore there exists a solution of the quadratic-exponential BSDE (*) and the optimal probability measure $\mathbb{Q}^*$ is characterized by its Radon-Nikodym density

$$dZ_t^{\mathbb{Q}^*} = Z_{t-}^{\mathbb{Q}^*} dL_t, \qquad dL_t = -dM_t^{V,c} + \sum_{i=1}^{d} \left(e^{-v_t^i} - 1 \right) dM_t^i.$$

(*) $dY_t = [\sum_{i=1}^d g(y_t^i)\lambda_t^i - U_t + \delta_t Y_t]dt + \frac{1}{2} d\langle M^{Y,c}\rangle_t + dM_t^{Y,c} + \sum_{i=1}^d y_t^i dM_t^i$

Step 3: In this step we prove that the solution $(Y, M^{Y,c}, y)$ of the BSDE(*) belongs to the required spaces.

(*) $dY_t = [\sum_{i=1}^{d} g(y_t^i)\lambda_t^i - U_t + \delta_t Y_t]dt + \frac{1}{2}d\langle M^{Y,c}\rangle_t + dM_t^{Y,c} + \sum_{i=1}^{d} y_t^i dM_t^i$

Comparison Theorem

Assume that for $k = 1, 2$, $(Y^k, M^{k,c}, y^k)$ is the solution of the BSDE (*) associated with $(U^k, \bar{U}_T^k)$. We denote $Y^{12} := Y^1 - Y^2$, $U^{12} := U^1 - U^2$ and $\bar{U}_T^{12} := \bar{U}_T^1 - \bar{U}_T^2$. Then,

$$S_t^\delta Y_t^{12} \leq \mathbb{E}^{\mathbb{Q}^{*,2}} \left[\int_t^T S_s^\delta U_s^{12} ds + S_T^\delta \bar{U}_T^{12} \middle| \mathcal{G}_t \right]$$

where $\mathbb{Q}^{*,2}$ is the solution of $\mathcal{P}(U^2, \bar{U}_T^2)$, i.e., the probability measure equivalent to $\mathbb{P}$ with Radon Nikodym density $Z^{\mathbb{Q}^{*,2}}$ given by

$$dZ_t^{\mathbb{Q}^{*,2}} = Z_{t^-}^{\mathbb{Q}^{*,2}} \left(-dM_t^{2,c} + \sum_{i=1}^d \left(e^{-y_t^{i,2}} - 1 \right) dM_t^i \right).$$

In particular, if $(U^1, \bar{U}_T^1) \leq (U^2, \bar{U}_T^2)$, one obtains $Y_t^1 \leq Y_t^2$, $dP \otimes dt$-a.e.

(*) $dY_t = [\sum_{i=1}^d g(y_t^i)\lambda_t^i - U_t + \delta_t Y_t]dt + \frac{1}{2} d\langle M^{Y,c} \rangle_t + dM_t^{Y,c} + \sum_{i=1}^d y_t^i dM_t^i$

Proof: Let $y^{i,12} := y^{i,1} - y^{i,2}$ and $M^{12,c} = M^{1,c} - M^{2,c}$. Then,

$$Y_t^{12} = \bar{U}_T^{12} + \int_t^T \left(U_s^{12} - \delta_s Y_s^{12}\right) ds - \sum_{i=1}^d \int_t^T y_s^{i,12} dM_s^i - \sum_{i=1}^d \int_t^T \left[g(y_s^{i,1}) - g(y_s^{i,2})\right] \lambda_s^i ds$$
$$+ \frac{1}{2} \int_t^T \left(d\langle M^{2,c}\rangle_s - d\langle M^{1,c}\rangle_s\right) - \int_t^T dM_s^{12,c}$$

Note that, since $M^{k,c}$ are continuous martingales,

$$-\langle M^{2,c}, M^{12,c}\rangle - \frac{1}{2}\langle M^{2,c}\rangle + \frac{1}{2}\langle M^{1,c}\rangle = \frac{1}{2}\langle M^{12,c}\rangle$$

Using the fact that the process $\langle M^{12,c}\rangle$ is increasing and that the function g is convex we get:

$$\begin{aligned} Y_t^{12} \quad \leq \quad & \bar{U}_T^{12} + \int_t^T \left(U_s^{12} - \delta_s Y_s^{12}\right) ds + \sum_{i=1}^d \int_t^T (e^{-y_s^{i,2}} - 1) y_s^{i,12} \lambda_s^i ds \\ & + \int_t^T d\langle M^{2,c}, M^{12,c}\rangle_s - \int_t^T dM_s^{12,c} - \sum_{i=1}^d \int_t^T y_s^{i,12} dM_s^i. \end{aligned}$$

$$
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Y_t^{12} \quad \leq \quad & \bar{U}_T^{12} + \int_t^T \left(U_s^{12} - \delta_s Y_s^{12}\right) ds + \sum_{i=1}^{d} \int_t^T (e^{-y_s^{i,2}} - 1) y_s^{i,12} \lambda_s^i ds \\
& + \int_t^T d\langle M^{2,c}, M^{12,c}\rangle_s \; - \int_t^T dM_s^{12,c} - \sum_{i=1}^{d} \int_t^T y_s^{i,12} dM_s^i.
\end{aligned}
$$

Let M^* and $M^{*,c}$ be the $\mathbb{Q}^{*,2}$-martingales obtained by Girsanov's transformation from M and $M^{12,c}$, where $d\mathbb{Q}^{*,2} = Z^{\mathbb{Q}^{*,2}} d\mathbb{P}$ and where $Z^{\mathbb{Q}^{*,2}}$ is given by (1), then:

$$
Y_t^{12} \leq \bar{U}_T^{12} + \int_t^T \left(U_s^{12} - \delta_s Y_s^{12}\right) ds - \sum_{i=1}^{d} \int_t^T y_s^{i,12} dM_s^{i*} - \int_t^T dM_s^{*,c}
$$

which implies that $Y_t^{12} \leq \mathbb{E}^{\mathbb{Q}^{*,2}}\left[\int_t^T e^{-\int_t^s \delta_r dr} U_s^{12} ds + e^{-\int_t^T \delta_r dr} \bar{U}_T^{12} \Big| \mathcal{G}_t \right]$.

In particular, if $(U^1, \bar{U}_T^1) \leq (U^2, \bar{U}_T^2)$, then $Y_t^1 \leq Y_t^2 \;\; d\mathbb{P} \otimes dt$-$a.e.$ □

Concavity property Define the map $F : D_1^{\exp} \times L^{\exp} \longrightarrow D_0^{\exp}$ as

$$F(U, \bar{U}) = V$$

where $(V, M^{V,c}, v)$ is the solution associated with $(U, \bar{U})$. Then F is concave, namely, for all $\theta \in (0,1)$ and $(U^1, \bar{U}_T^1), (U^2, \bar{U}_T^2) \in D_1^{\exp} \times L^{\exp}$:

$$F\left(\theta U^1 + (1-\theta)U^2, \theta \bar{U}_T^1 + (1-\theta)\bar{U}_T^2\right) \geq \theta F(U^1, \bar{U}_T^1) + (1-\theta)F(U^2, \bar{U}_T^2).$$

Consumption-investment problem

We consider a financial market consisting of $d+p+1$ assets. The *savings account* is assumed to be constant equal to 1, the prices of the $d+p$ *risky assets* are $\mathbb{G}$-semi-martingales given by

$$dS_t^j = S_{t^-}^j \left[\mu_t^j dt + \sum_{i=1}^{d} \varphi_t^{i,j} dM_t^i + \sum_{k=1}^{p} \sigma_t^{i,k} dW_t^k \right], j = 1, \dots, d+p$$

where W is a k-dimensional Brownian motion;

We suppose the market complete and arbitrage free, and we denote by $\widetilde{Z}$ the RN density of the unique emm $\widetilde{\mathbb{P}}$

$$d\widetilde{Z}_t = \widetilde{Z}_{t-}(\theta_t dW_t + \sum_{i=1}^{n} (e^{-z_t^i} - 1) dM_t^i), \; \widetilde{Z}_0 = 1 \, .$$

Given an initial wealth x and a policy (c, π), the wealth process $(X_t^{x,c,\pi})_{0 \leq t \leq T}$ associated to the triple (x, c, π) where x is the initial wealth, π is the portfolio strategy and c the consumption plan c, follows the dynamics given by:

$$dX_t^{x,c,\pi} = \pi_t dS_t - c_t dt, \quad X_0^{x,c,\pi} = x,$$

The set of consumption-investment strategies (c, π) satisfying the following no-bankruptcy condition is called the admissible strategies set and is denoted by $\mathcal{A}(x)$:

$$\mathbb{P} - a.s., \qquad X_t^{x,c,\pi} \geq 0, \qquad \forall t \in [0, T].$$

For this model, one assumes that all coefficiants are bounded. Then, the pair consumption-terminal wealth satisfies the budget constraint

$$\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[\int_0^T c_t dt + X_T^{x,c,\pi}\Big] \leq x$$

Let I and $\bar{I}$ be the inverse of the functions U' and $\bar{U}'$. The optimal plan (c^*, ψ^*) which solve the problem is given by:

$$c_t^* = I\left(\frac{\nu^*}{S_t^\delta}\frac{Z_t^{\widetilde{\mathbb{P}}}}{Z_t^*}\right) \qquad dt \otimes d\mathbb{P} \text{ a.s },\qquad \psi^* = \bar{I}\left(\frac{\nu^*}{S_T^\delta}\frac{Z_T^{\widetilde{\mathbb{P}}}}{Z_T^*}\right) \text{ a.s. }.$$

where $\nu^* > 0$ satisfies:

$$\mathbb{E}^{\widetilde{\mathbb{P}}}\left[\int_0^T I\left(\frac{\nu^*}{S_t^\delta}\frac{Z_t^{\widetilde{\mathbb{P}}}}{Z_t^*}\right) dt + \bar{I}\left(\frac{\nu^*}{S_T^\delta}\frac{Z_T^{\widetilde{\mathbb{P}}}}{Z_T^*}\right)\right] = x.$$

We restrict our attention to the logarithmic case $U(x) = \ln x, \bar{U}(c) = 0$ and $\delta = 0$.

The value function V has the form

$$V_t = \alpha(t)\ln(c_t^*) + (1+\alpha(t))J_t$$

where $\alpha(t) = T - t$ and $(J, \bar{M}^{J,c}, j)$ is the unique solution of the following Backward Stochastic Differential Equation:

$$\begin{aligned} dJ_t &= (1+k(t))J_t dt + \frac{1}{2}k(t)(1+k(t))\theta_t^2 dt \\ &+ d\bar{M}_t^{J,c} + \frac{1}{2}d\langle \bar{M}^{J,c}\rangle_t \\ &+ \sum_{i=1}^{d}\left(g(j_t^i)\bar{\lambda}_t^i + \left(k(t)(e^{-z_t^i}-1) + e^{k(t)z_t^i} - 1\right)\lambda_t^i\right)dt + \sum_{i=1}^{d} j_t^i d\bar{M}_t^i \\ J_T &= 0 \end{aligned}$$

where $k(t) = -\frac{\alpha(t)}{1+\alpha(t)}$.

Here, the processes $\bar{M}^{J,c}$ and $d\bar{M}_t^i = dN_t^i - \bar{\lambda}_t^i dt$ are $\bar{\mathbb{P}}$-martingales where $d\bar{\mathbb{P}}|_{\mathcal{G}_t} = \bar{Z}_t d\mathbb{P}|_{\mathcal{G}_t}$, $\bar{\lambda}_t^i = e^{k(t)z_t^i}\lambda_t^i$ and

$$d\bar{Z}_t \quad = \quad -\bar{Z}_{t^-}\left(k(t)\theta_t dW_t - \sum_{i=1}^{d}(e^{k(t)z_t^i} - 1)dM_t^i\right)$$

Thank you for your attention