Skorokhod's embedding problem and FBSDE

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Skorokhod's embedding problem

$(\Omega, \mathcal{F}, \mathbb{P})$ probability space with Brownian motion $W = (W_t)_{t \geq 0}$, Brownian standard filtration $(\mathcal{F}_t)_{t \geq 0}$

Classical version of the Skorokhod embedding problem:

given law ν with moment conditions;

find stopping time T with respect to filtration $(\mathcal{F}_t)_{t \geq 0}$ such that

$$W_T \quad \text{has law} \quad \nu.$$

Literature: Azema, Yor '79; Bass '83; Grandits, Falkner '00; Peskir '00; Skorokhod '65; Ankirchner, Heyne, I '08; Ankirchner, Strack '11; review Obloj '04

Bass' approach

Sketch of Bass' approach of the Skorokhod embedding problem for Brownian motion W:

Step 1:
find $\sigma(W_1)$-measurable random variable with law ν

F distribution function of ν, Φ distribution function of W_1; then for $y \in \mathbb{R}$

$$\begin{aligned} \mathbb{P}(F^{-1}(\Phi((W_1)) \leq y) &= \mathbb{P}(W_1 \leq \Phi^{-1}(F(y)) = \Phi(\Phi^{-1}(F(y)) = F(y), \quad \text{so} \\ \xi &= F^{-1}(\Phi(W_1)) = g(W_1) \quad \text{has law} \quad \nu \end{aligned}$$

Step 2:
use martingale representation on ξ to find square integrable process Z such that

$$\xi - \mathbb{E}(\xi) = \int_0^1 Z_s dW_s.$$

Let $M = \int_0^\cdot Z_s dW_s$. Here we need second moments for ν.

Bass' approach

Step 3:
change time to transform M into Brownian motion B

Let $\langle M\rangle = \int_0^\cdot Z_s^2 ds, \quad T = \langle M\rangle_1.$

Define the Dubins-Schwarz time change

$$\tau_s (= \langle M\rangle_s^{-1}) = \begin{cases} \inf\{t : \langle M\rangle_t > s\} & 0 \le s < T, \\ 1, & s \ge T. \end{cases}$$

Let $\tilde{B}$ be an independent Brownian motion. Then

$$B_t = M_{\tau_t} + \tilde{B}_t - \tilde{B}_{t\wedge T}, \quad t \ge 0,$$

is a Brownian motion such that

$$B_T = M_1 \quad \text{has law} \quad \nu.$$

This constitutes a *weak solution of the Skorokhod problem.*

Bass' approach

Step 4:
go from weak to strong solution

For this purpose, show that T is a stopping time w.r.t. B. This is done by describing carefully the functional dependence between the time change τ_s and the trajectories of the Brownian motion. For this purpose consider the martingale representation as the simplest form of a BSDE:

$$Y_t = \xi - \int_t^1 Z_s dW_s.$$

On the other hand let u be the solution of the boundary value problem

$$\frac{\partial}{\partial t} u(t,x) = \frac{1}{2} \frac{\partial^2}{\partial x^2} u(t,x), \quad u(1,x) = g(x).$$

Then we know that if p_t denotes the density of W_t

$$u(t,x) = \mathbb{E}(g(W_1)|\mathcal{F}_t)|_{W_t=x} = \int_{\mathbb{R}} p_{1-t}(y-x) g(y) dy,$$

Bass' approach

and in these terms

$$M_t = Y_t = u(t, W_t), \quad Z_t = \frac{\partial}{\partial x} u(t, W_t).$$

Therefore

$$t = \int_0^{\tau_t} (\frac{\partial}{\partial x} u(s, W_s))^2 ds,$$

and so

$$\frac{d}{dt}\tau_t = \frac{1}{(\frac{\partial}{\partial x} u(\tau_t, W_{\tau_t}))^2} = \frac{1}{(\frac{\partial}{\partial x} u(\tau_t, u^{-1}(\tau_t, B_t)))^2}$$

and τ_t is seen to be a $\mathcal{F}_t^B$-measurable. Hence for $s \geq 0$

$$\{T \leq s\} = \{\langle M \rangle_1 \leq s\} = \{1 \leq \tau_s\} \in \mathcal{F}_s^B$$

and so T is a $\mathcal{F}^B$-stopping time.

It remains to rewrite T as a function of W to get W_T with law ν.

Bass' approach for Brownian motion with linear drift

Sketch of the extension of Bass' approach to Brownian motion with linear drift

$$W_t + \kappa t, \quad t \geq 0.$$

Literature: Ankirchner, Heyne, I '08 and Ankirchner, Strack '11.

Step 1:
as before, $\xi = g(W_1)$ where $g = F^{-1} \circ \Phi$.

Step 2:
replace M by $M + \kappa\langle M \rangle$ in martingale representation; this just means to replace martingale representation by a more general BSDE

$$Y_t = \xi - \int_t^1 Z_s dW_s - \kappa \int_t^1 Z_s^2 ds, \quad t \in [0, 1].$$

Bass' approach for Brownian motion with linear drift

Step 3:
change time to transform $M = \int_0^\cdot Z_s dW_s$ into Brownian motion, and hence Y into Brownian motion plus linear drift via

$$\tau_s = \begin{cases} \inf\{t : \langle M \rangle_t > s\} & 0 \leq s < T, \\ 1, & s \geq T. \end{cases}$$

Then if as before $\tilde{B}$ is an independent Brownian motion,

$$B_t = M_{\tau_t} + \tilde{B}_t - \tilde{B}_{t \wedge T}, \quad t \geq 0,$$

is a Brownian motion and then according to step 2

$$B_T + \kappa T = Y_1 = \xi$$

has law ν. This gives a weak solution of the Skorokhod embedding problem for the Brownian motion with linear drift.

Bass' approach for Brownian motion with linear drift

Step 4:
this time we have to let v be the solution of the associated boundary value problem

$$\frac{\partial}{\partial t}v(t,x) = \kappa\frac{\partial}{\partial x}v(t,x) + \frac{1}{2}\frac{\partial^2}{\partial x^2}v(t,x), \quad v(1,x) = g(x).$$

Then by Feynman-Kac

$$M_t + \kappa\langle M\rangle_t = Y_t = v(t,W_t),$$

and the associated differential equation for the time change is given by

$$\frac{d}{dt}\tau_t = \frac{1}{(\frac{\partial}{\partial x}v(\tau_t,W_{\tau_t}))^2} = \frac{1}{(\frac{\partial}{\partial x}v(\tau_t,v^{-1}(\tau_t,B_t+\kappa\tau_t)))^2}.$$

Again τ_t is seen to be a B-stopping time. Hence it remains to write T as a function of W to get $W_T + \kappa T$ with law ν.

Bass' approach for Brownian motion with linear drift

More precisely, we obtain

Thm 1
Let $\kappa > 0$ and ν a probability measure on $\mathbb{R}$ such that $\int |x| d\nu(x) < \infty$ and $\int \exp(-2\kappa\, x)\, \nu(dx) < \infty$. Let W be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\mathcal{F}_t)_{t\geq 0}$ its right continuous and completed filtration.
Then there exists an $((\mathcal{F}_t)_{t\geq 0})$ stopping time T, with finite expectation, such that the process

$$W_T + \kappa T \quad \text{has law} \quad \nu.$$

Ankirchner, Strack '11 give conditions on ν under which this can be achieved with a bounded stopping time.

Brownian motion with more general drift

Goal:
Solve Shorokhod's embedding problem by means of BSDE techniques for Brownian motion with more general drift.

Consider

$$W_t + h(t), \quad t \geq 0,$$

with h deterministic, and differentiable.

Problem in Step 2:
Replacing M with $M + h(\langle M \rangle)$ want to reach

$$W_T + h(T) \quad \text{has law} \quad \nu.$$

This leads to BSDE

$$Y_t = \xi - \int_t^1 Z_s dW_s - \int_t^1 Z_s^2 h'(\int_0^s Z_u^2 du) ds, \quad t \in [0,1],$$

which is *path dependent*. Path dependent BSDE difficult to solve.

Brownian motion with more general drift

Idea:
Associate FBSDE to the problem:

$$\begin{aligned} X_t^1 &= W_t, \\ X_t^2 &= \int_0^t Z_s^2 ds, \\ Y_t &= g(X_1^1) + h(X_1^2) - \int_t^1 Z_s dW_s. \end{aligned}$$

FBSDE difficult to solve. Use *decoupling fields* according to J. Zhang (2010).

Decoupling fields

Definition
A function $u\colon [t,T]\times\Omega\times\mathbb{R}^n\to\mathbb{R}^m$ is called *decoupling field* for the FBSDE(g,f,μ,σ) if $u(T,\cdot)=g$ and if for all $t\le t_1\le t_2\le T$ and all $\mathcal{F}_{t_1}$-measurable random vectors $X\colon\Omega\to\mathbb{R}^n$ there are progressively measurable processes $X\colon[t_1,t_2]\times\Omega\to\mathbb{R}^n$, $Y\colon[t_1,t_2]\times\Omega\to\mathbb{R}^m$ and $Z\colon[t_1,t_2]\times\Omega\to\mathbb{R}^{m\times d}$ such that

$$X_s = X_{t_1} + \int_{t_1}^s \mu(r,X_r,Y_r,Z_r)dr + \int_{t_1}^s \sigma(r,X_r,Y_r,Z_r)dW_r,$$

$$Y_s = Y_{t_2} - \int_s^{t_2} f(r,X_r,Y_r,Z_r)dr - \int_s^{t_2} Z_r^\top dW_r,$$

$$Y_s = u(s,X_s)$$

for all $s\in[t_1,t_2]$.

Decoupling fields: local solution

We use more general results from Fromm, I. '13:

essential quantities: $L_{g,x}$, $L_{u,x}$, Lipschitz constants for g resp. u in x, $L_{\sigma,z}$, Lipschitz constant of σ in z

Thm 2
Suppose μ, σ and f are Lipschitz continuous in x and y and locally Lipschitz continuous in z i.e., for every compact set $B \subset \mathbb{R}^{m\times d}$ there exist a constant $L > 0$ such that

$$|\mu(t, x_1, y_1, z_1) - \mu(t, x_2, y_2, z_2)| \leq L(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|)$$

for all $t \in [0, T]$, σ and f, respectively. Moreover, assume g is Lipschitz continuous with $L_{g,x} < L_{\sigma,z}^{-1}$ and $||\mu(\cdot, 0, 0, 0)||_\infty$, $||f(\cdot, 0, 0, 0)||_\infty$, $||\sigma||_\infty < \infty$. Then there is $t \in [0, T)$ such that the FBSDE(g, f, μ, σ) has a unique decoupling field u on $[t, T]$ with $L_{u,x} < L_{\sigma,z}^{-1}$. u is regular, deterministic, continuous and satisfies

$$\sup_{t_1, t_2, X_{t_1}} ||Z||_\infty < \infty,$$

where $t \leq t_1 < t_2 \leq T]$ and X_{t_1} is an initial value.

Decoupling fields: local solution

Method of proof:

recursive a priori inequality for

$$
\begin{aligned}
X_s^1 &= X_{t_1} + \int_{t_1}^s \mu(r, X_r^0, Y_r^0, Z_r^0)dr + \int_{t_1}^s \sigma(r, X_r^0, Y_r^0, Z_r^0)dW_r, \\
Y_s^1 &= g(X_{t_2}^1) - \int_s^{t_2} f(r, X_r^1, Y_r^0, Z_r^0)dr - \int_s^{t_2} Z_r^1 dW_r,
\end{aligned}
$$

in norm

$$
\begin{aligned}
& ||(X,Y,Z)||_2 \\
= \quad & \max\left(\sup_{\{s\in[t,T]\}} \sqrt{\mathbb{E}_t(|X_s|^2)}, (1+L_{\sigma,z})\sqrt{\mathbb{E}_t(|Y_s|^2)}, (1+L_{\sigma,z})\sqrt{\mathbb{E}_t(\int_t^T |X_s|^2)} \right),
\end{aligned}
$$

$\mathbb{E}_t$ denoting conditional expectation w.r.t. $\mathcal{F}_t$.

Decoupling fields: local solution

For (X^1, Y^1, Z^1) related to (X^0, Y^0, Z^0) and $(\tilde{X}^1, \tilde{Y}^1, \tilde{Z}^1)$ related to $(\tilde{X}^0, \tilde{Y}^0, \tilde{Z}^0)$ we obtain

$$||(\tilde{X}^1 - X^1, \tilde{Y}^1 - Y^1, \tilde{Z}^1 - Z^1)||_2 \leq \gamma_t ||(\tilde{X}^0 - X^0, \tilde{Y}^0 - Y^0, \tilde{Z}^0 - Z^0)||_2.$$

with $\lim_{t \to T} \gamma_t = L_{g,x} \cdot L_{\sigma,z}$.

This inequality is used for defining decoupling field u via a fixed point argument.

A similar inequality is obtained for the formal derivatives with respect to a parameter such as the initial vector x of X:

$$(\nabla_x X^1, \nabla_x Y^1, \nabla_x Z^1)||_2 \leq C_X \vee (L_{g,x} + L(T-t))C_X + \gamma_t ||\nabla_x X^0, \nabla_x Y^0, \nabla_x Z^0)||_2$$

with $C_X = \text{esssup}_x \text{esssup}_{v \in S^{n-1}} ||\nabla_x X \cdot v||_\infty$.

This inequality is used for the control of the gradient of u.

Decoupling fields: global solution

By control of the gradient of u one obtains

Thm 3
Assume h and g are bounded and Lipschitz continuous. Then there is a unique decoupling field u on $[0,T]$ satisfying $L_{u,x} < L_{\sigma,z}^{-1}$ and $\|u\|_\infty < \infty$. This u is controlled and regular.
In particular the BSDE component has a unique bounded solution (Y,Z).

This can be applied to our particular FBSDE and yields the completion of Step 3. Step 4 is done analogously to the linear case. So our final result is

Thm 4
Suppose h and g are bounded and Lipschitz continuous. Then there is a stopping time T such that

$$W_T + h(T) \quad \text{has law} \quad \nu.$$

Outlook

This is just a report about work in progress.

Plans:

- extend the main result to diffusion processes of the form

$$X_t = \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s,$$

  with suitable conditions on b and σ.

- allow for non-smooth g
- solve the problem for $G-$Brownian motion