Systems of Variational Inequalities with Interconnected Bilateral Obstacles.

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Worshop 3 on BSDEs, Lebesgue Center, Rennes, May 22-24, 2013

0. Outlines

1. Setting of the problem and definitions ;
2. The case of one obstacles and its connections ;
3. Uniqueness of the solution ;
4. Existence of the solution.

1. Introduction

1. Γ^1 and Γ^2 be two finite sets ($\Lambda = \text{Card}(\Gamma^1 \times \Gamma^2)$) ;
2. For $i \in \Gamma^1$ and $j \in \Gamma^2$,

$$(\Gamma^1)^{-i} := \Gamma^1 - \{i\} \text{ and } (\Gamma^2)^{-j} := \Gamma^2 - \{j\}.$$

3. Let us consider: $\forall (i,j) \in \Gamma^1 \times \Gamma^2$,

$$\begin{cases} \min \Big[\big(v^{ij} - L^{ij}[\vec{v}]\big)(t,x), \\ \quad \max \Big\{ \big(v^{ij} - U^{ij}[\vec{v}]\big)(t,x), -\partial_t v^{i,j}(t,x) - \mathcal{L} v^{ij}(t,x) - \\ \qquad f^{ij}(t,x,(v^{kl}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2}, \sigma^\top(t,x) D_x v^{ij}(t,x)) \Big\} \Big] = 0, \\ v^{ij}(T,x) = h^{ij}(x) \end{cases} \quad (1)$$

where:

(i) the unknowns are the functions $(v^{ij}(t,x))_{(i,j)\in\Gamma^1\times\Gamma^2} := \vec{v}(t,x)$;

(ii) For $i \in \Gamma^1$ and $j \in \Gamma^2$,

$$L^{ij}[\vec{v}](t,x) := \max_{k\in(\Gamma^1)^{-i}} (v^{kj}(t,x) - \underline{g}_{kj}(t,x)\} \text{ (lower obstacle)}$$

$$U^{ij}[\vec{v}](t,x) := \min_{l\in(\Gamma^2)^{-j}} (v^{il}(t,x) - \bar{g}_{il}(t,x)) \text{ (upper obstacle)};$$

(iii)

$$\mathcal{L}\varphi(t,x) := b(t,x)D_x\varphi(t,x) + \frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(t,x)D^2_{xx}\varphi(t,x)],$$

(iv) $f^{ij}(t,x,(y^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2},z)$, $\underline{g}_{ij}(t,x)$, $\overline{g}_{ij}(t,x)$, $h_{ij}(x)$, $b(t,x)$ and $\sigma(t,x)$, (for $(t,x)\in[0,T]\times\mathbb{R}^k$, $y^{kl}\in\mathbb{R}$, $z\in\mathbb{R}^d$) are indata.

Motivation: System (1) is the HJB system associated with the zerosum switching game.

(i) Assume we have two players Π_1 (the maximizer) and Π_2 (the minimizer) who act on a system in switching it to the appropriate mode of working $(i,j) \in \Gamma^1 \times \Gamma^2$.
(ii) Γ^1 (resp. Γ^2) are the states to which Π_1 (resp. Π_2) is allowed to switch the system ;
(iii) f^{ij} and $h_{ij}(x)$ are related to the instantaneous and terminal payoffs respectively which are profits (resp. costs) for Π_1 (resp. Π_2) ;
(iv) $\overline{g}_{ij}(t,x)$ (resp. $\underline{g}_{ij}(t,x)$) is a function related to the switching cost for Π_1 (resp. Π_2).
(v) A strategy of switching consists of two components: the times when the decision to switch is made and to which state the system is switched ;

When Π_1 (resp. Π_2) implements a strategy δ (resp. θ) of siwtching the payoff is:

$$J^{ij}(\delta, \theta)$$

which is a profit (resp. cost) for Π_1 (resp. Π_2). And the main problems are the properties of

$$\inf_{\theta} \sup_{\delta} J^{ij}(\delta, \theta) \text{ or } \sup_{\delta} \inf_{\theta} J^{ij}(\delta, \theta).$$

Switching game problems got involved in the energy market, carbone market, etc. (Y.Hu and S.Tang, '08, M.Ludkovski '11).

■

Definition of a solution: viscosity sense

A. Subjet and superjet

(i) For a LSC (resp. USC) function $u : [0, T] \times \mathbb{R}^k \to \mathbb{R}$, we denote by $J^- u(t,x)$(resp. $J^+ u(t,x)$) the parabolic subjet (resp. superjet) of u at $(t,x) \in [0, T] \times \mathbb{R}^k$, as the set of triples $(p, q, M) \in \mathbb{R} \times \mathbb{R}^k \times \mathbb{S}^k$ satisfying

$$u(t', x') \geq \text{ (resp. } \leq \text{) } u(t,x) + p(t'-t) + q^\top (x'-x) + \frac{1}{2}(x'-x)^\top M(x'-x) + o\big(|t'-t| + |x'-x|^2\big)$$

where $\mathbb{S}^k$ is the set of symmetric real matrices of dimension k.

(ii) For a LSC (rep. USC) function $u : [0, T] \times \mathbb{R}^k \to \mathbb{R}$, we denote by $\overline{J}^- u(t,x)$ (resp. $\overline{J}^+ u(t,x)$) the parabolic limiting subjet (resp. superjet) of u at $(t,x) \in [0,T] \times \mathbb{R}^k$, as the set of triples $(p,q,M) \in \mathbb{R} \times \mathbb{R}^k \times \mathbb{S}^k$ such that:

(a) $(p,q,M) = \lim_{n\to\infty}(p_n, q_n, M_n)$; $(t,x) = \lim_{n\to\infty}(t_n, x_n)$;
(b) $(p_n, q_n, M_n) \in J^- u(t_n, x_n)\, (resp. J^+ u(t_n, x_n))$;
(c) $u(t,x) = \lim_n u(t_n, x_n)$.

(iii) For a given locally bounded $\mathbb{R}$-valued function $u(t,x)$, we denote by u_*(resp. u^*) its LSC (resp. USC) envelope defined by: $\forall\ (t,x) \in [0,T] \times \mathbb{R}^k$,

$$u_*(t,x) = \underline{\lim}_{(t',x')\to(t,x);\ t'<T}\, u(t',x') \text{ and}$$

$$u^*(t,x) = \overline{\lim}_{(t',x')\to(t,x);\ t'<T}\, u(t',x').$$

B. Definition

(i) A function $\vec{v} = (v^{kl}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2} : [0,T]\times\mathbb{R}^k \mapsto \mathbb{R}^\Lambda$ such that for any $(i,j)\in\Gamma^1\times\Gamma^2$, v^{ij} is LSC (resp. USC), is called a viscosity supersolution (resp. subsolution) to (1) if:

$\forall (i,j)\in\Gamma^1\times\Gamma^2$, $(t,x)\in[0,T)\times\mathbb{R}^k$, $(p,q,M)\in\overline{J}^- v^{ij}(t,x)$ (resp. $\overline{J}^+ v^{ij}(t,x)$) we have:

$$\begin{cases} \min\Big[v^{ij}(t,x) - L^{ij}[\vec{v}](t,x); \\ \quad \max\Big\{v^{ij}(t,x) - U^{ij}[\vec{v}](t,x); \\ \qquad -p - b(t,x)^\top.q - \frac{1}{2}Tr[(\sigma\sigma^\top)(t,x)M] \\ \qquad\qquad -f^{ij}(t,x,\vec{v}(t,x),\sigma^\top(t,x)q)\Big\}\Big] \geq (resp. \leq)\, 0\,; \\ v^{ij}(T,x) \geq (resp. \leq)\ h^{ij}(x). \end{cases}$$

(ii) A locally bounded function $\vec{v} = (v^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2}$ is called a viscosity solution of (1) if its associated LSC (resp. USC) envelope $(v_*^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2}$ (resp. $(v^{kl*})_{(k,l)\in\Gamma^1\times\Gamma^2}$) is a viscosity supersolution (resp. subsolution) of (1). ■

2. The case of one obstacle

A. Lower obstacle

It is obtained from (1) in making $\bar{g}_{ij} = +\infty$. It turns into the following system: $\forall i \in \Gamma^1$ (of card m_1),

$$
\begin{cases}
\min \Big[\big(v^i - \max_{k \in (\Gamma^1)^{-i}} (v^k - \underline{g}_{ik}) \big)(t,x); \\
\qquad -\partial_t v^i(t,x) - \mathcal{L} v^i(t,x) \\
\qquad\qquad -f^i(t,x,(v^k(t,x))_{k \in \Gamma^1}, \sigma^\top(t,x) D_x v^i(t,x)) \Big] = 0\,; \\
v^i(T,x) = h^i(x).
\end{cases}
\tag{2}
$$

B. Connection with BSDEs

For $(t,x) \in [0,T] \times \mathbb{R}^k$, let $X^{t,x}$ be s.t.:

$$\begin{cases} dX_s^{t,x} = b(s, X_s^{t,x})ds + \sigma(s, X_s^{t,x})dB_s, \; s \in [t,T]; \\ X_s^{t,x} = x \text{ for } s \leq t, \end{cases}$$

where:

(i) $(B_t)_{t \leq T}$ is a BM ;

(ii) b and σ are continuous, Lipschitz w.r.t x and of linear growth.

System of VIs (4) is related to the following system of reflected BSDEs:

$\forall i \in \Gamma^1$, $(t,x) \in [0,T] \times \mathbb{R}^k$,

$$\begin{cases} Y_s^{i;t,x} = h^i(X_T^{t,x}) + \\ \qquad \int_s^T f_i(r, X_r^{t,x}, Y_r^{1;t,x}, \ldots, Y_r^{m_1;t,x}, Z_r^{i;t,x})dr \\ \qquad\quad + K_T^{i;t,x} - K_s^{i;t,x} - \int_s^T Z_r^{i;t,x} dB_r, \ s \le T; \\ Y_s^{i;t,x} \ge \max_{j \in (\Gamma^1)^{-i}} \{ Y_s^{j;t,x} - \underline{g}_{ij}(s, X_s^{t,x}) \}, \ s \le T; \\ (Y_s^{i;t,x} - \max_{j \in (\Gamma^1)^{-i}} \{ Y_s^{j;t,x} - \underline{g}_{ij}(s, X_s^{t,x}) \}) dK_s^{i;t,x} = 0. \ \blacksquare \end{cases} \tag{3}$$

2.1 Existence and uniqueness of the solution in the case of one lower obstacle

Assumptions

[H-gh]:The main assumptions on $(h_i)_{i\in\Gamma^1}$, $(g_{ij})_{i,j\in\Gamma^1}$ are:

(i) Polynomial growth and continuity in (t, x).

(ii)

- the non free loop property, i.e., for any sequence of indices $i_1, \dots, i_k$ such that $i_1 = i_k$ and $card\{i_1, \dots, i_k\} = k - 1$ we have:

$$g_{i_1 i_2} + g_{i_2 i_3} + \cdots + g_{i_{k-1} i_k} + g_{i_k i_1} > 0.$$

- $g_{ij} \geq 0$.

(iii)

$$\forall x \in R, \ h_i(x) \geq \max_{j\in\Gamma^{1-i}} \{h_j(x) - g_{ij}(T, x)\}. \quad \blacksquare$$

[H-f] : Main assumptions on $(f_i)_{i\in\Gamma^1}$.

(i) f_i is uniformly Lipschitz continuous with respect to $(\overrightarrow{y},z) := (y^1,...,y^m,z)$

(ii) the mapping $(t,x) \mapsto f_i(t,x,0,\ldots,0)$ is uniformly continuous and of polynomial growth ;

(iii) *Monotonicity*: $\forall i \in \mathcal{J}$ and $k \in (\Gamma^1)^{-i}$, the mapping:

$$y_k \in R \mapsto f_i(t,x,y_1,...,y_{k-1},y_k,y_{k+1},...,y_m)$$

is non-decreasing whenever the other components $(t,x,y_1,...,y_{k-1},y_{k+1},...,y_m)$ are fixed.

Theorem (H.-J.Zhang, '10 and H.-MA Morlais '12)

Assume [H-gh], [H-f] are in force. Then the system of reflected BSDEs with oblique reflection (5) has a unique solution $(Y^{i;t,x}, Z^{i;t,x}, K^{i;t,x})_{i\in\Gamma^1}$. *Moreover there exists deterministic continuous functions* $(v^i(t,x))_{i\in\Gamma^1}$ *of polynomial growth such that:* $\forall (t,x) \in [0, T] \times \mathbb{R}^k$,

$$\forall s \in [t, T], Y_s^{i;t,x} = v^i(s, X_s^{t,x}).$$

Finally $(v^i(t,x))_{i\in\Gamma^1}$ *is the unique solution of the system of PDI with a lower interconnected obstacle (4) in the class of continuous functions with polynomial growth.*

Theorem (Comparison)

Let (f_i') (resp.(g_{ij}'), resp.(h_i')) be other data. If for any $i, j \in \Gamma^1$

$$f_i \le f_i', \;\; h_i \le h_i' \text{ and } g_{ij} \ge g_{ij}'$$

then for any $i \in \Gamma^1$,

$$Y^i \le Y'^i.$$

■

2.2 Existence and uniqueness of the solution in the case of one upper obstacle

In a symmetric way in taking $\underline{g}_{ij} \equiv \infty$ in (1) one obtains a system with upper obstacles which turns into: $\forall j \in \Gamma^2$ (of card m_2),

$$\begin{cases} \max\Big[(v^j - \min_{k\in(\Gamma^2)^{-j}}(v^k + \overline{g}_{jk}))(t,x); \\ \quad -\partial_t v^j(t,x) - \mathcal{L}v^j(t,x) \\ \qquad -f^j(t,x,(v^k(t,x))_{k\in\Gamma^2}, \sigma^\top(t,x)D_x v^j(t,x))\Big] = 0\,; \\ v^j(T,x) = h^j(x). \end{cases} \quad (4)$$

This system (4) is related to the following system of reflected BSDEs: $\forall j \in \Gamma^2$, $(t,x) \in [0,T] \times \mathbb{R}^k$,

$$\begin{cases} \bar{Y}_s^{j;t,x} = h^j(X_T^{t,x}) + \\ \qquad \int_s^T f_j(r, X_r^{t,x}, \bar{Y}_r^{1;t,x}, \ldots, \bar{Y}_r^{m_2;t,x}, \bar{Z}_r^{j;t,x}) dr \\ \qquad\quad + \bar{K}_T^{j;t,x} - \bar{K}_s^{j;t,x} - \int_s^T \bar{Z}_r^{j;t,x} dB_r, \ \ s \le T; \\ \bar{Y}_s^{j;t,x} \le \min_{k \in (\Gamma^2)^{-j}} \{ \bar{Y}_s^{k;t,x} + \overline{g}_{ij}(s, X_s^{t,x}) \}, \ s \le T; \\ (\bar{Y}_s^{j;t,x} - \min_{k \in (\Gamma^2)^{-j}} \{ \bar{Y}_s^{k;t,x} - \overline{g}_{ij}(s, X_s^{t,x}) \}) d\bar{K}_s^{j;t,x} = 0. \ \blacksquare \end{cases} \tag{5}$$

In a symmetric way we have:

Theorem

Under [H-gh], [H-f] , the system of reflected BSDEs with oblique reflection (5) has a unique solution $(\bar{Y}^{j;t,x}, \bar{Z}^{j;t,x}, \bar{K}^{j;t,x})_{i\in\Gamma^2}$. Moreover there exists deterministic continuous functions $(\bar{v}^j(t,x))_{j\in\Gamma^2}$ of polynomial growth such that: $\forall (t,x) \in [0,T]\times\mathbb{R}^k$,

$$\forall s \in [t,T], \bar{Y}_s^{j;t,x} = \bar{v}^j(s, X_s^{t,x}).$$

Finally $(\bar{v}^i(t,x))_{i\in\Gamma^1}$ is the unique solution of the system of PDI with a lower interconnected obstacle (4) in the class of continuous functions with polynomial growth.

3. System with bilateral obstacles

A. Assumptions

(H1) Each function f^{ij},

(i) is uniformly continuous in (t,x) uniformly w.r.t. the other variables $(\vec{y},z)$ and for any (t,x) and the mapping $(t,x)\to f^{ij}(t,x,0,0)$ is of polynomial growth.

(ii) is Lipschitz with respect to the variables $(\vec{y}:=(y^{ij})_{(i,j)\in\Gamma_1\times\Gamma_2},z)$;

(H2) Monotonicity: Let $\vec{y}=(y^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2}$, then for any $(i,j)\in\Gamma^1\times\Gamma^2$ and any $(k,l)\neq(i,j)$ the mapping $y^{k,l}\to f^{ij}(s,\vec{y},z)$ is non-decreasing.

(H3) The functions h^{ij} are continuous, of polynomial growth and satisfy

$$\max_{k\in(\Gamma^1)^{-i}}\left(h^{kj}(x)-\underline{g}_{ik}(T,x)\right)\leq h^{ij}(x)\leq\min_{l\in(\Gamma^2)^{-j}}\left(h^{il}(x)+\bar{g}_{jl}(T,x)\right).$$

(H4) The non-free loop property: The switching costs $\underline{g}_{ik}$ and $\bar{g}_{jl}$ are non-negative, jointly continuous in (t, x) and satisfy:

For any loop in $\Gamma^1 \times \Gamma^2$, i.e., pairs $(i_1, j_1), \ldots, (i_N, j_N)$ such that $(i_N, j_N) = (i_1, j_1)$, $\mathrm{card}\{(i_1, j_1), \ldots, (i_N, j_N)\} = N - 1$ and $\forall\ q = 1, \ldots, N-1$, either $i_{q+1} = i_q$ or $j_{q+1} = j_q$, we have:

$$\sum_{q=1,N-1} \varphi_{i_q i_{q+1}}(t, x) \neq 0, \tag{6}$$

where, $\forall\ \ q = 1, \ldots, N-1$,
$\varphi_{i_q i_{q+1}}(t, x) = -\underline{g}_{i_q i_{q+1}}(t, x)\mathbb{1}_{i_q \neq i_{q+1}} + \bar{g}_{j_q i_{q+1}}(t, x)\mathbb{1}_{j_q \neq j_{q+1}}$
(resp.
$\varphi_{i_q i_{q+1}}(t, x) = \underline{g}_{i_q, i_{q+1}}(t, x)\mathbb{1}_{i_q \neq i_{q+1}} - \bar{g}_{j_q, i_{q+1}}(t, x)\mathbb{1}_{j_q \neq j_{q+1}}$).

This assumption implies:

(i) For any loop $(i_1, \ldots, i_N) \in \Gamma^1$,

$$\sum_{p=1}^{N-1} \underline{g}_{i_k, i_{k+1}}(t, x) \quad > \quad 0 \tag{7}$$

(ii) The same happens with loops in Γ^2 with the functions $\bar{g}_{ij}$. $\blacksquare$

3.1. Uniqueness of the solution of system (1) by comparison

Lemma

Let $\vec{u} := (u^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ (resp. $\vec{w} := (w^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$) be an usc subsolution (resp. lsc supersolution) of (1). Let (t,x) be fixed and let

$$\tilde{\Gamma}(t,x) := \{(i,j)\in\Gamma^1\times\Gamma^2, u^{ij}(t,x) - w^{i,j}(t,x) = \max_{(k,l)\in\Gamma^1\times\Gamma^2}\{u^{kl}(t,x) - w^{kl}(t,x)\}.$$

Then, there exists $(i_0,j_0)\in\tilde{\Gamma}(t,x)$ such that

$$u^{i_0j_0}(t,x) > L^{i_0j_0}[\vec{u}](t,x) \quad \text{and} \quad w^{i_0j_0}(t,x) < U^{i_0j_0}[\vec{w}](t,x). \qquad (8)$$

Idea of the proof: By contradiction. If the conclusion does not hold, using the fact that $\vec{u}$ (resp. $\vec{w}$) is a subsolution (resp. a supersolution) one can find a free loop which is contradictory with (6). ∎

Comparison of sub. and supersolutions

Theorem

Assume that $\vec{u} = (u^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ *(resp.* $\vec{w} = (w^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$*) is an usc (resp. lsc) subsolution (resp. supersolution) of the system (1) such that, for any* $(i,j)\in\Gamma^1\times\Gamma^2$*, both* u^{ij} *and* w^{ij} *are of polynomial growth, i.e., there exist two constants* γ *and* C *such that*

$$|u^{ij}(t,x)| + |w^{ij}(t,x)| \leq C(1+|x|^{\gamma}). \tag{9}$$

Then, it holds that for any $(i,j)\in\Gamma^1\times\Gamma^2$ *and* $(t,x)\in[0,T]\times\mathbb{R}^k$*,*

$$u^{ij}(t,x) \leq w^{ij}(t,x).$$

Sketch of the proof: w.l.o.g we assume that there exists $\lambda > 0$ such that for any $(i,j) \in \Gamma^1 \times \Gamma^2$,

$$\lambda > c(f^{i,j})(\Lambda - 1)$$

and for any

$$(t, x, \vec{y}, z) \in [0, T] \times \mathbb{R}^k \times \mathbb{R}^{\Lambda+d}, \text{ and } (u, v) \in \mathbb{R}^2 \text{ s.t. } u \geq v,$$

$$f^{ij}(t, x, [\vec{y}^{-(ij)}, u], z) - f^{ij}(t, x, [\vec{y}^{-(ij)}, v], z) \leq -\lambda(u - v), \quad (10)$$

where $c(f^{ij})$ is the Lipschitz constant of f^{ij} w.r.t. $(y^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2}$.

The proof will be obtained by contradiction and by using Crandall-Ishii-Lions's Lemma.

Step 1: if the conclusion does not hold then there exists $\varepsilon_0 > 0$ and $(t_0, x_0) \in [0, T] \times \mathbb{R}^k$ such that

$$\max_{i,j} \left((u^{ij} - w^{ij})(t_0, x_0)\right) \geq \epsilon_0. \quad (11)$$

Next, w.l.o.g. we may assume that for any $(i,j) \in \Gamma^1 \times \Gamma^2$,

$$\lim_{|x|\to\infty} (u^{ij} - w^{ij})(t,x) = -\infty. \tag{12}$$

Indeed, one may replace w^{ij} with $w^{ij,\vartheta,\mu}$ defined by

$$w^{ij,\vartheta,\mu}(t,x) = w^{ij}(t,x) + \vartheta e^{-\mu t}|x|^{2\gamma+2},$$

which is still an *usc* supersolution of (1) for $\vartheta \le \vartheta_0$ and $\mu \ge \mu_0$ which satisfies (12). Therefore, it suffices to show that $u^{ij}(t,x) \le w^{ij,\vartheta,\mu}(t,x)$ and to take the limit as $\vartheta \to 0$.

Thus, assume that (11) and (12) are satisfied. Then, there is $R > 0$ such that:

$$\begin{aligned} &\max_{(t,x)\in[0,T]\times\mathbb{R}^k} \max_{i,j}\{(u^{ij} - w^{ij})(t,x)\} \\ &\quad = \max_{(t,x)\in[0,T]\times B(0,R)} \max_{i,j}\{(u^{ij} - w^{ij})(t,x)\} \\ &\quad = \max_{ij}(u^{ij} - w^{ij})(t^*,x^*) \ge \epsilon_0 > 0, \end{aligned} \tag{13}$$

where, $(t^*,x^*) \in [0,T) \times B(0,R)$ since $u^{ij}(T,x) \le w^{ij}(T,x)$, for all (i,j).

Step 2: Let (i_0, j_0) be an element of $\tilde{\Gamma}(t^*, x^*)$ that satisfies (8). For $n \geq 1$, let $\Phi_n^{i_0,j_0}$ be the function defined as follows.

$$\Phi_n^{i_0,j_0}(t,x,y) := (u^{i_0 j_0}(t,x) - w^{i_0 j_0}(t,y)) - \phi_n(t,x,y),$$

where,

$$\phi_n(t,x,y) := n|x-y|^{2\gamma+2} + |x-x^*|^{2\gamma+2} + (t-t^*)^2.$$

Since $\Phi_n^{i_0,j_0}$ is *usc* in (t,x,y), there exists $(t_n, x_n, y_n) \in [0,T] \times B(0,R)^2$ such that

$$\Phi_n^{i_0,j_0}(t_n,x_n,y_n) = \max_{(t,x,y)\in[0,T]\times B(0,R)^2} \Phi_n^{i_0,j_0}(t,x,y).$$

Moreover as $n \to \infty$ we have:

(i) $n|x_n - y_n|^{2\gamma+2} \to 0$ and $((t_n, x_n, y_n))_n \to (t^*, x^*, x^*)$;

(ii) $(u^{i_0 j_0}(t_n, x_n), w^{i_0 j_0}(t_n, y_n)) \to (u^{i_0 j_0}(t^*, x^*), w^{i_0 j_0}(t^*, x^*))$;

(iii) For n large enough we have:

$$u^{i_0 j_0}(t_n, x_n) > \max_{k \in (\Gamma^1)^{-i}} (u^{k j_0}(t_n, x_n) - g_{i_0 k}(t_n, x_n)), \qquad (14)$$

and

$$w^{i_0 j_0}(t_n, x_n) < \min_{l \in (\Gamma^2)^{-j}} (w^{i_0 l}(t_n, x_n) - g_{j_0 l}(t_n, x_n)). \qquad (15)$$

Next by CIL's Lemma there exist

(i) $(p_u^n, q_u^n, M_u^n) \in \bar{J}^{2,+}(u^{i_0 j_0})(t_n, x_n)$

(ii) $(p_w^n, q_w^n, M_w^n) \in \bar{J}^{2,-}(w^{i_0 j_0})(t_n, y_n)$

such that

$$p_u^n - p_w^n = \partial_t \tilde{\varphi}_n(t_n, x_n, y_n) = 2(t_n - t^*),$$

$$q_u^n \ (\text{resp. } q_w^n) = \partial_x \varphi_n(t_n, x_n, y_n) \quad (\text{resp.} - \partial_y \varphi_n(t_n, x_n, y_n))$$

and

$$\begin{pmatrix} M_u^n & 0 \\ 0 & -N_w^n \end{pmatrix} \leq A_n + \frac{1}{2n} A_n^2, \tag{16}$$

where

$$A_n = D^2_{(x,y)} \varphi_n(t_n, x_n, y_n).$$

But $(u^{ij})_{i,j}$ (resp. $(w^{ij})_{i,j}$) is a subsolution (resp. supersolution) of (1) and by (14) and (15) we get

$$-p_u^n - b(t_n,x_n)^\top . q_u^n - \frac{1}{2} Tr[(\sigma\sigma^\top)(t_n,x_n)M_u^n] \\ -f^{i_0 j_0}(t_n,x_n,(u^{ij}(t_n,x_n))_{(i,j)\in\Gamma^1\times\Gamma^2}, \sigma(t_n,x_n)^\top . q_u^n) \leq 0,$$

and

$$-p_w^n - b(t_n,y_n)^\top . q_w^n - \frac{1}{2} Tr[(\sigma\sigma^\top)(t_n,y_n)M_w^n] - \\ f^{i_0 j_0}(t_n,y_n,(w^{ij}(t_n,y_n))_{(i,j)\in\Gamma^1\times\Gamma^2}, \sigma(t_n,y_n)^\top . q_w^n) \geq 0.$$

Making the substraction between those two inequalities yields

$$-\Big\{ f^{i_0 j_0}(t_n,x_n,(u^{ij}(t_n,x_n))_{(i,j)\in\Gamma^1\times\Gamma^2}, \sigma(t_n,x_n)^\top . q_u^n) \\ -f^{i_0 j_0}(t_n,y_n,(w^{ij}(t_n,y_n))_{(i,j)\in\Gamma^1\times\Gamma^2}, \sigma(t_n,y_n)^\top . q_w^n) \Big\} \leq \varrho_n,$$

with $\overline{\lim}_{n\to\infty} \varrho_n \leq 0$.

Linearizing $f^{i_0 j_0}$ and using (10), we obtain

$$\lambda(u^{i_0 j_0}(t_n, x_n) - w^{i_0 j_0}(t_n, y_n)) \\ - \sum_{(i,j)\in\Gamma^1\times\Gamma^2,(i,j)\neq(i_0 j_0)} \Theta_n^{i,j}(u^{ij}(t_n, x_n) - w^{ij}(t_n, y_n)) \leq \varrho_n,$$

where, $\Theta_n^{i,j}$ is the increment rate of $f^{i_0 j_0}$ w.r.t. y^{ij} which is uniformly bounded (w.r.t. n) and non-negative thanks to the monotonicity assumption (H2). Therefore,

$$\lambda(u^{i_0 j_0}(t_n, x_n) - w^{i_0 j_0}(t_n, y_n)) \\ \leq \sum_{(i,j)\in\Gamma^1\times\Gamma^2,(i,j)\neq(i_0,j_0)} \Theta_n^{i,j}(u^{ij}(t_n, x_n) - w^{ij}(t_n, y_n)) + \varrho_n \\ \leq c(f^{i_0 j_0}) \times \sum_{(i,j)\in\Gamma^1\times\Gamma^2,(i,j)\neq(i_0,j_0)}((u^{ij}(t_n, x_n) - w^{ij}(t_n, y_n))^+ + \varrho_n.$$

Taking the limit as $n \to \infty$ we obtain

$$\begin{aligned}
&\lambda(u^{i_0j_0}(t^*,x^*) - w^{i_0j_0}(t^*,y^*)) \\
&\leq \overline{\lim}_{n\to\infty}\, c(f^{i_0j_0})\left(\sum_{(i,j)\in\Gamma^1\times\Gamma^2,(i,j)\neq(i_0,j_0)}(u^{ij}(t_n,x_n) - w^{ij}(t_n,y_n))^+\right) \\
&\leq c(f^{i_0j_0})\left(\sum_{(i,j)\in\Gamma^1\times\Gamma^2,(i,j)\neq(i_0,j_0)}\overline{\lim}_{n\to\infty}(u^{ij}(t_n,x_n) - w^{ij}(t_n,y_n))^+\right) \\
&\leq c(f^{i_0j_0})\left(\sum_{(i,j)\in\Gamma^1\times\Gamma^2,(i,j)\neq(i_0,j_0)}(u^{ij}(t^*,x^*) - w^{ij}(t^*,x^*))^+\right)
\end{aligned},$$

since u^{ij} (resp. w^{ij}) is *usc* (resp. *lsc*). As (i_0,j_0) belongs to $\tilde{\Gamma}(t^*,x^*)$, we obtain

$$\begin{aligned}
&\lambda(u^{i_0j_0}(t^*,x^*) - w^{i_0j_0}(t^*,y^*)) \\
&\leq c(f^{i_0j_0})\left((\Lambda-1)(u^{i_0j_0}(t^*,x^*) - w^{i_0j_0}(t^*,x^*))\right)
\end{aligned}$$

which is contradictory by (10) and (13). Thus, for any $(i,j)\in\Gamma^1\times\Gamma^2$, $u^{ij}\leq w^{ij}$. ■

As by-product we have:

Corollary

If system (1) admits a viscosity solution of polynomial growth, then it is unique and continuous.

4. Existence of the solution of (1)

Main idea: Using Perron's method. Thus we need (i) comparison ; (ii) a sub and supersolution.

A. The decreasing penalized scheme.

For $m \geq 0$ and $(i,j) \in \Gamma^1 \times \Gamma^2$ let,

$$
\begin{cases}
\bar{Y}_s^{ij,m} = h^{ij}(X_T^{t,x}) + \int_s^T \bar{f}^{ij,m}(r, X_r^{t,x}, (\bar{Y}_r^{kl,m})_{(k,l)\in\Gamma^1\times\Gamma^2}, \bar{Z}_r^{ij,m})dr + \\
\qquad \bar{K}_T^{ij,m} - \bar{K}_s^{ij,m} - \int_s^T \bar{Z}_r^{ij,m} dB_r, \\
\bar{Y}_s^{ij,m} \geq \max_{k\in(\Gamma^1)^{-i}} \{\bar{Y}_s^{kj,m} - \underline{g}_{ik}(s, X_s^{t,x})\}, \\
\int_0^T (\bar{Y}_s^{ij,m} - \max_{k\in(\Gamma^1)^{-i}}\{\bar{Y}_s^{kj,m} - \underline{g}_{ik}(s, X_s^{t,x})\})d\bar{K}_s^{ij,m} = 0;
\end{cases}
\tag{17}
$$

$$
\bar{f}^{ij,m}(s,x,\vec{y},z^{ij}) := f^{ij}(s,x,(y^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2}, z^{ij}) \\
-m\big(y^{ij} - \min_{l\in(\Gamma^2)^{-j}}(y^{il} + \bar{g}_{jl}(s,x))\big)^+.
$$

Theorem (i)

(i) The solution of (17) exists and is unique and $Y^{ij,m} \geq Y^{ij,m+1}$.
(ii) There exist $(\bar{v}^{i,j,m})_{(i,j)\in\Gamma^1\times\Gamma^2}$ *deterministic continuous functions of PG such that for any* $(i,j) \in \Gamma^1 \times \Gamma^2$, $(t,x) \in [0,T] \times \mathbb{R}^k$,

$$\forall s \in [t, T],\ \ \bar{Y}_s^{ij,m} = \bar{v}^{i,j,m}(s, X_s^{t,x}).$$

(iii) $\bar{v}^{ij,m} \geq \bar{v}^{ij,m+1}$.
(iv) $(v^{i,j,m})_{(i,j)\in\Gamma^1\times\Gamma^2}$ *is the unique solution of the following system of VIs with one interconnected obstacle:* $\forall\ (i,j) \in \Gamma^1 \times \Gamma^2$,

$$\begin{cases} \min\{\bar{v}^{ij,m}(t,x) - \max\limits_{k\in(\Gamma^1)^{-i}}(\bar{v}^{kj,m}(t,x) - \underline{g}_{ik}(t,x)); \\ \quad -\partial_t \bar{v}^{ij,m}(t,x) - b(t,x)D_x\bar{v}^{ij,m}(t,x) \\ \quad -\frac{1}{2}\,Tr(\sigma\sigma^\top(t,x)D^2_{xx}\bar{v}^{ij,m}(t,x)) \\ \quad -f^{ij,m}(t,x,(\bar{v}^{kl,m}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2},\sigma^\top(t,x)D_x\bar{v}^{ij,m}(t,x))\} = 0, \\ \\ \bar{v}^{ij,m}(T,x) = h^{ij}(x). \end{cases} \quad (18)$$

B. The increasing penalized scheme

For $n \geq 0$ and $(i,j) \in \Gamma^1 \times \Gamma^2$, let:

$$\begin{cases} \underline{Y}_s^{ij,n} = h^{ij}(X_T^{t,x}) + \int_s^T \underline{f}^{ij,n}(r, X_r^{t,x}, (\underline{Y}_r^{kl,n})_{(k,l)\in\Gamma^1\times\Gamma^2}, \underline{Z}_r^{ij,n})dr - \\ \qquad\qquad (\underline{K}_T^{ij,n} - \underline{K}_s^{ij,n}) - \int_s^T \underline{Z}_r^{ij,n} dB_r, \\ \underline{Y}_s^{ij,n} \leq \min_{l\in(\Gamma^2)^{-j}} \{\underline{Y}_s^{il,n} + \bar{g}_{jl}(s, X_s^{t,x})\}, \\ \int_0^T (\underline{Y}_s^{ij,n} - \min_{l\in(\Gamma^2)^{-j}} \{\underline{Y}_s^{il,n} + \bar{g}_{jl}(s, X_s^{t,x})\}) d\underline{K}_s^{ij,n} = 0 \end{cases} \tag{19}$$

$$\underline{f}^{ij,n}(s,x,\vec{y},z^{ij}) =$$
$$f^{ij}(s,x,(y^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2}, z^{ij}) + n\big(y^{ij} - \max_{k\in(\Gamma^1)^{-i}}\{y^{kj} - \underline{g}_{ik}(s,x)\}\big)^-.$$

Like the decreasing scheme, the increasing one enjoys:

Theorem (ii)

(i) The solution of (19) exists and is unique and $\underline{Y}^{ij,n} \leq \underline{Y}^{ij,n+1}$.
(ii) There exist $(\underline{v}^{i,j,m})_{(i,j)\in\Gamma^1\times\Gamma^2}$ *deterministic continuous functions of PG such that for any* $(i,j) \in \Gamma^1 \times \Gamma^2$, $(t,x) \in [0,T] \times \mathbb{R}^k$,

$$\forall s \in [t, T],\ \underline{Y}^{ij,m}_s = \underline{v}^{i,j,m}(s, X^{t,x}_s).$$

(iii) $\underline{v}^{ij,n}(t,x) \leq \underline{v}^{ij,n+1}(t,x)$.
(iv) $(\underline{v}^{i,j,m})_{(i,j)\in\Gamma^1\times\Gamma^2}$ *is the unique solution of the following system of VIs with one interconnected obstacle:* $\forall\ (i,j) \in \Gamma^1 \times \Gamma^2$,

$$\begin{cases} \max\Big\{\underline{v}^{ij,m}(t,x) - \min_{l\in(\Gamma^2)^{-j}}(\bar{v}^{kj,m}(t,x) + \bar{g}_{ik}(t,x)); \\ \quad -\partial_t \underline{v}^{ij,m}(t,x) - b(t,x)D_x\underline{v}^{ij,m}(t,x) \\ \quad -\frac{1}{2}\,Tr(\sigma\sigma^\top(t,x)D^2_{xx}\underline{v}^{ij,m}(t,x)) \\ \quad -\underline{f}^{ij,m}(t,x,(\underline{v}^{kl,m}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2}, \sigma^\top(t,x)D_x\underline{v}^{ij,m}(t,x))\Big\} = 0, \\ \underline{v}^{ij,m}(T,x) = h^{ij}(x). \end{cases} \tag{20}$$

C. Comparison of the schemes

$$\bar{v}^{ij}(t,x) := \lim_{m\to\infty} \bar{v}^{ij,m}(t,x) \quad \text{and} \quad \underline{v}^{ij}(t,x) := \lim_{n\to\infty} \underline{v}^{ij,n}(t,x).$$

Proposition (i)

For any $(i,j) \in \Gamma^1 \times \Gamma^2$ we have:

(i) $\forall n, m$, $\underline{v}^{ij,n} \leq \underline{v}^{ij} \leq \bar{v}^{ij} \leq \bar{v}^{ij,m}$.

(ii) $\underline{v}^{ij}$ (resp. $\bar{v}^{ij}$) is lsc (resp. usc) and of PG.

Hint: By construction of $\bar{Y}^{ij;m}$ and $\underline{Y}^{ij;n}$ via penalization of the obstacles. They are obtained as a limit wrt n (resp. m) of $Y^{ij;n,m}$ which satisfy:

$$Y^{ij;n,m+1} \leq Y^{ij;n,m} \leq Y^{ij;n+1,m}$$

i.e. increasing wrt n and decreasing wrt m (by Hu-Peng's result (Notes du CRAS '03) on comparison of multi-diemensional standard BSDEs). ■

Proposition (ii)

The functions $(\bar{v}^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ *is a viscosity subsolution of (1).* $\blacksquare$

Proposition (iii)

For any fixed m_0*,* $(\bar{v}^{ij,m_0})_{(i,j)\in\Gamma^1\times\Gamma^2}$ *is a viscosity supersolution of (1).*

Proof: The triples $((\bar{Y}^{ij,m_0}, \bar{Z}^{ij,m_0}, \bar{K}^{ij,m_0}))_{(i,j)\in\Gamma^1\times\Gamma^2}$ satify:

$$\begin{cases} \bar{Y}_s^{ij,m_0} = h^{ij}(X_T^{t,x}) + \int_s^T \bar{f}^{ij,m_0}(r, X_r^{t,x}, (\bar{Y}_r^{kl,m_0})_{(k,l)\in\Gamma^1\times\Gamma^2}, \bar{Z}_r^{ij,m_0})dr \\ \qquad +\bar{K}_T^{ij,m_0} - \bar{K}_s^{ij,m_0} - \int_s^T \bar{Z}_r^{ij,m_0} dB_r \\ \bar{Y}_s^{ij,m_0} \geq \max_{k\in(\Gamma^1)^{-i}} \{\bar{Y}_s^{kj,m_0} - \underline{g}_{ik}(s, X_s^{t,x})\}, \;\; s \leq T; \\ \int_0^T (\bar{Y}_s^{ij,m_0} - \max_{k\in(\Gamma^1)^{-i}} \{\bar{Y}_s^{kj,m_0} - \underline{g}_{ik}(s, X_s^{t,x})\}) d\bar{K}_s^{ij,m_0} = 0 \end{cases} \tag{21}$$

where

$$\bar{f}^{ij,m_0}(s, X_s^{t,x}, \vec{y}, z^{ij}) =$$
$$f^{i,j}(s, X_s^{t,x}, (y^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2}, z^{ij}) - m_0\big(y^{ij} - \min_{l\in(\Gamma^2)^{-j}}(y^{il} + \bar{g}_{jl}(s, X_s^{t,x}))$$

and

$$\forall (i,j) \in \Gamma^1 \times \Gamma^2, \;\; \forall (t,x), \;\; \forall s \in [t, T], \;\; \bar{Y}_s^{ij,m_0} = \bar{v}^{ij,m_0}(s, X_s^{t,x}).$$

But $\bar{Y}^{ij,m_0}$ is the value function of a zero-sum Dynkin game i.e., for $s \leq T$

$$\bar{Y}_s^{ij;m_0} = \text{ess sup}_{\sigma \geq s}\text{ess inf}_{\tau \geq s}\mathbf{E}[\int_s^{\sigma\wedge\tau} f^{ij}(r, X_r^{t,x}, (\bar{Y}_r^{ij;m_0})_{i\in\mathcal{J}}, \bar{Z}_r^{ij;m_0})dr$$

$$+\max_{k\in(\Gamma^1)^{-i}}\{\bar{Y}_\sigma^{kj;m_0} - \underline{g}_{ik}(\sigma, X_\sigma^{t,x})\}1\!\!1_{[\sigma<\tau]}+$$

$$\{\bar{Y}_\tau^{ij;m_0} \wedge \min_{l\in(\Gamma^2)^{-j}}\{\bar{Y}_\tau^{il;m_0} - \bar{g}_{jl}(\tau, X_\tau^{t,x})\}\}1\!\!1_{[\tau\leq\sigma<T]}$$

$$+h^{ij}(X_T^{t,x})1\!\!1_{[\tau=\sigma=T]}|\mathcal{F}_s]. \tag{22}$$

Thus $\bar{v}^{ij,m_0}$ is the unique viscosity solution (H.-Hassani, '05) of

$$\begin{cases} \min\big[\vartheta(t,x) - \max_{k\in(\Gamma^1)^{-i}}\{\bar{v}^{kj,m_0}(t,x) - \underline{g}_{ik}(t,x)\}, \\ \max\big\{\vartheta(t,x) - \bar{v}^{ij,m_0}(t,x) \vee \min_{l\in(\Gamma^2)^{-j}}(\bar{v}^{il,m_0}(t,x) - \bar{g}_{jl}(t,x)), \\ \quad -\partial_t\vartheta(t,x) - b(t,x)^\top D_x\vartheta(t,x) - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^\top(t,x)D^2_{xx}\vartheta(t,x)) - \\ \qquad f^{ij}(t,x,(\bar{v}^{kl,m_0}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2}, \sigma^\top(t,x).D_x\vartheta(t,x))\big\}\big] = 0\,; \\ \vartheta(T,x) = h^{ij}(x). \end{cases}$$

Let (t,x) and $(p,q,M) \in \bar{J}^{-}[\bar{v}^{ij,m_0}](t,x)$. As $\bar{v}^{ij,m_0}$ is a supersolution of the PDE then

$$\bar{v}^{ij,m_0}(t,x) \geq \max_{k \in (\Gamma^1)^{-i}} \{\bar{v}^{kj,m_0}(t,x) - \underline{g}_{ik}(t,x)\} \tag{23}$$

and

$$\begin{aligned} &\max\{\bar{v}^{ij,m_0}(t,x) - \bar{v}^{ij,m_0}(t,x) \vee \min_{l \in (\Gamma^2)^{-j}} (\bar{v}^{il,m_0}(t,x) - \bar{g}_{jl}(t,x))\,; \\ &\quad -p - b(t,x)^\top q - \tfrac{1}{2}\mathrm{Tr}(\sigma\sigma^\top(t,x)M) \\ &\quad -f^{ij}(t,x,(\bar{v}^{kl,m_0}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2}, \sigma^\top(t,x).q)\} \geq 0. \end{aligned} \tag{24}$$

But $a - (a \vee b) \leq a - b$ then

$$\max\{\bar{v}^{ij,m_0}(t,x) - \min_{l \in (\Gamma^2)^{-j}}(\bar{v}^{il,m_0}(t,x) - \bar{g}_{jl}(t,x))\,;$$

$$-p - b(t,x)^\top q - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^\top(t,x)M)$$

$$-f^{ij}(t,x,(\bar{v}^{kl,m_0}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2},\sigma^\top(t,x).q)\} \geq 0.$$

As $v^{ij,m_0}(T,x) = h^{ij}(x)$ and i,j arbitrary then $(v^{ij,m_0})_{i,j}$ is a viscosity supersolution of system (1). ∎

4.1 Perron's method

Let m_0 be fixed and let $\mathcal{U}_{m_0}$ defined as follows:

$$\mathcal{U} = \{\vec{u} := (u^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2} \text{ s.t. } \vec{u} \text{ is a subsolution of (1)} \\ \text{and } \forall\,(i,j)\in\Gamma^1\times\Gamma^2,\ \bar{v}^{i,j} \le u^{i,j} \le \bar{v}^{ij,m_0}\}.$$

$\mathcal{U}_{m_0}$ is not empty since it contains $(\bar{v}^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$. For (t,x) and $(i,j)\in\Gamma^1\times\Gamma^2$, let us set:

$$^{m_0}v^{ij}(t,x) = \sup\{u^{ij}(t,x),\ (u^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2}\in\mathcal{U}_{m_0}\}.$$

Theorem

The family $(^{m_0}v^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ does not depend on m_0 and is the unique continuous viscosity solution in the class of PG functions of system (1). Moreover for any $(i,j)\in\Gamma^1\times\Gamma^2$, $^{m_0}v^{ij} = \bar{v}^{ij}$.

Sketch of the proof:

(i) $({}^{m_0}v^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ is a subsolution thanks to monotonicity property of f^{ij};

(ii) If it is not a supersolution of (1) then it is not maximal, which is contradictory.

Therefore it is the unique solution for (1).

Once more uniqueness implies that $({}^{m_0}v^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ does not depend on m_0. Finally comparison implies that

$$\bar{v}^{ij} \leq^{m_0} v^{ij} \leq \bar{v}^{ij;m_0}$$

Finally making $m_0 \to \infty$ to obtain

$$\bar{v}^{ij} =^{m_0} v^{ij}$$

and thus $(\bar{v}^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ is the unique viscosity solution for (1). ■

Remark

The limit of the increasing scheme $(\underline{v}^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ *is continuous and is the unique viscosity solution (in the class PG) for the following system: For any* $(i,j)\in\Gamma^1\times\Gamma^2$,

$$\begin{cases} \max\Big\{\big(\underline{v}^{ij}-U^{ij}[\vec{\underline{v}}]\big)(t,x);\ \ \min\Big[\big(\underline{v}^{ij}(t,x)-L^{ij}[\vec{\underline{v}}]\big)(t,x), \\ -\partial_t\underline{v}^{ij}(t,x)-\mathcal{L}\underline{v}^{ij}(t,x) \\ -f^{ij}(t,x,(\underline{v}^{kl}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2},\sigma^\top(t,x)D_x\underline{v}^{ij}(t,x))\Big]\Big\}=0; \\ \underline{v}^{ij}(T,x)=h^{ij}(x). \end{cases} \tag{25}$$