Backward Stochastic Lyapunov Equation: Mild formulation by Domains of Fractional Powers

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- Let H be real separable Hilbert space.

- Let W be a one dimensional Wiener process defined on a probability basis $(\Omega, \mathcal{F}, \mathbb{P})$. We denote by $\mathcal{F}_t$ for $t \geq 0$ its natural filtration completed.

- Let $A : D(A) \subset H \to H$ be an unbounded operator that generates a C_0 semigroup.

We consider this Lyapunov equation

$$
\begin{cases}
-dP(t) = (A^*P(t) + P(t)A + [C^*(t)P(t)C(t) + C^*(t)Q(t) + Q(t)C(t)])\,dt \\
\qquad\qquad +L(t)\,dt + Q(t)\,dW(t), \quad t \in [0,T], \\
P(T) = P_T
\end{cases}
\tag{1}
$$

notice that $L \in L^{\infty}_{\mathcal{S},\mathcal{P}}((0,T) \times \Omega; L(H))$ and $P_T \in L^{\infty}_{\mathcal{S}}(\Omega, \mathcal{F}_T; L(H))$.

- It arise as the dual equation of the second variation in the maximum principle for optimal control problems for SPDEs: Tang-Li(LNPAM 1994), Fuhrman-Hu-Tessitore (CRAS 2012), Lu-Zang (Preprint 2012), Du-Meng (Preprints 2012)

- First step to solve the Riccati backward stochastic differential equation (BSRE), G. Tessitore (Sicon 2005)

$$\begin{cases} -dP(t) = (A^*P(t)+P(t)A+ C^*(t)P(t)C(t) + C^*(t)Q(t) + Q(t)C(t))\,dt \\ \qquad\qquad -(P(t)B(t)B^*(t)P(t) - L(t))\,dt + Q(t)\,dW(t) \qquad t \in [0,T] \\ P(T) = P_T \end{cases} \tag{2}$$

Main difficulty in the infinite dimensional case:

$L(H)$ that is not an Hilbert space.

New questions arise:

1 Is there a meaningful formulation for mild equation?

2 Characterization of Q? P has a natural characterization in terms of a stochastic quadratic form / value function

3 Once you find such a formulation, is the equation well posed? Which is the regularity for P and Q?

If the data are more regular, Hilbert Schmidt valued, then the Lyapunov equation is well posed.

Unfortunately the space $\Sigma_2(H)$, of Hilbert Schmidt operators from H to H, is far too small to cover significant applications.

IDEA: give meaning to the equation in $L(H)$ working in a bigger Hilbertian space close enough to it

Besides previous assumptions we ask

- A to be a self adjoint operator in H and there are a b.o.c $\{e_k : k \geq 1\}$ in H and $\omega > 0$, such that

$$Ae_k = -\lambda_k e_k, \qquad \text{with} \quad \omega \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \ldots,$$

- that there exists $\rho \in \left(\frac{1}{4}, \frac{1}{2}\right)$, such that

$$\sum_{k \geq 1} \lambda_k^{2\rho} < +\infty.$$

In particular A is the infinitesimal generator of an analytic semigroup in H.

These assumptions are satisfied if $H = L^2(0,1)$ and $A = \Delta +$ Dirichlet b.c.

Let us define

$$V := D((-A)^\rho) = \{x \in H : \sum_{k=1}^{+\infty} \lambda_k^{2\rho} \langle x, e_k \rangle_H^2 < +\infty\} = H_0^{2\rho}(0,1)$$

Clearly:

- $e_k \in V$ for every $k : 1, 2, \ldots$
- V is a separable Hilbert space and $\left\{\frac{e_k}{\lambda_k^\rho} : \ k \geq 1\right\}$ is a b.o.c. of V.
- the dual space V^* is an Hilbert space and $\left\{e_k \lambda_k^\rho : \ k \geq 1\right\}$ is a b.o.c. of V^*.
- there is a constant $M_A > 0$ such that $|e^{\sigma A}|_{L(H;V)} \leq \frac{M_A}{\sigma^\rho}$, $|e^{\sigma A}|_{L(V^*;H)} \leq \frac{M_A}{\sigma^\rho}$

we have following dense inclusions:

$$V \hookrightarrow_d H \simeq H^* \hookrightarrow_d V^*$$

Next we introduce the following space of operators

$$\mathcal{K} = L_2(V; H) \cap L_2(H; V^*)$$

Given two separable Hilbert spaces G and F, the space of operators $L_2(G; F)$ is the space of linear and bounded operators from G to F such that

$$\sum_{k=1}^{\infty} |Tg_k|_F^2 < \infty$$

where $\{g_k : k \geq 1\}$ is a complete orthonormal basis of G.

- $\mathcal{K}$ is a separable Hilbert space,
- $L(H) \subset \mathcal{K}$,
- $T \in \mathcal{K}$ iff $T \in L(V; H) \cap L(H; V^*)$ and $\sum_{k=1}^{\infty} \lambda_k^{-2\rho}(|Te_k|_H^2 + |T^* e_k|_H^2) < \infty$.

We can then prove the following result

Theorem 1 *There exists a unique solution* $(P, Q) \in L^2_{\mathcal{P},\mathcal{S}}(\Omega, C([0,T]; L(H)) \times L^2_{\mathcal{P}}(\Omega \times [0,T]; \mathcal{K}))$ *such that*

$$P(t) = e^{(T-t)A} P_T e^{(T-t)A} + \int_t^T e^{(s-t)A} (C^*(s)P(s)C(s) + \gamma(C(s))Q(s)) e^{(s-t)A}\, ds$$
$$+ \int_t^T e^{(s-t)A} L(s) e^{(s-t)A}\, ds + \int_t^T e^{(s-t)A} Q(s) e^{(s-t)A}\, dB_s$$

where $\gamma(C)G = C^*G + GC$ *for any* $C \in L(H)$ *and* $G \in \mathcal{K}$*.*

Moreover $(P_{|_{[0,T-\varepsilon]}}, Q_{|_{[0,T-\varepsilon]}}) \in L^2_{\mathcal{P}}(\Omega, C([0,T-\varepsilon]; \Sigma_2(H))) \times L^2_{\mathcal{P}}(\Omega \times [0,T-\varepsilon]; \Sigma_2(H)))$, *for any* $\varepsilon > 0$*.*

Proof (idea)

Main difficulty:

if $C \in L(H)$, the operator $\gamma(C)Q := C^*Q + QC$, that is bounded in $\Sigma_2(H)$ is not a bounded operator from $\mathcal{K}$ into itself

More precisely

$$\sum_{k\geq 1} \lambda_k^{-2\rho} |QCe_k|_H^2$$

may not be bounded:

even if $e_k \in V$ for every $k \geq 1$, Ce_k just belongs to H so that we only have $QCe_k \in V^*$

As a consequence we cannot use the result of Hu-Peng (1991) because we have an unbounded term in Q

Solution: We exploit the regularizing property of the semigroup e^{tA}.

For $Q \in \mathcal{K}$ we have

$$\sum_{k\geq 1} \lambda_k^{-2\rho} |e^{(s-t)A}(C^*(s)Q + QC(s))e^{(s-t)A}e_k|_H^2 \leq$$
$$\sum_{k\geq 1} \lambda_k^{-2\rho} e^{-2\lambda_k(s-t)}[|e^{(s-t)A}C^*(s)|_{L(H)}^2 |Qe_k|_H^2 + |e^{(s-t)A}|_{L(V^*;H)}^2 |QC(s)e_k|_H^2]$$
$$\leq e^{-2\lambda_1(t-s)} M_A^2 (|C|_{L^\infty(L(H))}^2 \sum_{k\geq 1} \lambda_k^{-2\rho} |Qe_k|_H^2 + (s-t)^{-2\rho} |Q|_{L_2(H;V^*)}^2 \sum_{k\geq 1} \lambda_k^{-2\rho} |C(s)e_k|_H^2)$$
$$\leq C'(M_A, \lambda_1, |C|_{L^\infty(L(H))}, T)(s-t)^{-2\rho} |Q|_{\mathcal{K}}^2$$

Fix now $Q \in L^2_{\mathcal{P}}(\Omega \times [T-\delta, T]; \mathcal{K}))$ and assume there exists a solution $(\widehat{P}, \widehat{Q}) \in L^2_{\mathcal{P}}(\Omega, C([T-\delta, T]; \mathcal{K})) \times L^2_{\mathcal{P}}(\Omega \times [T-\delta, T]; \mathcal{K}))$ of the mild equation:

$$\widehat{P}(t) = e^{(T-t)A} P_T e^{(T-t)A} + \int_t^T e^{(s-t)A}(C^*(s)Q(s) + Q(s)C(s))e^{(s-t)A}\, ds$$
$$+ \int_t^T e^{(s-t)A} L(s) e^{(s-t)A}\, ds + \int_t^T e^{(s-t)A} \widehat{Q} e^{(s-t)A}\, dW(s)$$

First we deduce the following estimate on P:

$$\mathbb{E}\sup_{t\in[T-\delta,T]}|\widehat{P}(t)|^2_{L(H)} \le C\Big[\mathbb{E}|P_T|^2_{L(H)} + \delta^{1-2\rho}\mathbb{E}\int_{T-\delta}^{T}|Q(s)|^2_{\mathcal{K}}\,ds + \delta^2|L|^2_{L^\infty}\Big]$$

Then we introduce the following dual equation

$$X(t) := \int_{T-\delta}^{t} e^{(s-t)A}G(s)e^{(s-t)A}dW(s)$$

where $G(s)e_k = \lambda_k^{-2\rho}\widehat{Q}(s)e_k, \quad k \ge 1.$

We have

$$\mathbb{E}\sum_{k=1}^{\infty}\lambda_k^{2\rho}|X(t)e_k|^2_H \le \mathbb{E}\int_{T-\delta}^{t}\sum_{k=1}^{\infty}\lambda_k^{-2\rho}|\widehat{Q}(s)e_k|^2_H\,ds$$

So by duality, we obtain the following estimate on $\widehat{Q}$:

$$\frac{1}{4}\mathbb{E}\int_{T-\delta}^{T}|\widehat{Q}(s)|^2_{\mathcal{K}}\,ds \le C\Big[\mathbb{E}|P_T|^2_{L(H)} + \delta^{1-2\rho}\mathbb{E}\int_{T-\delta}^{T}|Q(s)|^2_{\mathcal{K}}\,ds\Big]$$

Scheme of the proof

- Therefore can build a map Γ- using ad hoc approximations- from $L^2_{\mathcal{P}}(\Omega, C([T-\delta, T]; \mathcal{K})) \times L^2_{\mathcal{P}}(\Omega \times [T-\delta, T]; \mathcal{K}))$ into itself:

$$\Gamma(P, Q) := (\widehat{P}, \widehat{Q})$$

and we prove that there is a $\bar{\delta}$ such that Γ is a contraction.

- Global existence and uniqueness then follows easily.

- Typical parabolic regularity: exploit the regularizing property of the semi-group $t \to e^{tA}$ in H.

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