Analysis of a one-sided limit order book model

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Partly based on on-going joint work with **J. Reed** (NYU)

Limit order book

Limit order book: financial tool

- **Allows traders to place orders to be realized in the future**

Limit order (buy order):

- **Trader wants to buy asset at price p**
- **Nobody currently wants to sell at this price**
- **Order stocked in book, fulfilled as market fluctuates**

Market order (sell order)

- **Fulfills largest buy order in book**
- **Book determines price of asset**

Limit order book

×: limit order

Limit order ↔ new point

Market order ↔ rightmost point removed

Limit order book

×: limit order

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Market order ↔ rightmost point removed

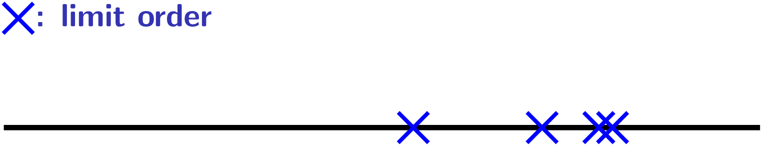
Limit order book

$\times$: limit order

Limit order $\leftrightarrow$ new point

Market order $\leftrightarrow$ rightmost point removed

Limit order book



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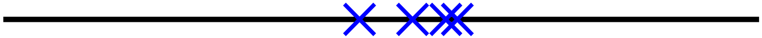

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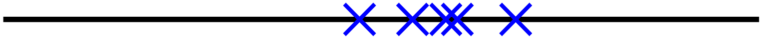

Limit order ↔ new point

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Limit order book

Complicated model

- Two-sided book: limit sell orders/market buy orders
- Cancellations
- Intricate arrival processes
- ...

Focus on one feature:

- State of the book influences arrivals

State of the book influences arrivals

Arrival of new order

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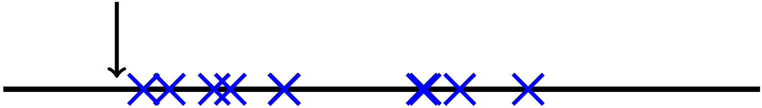
Arrival of new order

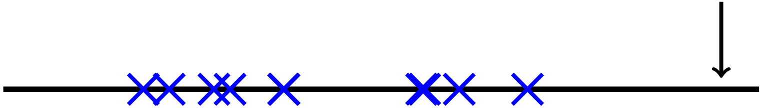


New order far to the left:

- **May never be fulfilled**

State of the book influences arrivals

Arrival of new order



New order far to the right:

- **Pays too much**

State of the book influences arrivals

Arrival of new order

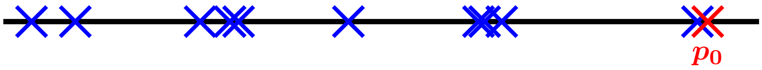
New order placed in the vicinity of **price**

Model

Asymptotic behavior of the price

Scaling limit

Model



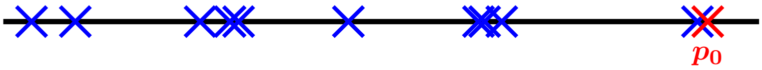
Discrete time Markov chain

- **State space: finite point processes on $\mathbb{R} = (-\infty, \infty)$**

Two parameters

- $\pi \in (0, 1)$: **probability of new order**
- $X \in \mathbb{R}$: **random variable, distance of new order with respect to current price**

Model



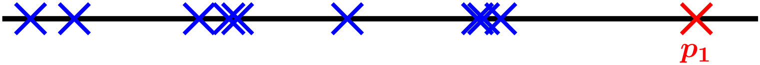
At each time step:

- **Remove rightmost order with probability $1 - \pi$**
- **Add order with probability π at $p + X$**
- **I.i.d. displacements $(X_k, k \geq 0)$**

Boundary condition

- **If book empty, start again with one order at 0**

Model



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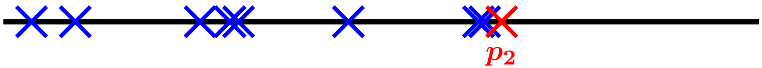

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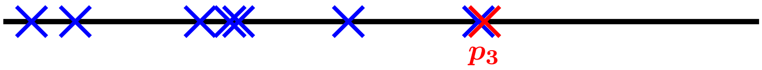

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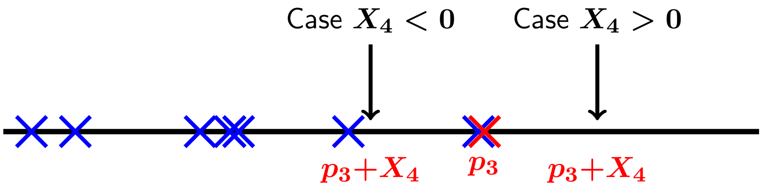

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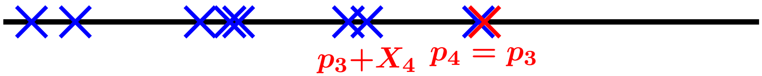

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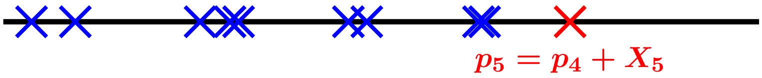

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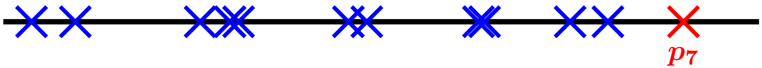

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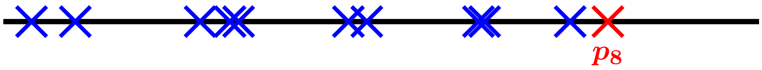

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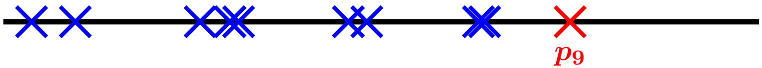

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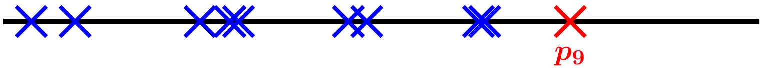
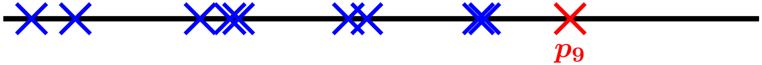


Point process on $\mathbb{R}$

- **Model on $(0, \infty)$ by taking exponential transformation**
- **Multiplicative rather than additive displacement (geometric Brownian motion)**

Model



Total number of orders in the book

- **Random walk reflected at 0**

Asymptotic behavior of the price

Price process

p_k: position of rightmost order at time k

Asymptotic behavior of p_k as $k \to +\infty$?

Assume $\pi > 1/2$

- $\pi \leq 1/2$: $p_k = 0$ infinitely often

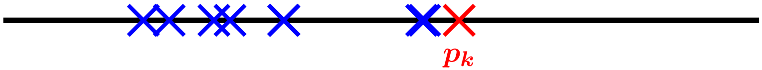
Asymmetric behavior

Price moves freely to the right

- **With probability $\pi\mathbb{P}(X > K)$, jump $> K$**

Price "slowed down" by orders sitting to its left

- **Orders to the left act as a barrier**



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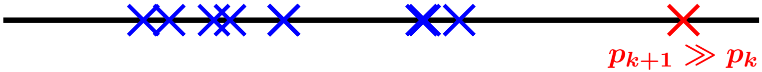

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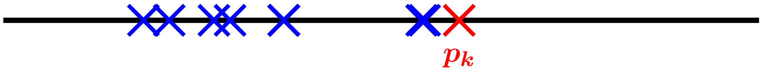

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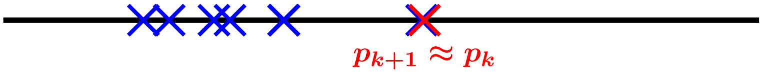

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$p_{k+1} \approx p_k$

$p_k \to +\infty$ for π large enough?

Main result

Theorem

If $\mathbb{E}X > 0$, then $p_k \to +\infty$.

Remarks

- $\mathbb{E}X > 0$: price drifts to the right, no barrier

Main result

Theorem

If $\mathbb{E}X > 0$, then $p_k \to +\infty$.

If $\mathbb{E}X < 0$ and $\mathbb{P}(X > 0) > 0$, then:

- $p_k \to +\infty$ if $\pi > \dfrac{1}{1+a}$;
- $p_k \to -\infty$ if $\pi < \dfrac{1}{1+a}$;

where $a = \inf_{\theta \geq 0} \mathbb{E}(e^{\theta X}) \in (0, 1]$.

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- $\mathbb{E}X > 0$: price drifts to the right, no barrier
- $\mathbb{E}X < 0$: $p_k \to +\infty$ if π large enough (barrier)

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- $\mathbb{E}X > 0$: price drifts to the right, no barrier
- $\mathbb{E}X < 0$: $p_k \to +\infty$ if π large enough (barrier)
- $p_k \to +\infty$ if $\mathbb{E}(e^{\theta X}) = +\infty$ for every $\theta > 0$ (!!)

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Proof: Coupling with branching random walk

Branching random walk

Each step:

- **Each particle removed and replaced by random number of particles**
- **Each new particle at random distance from “parent”**

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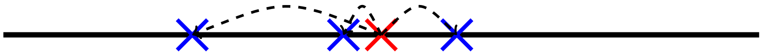

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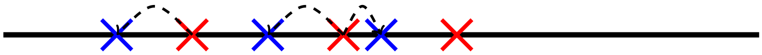

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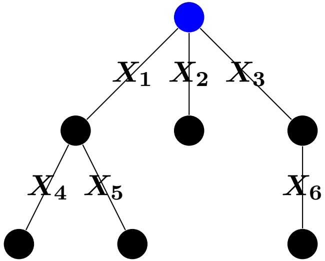
Coupling via tree representation

- **Add genealogy/filiation between particles/orders**

Branching random walk as dynamics on tree

Start from random tree $\mathcal{T}$

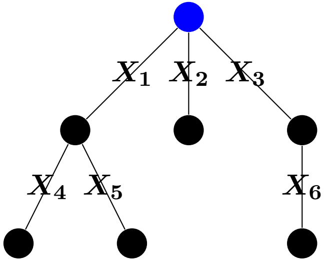
- **Geometric offspring distribution, parameter π**
- **I.i.d. labels $\sim X$**
- **Root is blue, other nodes black**



Branching random walk as dynamics on tree

Tree representation

- : alive
- : not born
- : dead



Representation on the line

Branching random walk as dynamics on tree

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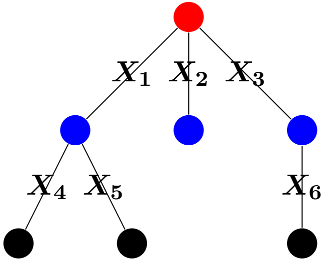

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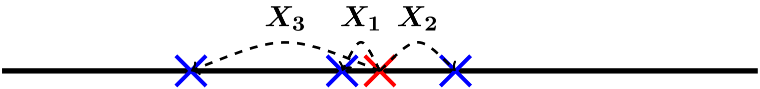

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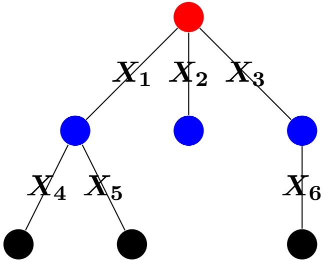

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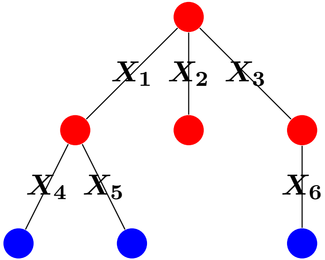

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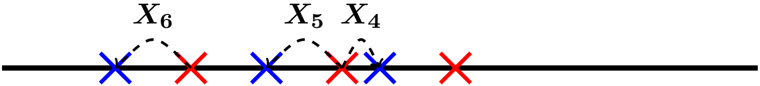

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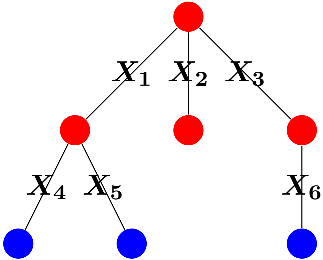


Representation on the line

Limit order book as dynamics on tree – 1/2

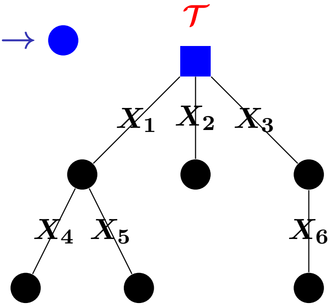
Run deterministic dynamic on $\mathcal{T}$

Only rightmost order "reproduces":

- ■: ● with largest label

If ■ has a black child: first ● → ●

Else, ■ → ●



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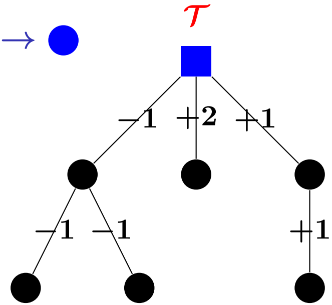

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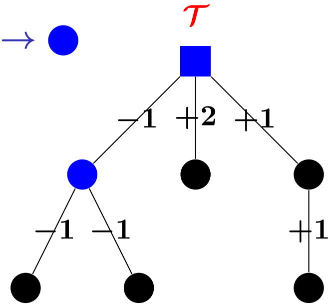

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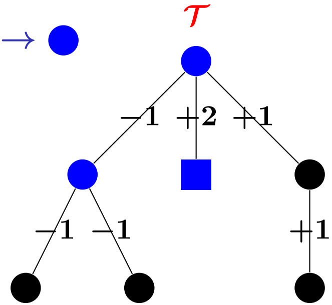

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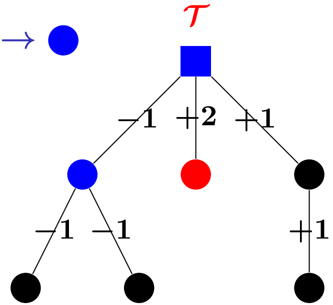

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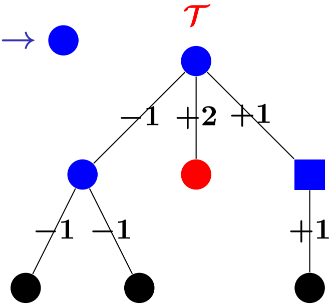

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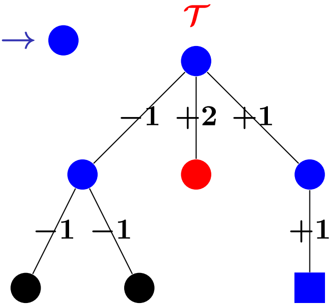

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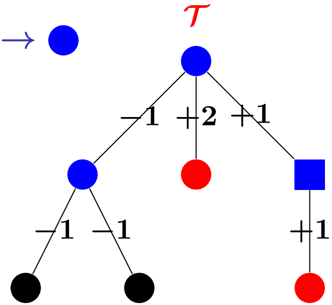

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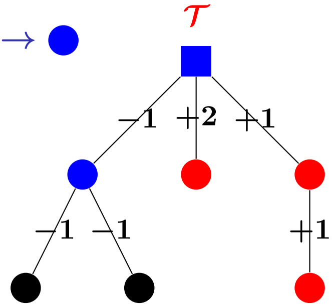


Else, ■ $\to$ ●

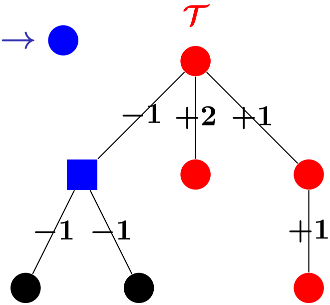
Limit order book as dynamics on tree – 1/2

Run deterministic dynamic on $\mathcal{T}$

Only rightmost order “reproduces”:

- ■: ● **with largest label**

If ■ has a black child: first ● $\rightarrow$ ●



Else, ■ $\rightarrow$ ●

Limit order book as dynamics on tree – 1/2

Run deterministic dynamic on $\mathcal{T}$

Only rightmost order "reproduces":

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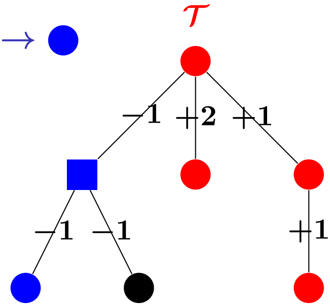
If ■ has a black child: first ● $\rightarrow$ ●

Else, ■ $\rightarrow$ ●



Limit order book as dynamics on tree – 1/2

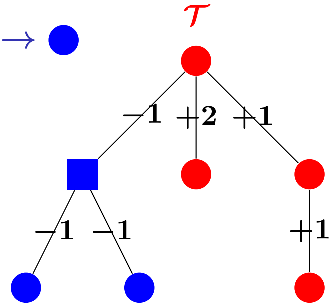
Run deterministic dynamic on $\mathcal{T}$

Only rightmost order “reproduces”:

- ■: ● **with largest label**

If ■ has a black child: first ● $\rightarrow$ ●

Else, ■ $\rightarrow$ ●



Limit order book as dynamics on tree – 2/2

Γ_k: labels of **blue nodes** at time k

Theorem

$(\Gamma_k, k \geq 0)$ is a realization of the limit order book process.

Limit order book as dynamics on tree – idea of proof

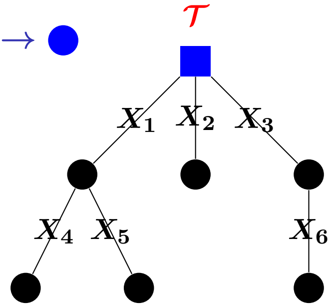
Run deterministic dynamic on $\mathcal{T}$

Only rightmost order "reproduces":

- ■: ● with largest label

If ■ has a black child: first ● $\to$ ●

Else, ■ $\to$ ●



Limit order book as dynamics on tree – idea of proof

Run deterministic dynamic on $\mathcal{T}$

Only rightmost order "reproduces":

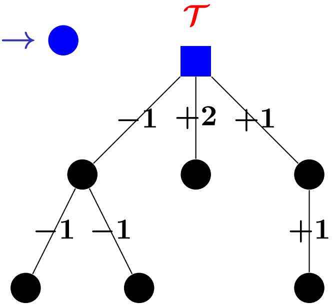
- ■: ● with largest label

If ■ has a black child: first ● $\to$ ●

- Probability π
- Displacement $\sim X$
- Limit order

Else, ■ $\to$ ●

- Probability $1 - \pi$
- Market order



Limit order book as dynamics on tree – idea of proof

Run deterministic dynamic on $\mathcal{T}$

Only rightmost order “reproduces”:

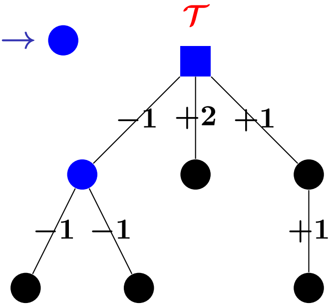
- ■ (blue): ● (blue) with largest label

If ■ (blue) has a black child: first ● (black) $\to$ ● (blue)

- Probability π
- Displacement $\sim X$
- Limit order

Else, ■ (blue) $\to$ ● (red)

- Probability $1 - \pi$
- Market order



Limit order book as dynamics on tree – idea of proof

Run deterministic dynamic on $\mathcal{T}$

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If ■ has a black child: first ● $\to$ ●

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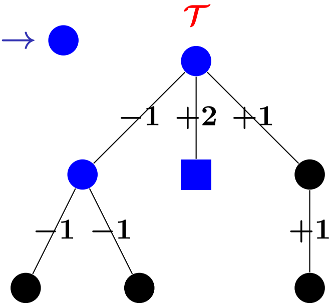

Else, ■ $\to$ ●

- Probability $1-\pi$
- Market order

Limit order book as dynamics on tree – idea of proof

Run deterministic dynamic on $\mathcal{T}$

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If ■ has a black child: first ● $\to$ ●

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Else, ■ $\to$ ●

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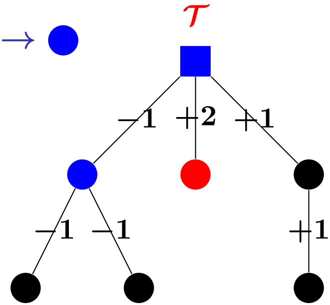

Limit order book as dynamics on tree – idea of proof

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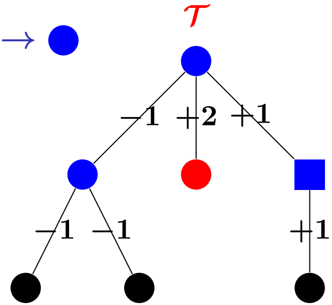

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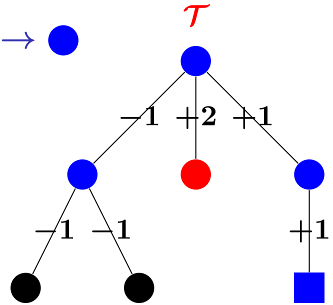

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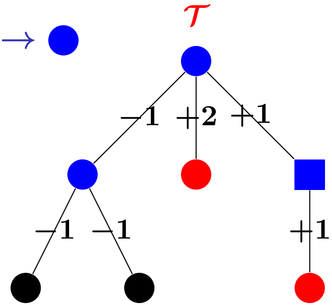

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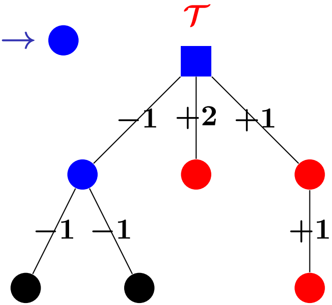

Limit order book as dynamics on tree – idea of proof

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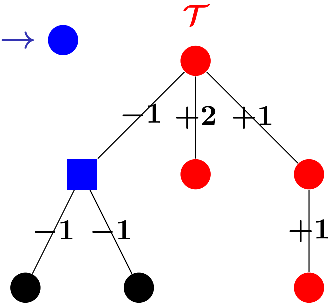

Limit order book as dynamics on tree – idea of proof

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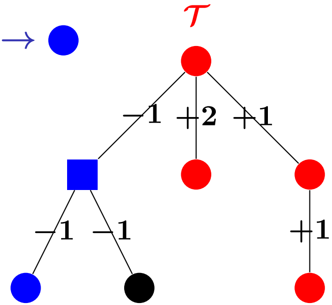

Limit order book as dynamics on tree – idea of proof

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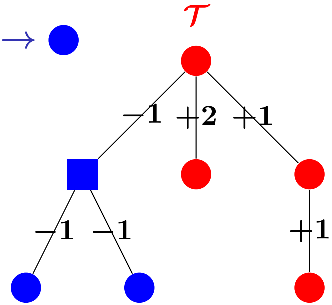
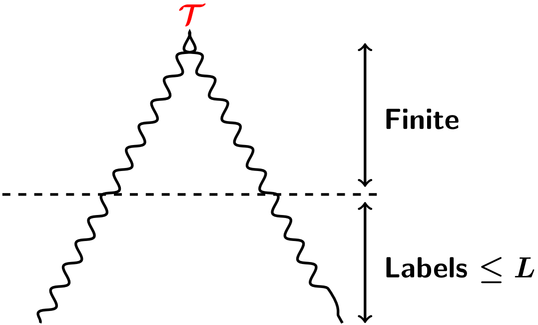


Case $p_k \to -\infty$: proof

Assume $\mathbb{E}X < 0$, $\mathbb{P}(X > 0) > 0$ and $\pi < 1/(1+a)$

Proof of $p_k \to -\infty$

- M_k: position of rightmost particle in BRW
- Well-known: $M_k \to -\infty$
- Fix some L: then $p_k \le L$ for k large enough



Case $p_k \to +\infty$: proof

Assume:

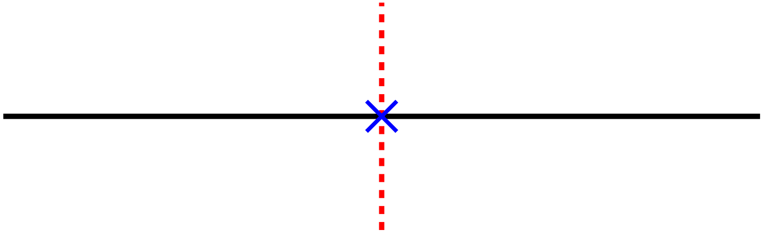
- $\mathbb{E}X < 0$, $\mathbb{P}(X > 0) > 0$ and $\pi > 1/(1 + a)$
- or $\mathbb{E}X > 0$

Goal: prove $p_k \to +\infty$

Case $p_k \to +\infty$: proof

Key observation

- Initial order at 0
- Event {Initial order stays forever in the book} independent of what happens in $(-\infty, 0)$
- Orders placed in $(-\infty, 0)$ may as well be instantaneously removed



Case $p_k \to +\infty$: proof

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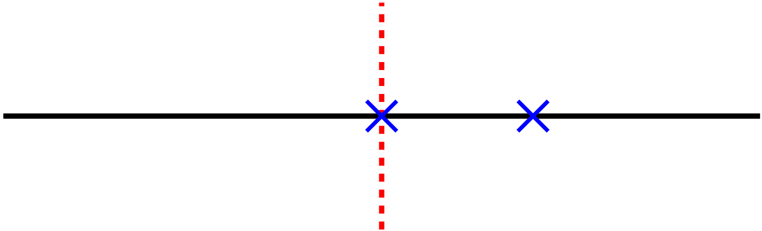

Case $p_k \to +\infty$: proof

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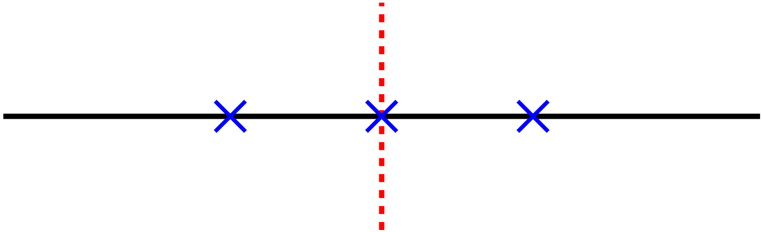
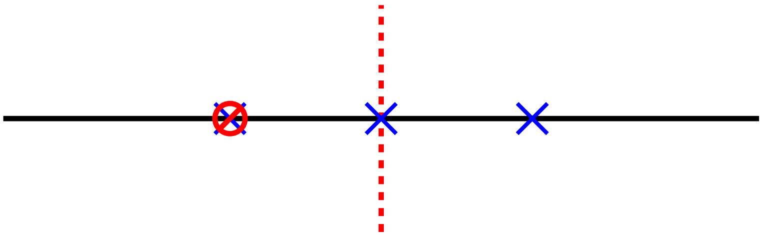


Case $p_k \to +\infty$: proof

Key observation

- **Initial order at 0**
- **Event {Initial order stays forever in the book} independent of what happens in $(-\infty, 0)$**
- **Orders placed in $(-\infty, 0)$ may as well be instantaneously removed**



Case $p_k \to +\infty$: proof

Key observation

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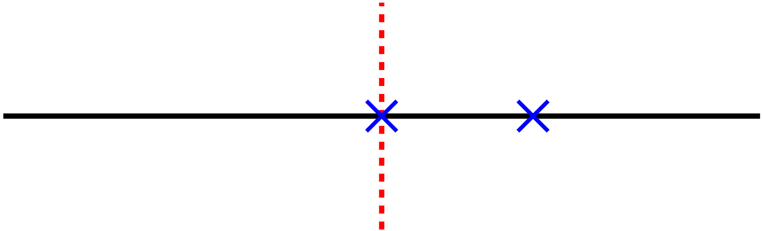


Case $p_k \to +\infty$: proof

Consequence

- $\mathcal{T}'$: remove from $\mathcal{T}$ all nodes with label in $(-\infty, 0)$ and their descendants

Initial order stays forever in the book

$\Updownarrow$

$\mathcal{T}'$ is infinite

$\Updownarrow$

Branching random walk with barrier at 0 survives

Theorem (BRW with a barrier)

$q = \mathbb{P}(\mathcal{T}' \text{ is infinite}) > 0$.

Case $p_k \to +\infty$: proof

Proof of $p_k \to +\infty$:

- **Each time, probability $q > 0$ that new order stays forever**
- **Renewal sequence that pushes the price to $+\infty$**

Scaling limit

Set-up

$(B(k), k \geq 0)$: critical limit order book

- Displacement distribution X: $\mathbb{E}X > 0$
- $\pi = \dfrac{1}{2}$: total number of orders = critical random walk

Renormalize B as follows:

$$\widehat{B}_n(t)([a,b]) = \frac{1}{n} B(n^2 t)([na, nb])$$

- Scale mass by n^{-1}, time by n^2 and space by n
- $(\widehat{p}_n(t), t \geq 0)$: renormalization of price process $(p(k))$

Different boundary condition:

- Always an order at 0

Conjecture

Conjecture

$(\widehat{B}_n, \widehat{p}_n) \Rightarrow (\widehat{B}, \widehat{p})$ as $n \to +\infty$, where $\widehat{p}$ is a reflected Brownian motion with variance $\mathbb{E}X$ and $\widehat{B}$ is Lebesgue measure on $[0, \widehat{p}]$:

$$\widehat{B}(t)(A) = \frac{1}{\mathbb{E}X} \int_0^{\widehat{p}(t)} \mathbb{1}_{\{x \in A\}} dx.$$

Proof

- Tightness + identification
- Tightness of $\widehat{B}_n$ "easy": martingale arguments
- If $\widehat{B}_n \Rightarrow \widehat{B}$, then $\widehat{p}_n = \sup \mathbf{supp}(\widehat{B}) \Rightarrow \sup \mathbf{supp}(\widehat{B}) = \widehat{p}$ (continuous mapping)

Proof strategy 1/3: continuous mapping

$B = \Phi(\mathcal{T})$: after scaling, $\widehat{B}_n = \Phi(\widehat{\mathcal{T}}_n)$

Well-known: $\widehat{\mathcal{T}}_n \Rightarrow \widehat{\mathcal{T}}$

- Genealogical structure: continuous random tree
- Labels of nodes: Brownian snake

$\widehat{B}_n \Rightarrow \Phi(\widehat{\mathcal{T}})$: meaning?

- Can be done for branching random walk (?)

Proof strategy 2/3: Laplace transform

"Classical" approach for superprocesses

Control convergence of $\mathbb{E}\left[\exp\left(\langle \widehat{B}_n(t), f\rangle\right)\right]$

- **If** $\widehat{B}_n \Rightarrow \widehat{B}$**, then**

$$\mathbb{E}\left(e^{\langle \widehat{B}(t), f\rangle}\right) = \mathbb{E}\left(e^{\langle \widehat{B}(0), f\rangle}\right) + \int_0^t \mathbb{E}\left[e^{\langle \widehat{B}(u), f\rangle}\left(f(\widehat{p}(u))^2 - \mathbb{E}X f'(\widehat{p}(u))\right)\right] du$$

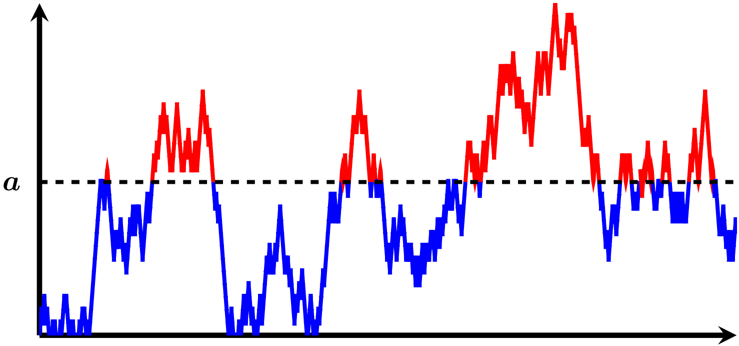
  with $\widehat{p} = \sup \textbf{supp}(\widehat{B})$
- $\widehat{B} = [0, \textbf{RBM}]$ **solves this**
- **Uniqueness?**

Proof strategy 3/3: regenerative trees

Assume $X \in \{1, 0, -1, -2, \ldots\}$

Key observation (same as before)

- **Excursions of $(p(k), k \geq 0)$ above level $a > 0$ are i.i.d.**



Proof strategy 3/3: regenerative trees

$\widehat{p}_n \Rightarrow \widehat{p}$: satisfies the same property and is continuous

Theorem (Weill'07)

$\widehat{p}$ codes a Lévy tree (scaling limit of Galton Watson tree).

Identify reflected Brownian motion through its length