The circular telegraph process

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The perstitent walk on $\mathbb{Z}/N\mathbb{Z}$

$(V_n)_{n\geq 0}$ markov chain on $\{\pm 1\}$ with transition matrix

$$\begin{pmatrix} \frac{1+\alpha}{2} & \frac{1-\alpha}{2} \\ \frac{1-\alpha}{2} & \frac{1+\alpha}{2} \end{pmatrix}$$

$(X_n)_{n\geq 0}$ second order markov chain on $\mathbb{Z}/N\mathbb{Z}$ with

$$X_{n+1} = X_n + V_n$$

Proposition

The maximal rate of convergence to equilibrium is obtain for

$$\alpha_{opt}^{(N)} = \frac{1 - \sin(\pi/N)}{1 + \sin(\pi/N)}$$

$$\mathbb{P}_{opt}(V_{n+1} = -V_n) \simeq \frac{\pi}{N}$$

The circular telegraph process

If $N_t \sim Poisson(a)$, let (X_t, V_t) be markov process on $\mathbb{T} \times \{\pm 1\}$ defined by

$$\begin{aligned} dX_t &= V_t dt \\ V_t &= (-1)^{N_t} \end{aligned}$$

or by its generator on $L^2(\mu)$ (μ being the uniform measure on $\mathbb{T} \times \{\pm 1\}$)

$$\begin{aligned} L_a f(x,v) &= v\partial_x f(x,v) + a(f(x,-v) - f(x,v)) \\ &= \overbrace{Tf(x,v)} + \overbrace{Kf(x,v)} \end{aligned}$$

and its semi-group P_t^a. We denote $f_t = P_t f$ and

$$\begin{aligned} F_t &= \|f_t - \mu f\|^2_{L^2(\mu)} \\ \|P_t - \mu\| &= \sup_{\|f\|=1} \sqrt{F_t} \end{aligned}$$

There's no randomness on the position variable :

$$\begin{aligned} F'_t &= 2 < Kf_t, f_t > \\ &= 0 \qquad \text{if } f_t \in KerK \end{aligned}$$

and no coercivity from the deterministic part of the process. Nevertheless,

Proposition

Let $\lambda_a = -\sup\{Re(\nu), \nu \in \sigma(L) \setminus \{0\}\}$. Then

$$\begin{aligned} \lambda_a &= a - \sqrt{(a^2-1)_+} \\ \|P_t - \mu\| &= g_a(t) e^{-\lambda_a t} \end{aligned}$$

with g_a an explicit prefactor such that, in particular,

$$||P_t - \mu|| = 1 - \frac{\min(a,1)}{3} t^3 + \underset{t \to 0}{o}(t^3)$$

L^2 norm of the semi-group

The $W_n = \text{span}(x, v \mapsto e^{inx})$ are stable orthogonal plans, whose span space is dense in $L^2(\mu)$. Thus

$$\|P_t - \mu\|_{L^2(\mu)} = \sup_{n\geq 0}\|P_t - \mu\|_{W_n}.$$

In dimension 2, everything is computable :

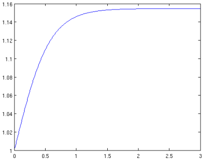
- $\sigma(L_{|W_n}) = \{-(a \pm \sqrt{a^2 - n^2})\}$
- Let f_+, f_- be respective eigenfunctions. Then

$$\begin{aligned} |< f_+, f_- >| &= \min\{| < g_+, g_- > | \text{ with } g_\pm \text{ eigenfunctions of } M \\ &\qquad \text{with } \|e^{tM}\| \text{ non-increasing and } \sigma(L_{|W_n}) = \sigma(M)\} \end{aligned}$$

- $\|P_t - \mu\|_{W_n} = g_n(t)e^{-t(a-\sqrt{(a^2-n^2)_+})}$

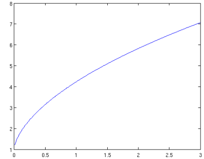
Behaviour of the prefactor

$a > n$ real spectrum



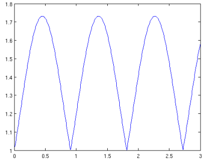
$$g_n(t) \longrightarrow \frac{\left(\frac{a}{n}\right)^2}{\left(\frac{a}{n}\right)^2 - 1}$$

$a = n$ Jordan block



$$g_n(t) \sim 2nt$$

$a < n$ complex spectrum



$$g_n(t) \in \left[1, \frac{1 + \frac{a}{n}}{1 - \frac{a}{n}}\right]$$

$$\nu = |\lambda_+ - \lambda_-|$$

Some details

- The renormalised process (X_n^N, V_n^N) converges in law to (X_t, V_t) but

$$\|P_n^N - \mu_N\| \underset{N\to\infty}{\longrightarrow} \sup_{n\in\frac{1}{2}\mathbb{N}} \{g_n(t)e^{-\lambda_n t}\} \neq \sup_{n\in\mathbb{N}}\{g_n(t)e^{-\lambda_n t}\} = \|P_t - \mu\|$$

- Moreover

$$\mathcal{L}_{opt}(V_N) \underset{N\to\infty}{\longrightarrow} \mathcal{E}(\frac{1}{2}) \neq \mathcal{E}(1) = \mathcal{L}_{opt}(V_t)$$

- Mostly $\|P_t - \mu\| = \|P_t - \mu\|_{W_1}$, but not necessarily if $a < 1$.
- If λ is the exponential rate of convergence on W_n then
  - if $a \geq n$, $F_t' + \lambda F_t$ is always non-negative.
  - if $a < n$, $F_t' + \lambda F_t = A\cos(\nu_n t + \phi)$.

3-order differential (in)equation

Now $f \in L^2(\mu)$ is fixed with $\mu f = 0$, and $F_t = \|f_t\|^2$. Then

$$\begin{aligned}
\partial_t F_t &= \; < (2K)f_t, f_t > \\
\partial_t^2 F_t &= \; < \left((2K)^2 + 2[K,T]\right) f_t, f_t > \\
\partial_t^3 F_t &= \; < \left((2K)^3 + 6[K^2,T] + 2\big[[K,T],T]\big]\right) f_t, f_t > .
\end{aligned}$$

In particular for $f_t \in KerK$ then $F_t' = 0$ but

$$\partial_t^3 F_t = -4a\|\partial_x f_t\|^2 \leq -4aF_t$$

In fact, for $f \in W_n$,

$$\left((\partial_t + 2a)^2 + 4(n^2 - a^2)\right)(\partial_t + 2a)F_t = 0$$

Nothing like $F_t''' + AF_t'' + BF_t' + CF_t \leq 0$ can be true for all f.

Let $h_t = e^{2at} F_t$ then either

- $h_t' \leq 0 \quad \longrightarrow$ exponential decay.
- Else $h_t''' + 4(1-a^2)h_t' \leq h_t''' + 4(n^2 - a^2)h_t' = 0 \quad \longrightarrow$ uniform w.r.t. n.

The inequality can be written

$$Q_3(\partial t)F_t + Q_1(\partial t)G_t \leq 0$$

with Q_i polynom of degree i and $G_t = \|\partial_x f_t\|^2$.

Proposition

Suppose now a depends on the position x, with $a(x) \geq a_* > 0$. Then one can still find Q_1 and Q_3 such that

$$Q_3(\partial t)F_t + Q_1(\partial t)G_t \leq 0$$

for all $f \in L^2(\mu)$.

From the inequality to exponential decay

G_t brings the coercivity missing in F'_t :

$$\begin{aligned} G_t - F'_t &= < (\partial_x^* \partial_x - 2K)\, f_t, f_t > \\ &:= k_t F_t \end{aligned}$$

with $k_t \geq k_* > 0$. Now we have

$$\tilde{Q}_3(\partial t) F_t + \tilde{Q}_1(\partial t)(k_t F_t) \leq 0$$

And, if $\tilde{Q}_1 | \tilde{Q}_3$ (which can be done),

$$(\partial_t + \lambda)\left((\partial_t + \eta)^2 + \nu_t\right) F_t \leq 0$$

With some $\nu_t \geq \nu_*$. By the Gronwall lemma

$$\left((\partial_t + \eta)^2 + \nu_*\right) F_t \leq \left((\partial_t + \eta)^2 + \nu_t\right) F_t \leq M_0 e^{-\lambda t}$$

Let ϕ be the solution of

$$\begin{cases} \Big[(\partial_t + \eta)^2 + \nu_*\Big]\phi = M_0 e^{-\lambda t} \\ \\ \phi(0) = F(0),\ \phi'(0) = F'(0). \end{cases}$$

so that $h_t = e^{\eta t}\left(F_t - \phi(t)\right)$ satisfy $h_t'' + \nu_* h_t \leq 0$. Then :

- Either $\nu_* \leq 0$ and applying twice the Gronwall lemma gives $h_t \leq 0$.
- Either $\nu_* > 0$; then while $h_t' > 0$

$$\frac{d}{dt}\left((h_t')^2 + \nu_* h_t^2\right) \leq 0$$

  so that $h_t \leq -\min_{s \leq t} h_s$. Moreover, by the Sturm theorem, $|\{h_t > 0\}| \leq \frac{\pi}{\nu_*}$: F and ϕ are interlaced.

To sum up :

The study of K, T, etc. gives $\lambda > 0$, η and a function $\nu_t \geq \nu_*$ with $\mathcal{R}e\left(\eta - \sqrt{\nu_*}\right) > 0$ s.t.

$$(\partial_t + \lambda)\left((\partial_t + \eta)^2 + \nu_t\right) F_t \leq 0$$

and this implies

- if $\nu_* \leq 0$, $F_t \leq \phi(t)$
- if $\nu_* > 0$, $F_t \leq \phi(t) + e^{-\eta t} \sup_{s \leq t} e^{\eta s}\left(\phi_s - F_s\right)$; furthermore the length of a time interval where $F > \phi$ is less than $\frac{\pi}{\nu_*}$

with in both cases ϕ solution of

$$(\partial_t + \lambda)\left[(\partial_t + \eta)^2 + \nu_*\right]\phi(t) = 0,$$

$\phi(0) = F_0$, $\phi'(0) = F_0'$ and $\phi''(0)$ explicit.

Some remarks :

- In particular,

$$\lim_{t\to\infty} \frac{1}{t} \ln F_t \leq -\min\left(\lambda, \mathcal{R}e(\eta - \sqrt{-\nu_*})\right) < 0$$

- As ϕ_0'' depends on F_0'' and G_0, this does not give directly a bound for $\|P_t - \mu\|$.
- F_t is the real L^2 norm of f_t, but G_t is needed to close the inequality.
- The method works with V_t an Ornstein-Uhlenbeck process, in other words with

$$L = v\partial_x - \partial_v + \partial_v^2$$

or with the kinetic Fokker-Planck dynamics (provided U'' is bounded)

$$L = v\partial_x + U'(x)\partial_v - v\partial_v + \partial_v^2$$

Application to other models ?

- If a depends on x and v the invariant measure is $e^{-U(x)}dx \otimes U_{\{\pm 1\}}$ where
$$a(x,1) - a(x,-1) = U'(x).$$
Can we still apply the same method ?
- Same question for the next-to-be presented ergodic variant of the telegraph process with generator
$$Lf = v\partial_x f + \left(a\mathbf{1}_{v=sign(x)} + b\mathbf{1}_{v=-sign(x)} \right) \left(f(x,-v) - f(x,v) \right)$$
with $b > a$. Problem : what is $\partial_x \left(a\mathbf{1}_{v=sign(x)} + b\mathbf{1}_{v=-sign(x)} \right)$?
- For some switching kinetic processes we also have
$$f_t \in KerK \Rightarrow F_t''' \leq -CF_t.$$

Thank you for your attention.

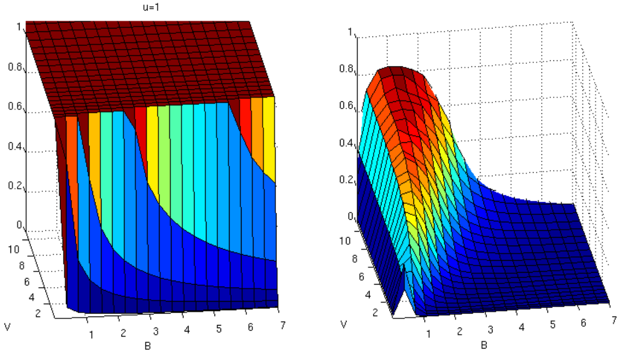


FIGURE: Theorical and numerical rate of relaxation for the kinetic Fokker-Planck