Long time behavior of an ergodic variant of the telegraph process with jump rates depending on the position

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Workshop : Piecewise Deterministic Markov Processes

Rennes, May 2013.

1. Motivation : study of a toy model of bacterial chemotaxis

Chemotaxis is the phenomenon in which cells or bacteria direct their movements according to certain chemicals in their environment. This is important for bacteria to find food (for example, glucose) by swimming towards the highest concentration of food molecules.

Generally, the motion of flagellated bacteria consists of a sequence of straight *run* phases and *tumble* phases.

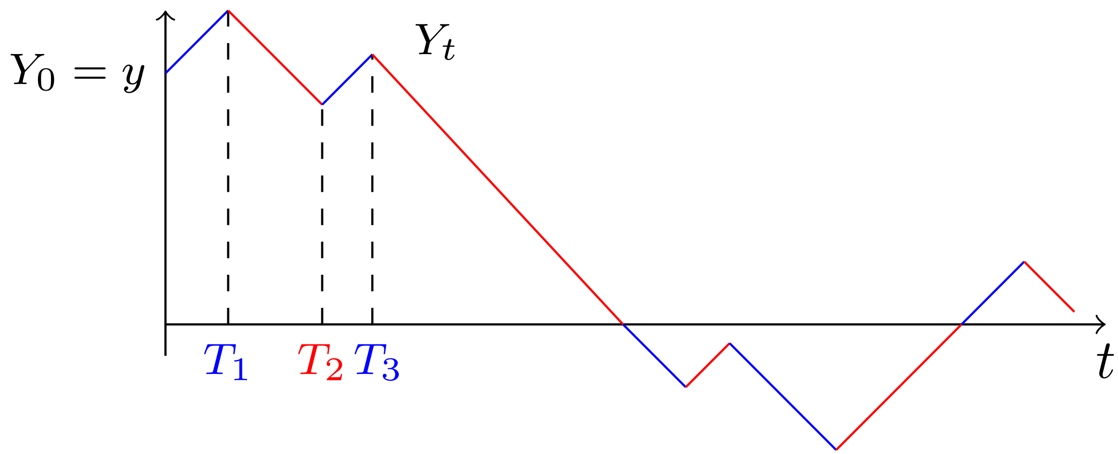
Othmer and Erban (2004) gave a first and complex model of this phenomenon, recently extended by Rousset and Samaev (2010).

Our aim : Study the long time behavior of a bacterium in a modified model

2. The model

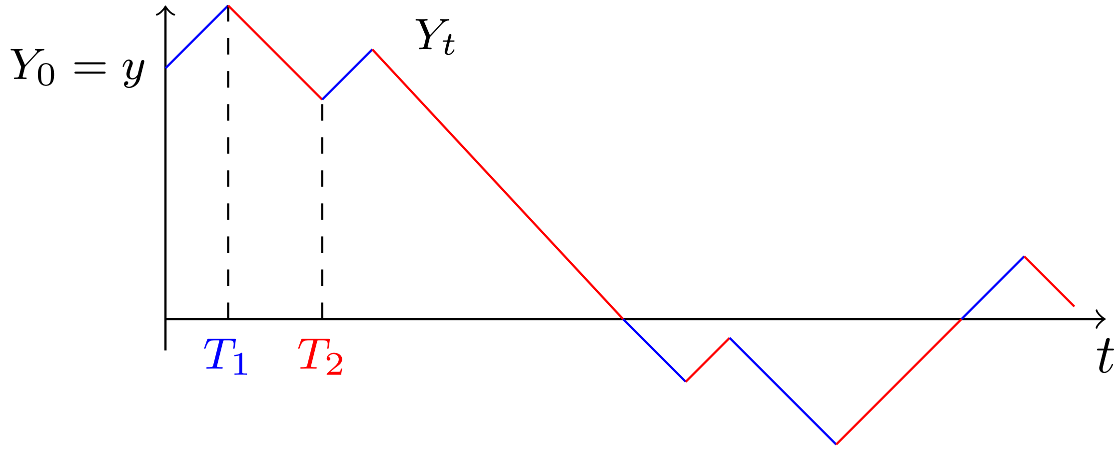
Let us imagine that food is located in 0.

Let Y_t be the position on the real line of a bacterium at time t, and $W_t = \dfrac{dY_t}{dt}$ its velocity.



Assume that $W_t \in \{-1, 1\}$. As long as $Y_t > 0$ the jump rate of W_t is equal to $b(Y_t)$ when $W = 1$ and is $a(Y_t)$ when $W = -1$. The situation is reversed when Y_t is negative.

The jump rates depend on the distance between the bacterium and the food, on the velocity and on the sign of the position.



Assumptions

- a is a non increasing function with $0 < \underline{a} \leq a(x) \leq a(0)$,
- b is a non decreasing function with $0 < \underline{b} \leq b(x)$,
- $0 < a(|y|) < b(|y|)$ for all $y \neq 0$.

We want to find the invariant measure of the ergodic Markov process (Y, W) and estimate the total variation distance of two processes starting from different initial data.

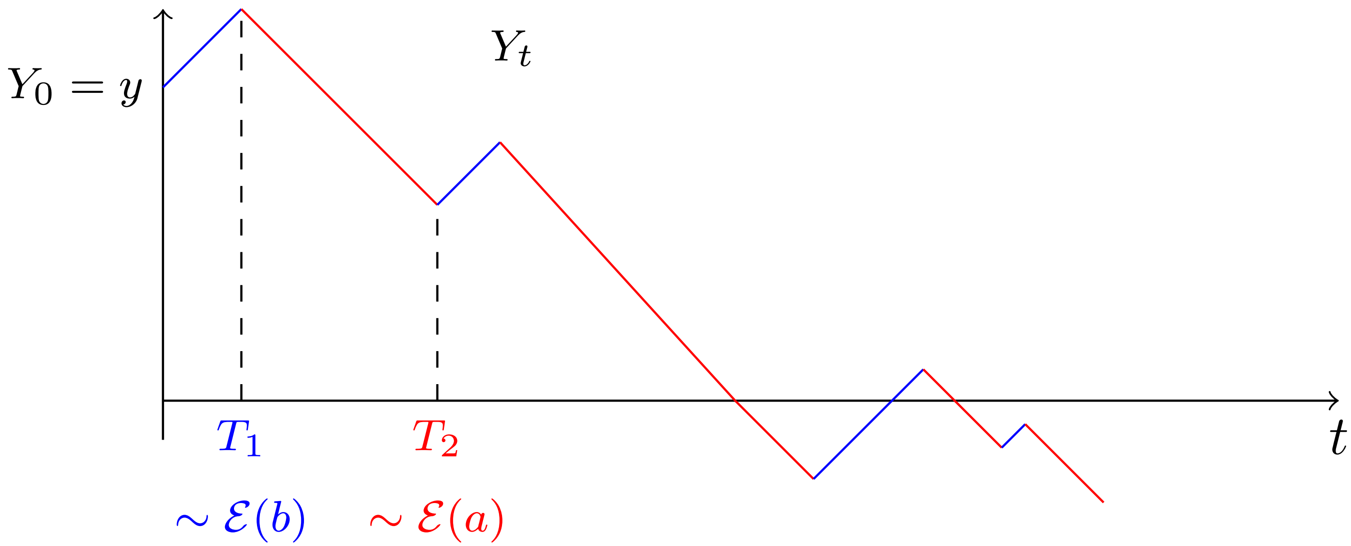
2. Some related processes

When $a = b$ are constant functions, we recognize the telegraph process introduced by Kac (1974), in which case the density of Y_t solves the damped wave equation

$$\frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial x^2} + a\frac{\partial p}{\partial t} = 0.$$

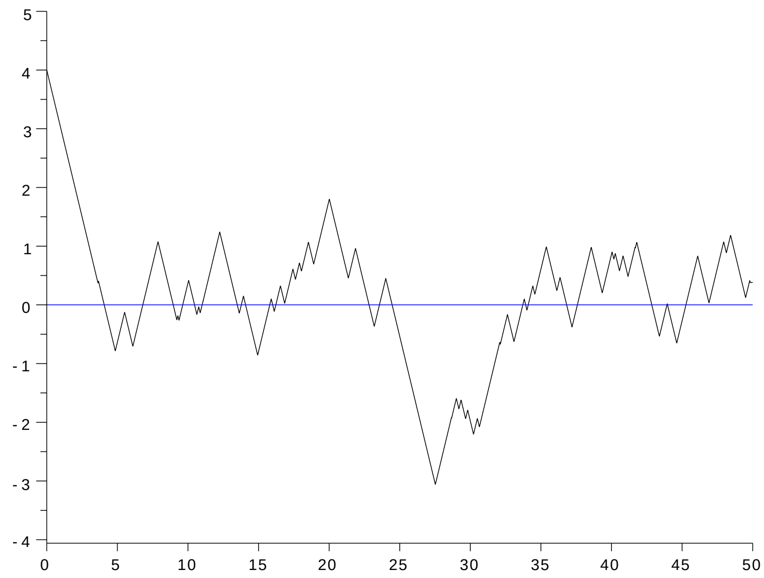
In this case, it is well known that $(Y_t)_{t\geq 0}$ converges to a Brownian motion in a suitable scaling limit.

A process with two different jump rates, but without dependence on the position, has been well studied by Herrmann and Vallois (2010).



They have computed the distribution of Y_t at each time in terms of modified Bessel functions.

When $a(|y|) < b(|y|)$, the bacterium spends in principle more time moving towards the origin than away from it. So a macroscopic attraction to the origin should then be expected in the long run.



Trajectory of $(Y_t)_{t\geq 0}$

3. The invariant measure

Theorem 1 *The invariant probability measure μ of (Y, W) is the product measure on $\mathbb{R} \times \{-1, +1\}$ given by*

$$\mu(dy, dw) = \frac{1}{C} e^{F(y)} dy \otimes \frac{1}{2}(\delta_{-1} + \delta_{+1})(dw).$$

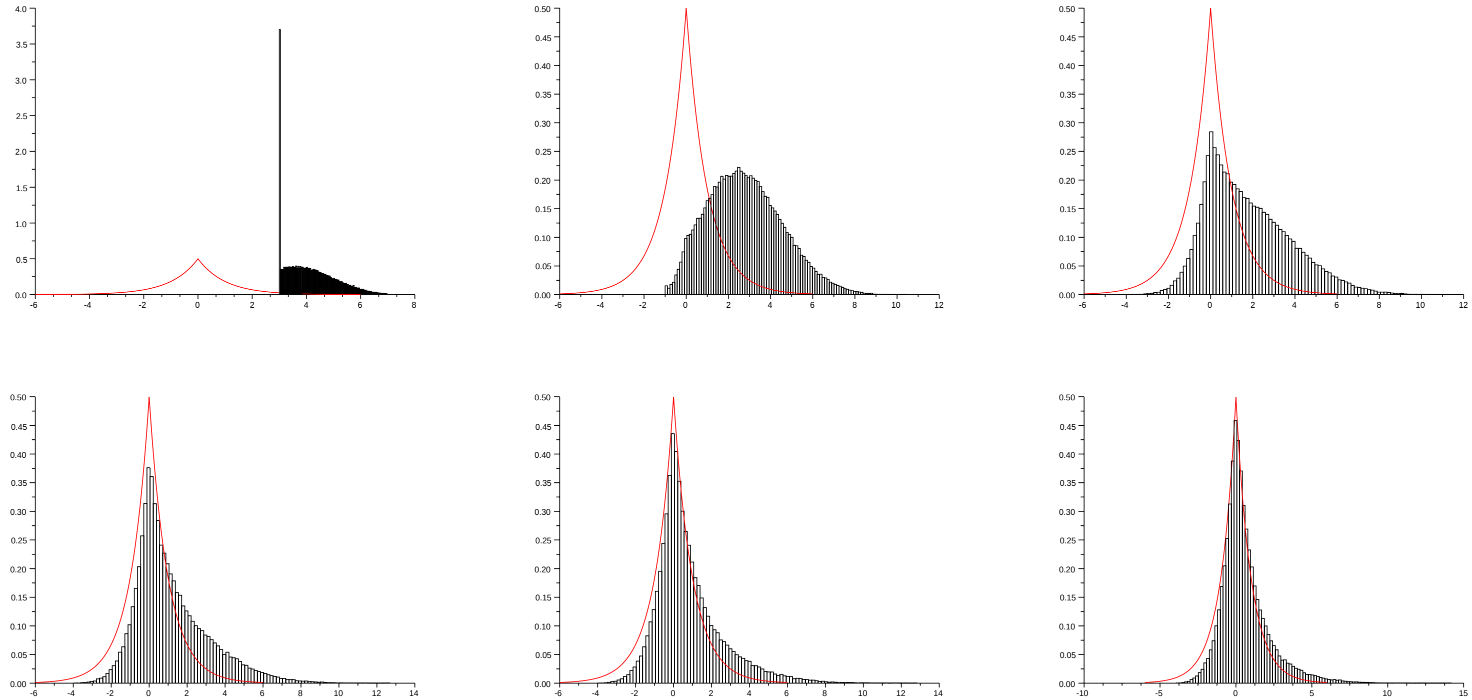
with $F(y)$ a primitive function of $f(y) = -sgn(y)(b(|y|) - a(|y|))$ and $C = \int_{\mathbb{R}} e^{F(y)} dy$.

Some examples

- Constant jump rates $0 < a < b$, then the invariant measure is

$$\mu(dy, dw) = \frac{b-a}{2} e^{-(b-a)|y|} dy \otimes \frac{1}{2}(\delta_{-1} + \delta_{+1})(dw).$$

Empirical distribution of Y_t starting from $(5, -1)$ for $t \in \{2, 6, 10, 14, 18, 22\}$ with $a = 1$ and $b = 2$.



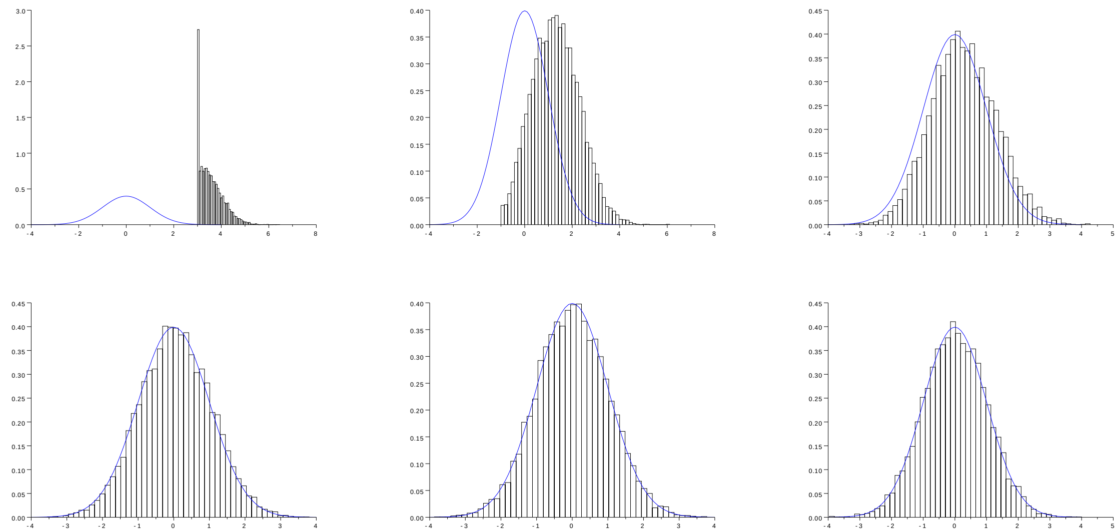
In a suitable scaling limit the process $(Y_t)_{t \geq 0}$ converges to the solution of the SDE

$$dZ_t = dB_t - c\,\mathrm{sgn}(Z_t)dt.$$

• $a > 0$ is a constant function and $b(|y|) = a + |y|$, then the invariant measure is

$$\mu(dy, dw) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \otimes \frac{1}{2}(\delta_{-1} + \delta_{+1})(dw).$$

Empirical distribution of Y_t starting from $(5, -1)$ for $t \in \{2, 6, 10, 14, 18, 22\}$ with $a = 1$.



In a suitable scaling limit the process $(Y_t)_{t\geq 0}$ converges to the solution of the SDE: $dZ_t = dB_t - \frac{1}{2} Z_t dt$.

4. Convergence to the equilibrium

Let $\mu_t^{y,w}$ the law of $Z_t = (Y_t, W_t)$ starting from $Z_0 = (y, w)$.

Our aim : Estimate for any $y, \tilde{y} \in \mathbb{R}$ and $w, \tilde{w} \in \{-1, +1\}$

$$\left\| \mu_t^{y,w} - \mu_t^{\tilde{y},\tilde{w}} \right\|_{\mathrm{TV}}$$

where $||\nu - \tilde{\nu}||_{\mathrm{TV}} = \inf\{\mathbb{P}(U \neq \tilde{U}) : U, \tilde{U}$ random variables

with $\mathcal{L}(U) = \nu$ and $\mathcal{L}(\tilde{U}) = \tilde{\nu}\}$.

A coalescent coupling $(U_t, \tilde{U}_t)_{t \geq 0}$ of two stochastic processes is such that $U_{t+T_*} = \tilde{U}_{t+T_*}$ for any $t \geq 0$ and T_* is called a coupling time. It follows in this case that

$$\left\| \mathcal{L}(U_t) - \mathcal{L}(\tilde{U}_t) \right\|_{\mathrm{TV}} \leq \mathbb{P}(T_* > t).$$

Consequently, we have to estimate the distribution of the coupling time.

We notice that for any nonnegative random variable T such that $T_* \leq_{\text{sto.}} T$, and any $\lambda \geq 0$ in the domain of the Laplace transform of T,

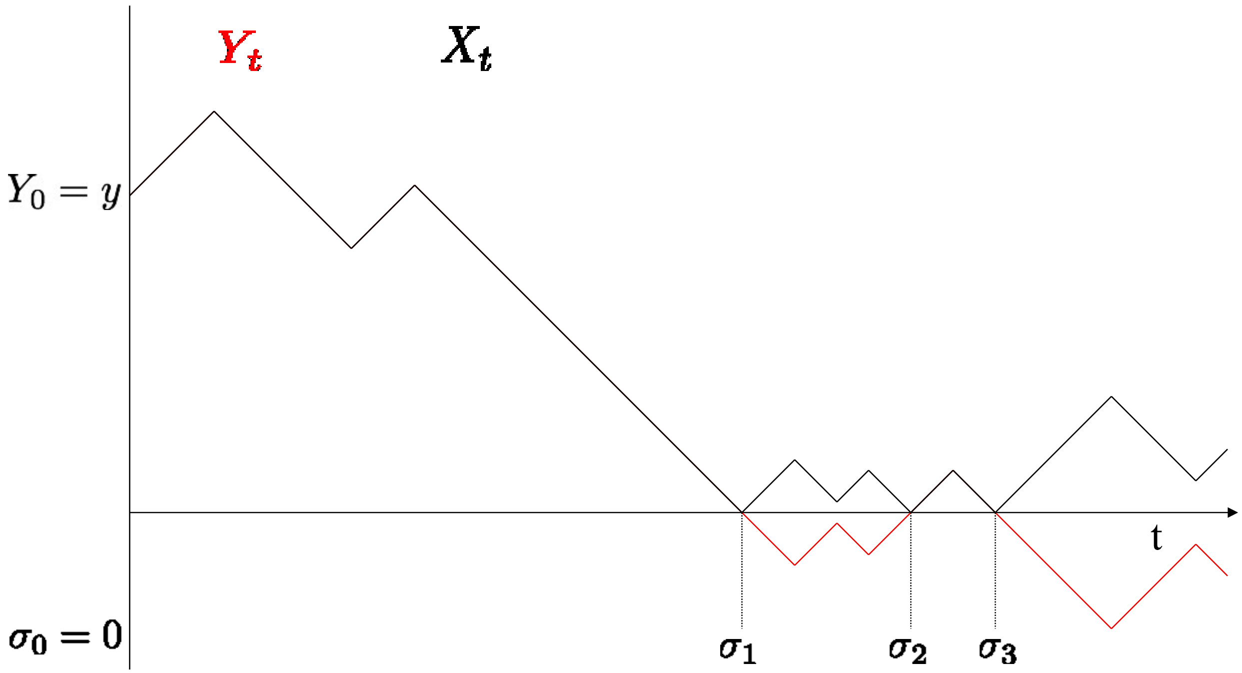
$$\begin{aligned} \left\| \mathcal{L}(U_t) - \mathcal{L}(\tilde{U}_t) \right\|_{\text{TV}} &\leq \mathbb{P}(T_* > t) \\ &\leq \mathbb{P}(T > t) \leq \mathbb{E}\left[e^{\lambda T}\right] e^{-\lambda t} \end{aligned}$$

by Chernoff's inequality.

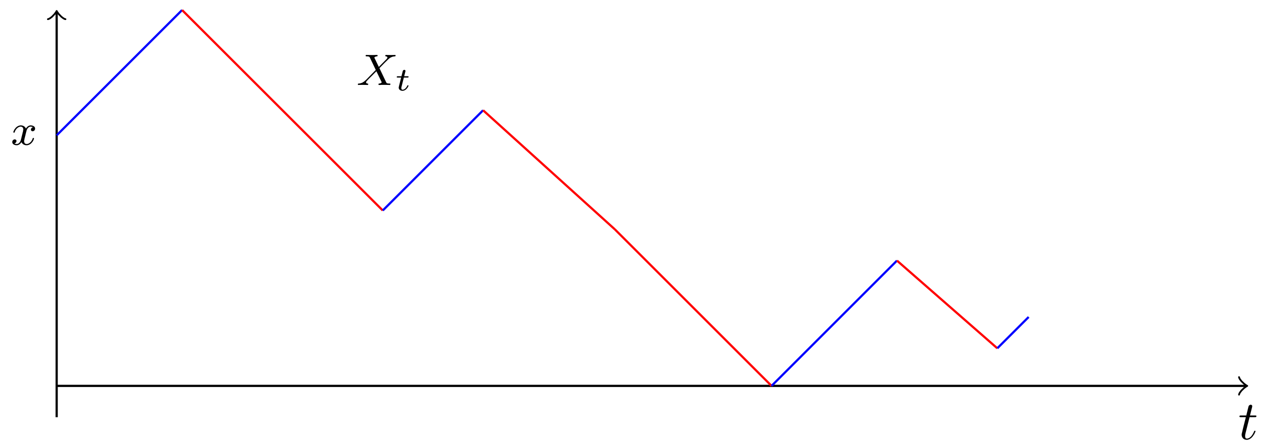
The main difficulty of our model is to deal with the dependence between the jump rate of the velocity W and the position Y.

The reflected process (X_t, V_t) with $X_t = |Y_t|$ and

$$\begin{cases} V_0 = \text{sgn}(Y_0) W_0 \\ \{t > 0 \,:\, \Delta V_t \neq 0\} = \{t > 0 \,:\, \Delta W_t \neq 0\} \cup \{t > 0 \,:\, Y_t = 0\}. \end{cases}$$



Notice that $(Y_t, W_t) = (-1)^i \text{sgn}(y)(X_t, V_t)$ fot $t \in [\sigma_i, \sigma_{i+1}]$.



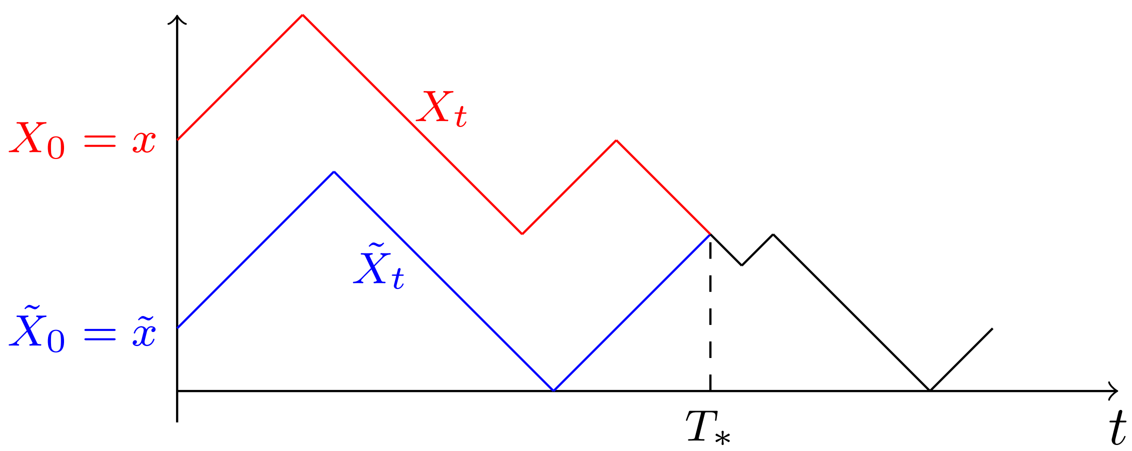
The infinitesimal generator of the process (X,V) is

$$Af(x,v) = v\partial_x f(x,v) + \left(a(x)\mathbb{1}_{\{v=-1\}} + b(x)\mathbb{1}_{\{v=1\}} + \frac{\mathbb{1}_{\{x=0\}}}{\mathbb{1}_{\{x>0\}}} \right) (f(x,-v) - f(x,v)).$$

5. Ideas of the proof

We consider two processes starting from different initial data $x, \tilde{x} \geq 0$ and $v, \tilde{v} \in \{-1, +1\}$.

We need to couple positions and velocities at the same time.



a) Properties of the jump times

Starting from (x, v), the first jump time $T_{(x,v)}$ with rate δ satisfies

$$T_{(x,v)} = \inf\left\{t \geq 0 : \int_0^t \delta(x + vs) \geq E\right\}$$

with E is an exponential variable with unit mean and

where $\delta = b$ if $v = +1$ and $\delta = a$ if $v = -1$.

Consequently,

$$T_{(x,+1)} = B^{-1}(E + B(x)) - x$$

$$T_{(x,-1)} = \big(x - A^{-1}(A(x) - E)\big) \wedge x$$

where A is the primitive function of a with $A(0) = 0$,
and B is the primitive function of b with $B(0) = 0$.

Let $0 < \tilde{x} < x$. We consider the same exponential variable to construct the jump times, then

$$T_{(x,+1)} \leq T_{(\tilde{x},+1)} \quad \text{and} \quad T_{(x,-1)} \geq T_{(\tilde{x},-1)}.$$

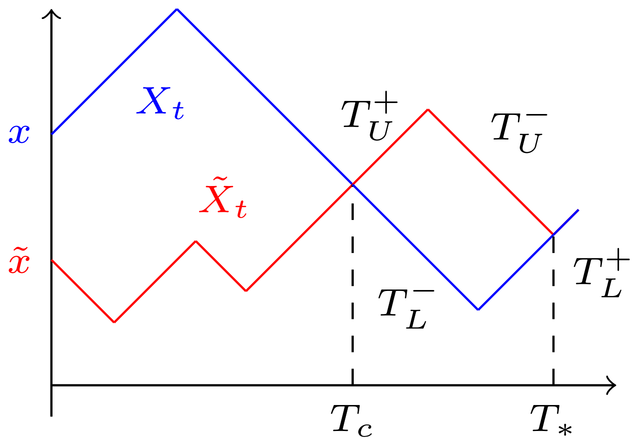
We also notice that

$$T_{(x,+1)} \overset{sto.}{\leq} E(b(x)) \quad \text{and} \quad E(a(x)) \wedge x \overset{sto.}{\leq} T_{(x,-1)} \overset{sto.}{\leq} E(\underline{a}) \wedge x.$$

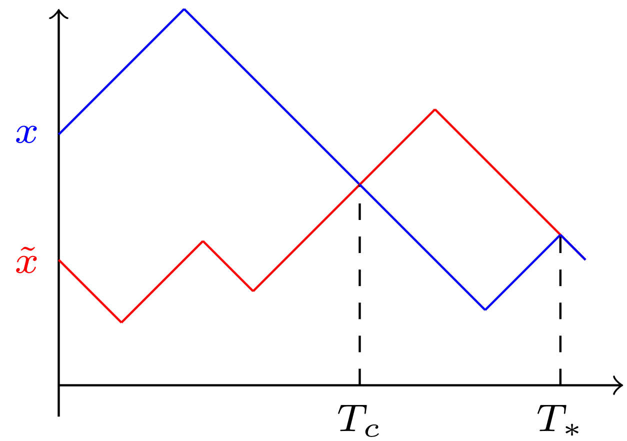
Consequently, the Laplace transform of $T_{(x,+1)}$ is controlled by the Laplace transform of $E(b(x))$ and the one of $T_{(x,-1)}$ is defined on $\mathbb{R}^+$.

Remark If $b(x) \underset{x \to +\infty}{\longrightarrow} +\infty$, then the Laplace transform of $T_{(x,+1)}$ is defined everywhere.

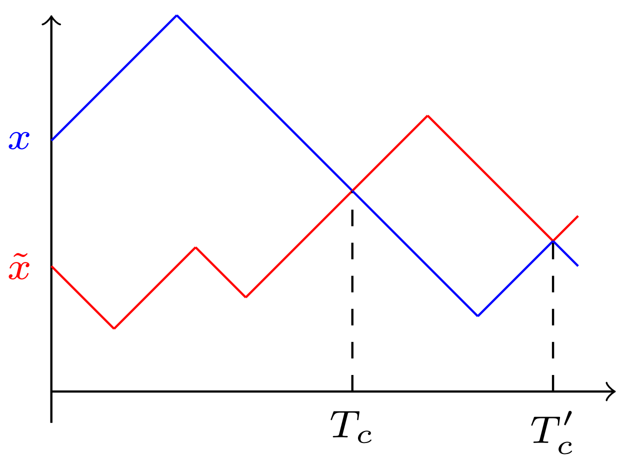
b) A way to stick both paths



$T_U^+ < T_L^+$ and $T_U^- = T_L^-$



$T_U^+ = T_L^+$ and $T_U^- > T_L^-$



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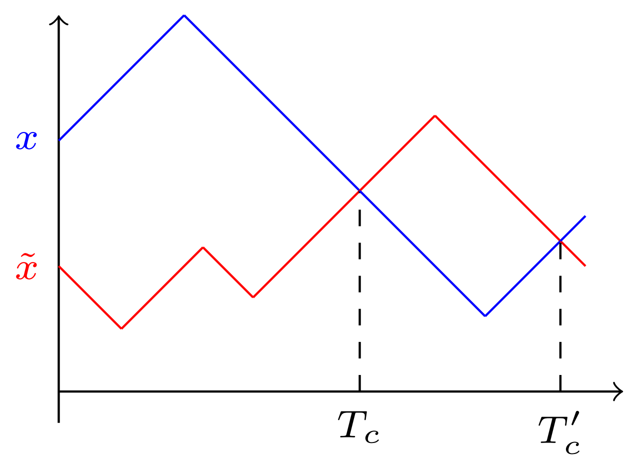


$T_U^+ < T_L^+$ and $T_U^- > T_L^-$

Let us recall that

$$\left\|\mu_t^{x,v} - \mu_t^{\tilde{x},\tilde{v}}\right\|_{\mathrm{TV}} \leq \mathbb{P}(T_* > t) \leq e^{-\lambda t}\mathbb{E}\left[e^{\lambda T}\right],$$

for any $T \geq T_*$.

The coupling time of two paths starting respectively from (x, v) and $(\tilde{x}, \tilde{v})$ with $x > \tilde{x}$ is

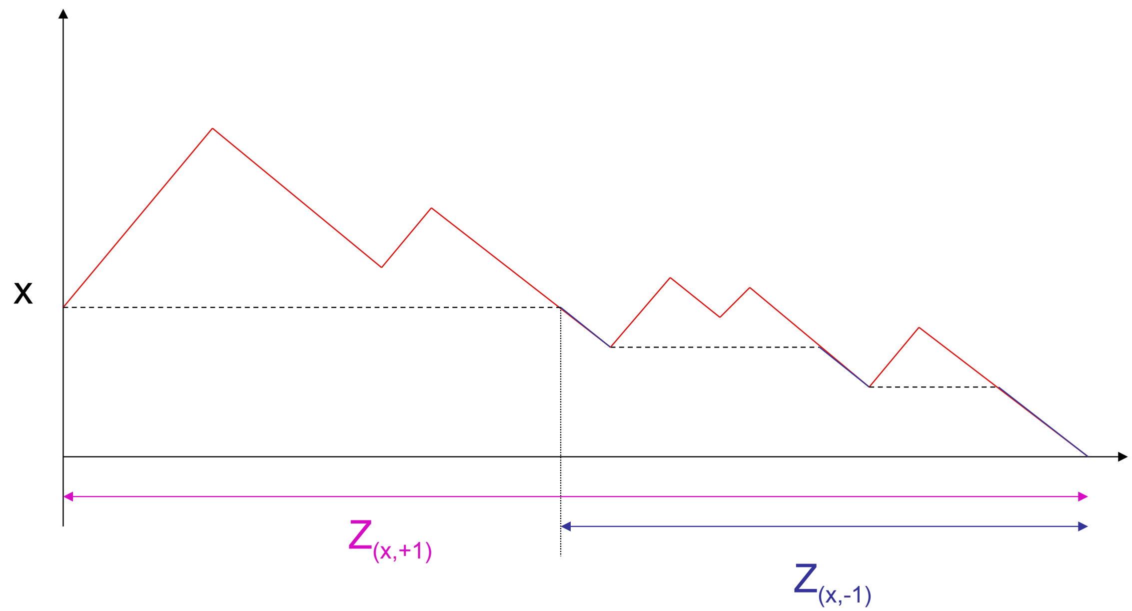
$$T_* = T_c + \text{ the spent time to stick both paths.}$$

We notice that $T_c \leq R_{(x,v)}$ where $R_{(x,v)}$ is the first hitting time of zero of the upper path.

c) Estimation of the Laplace transform of the first hitting time of zero

We consider $(X_0, V_0) = (x, v)$ and we denote by $R_{(x,v)}$ the first hitting time of zero of a path.

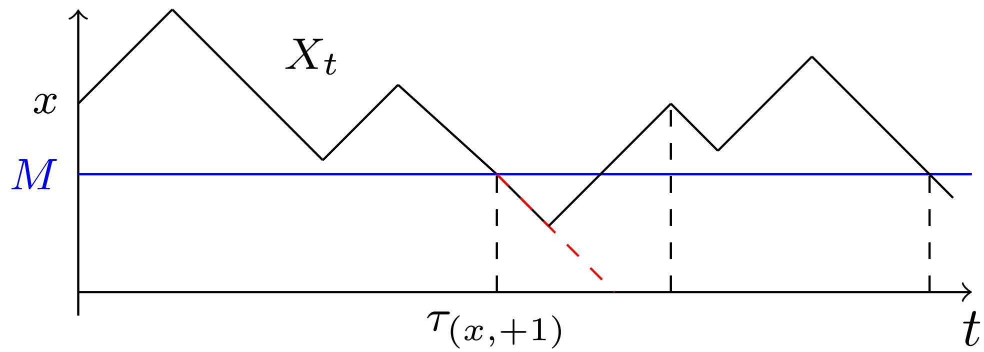
When a and b are constant functions,



$R_{x,+1} = S + x + \sum_{i=0}^{N} S^i$, with $N \sim \mathcal{P}ois(ax)$.

Let $X_0 = x$ and $V_0 = v$. Let $M > 0$ be fixed.

We introduce $\tau_{(x,v)} = \inf\{t \geq 0 : X_t = M\}$.



From $(x, v) = (M, -1)$ we can reach 0 in one step with probability

$$\mathbb{P}(T_{(M,-1)} > M) = e^{-A(M)}.$$

Then,

$$\begin{aligned} R_{x,+1} &= \tau_{(x,+1)} + R_{(M,-1)} \\ &\leq \tau_{(x+1)} + 2M + T_{(M,+1)} + \tau_{\left(M+T_{(M,+1)},+1\right)} + \dots \end{aligned}$$

Let G be a geometric variable with parameter

$$\mathbb{P}\big(T_{(M,-1)} > M\big) = e^{-A(M)}$$

independent of $\tau_{(x,+1)}$ which corresponds to the number of trials to hit zero in one step starting from M. There exists a sequence $(T^k)_{k\geq 1}$ of independent variables with the same distribution of $T_{(M,+1)}$, independent of $\tau_{(x,+1)}$ and of G such that

$$R_{(x,+1)} \leq \tau_{(x,+1)} + M + \sum_{k=1}^{G-1} \big(2M + T^k + \tau^k_{(M+T^k,-1)}\big)$$

where $\left(\tau^k_{(M+T^k,-1)}\right)_{k\geq 1}$ is a sequence of independent hitting times of the level M, independent of $\tau_{(x,+1)}$ and of G.

Since $a(x) < b(x)$, there exists $\rho = \rho(M) > 0$ such that for $f(x,v) = e^{\alpha x + \beta v}$, with $\alpha, \beta > 0$, we have

$$Af(x,v) \leq -\rho f(x,v) + c\mathbb{1}_{x \leq M}$$

and then

$$\mathbb{E}[e^{\rho \tau_{(x,v)}}] \leq e^{\alpha(x-M)+\beta(v+1)}.$$

Consequently, we have to choose M such that $\rho(M)$ and $e^{-A(M)}$ are big enough and then we have an estimation of the Laplace Transform of the hitting time of zero starting from (x,v).

Example When $a < b$ are constant functions, we have

$$\mathbb{E}\big(e^{\lambda R_{(x,-1)}}\big) = \begin{cases} e^{xc(\lambda)} & \text{if} \quad \lambda \leq \lambda_c \\ +\infty & \text{else} \end{cases}$$

with $c(\lambda) = \frac{b-a-\sqrt{(a+b-2\lambda)^2-4ab}}{2}$.

Consequently,

$$||\mu_t^{x,v} - \mu_t^{\tilde{x},\tilde{v}}||_{\text{TV}} \leq \frac{(a+b)b}{2a^2} e^{r(a,b)(x\vee\tilde{x})} e^{-\lambda_c t},$$

with $r(a,b) = \frac{3(b-a)}{4} \vee (b - \sqrt{ab})$ and $\lambda_c = \frac{(\sqrt{b}-\sqrt{a})^2}{2}$.

What's next ?

• Consider this process in higher dimension

• Consider more than one chemoattractant

• Consider the same model than Othmer and Erban and take into account the internal variable which solves an ODE.