Thermodynamics of Piecewise Deterministic Markov Processes

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Motivations

- **Biophysics**: modeling of molecular motors
- **Statistical physics**: the continuous variable in PDMPs follows a non–Markovian stochastic dynamics, on which to test **Macroscopic Fluctuation Theory**.

Macroscopic Fluctuation Theory: developed by Bertini, De Sole, Gabrielli, Jona-Lasinio, Landim; Bodineau, Derrida et al. It concerns large deviations in (Markov) stochastic systems and establishes relations of physical relevance for large deviations (as fluctuation–dissipation relations, Gallavotti-Cohen typed symmetries, generalized Onsager–Machlup symmetry)

Some targets

- characterize **stationary distribution**
- investigate **time–reversed process** under stationarity
- derive **averaging principle**, **static** and **dynamical large deviations**
- restricting to the continuous variable, check the validity of the **fluctuation–dissipation relation**, **Gallavotti–Cohen typed symmetry**

1. (FGR) *Non–equilibrium thermodynamics of piecewise deterministic Markov process.* J. Stat. Phys. (2009)
2. (FGR) *Averaging and Large Deviation Principles for fully–coupled PDMPs ...* MPRF (2010)
3. (FG) *A representation formula for the Freidlin-Wentzell quasipotential on the one dimensional torus.* AIHP (2012)

PDMPs (random switching fields)

State $(\mathbf{x}, \sigma) \in \mathbf{\Omega} \times \mathbf{\Gamma}$,
$\mathbf{\Omega} \subset \mathbb{R}^{\mathbf{d}}$ or $\mathbf{\Omega} = \mathbb{T}^{\mathbf{d}}$, $|\mathbf{\Gamma}| < +\infty$
Molecular motor: x **mechanical state**, σ **chemical state**
$\lambda > \mathbf{0}$ parameter
Markov generator:

$$\mathbf{Lf}(\mathbf{x}, \sigma) = \mathbf{F}_\sigma(\mathbf{x}) \cdot \nabla \mathbf{f}(\mathbf{x}, \sigma) + \lambda \sum_{\sigma' \in \mathbf{\Gamma}} \mathbf{r}(\sigma, \sigma' | \mathbf{x}) \left(\mathbf{f}(\mathbf{x}, \sigma') - \mathbf{f}(\mathbf{x}, \sigma) \right)$$
$$= \mathbf{F}_\sigma(\mathbf{x}) \cdot \nabla \mathbf{f}(\mathbf{x}, \sigma) + \lambda \mathbf{L_c}[\mathbf{x}] \mathbf{f}(\mathbf{x}, \sigma).$$

$F_\sigma(x) =$ vector fields, $r(\sigma, \sigma' | x) =$ rates of jumps.
Assumptions: (i) mechanical confinement, (ii) no explosion, (iii) $\mathbf{L_c}[\mathbf{x}]$ **is irreducible**

High jump frequency limit: averaging principle

$\mu(\sigma|\mathbf{x})$: invariant distribution for $L_c[x]$, quasi–stationary measure

$\bar{\mathbf{F}}(\mathbf{x}) = \sum_\sigma \mu(\sigma|\mathbf{x})\mathbf{F}_\sigma(\mathbf{x})$, averaged vector field

$\mathbb{P}^\lambda_{\mathbf{x_0},\sigma_\mathbf{0}}$ law of PDMP starting at (x_0, σ_0)

Theorem

Given $x_0 \in \Omega$ *let*

$$\dot{x}_\infty(t) = \bar{F}(x_\infty(t)), \qquad x_\infty(0) = x_0 .$$

Then, as $\lambda \to \infty$, *in* $\mathbb{P}^\lambda_{x_0,\sigma_0}$ *probability it holds:*

$$\begin{cases} x(t) \to x_\infty(t) \ \textit{in } C([0,T];\Omega),\ \|\cdot\|_\infty \\ \mathbb{1}(\sigma(t) = \sigma)dt \to \mu(\sigma|x_\infty(t))dt \ \textit{weakly, for all } \sigma \in \Gamma . \end{cases}$$

High jump frequency limit: LDP

Given $\sigma \in \mathbf{\Gamma}$, $\{\sigma(\mathbf{t})\}_{\mathbf{t}\in[\mathbf{0},\mathbf{T}]} \curvearrowright \mathbb{1}(\sigma(t)=\sigma)dt$
Space: $\mathbf{C}([\mathbf{0},\mathbf{T}];\mathbf{\Omega}) \times \mathcal{M}([\mathbf{0},\mathbf{T}])^{\mathbf{\Gamma}}$

Process concentrated on compact subspace $\mathcal{Y}_{\mathbf{x_0}}$:
$\mathbf{x}(\cdot) \in \mathbf{C}([\mathbf{0},\mathbf{T}];\mathbf{\Omega})$, $(\nu_\sigma)_{\sigma\in\mathbf{\Gamma}} \in \mathcal{M}([\mathbf{0},\mathbf{T}])^{\mathbf{\Gamma}}$
such that

1. $\nu_\sigma = \chi_\sigma(\mathbf{t})\mathbf{dt}$
2. $\sum_{\sigma\in\mathbf{\Gamma}} \chi_\sigma(\mathbf{t}) = \mathbf{1}$ a.e.
3. $\mathbf{x}(\mathbf{t}) = \mathbf{x_0} + \int_{\mathbf{0}}^{\mathbf{t}} \sum_{\sigma\in\mathbf{\Gamma}} \chi_\sigma(\mathbf{s})\mathbf{F}_\sigma(\mathbf{x}(\mathbf{s}))\,\mathbf{ds}$

From now on we restrict to $\mathcal{Y}_{\mathbf{x_0}}$; $\chi(\mathbf{t}) = \big(\chi_\sigma(\mathbf{t})\big)_{\sigma\in\mathbf{\Gamma}}$

Theorem

As $\lambda \to \infty$,

$$\mathbb{P}^{\lambda}_{\mathbf{x_0},\sigma_\mathbf{0}}\Big(\big\{\mathbf{x}(\mathbf{t}), \mathbb{1}(\sigma(\mathbf{t}) = \sigma)\mathbf{dt}\big\}_{\mathbf{t}\in[\mathbf{0},\mathbf{T}]\sigma\in\mathbf{\Gamma}} \approx \big\{\hat{\mathbf{x}}(\mathbf{t}), \hat{\chi}(\mathbf{t})\mathbf{dt}\big\}_{\mathbf{t}\in[\mathbf{0},\mathbf{T}]}\Big)$$

$$\sim \mathbf{e}^{-\lambda \mathbf{J}_{[\mathbf{0},\mathbf{T}]}\left(\left\{\hat{\mathbf{x}}(\mathbf{t}),\hat{\chi}(\mathbf{t})\mathbf{dt}\right\}_{\mathbf{t}\in[\mathbf{0},\mathbf{T}]}\right)}$$

$\mathbf{J}_{[\mathbf{0},\mathbf{T}]}$ lower semi-continuous, $\not\equiv \infty$, compact level set.
For each $\mathcal{A}$ open, $\mathcal{C}$ close:

$$\liminf_{\lambda\uparrow\infty} \frac{\mathbf{1}}{\lambda} \ln \mathbf{P}^{\lambda}_{\mathbf{x_0},\sigma_\mathbf{0}}\Big(\big\{\mathbf{x}(\mathbf{t}), \mathbb{1}(\sigma(\mathbf{t}) = \sigma)\mathbf{dt}\big\}_{\mathbf{t}\in[\mathbf{0},\mathbf{T}]\sigma\in\mathbf{\Gamma}} \in \mathcal{A}\Big) \geq -\inf_{\mathcal{A}} \mathbf{J}$$

$$\limsup_{\lambda\uparrow\infty} \frac{\mathbf{1}}{\lambda} \ln \mathbf{P}^{\lambda}_{\mathbf{x_0},\sigma_\mathbf{0}}\Big(\big\{\mathbf{x}(\mathbf{t}), \mathbb{1}(\sigma(\mathbf{t}) = \sigma)\mathbf{dt}\big\}_{\mathbf{t}\in[\mathbf{0},\mathbf{T}]\sigma\in\mathbf{\Gamma}} \in \mathcal{C}\Big) \leq -\inf_{\mathcal{C}} \mathbf{J}$$

Rate function

Integral representation:

$$\mathbf{J}_{[\mathbf{0},\mathbf{T}]}\big(\{\mathbf{x}(\mathbf{t}),\chi(\mathbf{t})\}_{\mathbf{t}\in[\mathbf{0},\mathbf{T}]}\big) := \int_{\mathbf{0}}^{\mathbf{T}} \mathbf{j}\big(\mathbf{x}(\mathbf{t}),\chi(\mathbf{t})\big)\mathbf{dt}\,. \qquad (1)$$

$\mathbf{j}(\mathbf{x},\cdot)$ is the Donsker–Varadhan LD rate functional for the empirical measure, time–homogeneous Markov chain (Z_t) on Γ with generator $L_c[x]$ (x frozen variable)

Recall: empirical measure $\pi_{\mathbf{T}} \in \mathcal{P}(\boldsymbol{\Gamma})$,

$$\pi_{\mathbf{T}}(\sigma) = \frac{\mathbf{1}}{\mathbf{T}} \int_{\mathbf{0}}^{\mathbf{T}} \mathbb{1}(\mathbf{Z_t} = \sigma)\mathbf{dt}$$

LDP mechanical trajectory

By contraction principle, LDP for mechanical trajectory $\mathbf{x}(\cdot)$.
Rate function: $\mathbf{J}^{\mathbf{m}}_{[\mathbf{0},\mathbf{T}]} : \mathbf{C}([\mathbf{0},\mathbf{T}];\boldsymbol{\Omega}) \to [\mathbf{0},\infty]$

$$\mathbf{J}^{\mathbf{m}}_{[\mathbf{0},\mathbf{T}]}\left(\{\mathbf{x}(\mathbf{t})\}_{\mathbf{t}\in[\mathbf{0},\mathbf{T}]}\right) = \int_{\mathbf{0}}^{\mathbf{T}} \mathbf{j}_{\mathbf{m}}\big(\mathbf{x}(\mathbf{t}),\dot{\mathbf{x}}(\mathbf{t})\big)\mathbf{dt}$$

$$\mathbf{j}_{\mathbf{m}}(\mathbf{x},\dot{\mathbf{x}}) = \inf\left\{\mathbf{j}(\mathbf{x},\chi)\ :\ \chi \in \mathcal{P}(\boldsymbol{\Gamma}) \text{ with } \dot{\mathbf{x}} = \textstyle\sum_\sigma \chi_\sigma \mathbf{F}_\sigma(\mathbf{x})\right\}.$$

Time–reversed process (very briefly)

Suppose ∃! regular stationary measure ρ_λ, consider time–reversed stationary process.
It is again PDMP but λ–dependence not necessarily linear:

$$\mathbf{F}^+_\sigma(\mathbf{x}) = -\mathbf{F}_\sigma(\mathbf{x})$$

$$\mathbf{r}^+(\sigma, \sigma'|\mathbf{x}) := \mathbf{r}(\sigma', \sigma|\mathbf{x})\frac{\rho_\lambda(\mathbf{x}, \sigma')}{\rho_\lambda(\mathbf{x}, \sigma)}$$

Rates can depend on λ.
Hence, previous LD results **cannot** be applied.

Time–reversed process (very briefly)

Isolated a class of exactly solvable PDMP with stationary distribution

$$\rho_\lambda = \mathbf{c}(\lambda)\mathbf{e}^{-\lambda \mathbf{S}(\mathbf{x})}\rho(\mathbf{x},\sigma)\mathbf{dx}\delta_\sigma$$

Rates of reversed process are λ–independent. Hence, previous LD results **can** be applied.

Comparison of LD of direct and reversed process gives deep insight into irreversibility (Onsager–Machlup symmetries, fluctuation–dissipation relation)

Fluctuation–dissipation relation

$$\mathbf{j_m}(\mathbf{x},\dot{\mathbf{x}}) = \nabla \mathbf{S}(\mathbf{x})\cdot\dot{\mathbf{x}} + \mathbf{j^+_m}(\mathbf{x},-\dot{\mathbf{x}}) \qquad\qquad \mathbf{x}\in\boldsymbol{\Omega},\ \dot{\mathbf{x}}\in\mathbb{R}^{\mathbf{d}}$$

$$\bar{\mathbf{F}}^+(\mathbf{x}) = \sum_\sigma \mu^+(\sigma|\mathbf{x})\mathbf{F}^+_\sigma(\mathbf{x}) = -\sum_\sigma \mu^+(\sigma|\mathbf{x})\mathbf{F}_\sigma(\mathbf{x})\,.$$

Start from equilibrium. For large T,λ, if $x(t)=x$, then with high probability

$$\big(\,\mathbf{x}(\mathbf{t})\,\big)_{\mathbf{t}\in[\mathbf{0},\mathbf{T}]} \sim \mathbf{x}^+_\infty(\mathbf{T}-\cdot)$$

where

$$\begin{cases}\dot{\mathbf{x}}^+_\infty(\mathbf{t}) = \bar{\mathbf{F}}^+\big(\mathbf{x}^+_\infty(\mathbf{t})\big)\\ \mathbf{x}^+_\infty(\mathbf{0}) = \mathbf{x}\end{cases}$$

Typically, $\mathbf{x}^+_\infty(\cdot)$ is not time–reversed of $\mathbf{x}_\infty(\cdot)$

PDMP on 1D torus

- state $(\mathbf{x}, \sigma)$, $x \in \mathbb{T}$, $\sigma \in \{0, 1\}$
- $\mathbf{F_0}, \mathbf{F_1} : \mathbb{T} \to \mathbb{R}$ Lipschitz
- $\mathbf{r}(\mathbf{0}, \mathbf{1}|\cdot), \mathbf{r}(\mathbf{1}, \mathbf{0}|\cdot)$ positive continuous jump rates on $\mathbb{T}$.
- $\lambda > \mathbf{0}$ parameter

Formal generator:

$$
\begin{aligned}
\mathbb{L}\mathbf{f}(\mathbf{x}, \sigma) &= \mathbf{F}_\sigma(\mathbf{x}) \cdot \nabla \mathbf{f}(\mathbf{x}, \sigma) + \lambda(\sigma, \mathbf{1} - \sigma|\mathbf{x}) \left(\mathbf{f}(\mathbf{x}, \mathbf{1} - \sigma) - \mathbf{f}(\mathbf{x}, \sigma) \right) \\
&= \mathbf{F}_\sigma(\mathbf{x}) \cdot \nabla \mathbf{f}(\mathbf{x}, \sigma) + \lambda \mathbf{L_c}[\mathbf{x}]\mathbf{f}(\mathbf{x}, \sigma)
\end{aligned}
$$

Invariant distribution

Theorem

Assume (i) nonvanishing fields F_0, F_1,
(ii) positive rates $r(0,1|x)$, $r(1,0|x)$.
Define

$$\mathbf{S(x)} = \int_0^{\mathbf{x}} \left(\frac{\mathbf{r(0,1|y)}}{\mathbf{F_0(y)}} + \frac{\mathbf{r(1,0|y)}}{\mathbf{F_1(y)}} \right) \mathbf{dy} \qquad \mathbf{x} \in \mathbb{R}\,.$$

Then:

- $\exists!$ *invariant distribution,*

$$\rho_\lambda = \rho_\lambda(\mathbf{x},\mathbf{0})\mathbf{dx}\delta_{\mathbf{0}} + \rho_\lambda(\mathbf{x},\mathbf{1})\mathbf{dx}\delta_{\mathbf{1}}$$

$$\begin{cases} \rho_\lambda(x,0) := \frac{k}{F_0(x)} \int_x^{x+1} \left[\frac{r(1,0|y)}{F_1(y)} e^{\lambda\left(S(y)-S(x)\right)} \right] dy\,, \\ \rho_\lambda(x,1) := \frac{k}{F_1(x)} \int_x^{x+1} \left[\frac{r(0,1|y)}{F_0(y)} e^{\lambda\left(S(y)-S(x)\right)} \right] dy\,. \end{cases}$$

Theorem

...

- *If* $\mathbf{S}(\mathbf{x}) = \int_{\mathbf{0}}^{\mathbf{x}} \left(\frac{\mathbf{r}(\mathbf{0},\mathbf{1}|\mathbf{y})}{\mathbf{F_0}(\mathbf{y})} + \frac{\mathbf{r}(\mathbf{1},\mathbf{0}|\mathbf{y})}{\mathbf{F_1}(\mathbf{y})} \right) \mathbf{dy}$ *is* **periodic** *with period* 1 *(*$S(1) = 0$*) then exactly solvable PDMP:*

$$
\begin{cases} \rho_\lambda(x,0) = \frac{c(\lambda)}{F_0(x)} e^{-\lambda S(x)}, \\ \rho_\lambda(x,1) = \frac{c(\lambda)}{F_1(x)} e^{-\lambda S(x)}. \end{cases} \qquad (2)
$$

- *Setting* $\mu_\lambda(\mathbf{x}) := \rho_\lambda(\mathbf{x}, \mathbf{0}) + \rho_\lambda(\mathbf{x}, \mathbf{1})$, *it holds*

$$
\lim_{\lambda \uparrow \infty} -\frac{\mathbf{1}}{\lambda} \mu_\lambda(\mathbf{x}) =
$$

$$
\min_{\mathbf{y} \in [\mathbf{x}, \mathbf{x}+\mathbf{1}]} (\mathbf{S}(\mathbf{x}) - \mathbf{S}(\mathbf{y})) - \min_{x' \in [0,1]} \min_{y' \in [x', x'+1]} (S(x') - S(y')) .
$$

Diffusion $(\mathbf{X}_t^\epsilon)_{t\geq 0}$ on $\mathbb{T}$

- Diffusion $\dot{\mathbf{X}}_t^\epsilon = \mathbf{b}(\mathbf{X}_t^\varepsilon) + \varepsilon \dot{\mathbf{w}}_t$
- $b : \mathbb{T} \to \mathbb{R}$ Lipschitz, w_t Wiener process, $\epsilon > 0$

Theorem (Freidlin–Wentzell)

Assume $\{x \in \mathbb{T} : b(x) = 0\}$ has finite number of connected components.
Then $\exists!$ invariant distribution $\mu_\varepsilon(\mathbf{x})\mathbf{dx}$ and

$$\lim_{\varepsilon \downarrow 0} -\varepsilon^2 \log \mu^\varepsilon(\mathbf{x}) = \mathbf{W}(\mathbf{x}) - \min_{\mathbf{y}\in\mathbb{T}} \mathbf{W}(\mathbf{y}), \qquad \mathbf{x} \in \mathbb{T},$$

W solving graph optimization problem.

Theorem

Define $\mathbf{S}(\mathbf{x}) = -\mathbf{2}\int_{\mathbf{0}}^{\mathbf{x}} \mathbf{b}(\mathbf{s})\mathbf{ds}$. *Then,*

$$\mu_\epsilon(\mathbf{x}) = \frac{\mathbf{1}}{\mathbf{c}(\epsilon)} \int_{\mathbf{x}}^{\mathbf{x}+\mathbf{1}} \mathbf{e}^{\epsilon^{-\mathbf{2}}(\mathbf{S}(\mathbf{y})-\mathbf{S}(\mathbf{x}))}\,\mathbf{dy}\,,$$

In particular, it holds

$$\lim_{\varepsilon\downarrow\mathbf{0}} -\varepsilon^{\mathbf{2}} \log \mu_\varepsilon(\mathbf{x}) =$$

$$\min_{\mathbf{y}\in[\mathbf{x},\mathbf{x}+\mathbf{1}]} \big(\mathbf{S}(\mathbf{x}) - \mathbf{S}(\mathbf{y})\big) - \min_{\mathbf{x}'\in[\mathbf{0},\mathbf{1}]} \min_{\mathbf{y}'\in[\mathbf{x}',\mathbf{x}'+\mathbf{1}]} \big(\mathbf{S}(\mathbf{x}') - \mathbf{S}(\mathbf{y}')\big)\,.$$

Maxwell–like construction

Theorem

Take $F : \mathbb{T} \to \mathbb{R}$ *continuous*

Define for $x \in \mathbb{R}$ $\begin{cases} S(x) = \int_0^x F(s)ds\,, \\ \Phi(x) = \min_{\mathbf{y}\in[\mathbf{x},\mathbf{x}+\mathbf{1}]}\big(\mathbf{S}(\mathbf{x}) - \mathbf{S}(\mathbf{y})\big)\,. \end{cases}$

Then

- Φ *Lipschitz and periodic of period* 1
- *Sunset at Alps (wait next slide)*
- *Suppose* $\{\mathbf{x} \in \mathbb{T} \,:\, \mathbf{F}(\mathbf{x}) = \mathbf{0}\}$ *has finite number of connected components. Then,*

$$\mathbf{\Phi}(\mathbf{x}) - \min_{\mathbf{x}'\in[\mathbf{0},\mathbf{1}]} \mathbf{\Phi}(\mathbf{x}') = \mathbf{W}(\mathbf{x}) - \min_{\mathbf{y}\in\mathbb{T}} \mathbf{W}(\mathbf{y})\,,$$

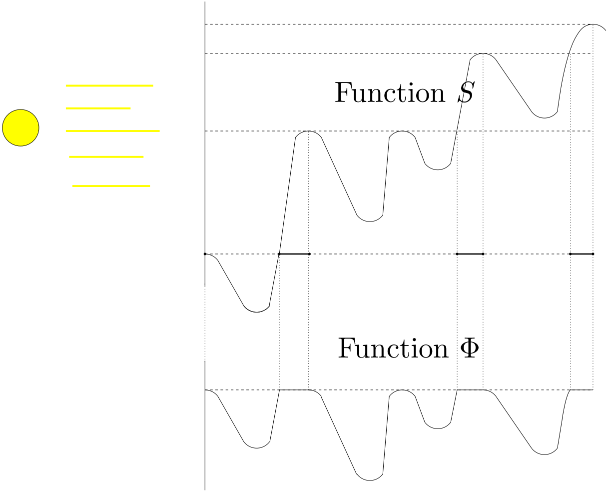
W *the function defined by [FW] with* $b(x) := -F(x)/2$.

Sunset at Alps

Graph of S: **mountain profile**
Sunset: light comes from the left

$$\begin{cases} x \text{ is in shadow} \Rightarrow \nabla\Phi = \nabla S \\ x \text{ is lightened} \Rightarrow \nabla\Phi = 0 \end{cases}$$

Note: $S(x+1) - S(x)$ constant


Function S
Function Φ

Hamilton–Jacobi equations and universality

$$\begin{cases} F : \mathbb{T} \to \mathbb{R} \text{ continuous} \\ S(x) = \int_0^x F(s)ds\,, \\ \Phi(x) = \min_{\mathbf{y}\in[\mathbf{x},\mathbf{x}+\mathbf{1}]}\big(\mathbf{S}(\mathbf{x}) - \mathbf{S}(\mathbf{y})\big)\,. \end{cases}$$

Theorem

Let $H(x,p)$, with $(x,p) \in \mathbb{T} \times \mathbb{R}$, such that

(A) $H(x,\cdot)$ is a convex function for any $x \in \mathbb{T}$,

(B) $H(x,0) = H(x,F(x)) = 0$ for any $x \in \mathbb{T}$.

Then $\Phi(x)$ as above is a **viscosity solution of the Hamilton-Jacobi equation**

$$H(x, \nabla\Phi(x)) = 0\,, \qquad x \in \mathbb{T}\,.$$

Viscosity solution

φ is a viscosity solution for HJ equation $H(x, \nabla\varphi(x)) = 0$ iff $\varphi \in C(\mathbb{T})$ and

$$\begin{aligned} \mathbf{H}(\mathbf{x},\mathbf{p}) \le \mathbf{0} \qquad & \forall \mathbf{x} \in \mathbb{T},\ \mathbf{p} \in \mathbf{D}^{+}\varphi(\mathbf{x})\,, \\ \mathbf{H}(\mathbf{x},\mathbf{p}) \ge \mathbf{0} \qquad & \forall \mathbf{x} \in \mathbb{T},\ \mathbf{p} \in \mathbf{D}^{-}\varphi(\mathbf{x})\,. \end{aligned}$$

Superdifferential $D^{+}\varphi(x)$, subdifferential $D^{-}\varphi(x)$ defined as

$$\mathbf{D}^{+}\varphi(\mathbf{x}) = \left\{ \mathbf{p} \in \mathbb{R} \,:\, \limsup_{\mathbf{y}\to\mathbf{x}} \frac{\varphi(\mathbf{y}) - \varphi(\mathbf{x}) - \mathbf{p}(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|} \le \mathbf{0} \right\},$$

$$\mathbf{D}^{-}\varphi(\mathbf{x}) = \left\{ \mathbf{p} \in \mathbb{R} \,:\, \liminf_{\mathbf{y}\to\mathbf{x}} \frac{\varphi(\mathbf{y}) - \varphi(\mathbf{x}) - \mathbf{p}(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|} \ge \mathbf{0} \right\}.$$

Hamiltonian and Quasipotential

Quasipotential theory gives Hamiltonians fulfilling main assumption.

$\mathbf{H}(\mathbf{x},\mathbf{p}) = \sup_{\mathbf{y}} \left[\mathbf{p}\mathbf{y} - \mathbf{j_m}(\mathbf{x},\mathbf{y})\right]$

$\mathbf{j_m}(\mathbf{x},\mathbf{y})$ density of the dynamical LDP (Lagrangian)

- Diffusion: $\mathbf{H}(\mathbf{x},\mathbf{p}) = \mathbf{p}\big(\mathbf{p} - \mathbf{F}(\mathbf{x})\big)$
- PDMP: much more complex expression (see [FGR][JSP 2010])