Piecewise Deterministic Markov Processes
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Stochastic billiard in an inhomogeneous medium

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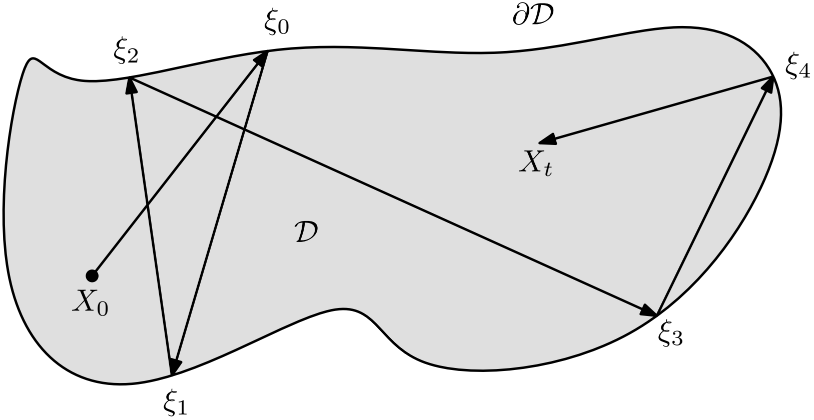
Introduction

Stochastic billiards on general tables: a particle moves according to its constant velocity inside some domain $\mathcal{D} \subset \mathbb{R}^d$ until it hits the boundary and bounces randomly inside according to some reflection law.



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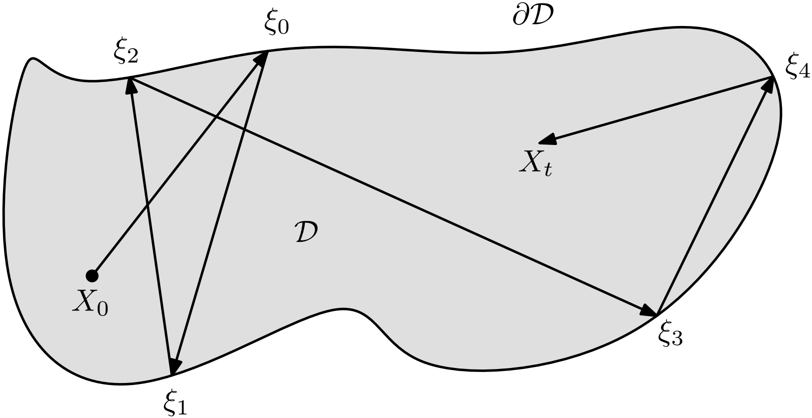

Piecewise Deterministic Markov Processes

Motivations

- Kinetic theory of gases
  Knudsen's book [1952]
  Diffusion in nanopores: Coppens-Malek [2003], Coppens-Dammers [2006]
  Goldstein-Kipnis-Ianiro [1985]: a mechanical particle system with stochastic boundary conditions
- Dynamical systems:
  Feres [2007-2013]: how stochasticity emerges from dynamical systems with microstructures
  S. Evans [2001]: C^1 boundary or polygon, uniform reflection law
- Monte Carlo Markov Chains, algorithms and games:
  Lalley-Robbins [1987, 1988]: convex $\mathcal{D}$ and cosine law. "princess and monster"
  Borovkov [1991, 1994], Romeijn [1998]: Monte Carlo Markov chains algorithm ("running shake-and-bake algorithm")
  Diaconis: Hit and Run algorithm

Outline

Billiard table

$\mathcal{D} \subset \mathbb{R}^d$ open connected domain, with boundary $\partial\mathcal{D}$ locally Lipschitz and almost everywhere continuously differentiable:

1– $\forall x \in \partial\mathcal{D}$, we can rotate $\partial\mathcal{D}$ so that it is locally the graph of a Lipschitz function.

2– $\exists\, \mathcal{R} \subset \partial\mathcal{D}$ open such that $\partial\mathcal{D}$ is continuously differentiable on $\mathcal{R}$ and the $(d-1)$-dimensional Hausdorff measure of $\partial\mathcal{D} \setminus \mathcal{R}$ is equal to zero.

Reflection law for stochastic billiard

Outgoing direction is random, with density (in the relative frame) γ on the open half sphere $\mathbb{S}_e = \{u \in \mathbb{R}^d : |u| = 1, u \cdot e > 0\}$, with $e =$ the first unit vector, such that

$$\inf_K \gamma > 0 \qquad \forall K \text{ compact } \subset \mathbb{S}_e$$

Main example for γ: cosine density,

$$\gamma(u) = \gamma_d \; e \cdot u \quad \text{on half sphere } \mathbb{S}_e$$

cf Knudsen [1952].

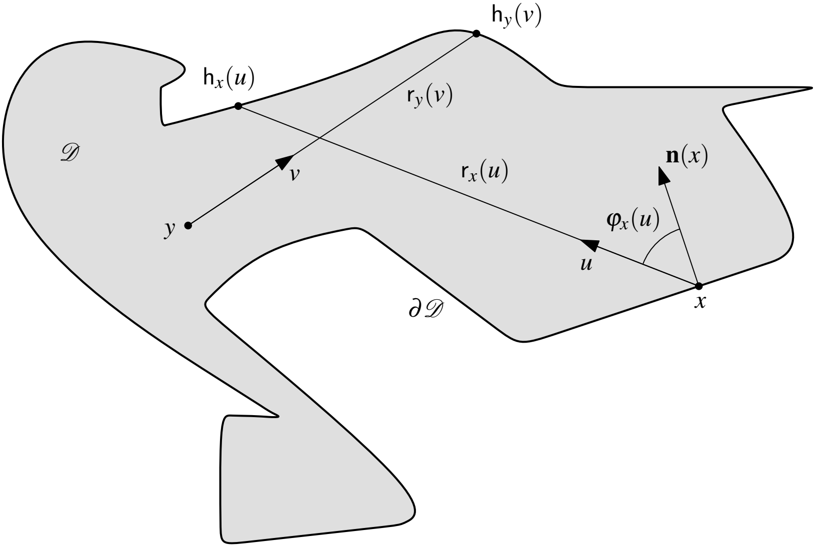

Figure: Bounce at $x \in \partial\mathcal{D}$ in dimension $d = 2$. The outgoing direction u is such that its angle $\varphi_x(u)$ with the normal $n(x)$ has density γ; independent of the ingoing direction.

Construction of KRW and KSB:

... standard way, with an i.i.d. sequence of law γ on $\mathbb{S}_e$.

- Knudsen Random Walk (KRW) $(\xi_n, n \geq 0) =$ sequence of impacts on the boundary.
  Markov chain in $\{\partial\mathcal{D}, \infty, \mathfrak{S}\}$.
  Note:: Start from $\xi_0 \in \mathcal{R}$. Then, with probability 1, ξ does not enter $\mathfrak{S}$.
- Knudsen Stochastic billiard: time-continuous process moving at speed 1.
  Is defined for all times, a.s..
  The couple (position, velocity) is Markov (PDMP).

Change the variable

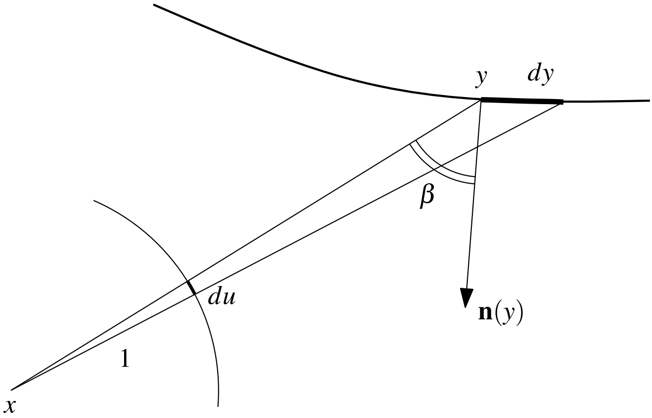

Figure: $du = \|x - y\|^{-(d-1)} \cos \beta \, dy$

Transition kernel for the random walk

Changing variable from $u \in \mathbb{S}_e$ to $y = h_x(U_x u)$, we get for $x \in \mathcal{R}$,

$$\mathbf{P}[\xi_{n+1} \in A \mid \xi_n = x] = \int_A K(x,y)\,dy\ ,$$

where dy is the surface measure on $\partial\mathcal{D}$ and

$$K(x,y) = \frac{\gamma(U_x^{-1}\frac{y-x}{\|y-x\|})\cos\big(\widehat{\mathbf{n}(y), y-x}\big)}{\|x-y\|^{d-1}}\mathbf{1}\{x,y \in \mathcal{R}, x \leftrightarrow y\}$$

where we write $x \leftrightarrow y$ *(see each other)* if the open segment $(x,y) \subset \mathcal{D}$.

Outline

Invariant measure for Knudsen random walk

Knudsen popularized and justified the choice $\gamma =$ Cosine law. Then, the transition density is

$$K(x,y) = \gamma_d \frac{((y-x)\cdot \mathbf{n}(x))\,((x-y)\cdot \mathbf{n}(y))}{\|x-y\|^{d+1}} \mathbf{1}\{x,y \in \mathcal{R}, x \leftrightarrow y\}$$

symmetric ! The surface measure dx on $\partial\mathcal{D}$ is reversible,

$$dx\ K(x,dy) = dy\ K(y,dx),$$

...and then invariant.

Asymptotics on a bounded table (for a general γ)

$$\text{Assumption :} \qquad \boxed{\text{diam}(\mathcal{D}) < \infty}$$

By the Lipschitz assumption, this implies that $|\partial\mathcal{D}| < \infty$.
The chain satisfies Döblin condition: there exist n, $\varepsilon > 0$ such that for all $x, y \in \mathcal{R}$

$$K^n(x, y) \geq \varepsilon \tag{1}$$

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Theorem

(i) *There exists a unique probability measure $\hat{\mu}$ on $\partial\mathcal{D}$ which is invariant for the random walk ξ_n. Moreover, $d\hat{\mu} << dx$.*

(ii) $\|\mathbf{P}[\xi_n \in \cdot] - \hat{\mu}\|_v \leq \beta_0 e^{-\beta_1 n}$ *($\|\cdot\|_v$ = total variation distance).*

(iii) *Central Limit Theorem: $\forall A \subset \partial\mathcal{D}$ measurable there exists σ_A ($\sigma_A > 0$ if $0 < |A| < |\partial\mathcal{D}|$) such that*

$$n^{-1/2}\left(\sum_{i=1}^{n} \mathbf{1}\{\xi_i \in A\} - n\hat{\mu}(A)\right) \xrightarrow{law} \mathcal{N}(0, \sigma_A^2)$$

For the cosine law, $d\hat{\mu} = |\partial\mathcal{D}|^{-1} dx$ uniform on $\partial\mathcal{D}$.

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Infinite horizontal “tube”

To understand large scale properties of billard, we consider a table $\mathcal{D} = \omega$, which is infinite in the first direction, write $x = (\alpha, u)$:

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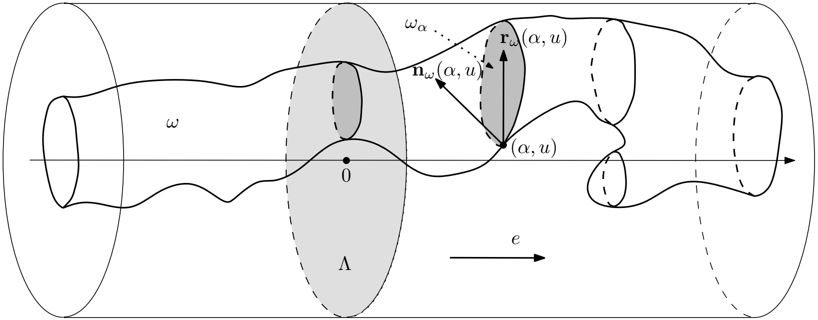

Figure: Infinite tube

Random tube = random environment

Any tube $\omega = (\omega_\alpha, \alpha \in \mathbb{R})$ is seen as the process of its sections

$$\omega = \{(\alpha, u) \in \mathbb{R}^d : u \in \omega_\alpha\}$$

Let $\mathfrak{E}$ to be the set of all open domains $A \subset \mathbb{R}^{d-1}$ contained in a fixed ball,

$$A \subset \Lambda := \{u \in \mathbb{R}^{d-1} : \|u\| \leq M\}.$$

Let $\Omega = \mathcal{C}(\mathbb{R} \to \mathfrak{E})$ “space of tubes” (equipped with the distance $\rho(A, B) = |(A \setminus B) \cup (B \setminus A)|$ on $\mathfrak{E}$ and cylinder sigma-algebra).

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Assume

$$\omega \sim \mathbb{P},$$

with $\mathbb{P}$ a probability measure on Ω, stationary and ergodic (w.r.t. shifts in α).

Random tube: assumptions, notations

Assumptions: $\mathbb{P}$-a.s., ω is open, connected, and:

(L) $\partial\omega$ is Lipschitz with uniform constants

(R) $\{x \in \partial\omega : \partial\omega \text{ is } C^1 \text{ in } x, |\mathbf{n}_\omega(x) \cdot e| \neq 1\}$ has full measure $\mathcal{H}_{d-1}$-measure

(P) Points on the boundary which are close, communicate "well" and "quickly":
$\exists N, \varepsilon, \delta$: $\mathbb{P}$-a.s., $\forall x, y \in \mathcal{R}$ with $|(x-y)\cdot e| \leq 2, \exists B_1, \dots, B_n \subset \partial\omega$, $n \leq N$ with $\nu^\omega(B_i) \geq \delta (i = 1, \dots, n)$, s.t.

- $K(x,z) \geq \varepsilon$ for all $z \in B_1$, $K(y,z) \geq \varepsilon$ for all $z \in B_n$,
- $K(z,z') \geq \varepsilon$ for all $z \in B_i$, $z' \in B_{i+1}$, $i = 1, \dots, n-1$

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Notation:
ν^ω = restriction of $(d-1)$-dimensional Hausdorff measure on $\partial\omega$

Stochastic billiard with drift in a random tube

Same assumptions as above on the random tube.

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Dynamics for KRW with drift of intensity $\lambda > 0$: acceptance/rejection.
If $\xi_n = x$,

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- Then,
  - if $(y - x) \cdot \mathbf{e} > 0$, set $\xi_{n+1} = y$,
  - if $(y - x) \cdot \mathbf{e} < 0$,
    set $\xi_{n+1} = y$ with probability $\exp\{-\lambda|(y - x) \cdot \mathbf{e}|\}$, and $\xi_{n+1} = x$ otherwise.

Typical path of the random walk (rejected jumps are shown as dotted lines).

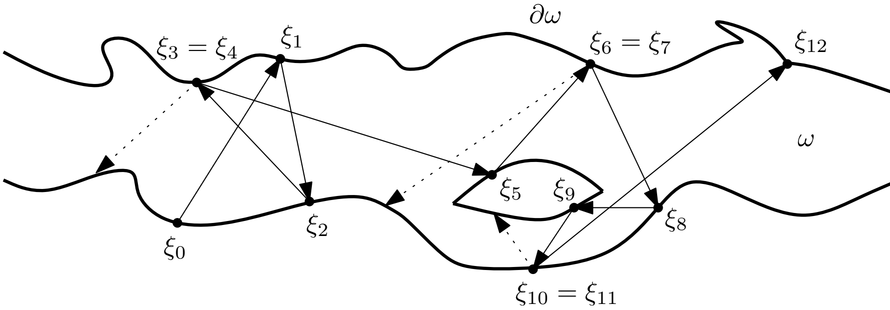

Figure: Knudsen random walk with drift

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Then, the measure ν_λ^ω with

$$\frac{d\nu_\lambda^\omega}{d\nu^\omega}(x) = \exp\{\lambda x \cdot \mathbf{e}\}$$

is invariant and reversible for ξ_n.

Law of large numbers

Theorem

Assume $d \geq 3$*. There exists* $\hat{v} > 0$ *deterministic such that, a.s.,*

$$\frac{\xi_n \cdot \mathbf{e}}{n} \to \hat{v} \qquad \text{as } n \to \infty$$

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Idea of proof: using condition (P), we make a coupling of ξ in a fixed ω, with a Random Walk in Random Environment (RWRE) on $\mathbb{Z}$, with unbounded jumps and stationary ergodic environment. We use a (new) Law of Large Numbers for the latter.

Coupling Stochastic Billiards with Random Walk

Let $(\eta_i; i \geq 1)$ i.i.d. uniform on $\{1, 2, \dots, N\}$, $J(n) = \eta_1 + \cdots + \eta_n$.

Condition P implies that: $\exists \delta > 0$ s.t.

$$\mathtt{P}_\omega{}^x[\xi_{\eta_1} \in B] \geq \delta \nu^\omega(B),$$

for all $x \in \partial\omega$ and $B \subset \{y \in \partial\omega : |(y - x) \cdot \mathbf{e}| \leq 1\}$.

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Let $U_j = \{x \in \partial\omega : x \cdot \mathbf{e} \in (j, j+1]\}$, and $\pi^j = \nu^\omega(\,\cdot\,|U_j)$ the uniform distribution on U_j.

We couple the process $(\xi_{J(n)}, n \geq 0)$ with i.i.d. Bernoulli $(\zeta'_n, n \geq 1)$ (independent of ω) of parameter δ,

$$P[\zeta'_n = 1] = 1 - P[\zeta'_n = 0] = \delta,$$

so that

on the event $\{\zeta'_n = 1\}$, $\xi_{J(n)}$ has distribution $\pi^{[\xi_{J(n-1)} \cdot \mathbf{e}]}$ on $U_{[\xi_{J(n-1)} \cdot \mathbf{e}]}$.

Set $\kappa_0 = 0$, and

$$\kappa_{m+1} = \min\{k > \kappa_m : \zeta'_k = 1\}, \qquad m \geq 1.$$

Then, under $P^{\zeta'} \otimes \mathrm{P}^x_{\omega,\zeta'}$, the sequence $(\xi_{J(\kappa_m)}, m \geq 0)$ is a Markov chain, with law of the form $\sum_{i\in\mathbb{Z}} a_i \pi^i$. The Markov chain is weakly lumpable.

Lemma

Under $P^{\zeta'} \otimes \mathrm{P}^{U_0}_{\omega,\zeta'}$, the sequence $([\xi_{J(\kappa_m)} \cdot \mathbf{e}], m \geq 0)$ is a RWRE on $\mathbb{Z}$, with transition probabilities

$$Q_\omega(i,j) = P^{\zeta'} \otimes \mathrm{P}^{U_i}_{\omega,\zeta'}[\xi_{J(\kappa_1)} \in U_j].$$

This is the bridge between SB in Random Tube and RWRE.
With some extra estimates on hitting times of sets by SB, it is enough to get a LLN for RWRE.

Outline

1. Stochastic billiard on a general table
2. Long time behavior in the compact case
3. Ballistic regime for Stochastic billiard with a drift
4. Law of Large Numbers for ballistic RWRE with unbounded jumps

Random walk in random environment with unbounded jumps on $\mathbb{Z}$

Attention: in this section,

$$\omega = (\omega_{x,y}; x, y \in \mathbb{Z}), \quad \omega_{x,y} \geq 0, \sum_y \omega_{x,y} = 1.$$

Let S_n be the RWRE in $\mathbb{Z}$ with $P_\omega(S_{n+1} = x + y | S_n = x) = \omega_{x,y}$.

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Assume $(\omega_{x,\cdot})_x$ is stationary and ergodic under some $\mathbb{P}$.

Consider also the RW in the truncated environment ω^ϱ ($\varrho \geq 1$ truncation parameter)

$$\omega^\varrho_{xy} = \begin{cases} \omega_{xy}, & \text{if } 0 < |y| < \varrho, \\ 0, & \text{if } |y| \geq \varrho, \\ \omega_{x0} + \sum_{y:|y|\geq\varrho} \omega_{xy}, & \text{if } y = 0, \end{cases}$$

RWRE: assumptions

Assume uniform ellipticity, uniform tails, strong transience (no traps):

Condition E. There exists $\tilde{\varepsilon}$ such that $\mathbb{P}[\omega_{01} \geq \tilde{\varepsilon}] = 1$.

Condition C. $\exists \alpha > 1, \gamma_1 > 0$ s.t. for all $s \geq 1$,

$$\sum_{y:|y|\geq s} \omega_{0y} \leq \gamma_1 s^{-\alpha}, \qquad \mathbb{P}\text{-a.s.}$$

Condition D. $\exists g_1 \geq 0$ with $\sum_{k=1}^{\infty} k g_1(k) < \infty, \exists \varrho_0 < \infty$, such that $\forall x \leq 0, \varrho \geq \varrho_0$,

$$E_\omega^0 N_\infty^\varrho(x) \leq g_1(|x|), \qquad \mathbb{P} - \text{a.s.}$$

with $N_n^\varrho(x) = \sum_{k\leq n} \mathbf{1}\{S_k^\varrho = x\}$.

Law of Large Numbers for ballistic RWRE with unbounded jumps

Proposition

Then, $\forall \varrho \in [\varrho_0, \infty], \exists v_\varrho > 0$ *s.t. we have*

$$\frac{S_n^\varrho}{n} \to v_\varrho, \quad n \to \infty, \quad \text{a.s.}$$

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- ✍ No reversibility is assumed.
- ✍ RWRE on $\mathbb{Z}$ with bounded jumps: long-time behavior determined by middle Lyapunov exponents of random matrices. Transience/recurrence by Key [1984], LLN by Goldsheid [2003, 2008], Brémont [2009]; lingering "à la Sinai" by Bolthausen and Goldsheid [2008].
- ✍ Only reference for unbounded jumps: 0-1 law by Andjel [1988].

Law of Large Numbers for RWRE with unbounded jumps: ideas of proof

□ Fix $\varrho_0 \leq \varrho < \infty$. Let $T_z^\varrho = \min\{k \geq 0 : S_k^\varrho \geq z\}$ first hitting time of $[z, \infty)$ by RWRE.

Lemma

Conditions E, C, D. imply there exists $\varepsilon_1 > 0$ such that, $\mathbb{P}$-a.s.,

$$\mathtt{P}_\omega{}^x[S^\varrho_{T^\varrho_0} = 0] \geq 2\varepsilon_1$$

for all $x \leq 0$ and for all $\varrho \in [\varrho_0, \infty]$.

We can couple RWRE S^ϱ with an i.i.d. Bernoulli (ε_1) sequence $\zeta = (\zeta_1, \zeta_2, \zeta_3, \dots)$ in such a way that

$$\zeta_j = 1 \quad \Longrightarrow \quad S^\varrho_{T^\varrho_{j\varrho}} = j\varrho.$$

Denote by ℓ_k the time of k-th success of ζ..

Lemma

The pair $(\theta_{S_k^\varrho}\omega, T^\varrho_{\ell_k\varrho})$ is cycle-stationary and ergodic.

In particular, $\theta_{\ell_k\varrho}\omega \overset{\text{law}}{=} \omega.$

Hence, for finite ϱ, we derive:

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Hence, for finite ϱ, we derive:

- the proposition using the ergodic theorem.
- There exists an invariant measure $\mathbb{Q}^\varrho$ for the environment seen from the walker.
- By condition (D),

$$\gamma \leq \frac{d\mathbb{Q}^\varrho}{d\mathbb{P}}(\omega) \leq 1/\gamma.$$

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- By condition (D),

$$\gamma \leq \frac{d\mathbb{Q}^\varrho}{d\mathbb{P}}(\omega) \leq 1/\gamma.$$

- Any weak limit $\mathbb{Q}^\infty$ is invariant for S, and $v_\infty = \lim v_\varrho$ is its speed.

Past and Future

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Open Questions

1. Compact Domain: what geometric feature of the domain $\mathcal{D}$ determines the rate of approach to equilibrium ? Estimate the spectral gap for the cosine law ?
   Feres-Zhang 2010,2012, Cook-Feres 2012
2. Infinite random tube: Study the sub-ballistic regime ? Slowdowns and traps ?