Random Polynomials and their Zeros†

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† Joint work with Christopher Corley and Andrew Ledoan

Definitions/Brief History

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where $\{\eta_j\}$ are standard normal complex-valued random variables, and $\{\phi_j\}$ are polynomials such that $\deg \phi_j = j$ with $\phi_0 = 1$.

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4. general setting: Corley and Ledoan (2020)

Overview of our results

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3. Asymptotics for $\mathbb{E}[N_{\mathbf{v}}^{\mathbf{\Phi_n}}(\Theta)]$ and its related density function, as $n \to \infty$, in $D \subseteq \mathbb{D}$ when ϕ_j are orthogonal polynomials on the unit circle or Bergman polynomials

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4. Numerical simulations

Main Result #1: The Contour Integral Formula

For $z, w \in \mathbb{C}$, and k and l nonnegative integers, let

$$K_n(z,w) = \sum_{j=0}^{n} \phi_j(z)\overline{\phi_j(w)}, \quad \text{and} \quad K_n^{(k,l)}(z,w) = \sum_{j=0}^{n} \phi_j^{(k)}(z)\overline{\phi_j^{(l)}(w)}.$$

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Theorem (Contour Integral Formula)

Let $\mathbf{\Phi}_n = \sum_{j=0}^{n} \eta_j \phi_j(z)$, where $\{\eta_j\}$ are standard normal complex-valued random variables, and $\{\phi_j\}$ are polynomials such that $\deg \phi_j = j$ with $\phi_0 = 1$. For every domain $\Theta \subset \mathbb{C}$ and every fixed positive integer n, if $N_{\mathbf{v}}^{\mathbf{\Phi_n}}(\Theta)$ denotes the number of complex roots in Θ of $\mathbf{\Phi}_n = \mathbf{v}$, then

$$\mathbb{E}\left[N_{\mathbf{v}}^{\mathbf{\Phi_n}}(\Theta)\right] = \frac{1}{2\pi i} \oint_{\partial\Theta} \frac{\overline{K_n^{(0,1)}(z,z)}}{K_n(z,z)} \exp\left[-\frac{|\mathbf{v}|^2}{K_n(z,z)}\right] dz.$$

Main Ideas of the Proof of the Contour Integral Formula

Assume for simplicity that $\mathbf{\Phi}_n - \mathbf{v}$ has no zeros on $\partial\Theta$. Since $\mathbf{\Phi}_n - \mathbf{v}$ is holomorphic within and on Θ, by the argument principle, we have

$$N_{\mathbf{v}}^{\mathbf{\Phi}_n}(\Theta) = \frac{1}{2\pi i} \oint_{\partial\Theta} \left(\frac{\mathbf{\Phi}_n'(z)}{\mathbf{\Phi}_n(z) - \mathbf{v}} \right) dz.$$

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$$\mathbb{E}\left[N_{\mathbf{v}}^{\mathbf{\Phi}_n}(\Theta)\right] = \mathbb{E}\left[\frac{1}{2\pi i} \oint_{\partial\Theta} \left(\frac{\mathbf{\Phi}_n'(z)}{\mathbf{\Phi}_n(z) - \mathbf{v}} \right) dz\right]$$

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1. Justification of the exchange of the expectation and contour integral

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1. Justification of the exchange of the expectation and contour integral
2. Computation of the integrand: Apply the mean ratio of complex normal random variables method given by Shaohan Wu (2019)

Main Result #2: The Area Integral Formula

From

$$\mathbb{E}\left[N_{\mathbf{v}}^{\mathbf{\Phi_n}}(\Theta)\right] = \frac{1}{2\pi i}\oint_{\partial\Theta} \frac{\overline{K_n^{(0,1)}(z,z)}}{K_n(z,z)} \exp\left[-\frac{|\mathbf{v}|^2}{K_n(z,z)}\right] dz,$$

by Green's Theorem we the following:

Theorem (Area Integral Formula)

For every domain $\Theta \subset \mathbb{C}$ and every fixed positive integer n,

$$\mathbb{E}\left[N_{\mathbf{v}}^{\mathbf{\Phi}_n}(\Theta)\right] = \iint_{\Theta} \rho_{n,\mathbf{v}}(z)\, dx\, dy,$$

where

$$\rho_{n,\mathbf{v}}(z) = \frac{1}{\pi}\left[\frac{K_n^{(1,1)}(z,z)}{K_n(z,z)} - \frac{\left|K_n^{(0,1)}(z,z)\right|^2}{K_n(z,z)^2}\left(1 - \frac{|\mathbf{v}|^2}{K_n(z,z)}\right)\right] \cdot \exp\left[-\frac{|\mathbf{v}|^2}{K_n(z,z)}\right].$$

Corollary #1

Corollary

For every domain $\Theta \subset \mathbb{C}$ and every fixed positive integer n,

$$\lim_{|\mathbf{v}|\to\infty} \mathbb{E}\left[N_{\mathbf{v}}^{\mathbf{\Phi}_n}(\Theta)\right] = 0$$

and

$$\lim_{|\mathbf{v}|\to\infty} \rho_{n,\mathbf{v}}(z) = 0.$$

Corollary #2

Corollary

For every domain $\Theta \subset \mathbb{C}$ and every fixed positive integer n, if $\mathbf{v} = 0$, then the mean number of zeros in Θ of $\mathbf{\Phi}_n$ is

$$\frac{1}{2\pi i} \oint_{\partial\Theta} \frac{\overline{K_n^{(0,1)}(z,z)}}{K_n(z,z)}\, dz$$

or, equivalently,

$$\iint_{\Theta} \rho_{n,0}(z)\, dx\, dy,$$

where

$$\rho_{n,0}(z) = \frac{K_n(z,z)K_n^{(1,1)}(z,z) - \left|K_n^{(0,1)}(z,z)\right|^2}{\pi K_n(z,z)^2}.$$

Applications: Random Orthogonal Polynomials (ROP)

For $\mathbf{\Phi}_n = \sum_{j=0}^{n} \eta_j \phi_j(z)$ we take $\{\phi_j\}$ to be either

1. Orthogonal Polynomials on the Unit Circle (OPUC); $\{\phi_j\}$ such that

$$\int_{\mathbb{T}} \phi_j(e^{i\theta}) \overline{\phi_k(e^{i\theta})} \, d\mu(e^{i\theta}) = \delta_{jk}, \quad j, k \in \mathbb{N} \cup \{0\}.$$

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   Ex.) $\{\sqrt{(j+1)(j+k+1)/(k\pi)}z^j\}$, $k>0$, $d\mu = (1-|z|^{2k})dA(z)$

Classes of OPUC we take as basis functions

1. $\{\phi_j\}$ are said to be from the Szegő class if $d\mu = W(\theta)d\theta$, where $W(\theta) \geq 0$ on $[-\pi, \pi]$, and $\int_{-\pi}^{\pi} W(\theta)\, d\theta$ and $\int_{-\pi}^{\pi} |\log W(\theta)|\, d\theta$ both exist.

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$$\lim_{n\to\infty} \phi_{n+1}^*(z) = \frac{1}{D(z)}, \quad \text{where} \quad \phi_n^*(z) = z^n \overline{\phi_n(1/\bar{z})},$$

and

$$D(\xi) = \exp\left[\frac{1}{4\pi}\int_{-\pi}^{\pi} \log W(\theta) \left(\frac{1+\xi e^{-i\theta}}{1-\xi e^{-i\theta}}\right) d\theta\right]$$

is uniquely determined by W, analytic and nonzero whenever $|\xi| < 1$, and $D(0) > 0$.

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3. $\{\phi_j\}$ are said be Sthal, Totik, and Ullman (STU) regular when the leading coefficient κ_j of ϕ_j satisfies

$$\lim_{j\to\infty} \sqrt[j]{\kappa_j} = 1.$$

Main Tool used on ROP spanned by OPUC

For $z, w \in \mathbb{C}$ with $\overline{w}z \neq 1$, the Christoffel-Darboux formula gives

$$\sum_{j=0}^{n} \phi_j(z)\overline{\phi_j(w)} = \frac{\overline{\phi_{n+1}^*(w)}\,\phi_{n+1}^*(z) - \overline{\phi_{n+1}(w)}\,\phi_{n+1}(z)}{1 - \overline{w}z}$$

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Main Tool used on ROP spanned by OPUC

For $z, w \in \mathbb{C}$ with $\overline{w}z \neq 1$, the Christoffel-Darboux formula gives

$$\sum_{j=0}^{n} \phi_j(z)\overline{\phi_j(w)} = \frac{\overline{\phi_{n+1}^*(w)}\,\phi_{n+1}^*(z) - \overline{\phi_{n+1}(w)}\,\phi_{n+1}(z)}{1 - \overline{w}z} = K_n(z, w)$$

Thus

$$\begin{aligned} K_n(z,z) &= \frac{|\phi_{n+1}^*(z)|^2 - |\phi_{n+1}(z)|^2}{1 - |z|^2} \\ &= \frac{|\phi_{n+1}^*(z)|^2 \left(1 - |\phi_{n+1}(z)|^2/|\phi_{n+1}^*(z)|^2\right)}{1 - |z|^2} \\ &= \frac{|\phi_{n+1}^*(z)|^2 \,(1 - o(1))}{1 - |z|^2}, \end{aligned}$$

locally uniformly if $z \in \mathbb{D}$ and $\{\phi_j\}$ are from the Nevai Class.

ROP spanned by OPUC from the Nevai Class

Theorem

Let the basis functions for $\mathbf{\Phi}_n$ *be OPUC* $\{\phi_j\}_{j=0}^{\infty}$ *obeying the Nevai class. Then for* $z \in \mathbb{D}$*, locally uniformly as* $n \to \infty$ *we have*

$$\rho_{n,\mathbf{v}}(z) = \frac{1}{\pi}\left(\frac{1}{(1-|z|^2)^2} + \frac{|\mathbf{v}|^2[1-|z|^2+o(1)]}{|\phi_{n+1}^*(z)|^2}\left|\frac{z}{1-|z|^2} + \frac{\overline{\phi_{n+1}^{*\,\prime}(z)}+o(1)}{\overline{\phi_{n+1}^*(z)}}\right|^2\right) \cdot \exp\left[-\frac{|\mathbf{v}|^2(1-|z|^2)}{|\phi_{n+1}^*(z)|^2}+o(1)\right].$$

ROP spanned by OPUC from the Szegő Class

Theorem

Let the basis functions for $\mathbf{\Phi}_n$ be OPUC $\{\phi_j\}_{j=0}^{\infty}$ obeying the Szegő class. Then locally uniformly whenever $|z| < 1$,

$$\lim_{n\to\infty} \rho_{n,\mathbf{v}}(z) = \frac{1}{\pi}\left(\frac{1}{(1-|z|^2)^2} + |\mathbf{v}|^2(1-|z|^2)|D(z)|^2 \left|\frac{z}{1-|z|^2} - \frac{\overline{D'(z)}}{\overline{D(z)}}\right|^2\right) \cdot \exp\left[-|\mathbf{v}|^2(1-|z|^2)|D(z)|^2\right].$$

ROP spanned by the monomials

Corollary

Let the basis functions for $\mathbf{\Phi}_n$ *be* $\phi_j(z) = z^j$*, then, for every open circular disk* $D_\varrho \subset \mathbb{D}$*,*

$$\lim_{n\to\infty} \mathbb{E}\left[N_{\mathbf{v}}^{\mathbf{\Phi}_n}(D_\varrho)\right] = \frac{\varrho^2}{1-\varrho^2} \exp\left[-|\mathbf{v}|^2(1-\varrho^2)\right].$$

ROP spanned by STU regular OPUC

Theorem

Let the measure μ *of orthogonality for OPUC* $\{\phi_j\}_{j=0}^{\infty}$ *be a strictly positive Borel measure on* $[-\pi, \pi)$*, absolutely continuous with respect to the Lebesgue measure, and regular in the sense of Sthal, Totik, and Ullman. Assume that* μ *has a positive weight function that is continuous on* $\mathbb{T}$*. When* $\mathbf{\Phi}_n$ *has such basis functions* ϕ_j*, it follows that*

$$\lim_{n\to\infty} \frac{\mathbb{E}\left[N_{\mathbf{v}}^{\mathbf{\Phi}_n}(\mathbb{D})\right]}{n} = \frac{1}{2}.$$

ROP spanned by Bergman Polynomials

When $\{\phi_j\}$ are Bergman Polynomials on the unit disk, it is known that locally uniformly for $z, w \in \mathbb{D}$ we have

$$\lim_{n\to\infty} K_n(z, w) = \frac{1}{\pi(1 - z\overline{w})^2}.$$

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Taking respective derivatives of the above and then evaluating on the diagonal, after much algebraic simplification we achieve

Theorem

Let the basis functions for $\mathbf{\Phi}_n$ be Bergman polynomials $\{\phi_j\}_{j=0}^{\infty}$ on $\mathbb{D}$. Then locally uniformly for every $z \in \mathbb{D}$,

$$\lim_{n\to\infty} \rho_{n,\mathbf{v}}(z) = \frac{2}{\pi}\left(\frac{1}{(1-|z|^2)^2} + 2\pi|\mathbf{v}z|^2\right)\exp\left[-\pi|\mathbf{v}|^2(1-|z|^2)^2\right].$$

ROP spanned by Bergman Polynomials

Corollary

Let the basis functions for $\mathbf{\Phi}_n$ *be Bergman polynomials* $\{\phi_j\}_{j=0}^{\infty}$ *on* $\mathbb{D}$*. Then for every open circular disk* $D_\varrho \subset \mathbb{D}$ *with* $\varrho < 1$*,*

$$\lim_{n\to\infty} \mathbb{E}\left[N_{\mathbf{v}}^{\mathbf{\Phi}_n}(D_\varrho)\right] = \frac{2\varrho^2}{1-\varrho^2} \exp\left[-\pi|\mathbf{v}|^2(1-|z|^2)^2\right].$$

ROP spanned by STU regular Bergman Polynomials

Theorem

Let μ be the measure of orthogonality for Bergman polynomials $\{\phi_j\}_{j=0}^{\infty}$ be a strictly positive Borel measure on $\mathbb{D}$, absolutely continuous with respect to the Lebesgue measure, and regular in the sense of Sthal, Totik, and Ullman. Assume that μ has a positive weight function that is continuous on $\mathbb{T}$. When $\mathbf{\Phi}_n$ has such basis functions ϕ_j, it follows that

$$\lim_{n\to\infty} \frac{\mathbb{E}\left[N_{\mathbf{v}}^{\mathbf{\Phi}_n}(\mathbb{D})\right]}{n} = \frac{2}{3}.$$

Numerical Simulations

We now show some examples of numerical simulation of $\mathbf{\Phi}_n$ spanned by Bergman polynomials ϕ_j at various $\mathbf{v}$ level-crossings. Consider the Bergman polynomials

$$\phi_j(z) = \sqrt{\frac{(j+1)(j+2+1)}{2\pi}} z^j, \quad j = 0, \ldots, n,$$

with weight function $w(z) = 1 - |z|^{2\cdot 2}$

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In the following numerical simulations we will take a close examination at the relationship between the mean $\mathbb{E}\left[N_{\mathbf{v}}^{\mathbf{\Phi}_n}(\Theta)\right]$ and $\mathbf{v}$ level-crossings, to study the profiles of the density $\rho_{n,\mathbf{v}}$ and roots of $\mathbf{\Phi}_n = \mathbf{v}$ for some values of the parameters n and $\mathbf{v}$, and to compare the analytical results to numerical evidence

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In what follows, all images were made in Wolfram Mathematica® version 12.3.1

Numerical Simulations

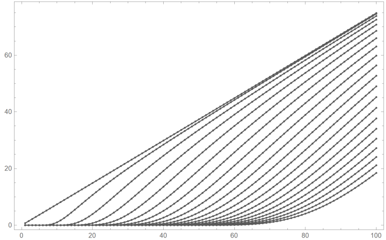
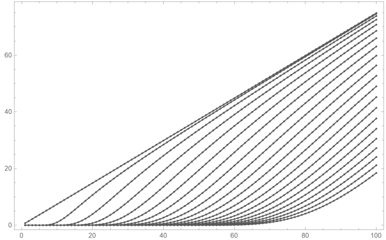


Figure: The effect of selection of different $\mathbf{v}$ level-crossings on $\mathbb{E}\left[N_{\mathbf{v}}^{\Phi_n}(\Theta)\right]$ represented by small dots with $\mathbf{v} = 0 + i0,\ 10 + i10,\ \ldots,\ 200 + i200$ in the order left-to-right.

As $\mathbf{v}$ increases with n, we can visualize the relationship between $\mathbb{E}\left[N_{\mathbf{v}}^{\Phi_n}(\Theta)\right]$ and $\mathbf{v}$ using strings of numerical values, which initially shift to the right rather than moving upward due to the fewer number of roots being counted.

Numerical Simulations



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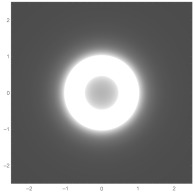
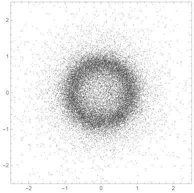
The differences between the numerical values about $\mathbf{v} = 0 + i0$ and those about the other levels seem to decrease to 0 when $\mathbf{v}$ is an increasing function of n. This is in agreement with our corollary on $|\mathbf{v}| \to \infty$.

Zeros of 30,000 different $\boldsymbol{\Phi}_{10}(z)$: Analytical vs Numerical

$\mathbf{v} = 0 + i0$





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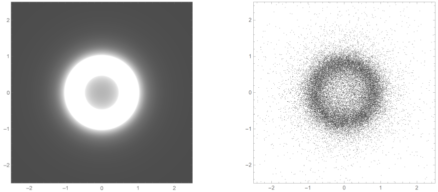
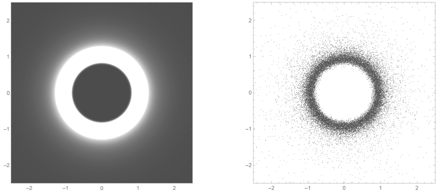


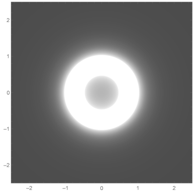
$\mathbf{v} = 0 + i0$

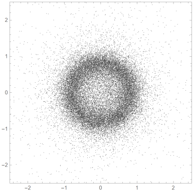


$\mathbf{v} = 5 + i5$

Zeros of 30,000 different $\mathbf{\Phi}_{10}(z)$: Analytical vs Numerical

$\mathbf{v} = 0 + i0$





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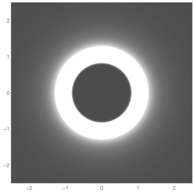
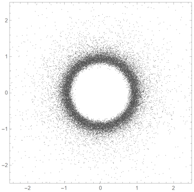
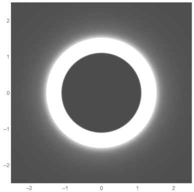
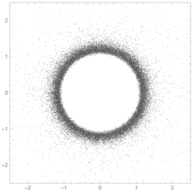




$\mathbf{v} = 40 + i40$





Work in Progress

We are currently working on analogs some of the results given in the presentation when $\{\eta_j\}$ are real-valued standard normal random variables, as well as the variance of the number of level crossings for $\mathbf{\Phi}_n(z)$

Acknowledgments

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2. the organizers of the Geometry of Random Nodal Domains Conference
3. the audience for their attention