Expected nodal volume for non-Gaussian random band-limited functions

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Rennes, September 6, 2021

1. Statement of the problem

General Setup

- (M, g) – Compact smooth n-manifold with no boundary
- $\Delta = div \circ grad$ Laplace-Beltrami on M
- <u>Eigenfunctions:</u>

  $\lambda_j \geq 0 \qquad \Delta\varphi_j + \lambda_j^2 \varphi_j = 0$
- Orthonormal basis of L^2(M,dVol), $\lambda_j \to \infty$
- Weyl law $\left|\{\lambda_j \leq T\}\right| \sim c \cdot T^n$
- Error $o(T^{n-1})$? Further restriction.

Band-limited functions

- $\lambda_j \geq 0 \qquad \Delta\varphi_j + \lambda_j^2 \varphi_j = 0$
- Random band-limited functions:

  b ∈ [0,1] fixed ("band"), $T \gg 0$

$$f_{b;T}(x) = \sum_{b \cdot T \leq \lambda_j \leq T} a_j \cdot \varphi_j(x)$$

- a_j - i.i.d. r.v. (univariate)
- Most interesting (&difficult) $b = 1$
- $f_T(x) = \sum_{T-\rho(T) \leq \lambda_j \leq T} a_j \cdot \varphi_j(x)$

  $\rho(T) = o(T)$ "monochromatic", $\rho \geq 1$ ($\rho \geq c_0$).

Nodal volume

- $f_T(x) = \sum_{T-\rho(T)\leq\lambda_j\leq T} a_j \cdot \varphi_j(x)$
- Central objective: nodal hypersurface volume of f_T
  $\mathcal{V}(f_T) = \mathcal{H}^{n-1}\left(f_T^{-1}(0)\right).$
- Want M general as possible, minimal assumptions on a_j, minimal possible $\rho \geq 1$.
- Most fundamental: expectation

$$\mathbb{E}[\mathcal{V}(f_T)], \text{ as } T \to \infty \quad ?$$

- Obeys Berry's Random Wave model conjecture?

II. Background

Deterministic

- Yau's conjecture: $c_2 > c_1 > 0$

$$c_1 \cdot \lambda_j \leq \mathcal{H}^{n-1}\left(\varphi_j^{-1}(0)\right) \leq c_2 \cdot \lambda_j$$

- Real analytic: Bruning, Bruning-Gromes (lower), Donnelly-Fefferman (upper)
- Upper bound valid combinations (Lin `91, Jerison-Lebeau `01):

$$\mathcal{H}^{n-1}\left(f_T^{-1}(0)\right) \leq c_2 \cdot T$$

- Smooth: Logunov, Logunov-Malinnikova (lower & polynomial upper bound).
- Conjecture (Zelditch): M uniformly hyperbolic

$$\mathcal{H}^{n-1}\left(\varphi_j^{-1}(0)\right) \sim c_0 \cdot \lambda_j$$

Random Gaussian

- Vast & expanding (Kac-Rice)
- $f_T(x) = \sum_{T-\rho(T)\le\lambda_j\le T} a_j \cdot \varphi_j(x)$; a_j i.i.d. $\mathcal{N}(0,1)$
- Zelditch $\rho = 1$ $(\rho \ge 1)$, M real analytic $\mathbb{E}[\mathcal{V}(f_T)] \sim * T$.
- Different treatment depending M aperiodic or periodic (Zoll).
- Canzani-Hanin $Var\left(\frac{\mathcal{V}(f_T)}{T}\right) = O(T^{-\alpha})$ $\alpha = \alpha(n) > 0$.
- Spherical harmonics: Precise expression for the variance (W `09). CLT (Marinucci-Rossi-W `17).
- Toral eigenfunctions ("Arithmetic random waves"): Precise variance (Krishnapur-Kurlberg-W `13). NCLT (Marinucci-Rossi-W-Peccati `16).

Random non-Gaussian

- Angst-Pham-Poly (`18): 2d trigonometric polynomials. Universality of expected nodal length.
- Comparable to band-limited $b = 0$ (separation of variables).
- Seems to work non-Gaussian arithmetic random waves
- Bally-Caramellino-Poly (`17): 1d trigonometric polynomials. Non-universality of variance.
- O. Nguyen-Vu (`20), H. Nguyen-O. Nguyen-Vu (`20), Do, O. Nguyen-Vu: Models of 1d non-Gaussian random polynomials. Variance +CLT.

III. Statement of the main result

Principal result

- Theorem (Z. Kabluchko-A. Sartori-IW `21): Let M be real analytic n-manifold (empty boundary)

$$f_T(x) = \sum_{T-\rho(T)\le\lambda_j\le T} a_j \cdot \varphi_j(x) \quad \rho(T) = o(T)$$

a_j centred i.i.d. $\mathbb{E}\left[a_j^2\right] = 1.\, \mathcal{V}(f_T) = \mathcal{H}^{n-1}\left(f_T^{-1}(0)\right)$

- Assume either: (1) $n = 3, \quad \rho(T) \geq 1$

(2) $n = 2, \quad \rho(T) \geq \frac{T}{\log(T)}$

Then $\mathbb{E}[\mathcal{V}(f_T)] = c_M \cdot T + o_{T\to\infty}(T)$

- $c_M =* \, vol(M)$ **explicit** (consistent Zelditch, RWM)
- Do not assume anything beyond analyticity of M.

Spherical harmonics

- General result does not apply 2d spherical harmonics.
- Let $\{Y_{lm}\}_{l\geq 1, -l\leq m\leq l}$ the Laplace spherical harmonics.
- $\varphi_l(x) = \sum_{m=-l}^{l} a_m \cdot Y_{lm}(x)$ (Can think $\rho(T) = 1$)
- Not invariant w.r.t. choice of basis (unless Gaussian)
- Theorem (Kabluchko-Sartori-W, `21): Let a_m be Bernoulli ± 1 r.v.
$$\mathbb{E}[\mathcal{V}(\varphi_l)] = \sqrt{2}\pi \cdot l + o_{l\to\infty}(l)$$
- Method specialized for Bernoulli r.v., but likely to work in general (with "slight" modifications).

Conjectures variance

- $f_T(x) = \sum_{T-\rho(T)\leq\lambda_j\leq T} a_j \cdot \varphi_j(x)$
- $\mathbb{E}[\mathcal{V}(f_T)] = c_M \cdot T + o_{T\to\infty}(T)$
- Our arguments likely imply $Var\left(\frac{\mathcal{V}(f_T)}{T}\right) = O\left(T^{-\delta}\right)$ some small $\delta > 0$ (perhaps more restrictive a_j).
- Precise asymptotic law **open** (even for Gaussian).
- Conjecture 1: $\varphi_l(x) = \sum_{m=-l}^{l} a_m \cdot Y_{lm}(x)$ spherical harmonics $Var\big(\mathcal{V}(\varphi_l)\big) = c_1 \cdot l + c_2 \cdot \log(l) + O(1)$

  $c_1 \geq 0, c_2$ depend on law of a_j.
- c_1 vanishes for Gaussian, does not vanish identically.
- Reminiscent Bally-Caramellino-Poly 1d (no cancellation)

Conjectures variance (cont.)

- **Conjecture 1:** $\varphi_l(x) = \sum_{m=-l}^{l} a_m \cdot Y_{lm}(x)$ spherical harmonics $Var\big(\mathcal{V}(\varphi_l)\big) = c_1 \cdot l + c_2 \cdot \log(l) + O(1)$

  $c_1 \geq 0, c_2$ depend on law of a_j.
- c_1 vanishes for Gaussian, does not vanish identically.
- **Conjecture 2:** Arithmetic random waves $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ $x = (x_1, x_2) \in \mathbb{T}^2$ $g_n(x) = \sum_{\mu \in \mathbb{Z}^2 : \|\mu\|^2 = n} a_\mu \cdot e(\langle \mu, x \rangle)$

  $$Var\big(\mathcal{V}(g_n)\big) \sim \widetilde{c_1} \cdot \frac{n}{r_2(n)} + \widetilde{c_2} \cdot \frac{n}{r_2(n)^2}$$
- $\widetilde{c_1} \geq 0, \widetilde{c_2}$ also depend on **arithmetics**.

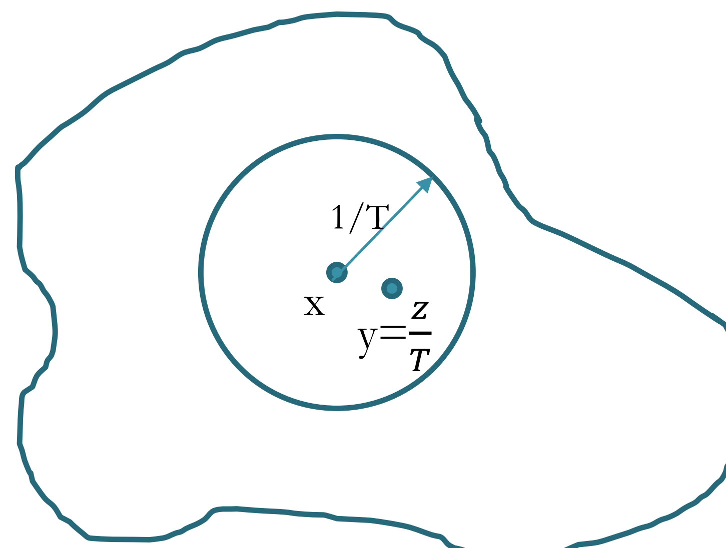
IV. Outline of the proof

Local limit law

- Fix reference point $x \in M$, rescale.
- Assume $T \gg 0$ so $\frac{1}{T} <$ injectivity radius of M.
- Local bijection y $\mapsto exp_x(y)$ y $\in T_x M \cong \mathbb{R}^2$
- $F_x(z) = f_T\left(exp_x\left(\frac{z}{T}\right)\right), \qquad z \in B(1)$
- **Local Weyl law** $(n = 2)$: covariance⇝**Berry's RWM**

  $\mathbb{E}\left[[F_x(z) \cdot F_x(z')]\right] \sim J_0(\|z - z'\|)$
- **Almost** all $x \in M$,

  uniform z, $z' \in B(1)$



Local Weyl law

- Spectral projector $\sum_{T-\rho(T)\le\lambda_j\le T}\varphi_j(y)\cdot\varphi_j(y')$
- Separate: geodesic flow M **periodic** or **aperiodic**.
- Periodic – closed geodesics full measure. **Zoll surface**; eigenvalues cluster. Assume T doesn't break clusters.
- Aperiodic: closed geodesics measure 0.
- Fact: M analytic⇒geodesic flow **aperiodic** or **periodic**
- $x \in M$ **self-focal** if loops on x of measure 0.
- Zelditch: aperiodic ⇒ self-focal points measure 0.
- Work with x not self-focal (aperiodic case).

Limit law for local functions

- $F_x(z) = f_T\left(exp_x\left(\frac{z}{T}\right)\right), \quad z \in B(1)$
- $f_T(\cdot) = \sum_{T-\rho(T)\leq\lambda_j\leq T} a_j \cdot \varphi_j(\cdot)$
- E.g. $F_x(0) = \sum a_j \cdot \varphi_j(x)$ increasing number of summands
- Want: the law of $F_x(\cdot)$ converge to RWM.
- E.g. $F_x(0)$ converges in distribution to $\mathcal{N}(0,1)$?
- Lindeberg condition satisfied if $\left\|\varphi_j(x)\right\|_\infty$ is not too large.
- Problem: $\left\|\varphi_j(x)\right\|_\infty$ might grow beyond allowed.
- Vast literature on growth of $\left\|\varphi_j(x)\right\|_\infty$, moments $\varphi_j(x)$

Expected nodal volume

- Sogge's bound $\|\varphi_j\|_p = O\left(\lambda_j^{\sigma(p)}\right)$ explicit $\sigma(p)$>0
- Deduce $F_x(\cdot) \longrightarrow RWM$ outside "**bad**" subdomain measure $O\left(\frac{\log(T)}{T}\right)$. Convergence in law.
- First finite-dimensional distribution (&derivatives) using multidimentional CLT, then prove tightness.
- Max bound values ~ functions.
- Can deduce here $\mathcal{V}(F_x) \longrightarrow \mathcal{V}(RWM)$ outside bad
- Deduce $\mathbb{E}[\mathcal{V}(F_x)] \longrightarrow \mathbb{E}[\mathcal{V}(RWM)]$ outside bad.
- Tail estimate $\mathcal{P}(\mathcal{V}(F_x) > K)$ ("Anti-concentration I").
- Heavy use analyticity. Crux/bulk of the argument. Modern techniques: complexification, doubling index,… (Logunov).

Anti-concentration II

- Glue partial nodal length to global.
- No control over "bad subdomain" of M
- Domain of measure $O\left(\frac{\log(T)}{T}\right)$ no $F_x \longrightarrow RWM$
- The bound $\mathcal{V}(F_x) = O(1)$ **would** be useful,
- Counter-example: (sectorial) spherical harmonics ("hedgehog").

  Excluded since $\rho \geq \frac{T}{\log(T)}$
- Radius l Around N: nodal length

$\approx (l \;\; lines) \times \left(\frac{1}{l} \; each\right) = 1$

- $\approx l$ after scaling to F_x on $B(1)$
- Nodal length unbounded RKHS

N

$Y_{lm} \; m = l \;\; (l = 10)$

Anti-concentration II (cont.)

- Settle for a weaker (average w.r.t. x) statement.
- Techniques Donnelly-Fefferman $\mathcal{V}(F_x) = O(T)$ still useful
- $O(1)$ (on M) before scaling. Local refinement Yau.
- Bad measure $O\left(\frac{\log(T)}{T}\right)$. Need (logarithmic) improvement.
- All F_x are coming from the same f_T.
- **Result(KSW):** $\mathcal{V}(F_x) = O\left(T \cdot 2^{-\log(T)/\log\log(T)}\right)$ "a.a." x.
- Heavy use analyticity. Crux/bulk of the argument.
- Not easy even for Gaussian on the square. Showed $\mathbb{E}[\mathcal{V}(F_x)] = O(\log(T)^*)$ **expectation. Number theory** (Cammarota-Klurman-W `19). Deterministic (Sartori `20).

Merci Beaucoup!