Random Spherical Harmonics: Overview and Recent Results

Anna Paola Todino

Politecnico di Torino

Conference - Geometry of random nodal domains, Rennes

06 -10 September 2021

Random Spherical Harmonics

Given $\ell \in \mathbb{N}$, $T_\ell : \mathbb{S}^2 \to \mathbb{R}$

$$T_\ell(x) := \sqrt{\frac{4\pi}{2\ell+1}} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(x)$$

where $\{Y_{\ell m}(x), m = -\ell, \ldots, \ell\}$ are spherical harmonics. $\{a_{\ell m}\}$ are Gaussian r.v. i.i.d. with the condition $\bar{a}_{\ell m} = (-1)^m a_{\ell m}$ such that

$$\mathbb{E}[a_{\ell m} \bar{a}_{\ell' m'}] = \delta_\ell^{\ell'} \delta_m^{m'} \quad \ell = 1, 2, \ldots, \ m = -\ell, \ldots, \ell$$

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$\Rightarrow T_\ell$ are real-valued, independent and isotropic centered Gaussian fields and

$$Cov(T_\ell(x), T_\ell(y)) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y_{\ell m}(x) \bar{Y}_{\ell m}(y) = P_\ell(\cos d(x, y))$$

$x, y \in \mathbb{S}^2$, where P_ℓ are the Legendre Polynomials.

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$x, y \in \mathbb{S}^2$, where P_ℓ are the Legendre Polynomials. $T_\ell(x)$ are called the $\ell-$th Gaussian spherical eigenfunctions or random spherical harmonics and they satisfy for $\ell \in \mathbb{N}$

$$\Delta_{\mathbb{S}^2} T_\ell + \ell(\ell+1) T_\ell = 0 \quad \text{a.s.}$$

Excursion Sets

Definition

For $u \in \mathbb{R}$

$$A_u(T_\ell, \mathbb{S}^2) := \left\{ x \in \mathbb{S}^2 : T_\ell(x) \geq u \right\}.$$

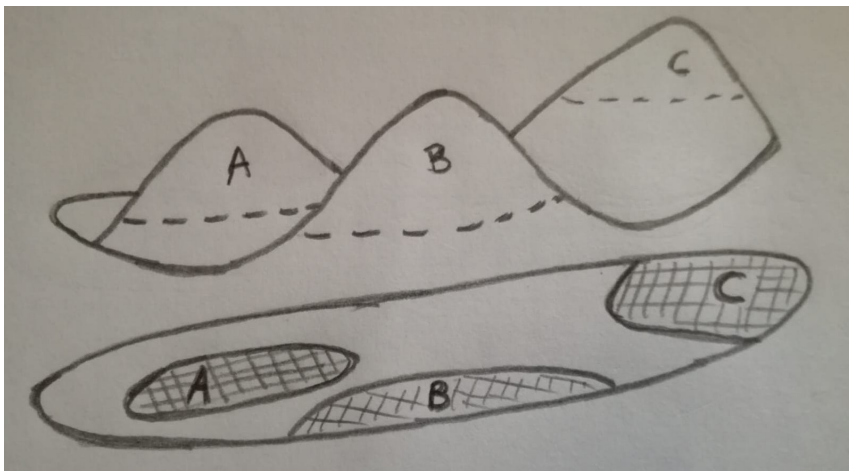
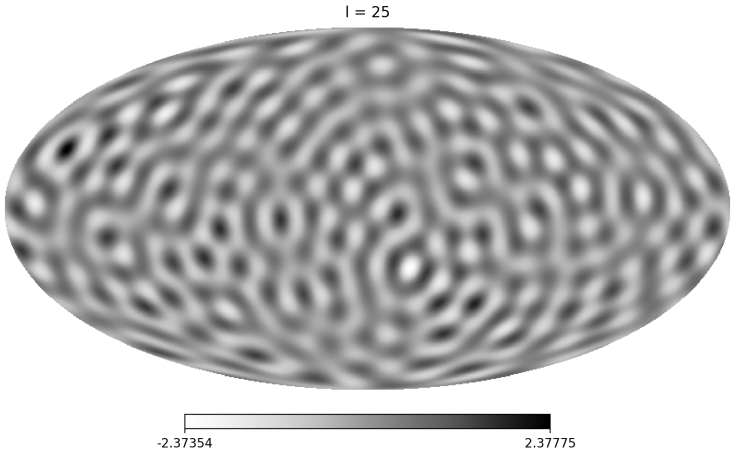
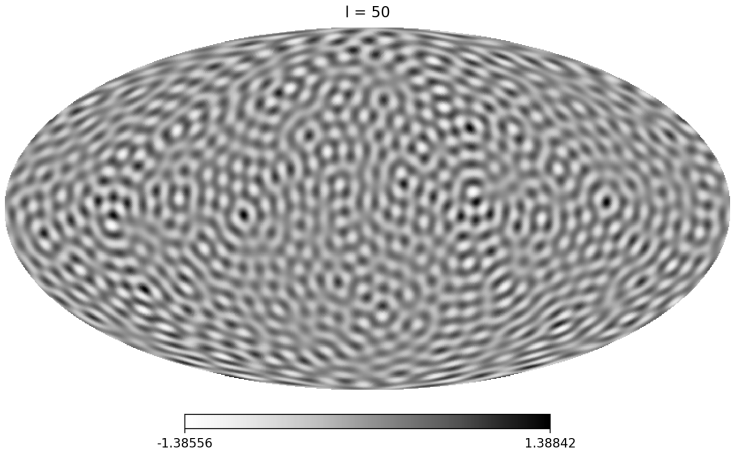


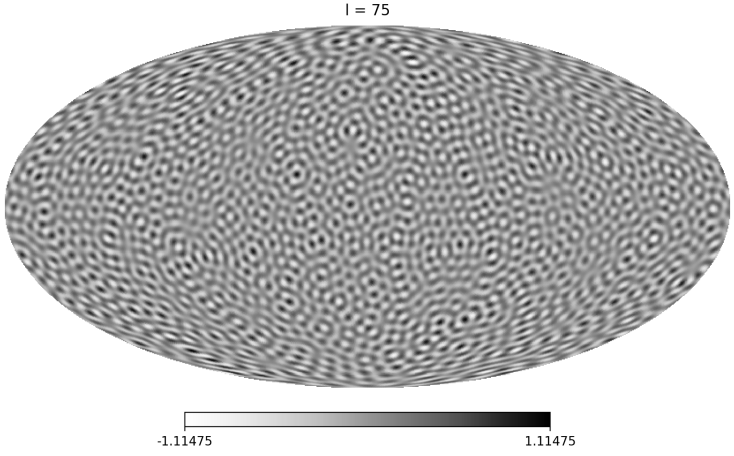
Figure: excursion sets of a smooth function defined over a manifold

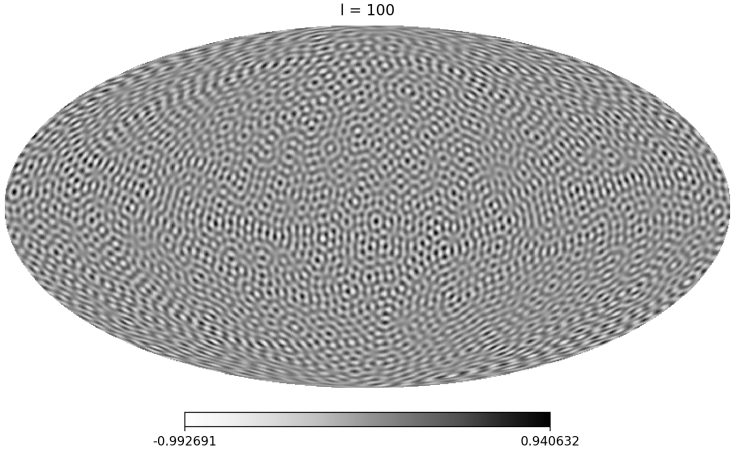
- High-energy behavior of the eigenfunction T_ℓ ($\ell \to \infty$): Geometry of excursion sets $A_u(T_\ell, \mathbb{S}^2)$ for large ℓ

EXCURSION SETS OF T_ℓ









Lipschitz Killing Curvatures in 2-dimensions

In 2-dimensions one has:

- $\mathcal{L}_0(A_u(T_\ell, \mathbb{S}^2))$ is the Euler-Poincaré characteristic of the excursion regions
  $(\mathcal{L}_0(\mathbb{S}^2) = 2)$
- $\mathcal{L}_1(A_u(T_\ell, \mathbb{S}^2))$ is half the boundary length of the excursion regions (that is the length of level curves $T^{-1}(u)$)
  $(\mathcal{L}_1(\mathbb{S}^2) = 0)$
- $\mathcal{L}_2(A_u(T_\ell, \mathbb{S}^2))$ is the area of the excursion regions
  $(\mathcal{L}_2(\mathbb{S}^2) = 4\pi)$

Lipschitz Killing Curvatures in 2-dimensions

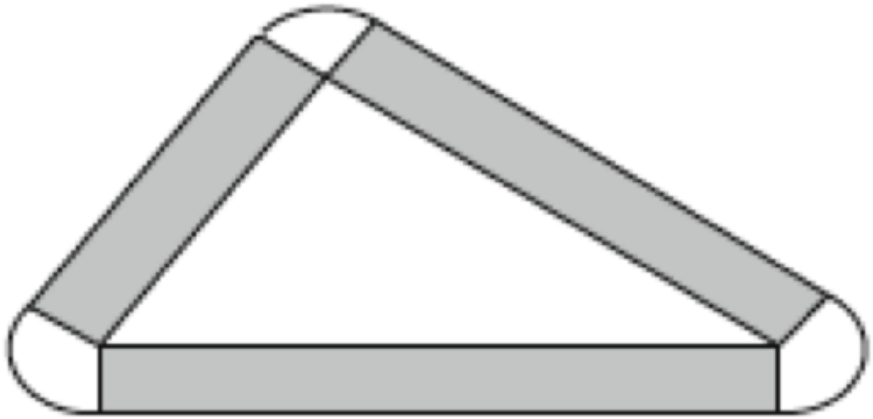


Figure: The tube around a triangle (figure of Adler and Taylor (2007)).

$$\text{area}(\mathit{Tube}(\mathbb{M}, \rho)) = \pi\rho^2\mathcal{L}_0(\mathbb{M}) + 2\rho\mathcal{L}_1(\mathbb{M}) + \mathcal{L}_2(\mathbb{M}).$$

Results in the Literature

GAUSSIAN KINEMATIC FORMULA (GKF)

By Adler and Taylor (2007)

$$\mathbb{E}[\mathcal{L}_i^{T_\ell}(A_u(T_\ell, \mathbb{S}^2))] = \sum_{k=0}^{\dim(\mathbb{S}^2)-i} \binom{i+k}{k} \frac{\omega_{i+k}}{\omega_i \omega_k} \mathcal{L}_{i+k}^{T_\ell}(\mathbb{S}^2) \mathcal{M}_k^2([u, \infty))$$

$\mathcal{M}_k$ are the Gaussian Minkowski functionals. They are defined by

$$\mathcal{M}_j([u, \infty)) = (2\pi)^{-(j+1)/2} H_{j-1}(u) e^{-u^2/2}$$

where H_j denotes the Hermite polynomials.

Hermite Polynomials

Hermite polynomials satisfy

$$H_{-1}(u)e^{-u^2/2} := (2\pi)^{-1/2}(1-\Phi(u))$$

$$H_q(x) = (-1)^q \phi(x)^{-1} \left\{ \frac{d^q}{dx^q}\phi(x) \right\} \quad j = 0, 1, \ldots$$

$\Phi(u)$ and $\phi(u)$ are the standard Gaussian distribution function and the standard Gaussian density.

The first few polynomials are provided by

$$H_1(x) = x, \; H_2(x) = x^2 - 1, \; H_3(x) = x^3 - 3x, \ldots.$$

Comments

- $\mathcal{M}_k([u, \infty))$ depends only on u
- The $\mathcal{L}_k^{T_\ell}(\mathbb{S}^2)$ part depends only on metric properties, and on correlation; if the metric is scaled by λ, $\mathcal{L}_k(\mathbb{S}^2)$ scales by $\lambda^{dim(\mathbb{S}^2)-k}$.
- In the isotropic case λ is the square root of the derivatives of the covariance function at the origin.

$$\lambda = \sqrt{\frac{\ell(\ell+1)}{2}}$$

Results in the Literature

For $u \neq 0$

LKC	Mean	Variance
$\mathcal{L}_0(A_u(T_\ell, \mathbb{S}^2))$	$2\{1-\Phi(u)\} + \lambda^2 \frac{ue^{-u^2/2}}{\sqrt{(2\pi)^3}} 4\pi$	$(H_3(u) + H_2'(u))^2 \phi(u)^2 \times \frac{\lambda^2}{\ell}$
$\mathcal{L}_1(A_u(T_\ell, \mathbb{S}^2))$	$\frac{\pi}{\sqrt{2}} \times \lambda e^{-u^2/2}$	$(H_2(u) + H_1'(u))^2 \phi(u)^2 \times \frac{\lambda}{\ell}$
$\mathcal{L}_2(A_u(T_\ell, \mathbb{S}^2))$	$4\pi \times \{1-\Phi(u)\}$	$(H_1(u) + H_0'(u))^2 \phi(u)^2 \times \frac{\lambda^0}{\ell}$

For $u = 0$

LKC	Mean	Variance
$\mathcal{L}_0(A_0(T_\ell, \mathbb{S}^2))$	1	$?O\left(\ell^2 \log \ell\right)$
$\mathcal{L}_1(A_0(T_\ell, \mathbb{S}^2))$	$\frac{\pi}{\sqrt{2}} \times \lambda$	$\frac{1}{32} \log \ell$
$\mathcal{L}_2(A_0(T_\ell, \mathbb{S}^2))$	2π	$\frac{16\pi^2 0.0188}{\ell^2}$

Results in the Literature

Excursion Sets of Random Spherical Harmonics T_ℓ:

- **Mean** $\rightarrow$ GKF by Adler Taylor
- **Asymptotic Variance & Quantitative CLT** $\rightarrow$ Marinucci-Wigman (CMP2014), Wigman (CMP2010), Marinucci-Rossi (JFA2015), Marinucci-Rossi-Wigman (AIHP(B)2020), Cammarota-Marinucci (AoP2015)

Extension to Subdomains

$B_r \subset \mathbb{S}^2$ the ball of radius r.

LKC	Mean	Variance
$\mathcal{L}_0(A_u(T_\ell, B_r))$	?	?
$\mathcal{L}_1(A_0(T_\ell, B_{r_\ell}))$	$\frac{2\pi(1-\cos r_\ell)}{2} \times \lambda,$	$\frac{1}{256} r_\ell^2 \log r_\ell \ell$
$\mathcal{L}_2(A_u(T_\ell, B_r))$	$(1-\phi(u))2\pi(1-\cos r)$	$\frac{C}{\ell}$

- Excursion Area in Spherical caps $\mathcal{L}_2(A_u(T_\ell, B_r)) \rightarrow$ **T., JMP2019**
- Nodal length in Shrinking domains $\mathcal{L}_1(A_0(T_\ell, B_{r_\ell}))$ $(r_\ell \rightarrow 0, r_\ell \ell \rightarrow \infty$ as $\ell \rightarrow \infty) \rightarrow$ **T., Bernoulli2020**

Main idea: to write these geometric quantities as integral functionals of T_ℓ in $L^2(\mathbb{P})$ and use the Wiener-Ito chaos expansion. The area by

$$\mathcal{L}_2(A_u(T_\ell, \mathbb{S}^2)) := \int_{\mathbb{S}^2} 1_{\{T_\ell(x) \geq u\}} \, dx,$$

the boundary length by

$$\mathcal{L}_1(A_u(T_\ell, \mathbb{S}^2)) := \frac{1}{2} \int_{\mathbb{S}^2} \delta_u(T_\ell(x)) ||\nabla T_\ell(x)|| \, dx$$

and the EPC by

$$\mathcal{L}_0(A_u(T_\ell, \mathbb{S}^2)) = \int_{\mathbb{S}^2} det(\nabla^2(T_\ell(x))) 1_{\{T_\ell(x) \geq u\}} \delta_u(\nabla(T_\ell)) \, dx$$

Nodal Length in Shrinking Domains

Let $B_{r_\ell} \subset \mathbb{S}^2$ be a **(shrinking) spherical cap** of radius

$$r_\ell \to 0,$$

as $\ell \to \infty$, centered in the "North Pole" N and such that $r_\ell \ell \to \infty$.

We study the **nodal length** of T_ℓ restricted to B_{r_ℓ}, i.e.,

$$\mathcal{Z}_{\ell, r_\ell} := len(\{x \in \mathbb{S}^2 \cap B_{r_\ell} : T_\ell(x) = 0\}) = 2\mathcal{L}_1(A_0(T_\ell, B_{r_\ell}))$$

Theorem

For all $r_\ell > 0$

$$\mathbb{E}[\mathcal{Z}_{\ell, r_\ell}] = \frac{2\pi(1 - \cos r_\ell)}{2\sqrt{2}} \sqrt{\ell(\ell+1)}$$

and for $r_\ell \to 0, r_\ell \ell \to \infty$

$$\mathrm{Var}(\mathcal{Z}_{\ell, r_\ell}) = \frac{1}{256} \cdot r_\ell^2 \log(r_\ell \ell) + O(r_\ell^2).$$

Limit Theorems

Remark (LLN)

$$\frac{\mathcal{Z}_{\ell,r_\ell}}{\mathbb{E}[\mathcal{Z}_{\ell,r_\ell}]} \to_{\ell\to+\infty} 1 \text{ in probability.}$$

Theorem (Quantitative CLT, T. 2020)

As $\ell \to \infty$ ($r_\ell \to 0$ and $\ell r_\ell \to \infty$),

$$d_W\left(\frac{\mathcal{Z}_{\ell,r_\ell} - \mathbb{E}[\mathcal{Z}_{\ell,r_\ell}]}{\sqrt{\mathrm{Var}(\mathcal{Z}_{\ell,r_\ell})}}, Z\right) = O((\log r_\ell \ell)^{-1/2}),$$

where $Z \sim \mathcal{N}(0,1)$.

$$d_W(A,B) := \sup_{h\in Lip(1)} |\mathbb{E}[h(A)] - \mathbb{E}[h(B)]|$$

Remind: Nodal length

$$T_\ell^{-1}(0) := \{x \in \mathbb{S}^2 : T_\ell(x) = 0\} \quad \mathcal{Z}_\ell := len(T_\ell^{-1}(0))$$

Theorem (Wigman 2010)

$$\forall \ell \qquad \mathbb{E}[\mathcal{Z}_\ell] = \frac{4\pi}{2\sqrt{2}}\sqrt{\ell(\ell+1)}$$

$$As\ \ell \to \infty \qquad Var(\mathcal{Z}_\ell) = \frac{1}{32}\log \ell(1 + o(1))$$

Remark (LLN)

$$\frac{\mathcal{Z}_\ell}{\mathbb{E}[\mathcal{Z}_\ell]} \to_{\ell\to+\infty} 1 \ a.s.$$

Theorem (Marinucci, Rossi, Wigman 2020)

$$d_W\left(\frac{\mathcal{Z}_\ell - \mathbb{E}[\mathcal{Z}_\ell]}{\sqrt{Var(\mathcal{Z}_\ell)}}, Z\right) = O((\log \ell)^{-1/2}) \qquad Z \sim N(0,1)$$

Idea of the Proof

It can be formally written as

$$\mathcal{Z}_{\ell,r_\ell} = \int_{B_{r_\ell}} \delta_0(T_\ell(x))||\nabla T_\ell(x)||\, dx$$

Wiener-Ito chaos decomposition

$$\mathbb{L}^2(\mathbb{P}) = \oplus_{q=0}^{\infty} C_q, \quad C_q \text{ q-th Wiener-Chaos } \quad C_q \perp C_q'$$

$$\mathcal{Z}_{\ell,r_\ell} = \mathbb{E}[\mathcal{Z}_{\ell,r_\ell}] + \sum_{q=1}^{+\infty} \mathcal{Z}_{\ell,r_\ell}[q] \quad \text{ in } \mathbb{L}^2(\mathbb{P})$$

$$\mathcal{Z}_{\ell,r_\ell}[q] = Proj(\mathcal{Z}_{\ell,r_\ell}|C_q).$$

Wiener-Ito Chaos Expansion

$$\mathcal{Z}_{\ell,r_\ell} - \mathbb{E}[\mathcal{Z}_{\ell,r_\ell}] = \sum_{q=2}^{+\infty} \mathcal{Z}_\ell[2q]$$

$$= \sqrt{\frac{\ell(\ell+1)}{2}} \sum_{q=2}^{\infty} \sum_{u=0}^{q} \sum_{k=0}^{u} \frac{\alpha_{k,u-k}\beta_{q-u}}{k!(u-k)!(q-u)!}$$

$$\times \int_{B_{r_\ell}} H_{q-u}(T_\ell(x)) H_k\left(\frac{\nabla_1 T_\ell(x)}{\sqrt{\ell(\ell+1)/2}}\right) H_{u-k}\left(\frac{\nabla_2 T_\ell(x)}{\sqrt{\ell(\ell+1)/2}}\right) dx$$

$$T_\ell(x), \frac{\nabla_1 T_\ell(x)}{\sqrt{\ell(\ell+1)/2}}, \frac{\nabla_2 T_\ell(x)}{\sqrt{\ell(\ell+1)/2}} \text{ i.i.d. } \sim N(0,1)$$

$$\mathcal{Z}_\ell[0] = \sqrt{\frac{\ell(\ell+1)}{2}} \beta_0 \alpha_{0,0} 4\pi = \frac{4\pi}{2\sqrt{2}} \sqrt{\ell(\ell+1)}, \qquad \forall \ell$$

$$\mathcal{Z}_{\ell,r_\ell} - \mathbb{E}\mathcal{Z}_{\ell,r_\ell} = \mathcal{Z}_{\ell,r_\ell}[4] + \sum_{q=1}^{+\infty} \mathcal{Z}_{\ell,r_\ell}[2q]$$

Fourth Chaotic component

$$
\begin{aligned}
&Proj(\mathcal{Z}_{\ell,r_\ell} | C_4) \\
&= \sqrt{\frac{\ell(\ell+1)}{2}} \Bigg\{ \frac{\alpha_{0,0}\beta_4}{4!} \int_{B_{r_\ell}} H_4(T_\ell(x))\, dx \\
&+ \frac{\alpha_{2,0}\beta_2}{2!2!} \int_{B_{r_\ell}} H_2(T_\ell(x)) H_2\left(\frac{\partial_{1;x} T_\ell(x)}{\sqrt{\ell(\ell+1)/2}} \right) dx \\
&+ \frac{\alpha_{4,0}\beta_4}{4!} \int_{B_{r_\ell}} H_4\left(\frac{\partial_{1;x} T_\ell(x)}{\sqrt{\ell(\ell+1)/2}} \right) dx \\
&+ \frac{\alpha_{2,2}\beta_0}{2!2!} \int_{B_{r_\ell}} H_2\left(\frac{\partial_{1;x} T_\ell(x)}{\sqrt{\ell(\ell+1)/2}} \right) H_2\left(\frac{\partial_{2;x} T_\ell(x)}{\sqrt{\ell(\ell+1)/2}} \right) dx \\
&+ \frac{\alpha_{0,2}\beta_2}{2!2!} \int_{B_{r_\ell}} H_2(T_\ell(x)) H_2\left(\frac{\partial_{2;x} T_\ell(x)}{\sqrt{\ell(\ell+1)/2}} \right) dx \\
&+ \frac{\alpha_{0,4}\beta_0}{4!} \int_{B_{r_\ell}} H_4\left(\frac{\partial_{2;x} T_\ell(x)}{\sqrt{\ell(\ell+1)/2}} \right) dx \Bigg\}.
\end{aligned}
$$

"Local" Sample Trispectrum

- We define first the sequence of centered random variables ("local sample trispectrum"):

$$\mathcal{M}_{\ell,r_\ell} := -\frac{1}{4}\sqrt{\frac{\ell(\ell+1)}{2}}\frac{1}{4!}\int_{B_{r_\ell}} H_4(T_\ell(x))\,dx$$

- Asymptotic full correlation between the "local" nodal length and the "local sample trispectrum":

Proposition

The correlation between $\mathcal{Z}_{\ell,r_\ell}$ and $\mathcal{M}_{\ell,r_\ell}$ is given by

$$Corr(\mathcal{Z}_{\ell,r_\ell};\mathcal{M}_{\ell,r_\ell}) = 1 + O\left(\frac{1}{\log r_\ell \ell}\right) = 1 + o(1),$$

in the high energy limit $\ell \to \infty$.

Fourth Cumulant of the sample Trispectrum

$\{\mathcal{M}_{\ell,r_\ell}\} \subset C_4$
$h_{\ell,r_\ell;4} = \int_{B_{r_\ell}} H_4(T_\ell(x))\,dx$ $\qquad$ $cum_4(h_{\ell,r_\ell;4}) = \mathbb{E}[h^4_{\ell,r_\ell;4}] - 3\mathbb{E}[h^2_{\ell,r_\ell;4}]^2$

Lemma

As $\ell \to \infty$,

$$cum_4(h_{\ell,r_\ell,4}) = O\left(\frac{r_\ell^4}{\ell^4}\log r_\ell \ell\right).$$

$$\tilde{\mathcal{M}}_{\ell,r_\ell} := \frac{\mathcal{M}_{\ell,r_\ell}}{\sqrt{Var(\mathcal{M}_{\ell,r_\ell})}}$$

$$\Rightarrow cum_4(\tilde{\mathcal{M}}_{\ell,r_\ell}) = O\left(\left(\frac{1}{\log r_\ell \ell}\right)\right)$$

and the Fourth Moment Theorem (Nourdin-Peccati 2009) implies

$$d_W(\tilde{\mathcal{M}}_{\ell,r_\ell}, Z) = O\left(\left(\frac{1}{\log r_\ell \ell}\right)^{1/2}\right) \quad Z \sim \mathcal{N}(0,1).$$

Central Limit Theorem

$\tilde{\mathcal{Z}}_\ell := \frac{\mathcal{Z}_\ell - \mathbb{E}[\mathcal{Z}_\ell]}{\sqrt{Var(\mathcal{Z}_\ell)}}$

Proposition

$$\mathbb{E}|\tilde{\mathcal{Z}}_{\ell,r_\ell} - \tilde{\mathcal{M}}_{\ell,r_\ell}|^2 = O\left(\frac{1}{\log r_\ell \ell}\right).$$

$$\Rightarrow d_W(\tilde{\mathcal{Z}}_{\ell,r_\ell}, \mathcal{N}(0,1)) \leq d_W(\tilde{\mathcal{M}}_{\ell,r_\ell}, \mathcal{N}(0,1)) + \sqrt{\mathbb{E}[\tilde{\mathcal{Z}}_{\ell,r_\ell} - \tilde{\mathcal{M}}_{\ell,r_\ell}]^2}$$

$$= O\left(\left(\frac{1}{\log r_\ell \ell}\right)^{1/2}\right)$$

Other LKCs: $u \neq 0$

$$\mathcal{L}_2(A_u(T_\ell, \mathbb{S}^2)) - \mathbb{E}[\mathcal{L}_2(A_u(T_\ell, \mathbb{S}^2))] = C_2 \int_{\mathbb{S}^2} H_2(T_\ell(x))\, dx + R_2(u)$$

$$\mathcal{L}_1(A_u(T_\ell, \mathbb{S}^2)) - \mathbb{E}[\mathcal{L}_1(A_u(T_\ell, \mathbb{S}^2))] = C_1\sqrt{\ell(\ell+1)} \int_{\mathbb{S}^2} H_2(T_\ell(x))\, dx + R_1(u)$$

$$= \mathcal{Z}_\ell(u)[2] + R_1(u)$$

$$\mathcal{L}_0(A_u(T_\ell, \mathbb{S}^2)) - \mathbb{E}[\mathcal{L}_0(A_u(T_\ell, \mathbb{S}^2))] = C_0\ell(\ell+1) \int_{\mathbb{S}^2} H_2(T_\ell(x))\, dx + R_0(u)$$

Remark

Berry's cancellation

$$\mathcal{Z}_\ell[2] = 0$$

Proposition

The second component of the chaos expansion of $\mathcal{Z}_{\ell, r_\ell}$ is, as $\ell \to \infty$,

$$Proj(\mathcal{Z}_{\ell, r_\ell}|2) = O(r_\ell^2)$$

LLN

$$\frac{\mathcal{Z}_\ell}{\sqrt{\ell(\ell+1)}} \to_{\ell\to\infty} \frac{4\pi}{2\sqrt{2}} \text{ a.s.} \quad ; \frac{\mathcal{Z}_{\ell,r_\ell}}{\sqrt{\ell(\ell+1)}} \to_{\ell\to\infty} \frac{2\pi(1-\cos r_\ell)}{2\sqrt{2}} \text{ in prob.}$$

CLT (quantitative) [Marinucci-Rossi-Wigman 2020], [T. 2020]

$$\tilde{\mathcal{Z}}_\ell := \frac{\mathcal{Z}_\ell - \mathbb{E}[\mathcal{Z}_\ell]}{\sqrt{Var(\mathcal{Z}_\ell)}} \to^d_{\ell\to\infty} Z \quad ; \tilde{\mathcal{Z}}_{\ell,r_\ell} = \frac{\mathcal{Z}_{\ell,r_\ell} - \mathbb{E}[\mathcal{Z}_{\ell,r_\ell}]}{\sqrt{Var(\mathcal{Z}_{\ell,r_\ell})}} \to^d_{\ell\to\infty} Z \sim N(0,1)$$

The CLT implies that $\forall z > 0$ (also $z \leq 0$)

$$\mathbb{P}\left(\frac{\mathcal{Z}_\ell - \mathbb{E}[\mathcal{Z}_\ell]}{\sqrt{Var(\mathcal{Z}_\ell)}} \geq z\right) \to \mathbb{P}(Z \geq z).$$

What happens when $z = z_\ell \to \infty$ as $\ell \to \infty$? Let us write $z_\ell = z \cdot a_\ell$ with $z > 0$ and $a_\ell \to \infty$.
MDP answers this question at a logarithmic scale. The idea is to study $\mathbb{P}(\tilde{\mathcal{Z}}_\ell / a_\ell \geq z)$ as $\ell \to \infty$ at a logarithmic scale

Large Deviations Principle

$\{X_n, n \in \mathbb{N}\}$ a sequence of r.v. taking values in $\mathbb{R}$.

Definition

$\{X_n, n \in \mathbb{N}\}$ satisfies the LDP with speed $0 \leq s_n \to \infty$ and good rate function I if the level sets $\{\{x : I(x) \leq \alpha\}, \alpha \geq 0\}$ are compact and for all $B \in \mathbb{B}(\mathbb{R})$ we have

$$- \inf_{x \in int(B)} I(x) \leq \liminf_{n \to \infty} \frac{1}{s_n} \log \mathbb{P}(X_n \in B)$$

$$\limsup_{n \to \infty} \frac{1}{s_n} \log \mathbb{P}(X_n \in B) \leq - \inf_{x \in cl(B)} I(x)$$

Moderate Deviation Principle

Theorem (Macci, Rossi, T. ALEA2021)

Let $\{a_\ell\}_\ell$ *sequence of positive numbers s.t.* $a_\ell \to \infty$ *and* $\frac{a_\ell}{\sqrt{\log\log \ell}} \to 0$ *as* $\ell \to \infty$. *Then for* $B \subseteq \mathbb{R}$ *measurable, as* $\ell \to \infty$

$$\mathbb{P}\left(\tilde{\mathcal{Z}}_\ell / a_\ell \in B\right) \simeq e^{-a_\ell^2 \inf_{x\in B} I(x)}$$

$I(x) = \frac{x^2}{2}$, $x \in \mathbb{R}$.

Theorem (Macci, Rossi, T. ALEA2021)

Let $\{a_\ell\}_\ell$ *sequence of positive numbers s.t.* $a_\ell \to \infty$ *and* $\frac{a_\ell}{\sqrt{\log\log r_\ell \ell}} \to 0$ *as* $\ell \to \infty$. *Then for* $B \subseteq \mathbb{R}$ *measurable, as* $\ell \to \infty$

$$\mathbb{P}\left(\tilde{\mathcal{Z}}_{\ell,r_\ell} / a_\ell \in B\right) \simeq e^{-a_\ell^2 \inf_{x\in B} I(x)}$$

$I(x) = \frac{x^2}{2}$, $x \in \mathbb{R}$.

Definition

$\{X_n, n \in \mathbb{N}\}$ and $\{Y_n, n \in \mathbb{N}\}$ are exponentially equivalent at speed $0 \leq s_n \to \infty$ if for every $\delta > 0$ $\limsup_{n\to\infty} \frac{1}{s_n} \log \mathbb{P}(|X_n - Y_n| > \delta) = -\infty$

Lemma (Dembo and Zeitouni 1998)

Assume that $\{X_n, n \in \mathbb{N}\}$ satisfies the LDP with speed s_n and good rate function I. Then, if $\{X_n, n \in \mathbb{N}\}$ and $\{Y_n, n \in \mathbb{N}\}$ are exponentially equivalent at speed s_n, the same LDP holds for $\{Y_n, \in \mathbb{N}\}$.

Theorem (Schulte-Thäle 2016)

$F_n, n \geq 1$ be a sequence of r.v. belonging to the $q-$th Wiener Chaos, $\lim_{n\to\infty} cum_4(F_n) = 0$.
$L_n := (qq!)^{-1}\sqrt{cum_4(F_n)}$ *and* $\Delta_n = (q^{3q/2} L_n^{\alpha(q)})^{-1}$.
$(a_n : n \geq 1)$ *real sequence s.t.*

$$\lim_{n\to\infty} a_n = \infty \text{ and } \lim_{n\to\infty} a_n/\Delta_n^{1/(q-1)} = 0.$$

$\Rightarrow (a_n^{-1} F_n : n \geq 1)$ *satisfies a MDP with speed a_n^2 and Gaussian rate function $I(z) = z^2/(2q!)$.*

Moderate Deviation Principle

Proposition

a_ℓ sequence of positive numbers s.t. $a_\ell \to \infty$, $\frac{a_\ell}{(\log r_\ell \ell)^{1/14}} \to \infty$ as $\ell \to \infty$. Then for all $B \subset \mathbb{R}$ measurable, as $\ell \to \infty$

$$\mathbb{P}\left(\tilde{\mathcal{M}}_{\ell,r_\ell}/a_\ell \in B\right) \simeq e^{-a_\ell^2 \inf_{x\in B} I(x)}$$

$cum_4(\tilde{\mathcal{M}}_{\ell,r_\ell}) = O\left(\frac{1}{\log r_\ell \ell}\right)$. Let $a_\ell \to \infty$ s.t. $a_\ell/\Delta_\ell^{1/3} \to 0$ where

$$\Delta_{\ell,r_\ell}^{-1} := (\sqrt{cum_4(\tilde{\mathcal{M}}_{\ell,r_\ell})})^{-3/7} \sim (\log r_\ell \ell)^{-3/14}.$$

$a_\ell \to \infty$ $\frac{a_\ell}{(\log \ell)^{1/7}} \to \infty$ as $\ell \to \infty$ $cum_4(\tilde{\mathcal{M}}_\ell) = O\left(\frac{1}{(\log \ell)^2}\right)$

$$\Delta_\ell := (\sqrt{cum_4(\tilde{\mathcal{M}}_\ell)})^{-3/7} \sim (\log \ell)^{3/7}.$$

Exp. Equivalence: $\{\tilde{\mathcal{M}}_{\ell,r_\ell}/a_\ell, \ell \in \mathbb{N}\}$ is exp. equ.to $\{\tilde{\mathcal{Z}}_{\ell,r_\ell}/a_\ell, \ell \in \mathbb{N}\}$ as $a_\ell \to \infty$ $\frac{a_\ell}{\sqrt{\log\log r_\ell \ell}} \to 0$.

$\forall \delta$, as $\ell \to \infty$,

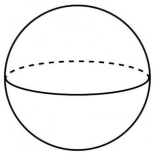
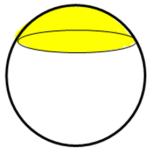
$$\limsup_{\ell\to\infty} \frac{1}{a_\ell^2} \log \mathbb{P}\left(\left| \frac{\tilde{\mathcal{Z}}_{\ell,r_\ell}}{a_\ell} - \frac{\tilde{\mathcal{M}}_{\ell,r_\ell}}{a_\ell} \right| > \delta \right)$$

$$\leq \limsup_{n\to\infty} a_\ell^{-2} \log \frac{a_\ell^{-2}\mathbb{E}[|\tilde{\mathcal{Z}}_{\ell,r_\ell} - \tilde{\mathcal{M}}_{\ell,r_\ell}|^2]}{\delta^2}$$

$$\mathbb{E}|\tilde{\mathcal{Z}}_{\ell,r_\ell} - \tilde{\mathcal{M}}_{\ell,r_\ell}|^2 = O\left(\frac{1}{\log r_\ell \ell}\right)$$

$\Rightarrow$ MDP whenever $a_\ell = o(\sqrt{\log\log r_\ell \ell})$

Comments

nodal length $\mathcal{Z}_\ell, \mathcal{Z}_{\ell,r_\ell}$	$\mathbb{S}^2$ 	B_{r_ℓ} 
Leading term	$\int_{\mathbb{S}^2} H_4(T_\ell(x))\,dx$	$\int_B H_4(T_\ell(x))\,dx$
2° Chaos	$Proj(\mathcal{Z}_\ell \mid C_2) = 0$	$Proj(\mathcal{Z}_{\ell,r_\ell} \mid C_2) \neq 0$
LLN	$\vee$	$\vee$
QCLT	$\vee$	$\vee$
MDP	$\vee$	$\vee$

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