Asymptotic nodal length and log-integrability of toral eigenfunctions

Geometry of Random nodal Domains. Rennes 2021

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September 6, 2021

General set-up

(M, g) compact smooth surface

$\Delta = \Delta_g$ Laplace-Beltrami operator

A sequence of eigenfunctions

$$(\Delta + \lambda_i)(f_{\lambda_i}) = 0$$

where $\{\lambda_i\} \subset \mathbb{R}$ discrete and $\lambda_i \longrightarrow \infty$ as $i \longrightarrow \infty$.

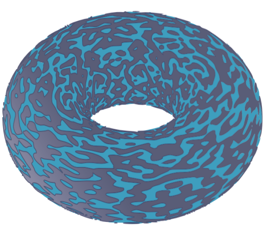

Figure: Picture borrowed from Rennes 2019

Yau's conjecture

The nodal set:

$$\mathscr{Z}(f_\lambda) = \{x \in M : f_\lambda(x) = 0\}.$$

Yau conjectured:

$$c_M\sqrt{\lambda} \leq \mathscr{L}(f_\lambda) = \mathrm{Vol}(\mathscr{Z}(f_\lambda)) \leq C_M\sqrt{\lambda}$$

- Brüning ('78) and Yau lower bound surfaces
- Donnelly-Fefferman ('88) for real analytic metrics
- Logunov-Malinnikova ('18) lower bound for smooth manifolds, polynomial upper bound

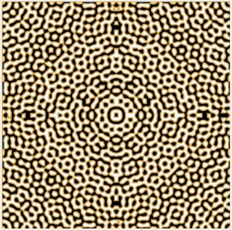

Figure: Toral eigenfunction

The random wave model

Berry ('77): Laplace eigenfunctions (on chaotic surfaces) should behave like Random Waves (RWM)

Berry's Random Waves Gaussian field on $\mathbb{R}^2$, spectral measure Lebesgue measure on $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$.

Equivalently, $\boldsymbol{E}[F_0(x)F_0(y)] = J_0(|x-y|)$

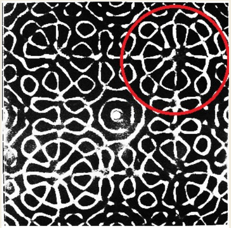

Figure: Planck-scale

RWM and nodal length I

$B \subset \mathbb{R}^2$ box of side 1, $R > 1$ fixed (large) constant

Berry:

$$\mathbb{E}[\mathscr{L}(F_0(R\cdot), B)] = \frac{R}{2\sqrt{2}}$$

Berry's cancellations

$$\mathrm{Var}[\mathscr{L}(F_0(R\cdot), B)] = \frac{\log R}{512\pi}(1 + o_{R\to\infty}(1))$$

-Also observed on the sphere (Wigman) and on the 2d-torus (KKW)

RWM and nodal length II

For "generic" eigenfunctions:

$$\mathscr{L}(f_\lambda) = \frac{\lambda^{1/2}}{2\sqrt{2}}(1 + o_{\lambda\to\infty}(1))$$

We might expect:

$$\mathscr{L}(f_\lambda, B) = \frac{\mathrm{Vol}(B)}{2\sqrt{2}}\lambda^{1/2}(1 + o_{\lambda\to\infty}(1)),$$

for any ball B of radius $r > 0$ larger than the Planck-scale $r\lambda^{1/2} \to \infty$.

Toral eigenfunctions

$f : \mathbb{T}^2 \to \mathbb{R}$ such that $\Delta f + \lambda f = 0$

$$f_\lambda(x) = \sum_{\substack{\xi \in \mathbb{Z}^2 \\ |\xi|^2 = \lambda}} a_\xi e(\langle x, \xi \rangle)$$

In this talk $a_\xi = 1$ (Bourgain's eigenfunctions)
Eigenvalue: $4\pi^2\lambda = 4\pi^2(\square + \square)$

$\{\lambda \leq X : \lambda = \square + \square\} = c\frac{X}{\sqrt{\log X}}(1 + o(1))$

Multiplicity: $N := N(\lambda) \approx \log^{O(1)}(\lambda)$, the number of lattice points in the circle of radius $\sqrt{\lambda}$.

Asymptotic nodal length above Planck-scale

Let $f : \mathbb{T}^2 \to \mathbb{R}$ be a deterministic Bourgain eigenfunction (coefficients all equal to 1)

Theorem (S.)

Let $\varepsilon > 0$, there exists a density one subset of eigenvalues such that

$$\mathscr{L}(f, B) = \frac{\mathrm{Vol}(B)}{2\sqrt{2}} \lambda^{1/2}(1 + o_{\lambda\to\infty}(1)),$$

uniformly for all ball $B \subset \mathbb{T}^2$ of radius $r > \lambda^{-1/2+\varepsilon}$.

Theorem holds (with different constant) for flat function: $|a_\xi|^2 \leq N^\varepsilon$.

Distribution at Planck-scale I

Let f be Bourgain, $R > 1$ write $F_{x,R}(y) = F_x(y) = f\left(x + \frac{Ry}{\lambda^{1/2}}\right)$,
Consider spacial average

$$\mathrm{Var}(\mathscr{L}(F_x)) = \int_{\mathbb{T}^2} \left(\mathscr{L}(F_x) - \frac{R}{2\sqrt{2}}\right)^2 dx$$

Theorem (S.)

There exists a density one subset of eigenvalues such that

$$\mathrm{Var}(F_x) = \frac{\log R}{512\pi}(1 + o_{R\to\infty}(1))(1 + o_{\lambda\to\infty}(1)),$$

where the order of limits is $\lambda \to \infty$ followed by $R \to \infty$.

Distribution at Planck-scale II

Corollary

Let f be Bourgain, there exists a density one subset of eigenvalues such that

$$\lim_{R\to\infty}\lim_{\lambda\to\infty}\mathrm{Vol}\left(\left\{x\in\mathbb{T}^2:\left|\frac{\mathscr{L}(F_x)}{R}-\frac{1}{2\sqrt{2}}\right|>\epsilon\right\}\right)\longrightarrow 0.$$

However, there also exist flat eigenfunctions $\tilde{f}$ so that

$$\mathrm{Var}[\mathscr{L}(\tilde{F}_x)]\gtrsim R^2.$$

Spectrum of toral eigenfunctions and correlations

$V = V(\lambda) := \{\xi \in \mathbb{Z}^2 : |\xi|^2 = \lambda\}$

Bombieri-Bourgain: let $\ell \in \mathbb{Z}_{>0}$, for a density one of λ's there are no non-trivial solutions to

$$\xi_1 + \ldots + \xi_\ell = 0 \qquad \xi_i \in V \quad \Delta(\ell)$$

Notice $\xi \in V$ then $-\xi \in V$

$$\xi_1 = -\xi_2 \quad \xi_3 = -\xi_4 \ldots$$

Cammarota-Klurman-Wigman: for a density one of λ's there are no non-trivial solutions to

$$\xi_1^j + \ldots + \xi_\ell^j = 0 \qquad \xi_i \in V \quad \xi = (\xi^1, \xi^2)$$

Bourgain's de-randomisation I

$\Delta(\ell)$ implies (asymptotic) independence of $\{e(\xi x)\}_{\xi\in V}$ for $x \in \mathbb{T}^2$ chosen uniformly at random. For $p_i \in \mathbb{Z}$

$$\int_{\mathbb{T}^2} \prod_{i=1}^{m} e(p_i \xi_i x) dx = \left| \left\{ (\xi_1, \ldots, \xi_m) : \sum_i p_i \xi_i = 0 \right\} \right| = 0$$

->Gaussianity together with derivatives

$$F_x(y) = f\left(x + \frac{Ry}{\sqrt{\lambda}}\right) = \sum_{\xi} e(\xi x) e(\xi Ry/\sqrt{\lambda})$$

Bourgain's de-randomisation II

Pseudo-spectral measure

$$\int_{\mathbb{T}^2} F_x(y)\overline{F_x(y')} = \sum_{\xi} e(\xi R(y-y')/\sqrt{\lambda}),$$

Erdös-Hall, Katai-Körnei: $V/\lambda^{1/2}$ equidistributes on $\mathbb{S}^1$ for a density one of λ:

$$F_x(y) \xrightarrow{d} F_0 \qquad\qquad C^2([-1/2, 1/2]^2).$$

Bourgain, Buckley-Wigman: asymptotic law for number of nodal domains.

Stability zero set

Stability of the zero set under C^2-perturbations

$$\mathscr{L}(F_x) \xrightarrow{d} \mathscr{L}(F_0)$$

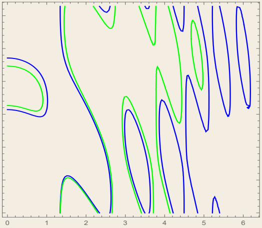

Figure: 100^{-1}-perturbation

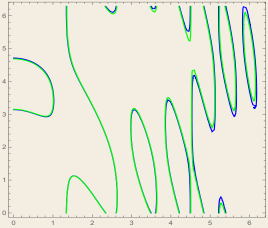

Figure: 1000^{-1}-perturbation

Anti-concentration

$$\mathscr{L}(f_\lambda) = \frac{\lambda^{1/2}}{R} \int_{\mathbb{T}^2} \mathscr{L}(F_x)dx$$

We would like

$$\int_{\mathbb{T}^2} \mathscr{L}(F_x)dx \xrightarrow{?} \mathbb{E}[\mathscr{L}(F_0)] = \frac{R}{2\sqrt{2}},$$

We need $p > 1$,

$$\int_{\mathbb{T}^2} \mathscr{L}(F_x)^p dx \stackrel{?}{=} O_{p,R}(1)$$

Doubling index and log-integrability

Donnelly-Fefferman estimate

$$\mathscr{L}(F_x) \lesssim_R \log \left| \frac{\sup_{2B} |f|}{\sup_B |f|} \right| \leq \log |\sup_{2B} |f|| + |\log |f(x)||,$$

$B = B(x, R/\sqrt{\lambda})$ Since $||f||_{L^2} = 1$, f cannot assume large values too often, thus

$$\int_{\mathbb{T}^2} \mathscr{L}(F_x)^p dx \lesssim_{R,p} \int_{\mathbb{T}^2} |\log |f(x)||^p dx.$$

A theorem of Nazarov

Let $g \in L^2(\mathbb{T})$

$$\mathrm{Spec}(g) = \left\{ n \in \mathbb{Z} : \int g(x) e(nx) dx \neq 0 \right\}.$$

Theorem (Nazarov)

Let $\varepsilon > 0$, suppose that $S \subset \mathbb{Z}$ satisfies $\Delta(\ell)$ for some $\ell > 4$. Then for all $g \in L^2(\mathbb{T})$ with $\mathrm{Spec}(g) \subset S$ and $||g||_{L^2} = 1$, we have

$$\int_{\mathbb{T}} |\log|g(x)||^{\frac{\ell}{4} - \varepsilon} dx \leq C(\ell, \varepsilon)$$

$\Delta(\ell)$: no non-trivial solutions to

$$n_1 + \ldots . + n_\ell = 0, \qquad n_i \in V$$

Concluding the proof of anti-concentration

Recall

$$\int_{\mathbb{T}^2} \mathscr{L}(F_x)^p dx \lesssim_{R,p} \int_{\mathbb{T}^2} |\log|f(x)||^p dx = \int_{\mathbb{T}} \int_{\mathbb{T}} |\log|f(x_1, x_2)||^p dx_1 dx_2$$

Nazarov's Theorem and $\Delta(\ell = 4p + 2)$ condition for $S^i = \{\xi^i : \xi = (\xi^1, \xi^2), |\xi|^2 = \lambda\}, i = 1, 2$

$$\int_{\mathbb{T}^2} |\log|f(x)||^p dx \leq C(p).$$

Proof of Theorems (at macroscopic scale)

Anti-concentration and locality

$$\mathscr{L}(f_\lambda) = \frac{\lambda}{R}\int_{\mathbb{T}^2} \mathscr{L}(F_x)dx \longrightarrow \frac{\lambda}{R}\mathbb{E}[\mathscr{L}(F_o)] = \frac{\lambda}{2\sqrt{2}}$$

$$\mathrm{Var}(\mathscr{L}(F_x)) \longrightarrow \mathrm{Var}(\mathscr{L}(F_0)) = \frac{\log R}{512\pi}(1 + o_{R\to\infty}(1)).$$

Berry:

$$\mathrm{Var}(\mathscr{L}(F_o)) = \frac{\log R}{512\pi}(1 + o_{R\to\infty}(1)).$$

Thank you!